A Brief Introduction to Integrability in $\mathcal{N} = 4$ SYM

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Zur Theorie der Metalle.

I. Eigenwerte und Eigenfunktionen der linearen Atomkette.

Von **H. Bethe** in Rom.

(Eingegangen am 17. Juni 1931.)

Es wird eine Methode angegeben, um die Eigenfunktionen nullter und Eigenwerte erster Näherung (im Sinne des Approximationsverfahrens von London und Heitler) für ein „eindimensionales Metall" zu berechnen, bestehend aus einer linearen Kette von sehr vielen Atomen, von denen jedes außer abgeschlossenen Schalen ein *s*-Elektron mit Spin besitzt. Neben den „Spinwellen" von Bloch treten Eigenfunktionen auf, bei denen die nach einer Richtung weisenden Spins möglichst an dicht benachbarten Atomen zu sitzen suchen; diese dürften für die Theorie des Ferromagnetismus von Wichtigkeit sein.

§ 1. In der Theorie der Metalle hat man sich bis vor einiger Zeit darauf beschränkt, die Bewegung der einzelnen Leitungselektronen im Potentialfeld der Metallatome zu untersuchen (Sommerfeld, Bloch). Von der Wechselwirkung der Elektronen untereinander wurde abgesehen, wenigstens soweit sie nicht summarisch in dem auf die Elektronen wirkenden Potential untergebracht werden kann. Dieses Verfahren war für die Probleme der metallischen Leitfähigkeit (mit Ausnahme der Supraleitung) sehr fruchtbar, ließ aber ein tieferes Eindringen etwa in das Problem des Ferromagnetismus nicht zu[1]) und machte z. B. die Berechnung der Kohäsionskräfte im Metall zu einem ganz hoffnungslosen Unternehmen: Die für die Störungsenergie erster Näherung maßgebenden Austauschkräfte zwischen den Leitungselektronen sind von gleicher Größenordnung wie die Nullpunktsenergie des Elektronengases (Energie nullter Näherung), und dementsprechend kann man abschätzen, daß die zweite Näherung wieder von derselben Größenordnung wird usw. Dieser Umstand stimmt etwas skeptisch gegen die ganze Approximation, bei der die Bewegung des einzelnen Elektrons (kinetische Nullpunktsenergie) als überragend wichtig gegenüber der Wechselwirkung (Austauschenergie) angesehen wird.

Deshalb ist in letzter Zeit von Slater[2]) und Bloch[3]) versucht worden, das Problem von der anderen Seite her zu approximieren, d. h. die Atome als fest vorgebildet anzunehmen und ihre Wechselwirkung als Störung zu betrachten, wie es der London-Heitlerschen Approximation für Moleküle

[1]) F. Bloch, ZS. f. Phys. **57**, 545, 1929 zeigte, daß unter gewissen Voraussetzungen auch „freie Elektronen" Ferromagnetismus zeigen können.
[2]) J. C. Slater, Phys. Rev. **35**, 509, 1930.
[3]) F. Bloch, ZS. f. Phys. **61**, 206, 1930 (im folgenden mit l. c. zitiert).

SU(2) Heisenberg Spin Chain

Hamiltonian

$$\mathbf{H} = \sum_{l=1}^{L} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}) \qquad \text{with} \qquad \vec{\sigma}_{L+1} := \vec{\sigma}_1 \,.$$

Acts on

$$\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L-\text{ times}}$$

as "one minus permutation"

$$\mathbf{H} = 2 \sum_{l=1}^{L} (\mathbb{I}_{l,l+1} - \mathbb{P}_{l,l+1}) \,.$$

Spectral problem:

$$\mathbf{H} \cdot |\psi\rangle = E \, |\psi\rangle \,.$$

Bethe's Ansatz of 1931

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}\,,$$

— — — — — — — — — — — — — — —

$$E = \sum_{j=1}^{K} \frac{2}{u_j^2 + \frac{1}{4}}\,.$$

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SUPERSYMMETRIC YANG-MILLS THEORIES *

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Yang-Mills theories with simple supersymmetry are constructed in 2, 4, 6, and 10 dimensions, and it is argued that these are essentially the only cases possible. The method of dimensional reduction is then applied to obtain various Yang-Mills theories with extended supersymmetry in two and four dimensions. It is found that all possible four-dimensional Yang-Mills theories with extended supersymmetry are obtained in this way.

1. Introduction

Three types of supersymmetric Yang-Mills theories in four dimensions are known. In the first one that was found [1] the infinitesimal parameter of the supersymmetry transformation is a Majorana spinor ("simple" supersymmetry). In the second one [2] it is a Dirac spinor ("complex" supersymmetry). In the third case it consists of four Majorana (or Weyl) spinors [3]. This last model was obtained recently by applying the method of dimensional reduction to a supersymmetric Yang-Mills theory in ten-dimensional space-time.

The goal of this paper is to classify all the possible supersymmetric Yang-Mills theories in both two and four dimensions. The interest in four dimensions is obvious, of course, as one of these schemes may be part of a correct theory. The two-dimensional cases are also emphasized because of the possibility of coupling such Yang-Mills multiplets to a corresponding two-dimensional supergravity theory [4] in order to get a modified string theory. Our technique consists of two stages. In the first stage Yang-Mills theories with simple supersymmetry are constructed for all space-time dimensions in which it is possible. Then in the second stage each of the higher-

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** On leave of absence from Institute of Theoretical Physics, Göteborg, Sweden.

$\mathsf{PSU}(2,2|4)$ Super-Yang-Mills Theory

$\mathcal{N}=4$ SYM, unique action: [Brink, Schwarz, Scherk '76; Gliozzi, Scherk, Olive '77]

$$S=\frac{N}{\lambda}\int d^4x\,2\,\mathrm{Tr}\Big(\tfrac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+\tfrac{1}{2}\mathcal{D}^\mu\Phi^m\,\mathcal{D}_\mu\Phi_m-\tfrac{1}{4}[\Phi^m,\Phi^n][\Phi_m,\Phi_n]$$

$$+\dot\Psi^a_{\dot\alpha}\sigma_\mu^{\dot\alpha\beta}\mathcal{D}^\mu\Psi_{\beta a}-\tfrac{i}{2}\,\Psi_{\alpha a}\sigma_m^{ab}\epsilon^{\alpha\beta}[\Phi^m,\Psi_{\beta b}]-\tfrac{i}{2}\,\dot\Psi^a_{\dot\alpha}\sigma^m_{ab}\epsilon^{\dot\alpha\dot\beta}[\Phi_m,\dot\Psi^b_{\dot\beta}]\Big)$$

$$+\frac{\theta_{\mathrm{YM}}}{16\,\pi^2}\int d^4x\,2\,\mathrm{Tr}\,\mathcal{F}^{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu}$$

Free parameters: $\lambda=Ng^2_{\mathrm{YM}}$ and N and theta angle θ_{YM}.

Non-trivial CFT_4. [Sohnius, West '81; Brink, Lindgren, Nilsson '83; Mandelstam '83; Howe, Stelle, Townsend '84]

$\mathfrak{su}(N)$ gauge symmetry, Olive-Montonen symmetry, and $\mathsf{PSU}(2,2|4)$.

Hidden string description (AdS/CFT). [Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98]

AdS/CFT Bethe Equations of 2006

[Beisert, Staudacher '05,'06]

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}} \,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}} \,,$$

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}} \,,$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}} \,,$$

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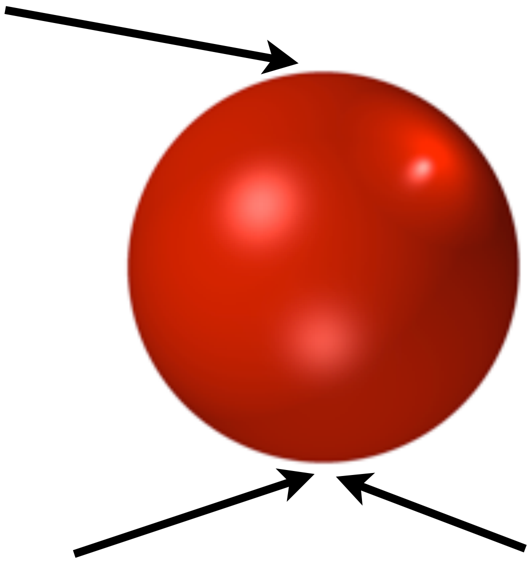
$$E(g) = 2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right) = \frac{1}{g^2}\sum_{j=1}^{K_4}\left(\sqrt{1+16\,g^2\,\sin\frac{p_j}{2}}-1\right), \qquad \Delta = \Delta_0 + g^2\,E(g)\,, \qquad K_4 = K$$

$$1 = \prod_{j=1}^{K_4}\left(\frac{x^+_{4,j}}{x^-_{4,j}}\right) = \prod_{j=1}^{K_4} e^{ip_j}\,, \qquad u_k = x_k + \frac{g^2}{x_k}\,, \qquad u_k \pm \tfrac{i}{2} = x^\pm_k + \frac{g^2}{x^\pm_k}\,.$$

Yang-Mills Theory on the Riemann Sphere

Yang-Mills Theory depends on a coupling constant λ. Let us assume that λ takes on complex values, and that the theory thus lives on $\mathbb{C} \cup \infty$:

Strong Coupling $\lambda = \infty$



Weak Coupling $\lambda = 0$

Radius of convergence zero!

Planar Yang-Mills Theory and Gauge-String Duality

Two important ideas for making progress:

- The planar Limit (also known as ‘t Hooft’s large N Limit).
- Gauge-string duality, and in particular the AdS/CFT correspondence.

These ideas are closely related, but should be distinguished.

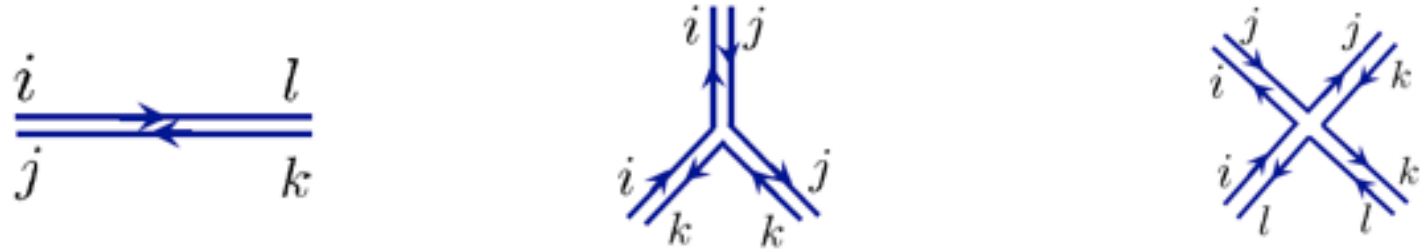
Planar Limit of Yang-Mills Theory

Once more the action of $\mathcal{N}=4$ theory with $\mathfrak{su}(N)$ gauge symmetry:

$$S=\frac{N}{\lambda}\int\frac{d^4x}{4\pi^2}\,\mathrm{Tr}\Bigg(\tfrac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+\tfrac{1}{2}\mathcal{D}^\mu\Phi^m\mathcal{D}_\mu\Phi_m-\tfrac{1}{4}[\Phi^m,\Phi^n][\Phi_m,\Phi_n]$$

$$+\dot{\Psi}^a_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu\mathcal{D}^\mu\Psi_{\beta a}-\tfrac{i}{2}\Psi_{\alpha a}\sigma^{ab}_m\epsilon^{\alpha\beta}[\Phi^m,\Psi_{\beta b}]-\tfrac{i}{2}\dot{\Psi}^a_{\dot{\alpha}}\sigma^m_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}[\Phi_m,\dot{\Psi}^b_{\dot{\beta}}]\Bigg)$$

'T Hooft showed, that in the large N limit only planar Feynman graphs survive (related to the mathematical notion of ,,fat graphs"):



Turns the asymptotic series of perturbation theory into convergent series.

The AdS/CFT Correspondence

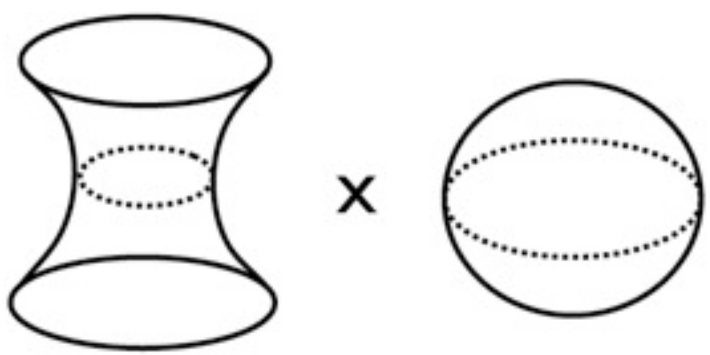
[Maldacena '97]

$\mathcal{N} = 4$ $SU(N)$ Yang-Mills Theory

't Hooft coupling: $\lambda = N g_{\mathrm{YM}}^2$ 1/number-of-colors: $\frac{1}{N}$

is dual to:

IIB superstrings on $AdS_5 \times S^5$



string tension: $\frac{R^2}{\alpha'} = \sqrt{\lambda}$ string coupling: $g_s = \frac{\lambda}{4\pi N}$

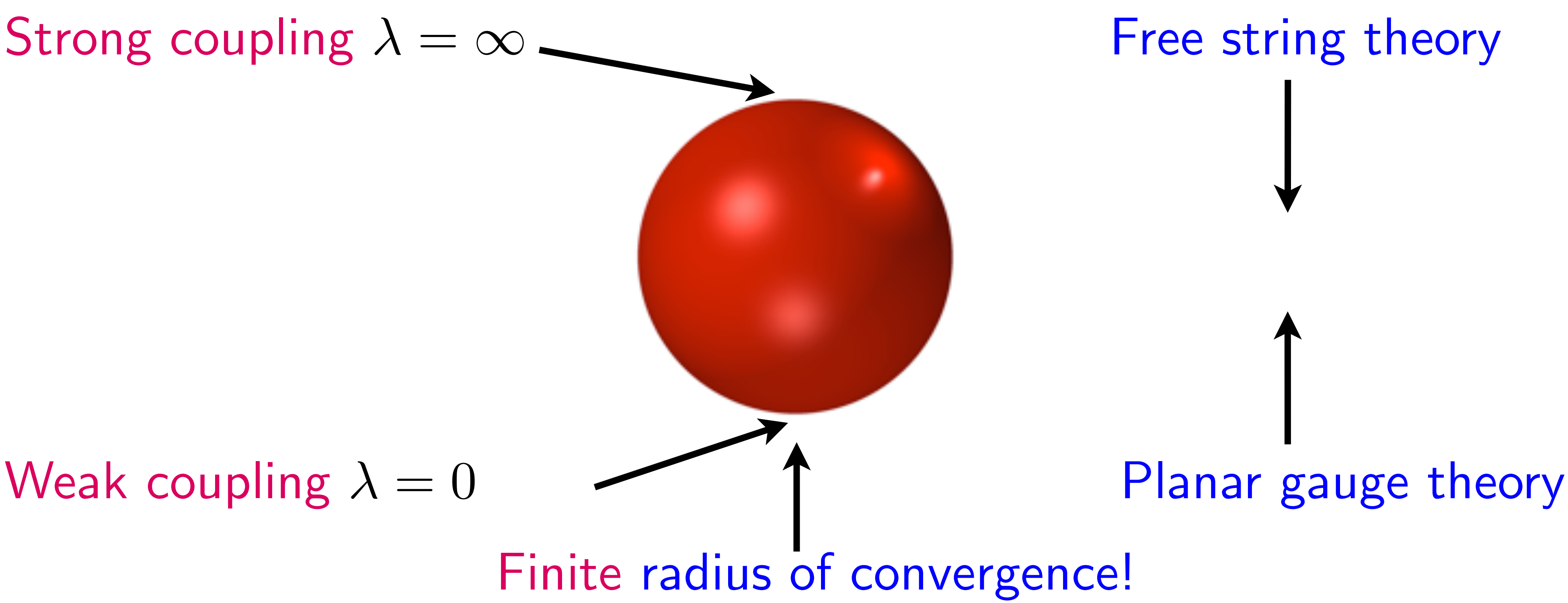
$AdS_5 \times S^5$

Five-dim. Anti-De-Sitter Space

Five-dimensional Sphere

Gauge-String Duality on the Riemann Sphere

Yang-Mills Theory depends on a coupling constant λ. Let us assume that λ takes on complex values, and that the theory thus lives on $\mathbb{C} \cup \infty$:

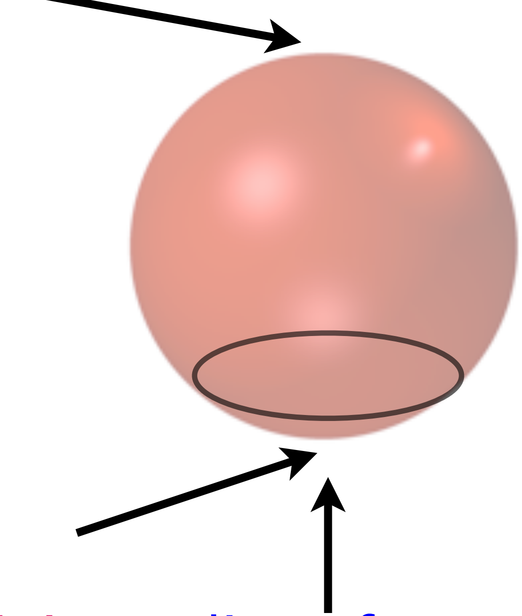


Gauge-String Duality on the Riemann Sphere

Yang-Mills Theory depends on a coupling constant λ. Let us assume that λ takes on complex values, and that the theory thus lives on $\mathbb{C} \cup \infty$:

Strong coupling $\lambda = \infty$

Free string theory



Weak coupling $\lambda = 0$

Planar gauge theory

Finite radius of convergence!

Exact Solvability of $\mathcal{N} = 4$ Yang-Mills Theory

Mathematically arguably the “most beautiful Yang-Mills theory”!
There are striking hints, that it is exactly solvable.

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PSU$(2, 2|4)$ Integrability

Conjecture: Planar theory enjoys Yangian symmetry at finite coupling.

The Asymptotic All-Loop AdS/CFT Bethe Equations

[Beisert, Staudacher '05,'06]

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}\,,$$

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}\,,$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}\,,$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\,,$$

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$$E(g) = 2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right) = \frac{1}{g^2}\sum_{j=1}^{K_4}\left(\sqrt{1+16\,g^2\,\sin\frac{p_j}{2}}-1\right), \qquad \Delta = \Delta_0 + g^2\,E(g)\,, \qquad K_4 = K\,.$$

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The Spectral Problem of $\mathcal{N} = 4$ SYM

Conformal invariance restricts the structure of two-point functions:

$$\langle \mathcal{O}_n(x)\,\mathcal{O}_m(0)\rangle = \frac{\delta_{nm}}{x^{2\Delta_n}}\,.$$

Δ_n is the anomalous scaling dimension of the composite operator $\mathcal{O}_n$.
This leads to the mixing problem of $\mathcal{N} = 4$:

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi_\alpha^A(\mathcal{D}_\mu\mathcal{Z})\ldots\right)\mathrm{Tr}\,(\ldots\ldots)\ \ldots + \ldots$$

The partons carry additive, protected Lorentz and R-symmetry charges S_1, S_2, J_1, J_2, J_3. Here Δ_n is related to the dilatation generator $\mathfrak{D}$:

$$\left[\,\mathfrak{D}, \mathcal{O}_n(0)\,\right] = i\,\Delta_n\,\mathcal{O}_n(0)\,.$$

$\Delta_n(\lambda)$ is not protected, it generically depends on the 't Hooft coupling λ.

Mixing Problem in $\mathcal{N} = 4$ SYM and Spin Chains

Consider twist operators:

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{D}^{S_1} \mathcal{Z}^{J_3}\right) + \ldots .$$

$\mathcal{D} = \mathcal{D}_1 + i\,\mathcal{D}_2$ mit $\mathcal{D}_\mu = \partial_\mu + i\,A_\mu$ is a covariant lightcone derivative.

The dilatation operator is regarded as the Hamiltonian of a spin chain.

The spin chain is

$$\mathrm{Tr}\left((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\ldots(\mathcal{D}^{s_{J_3-1}}\mathcal{Z})(\mathcal{D}^{s_{J_3}}\mathcal{Z})\right),$$

where $S_1 = s_1 + s_2 + \ldots + s_{J_3-1} + s_{J_3} := M =$ Magnon number.

The Interpolating Scaling Function

The scaling dimension of operators of low twist J_3 behaves in a very interesting logarithmic way at large spin $S_1 \to \infty$:

$$\Delta - S_1 - J_3 \; = \; f(g) \; \log S_1 \; + \; O(S_1^0) \, .$$

$f(g)$ is the universal scaling function, where $g^2 = \lambda / 16 \, \pi^2$.

Also appears in the structure of MHV-amplitudes und in lightcone segmented Wilson loops $\mathcal{W}$! Gluon 4-point function in $4 - 2 \, \epsilon$ dimensions:

[Bern, Dixon, Smirnov]

$$\mathcal{M}_4^{\mathrm{All-Loop}} \simeq \exp \left[f(g) \, \mathcal{M}_4^{\mathrm{One-Loop}} \right] \, , \qquad \mathcal{M}_4^{\mathrm{All-Loop}} \simeq \langle \mathcal{W} \rangle \, .$$

The Interpolating Integral Equation

The non-linear asymptotic Bethe equations reduce in the limit $S_1 \to \infty$, where $L \to \infty$ with $L << \log S_1$, to a linear integral equation for the density $\hat{\sigma}$ of Bethe roots. These describe the one-dimensional "motion" of the covariant derivatives of the twist operators: [Beisert, Eden, MS '06]

$$\hat{\sigma}(t) = \frac{t}{e^{\,t} - 1} \left[\hat{K}(2\,g\,t, 0) - 4\,g^2 \int_0^\infty dt' \, \hat{K}(2\,g\,t, 2\,g\,t') \, \hat{\sigma}(t') \right].$$

The universal scaling function $f(g)$ is then given by

$$f(g) = 16\,g^2\,\hat{\sigma}(0)\,.$$

The kernel $\hat{K}$ is of a rather involved structure, it will not be written here.

Gauge Theory Meets String Theory

Integrability allows to derive an exact equation for the so-called anomalous cusp dimension of $\mathcal{N} = 4$ theory. It is valid, because of the mentioned analyticity at small $g = 4\pi\sqrt{\lambda}$, for arbitrary values of g. [Beisert, Eden, MS '06]

The equation was tested in gauge theory at weak coupling in perturbation theory to four loop order: [Bern, Czakon, Dixon, Kosower, Smirnov, '06]

$$f(g) = 8\,g^2 - \tfrac{8}{3}\,\pi^2 g^4 + \tfrac{88}{45}\,\pi^4 g^6 - 16\left(\tfrac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \pm \ldots\,.$$

At strong coupling the scaling function $f(g)$ agrees to the three known orders of perturbation theory: [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin '02],

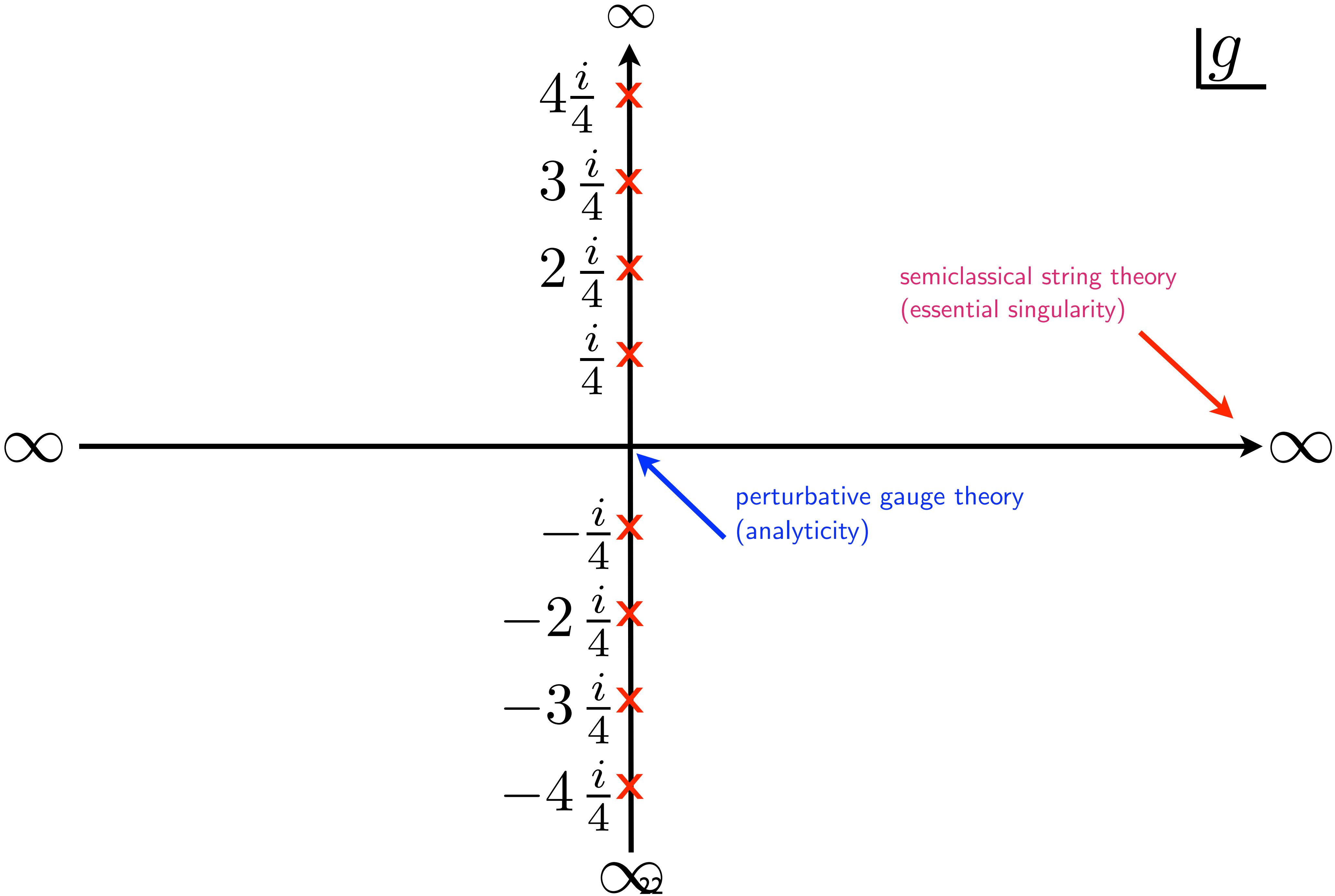
[Roiban, Tirziu, Tseytlin '07; Roiban, Tseytlin '07] [Basso, Korchemsky, Kotański '07]

$$f(g) = 4\,g - \frac{3\log 2}{\pi} - \frac{\mathrm{K}}{4\pi^2}\,\frac{1}{g} - \ldots$$

Conclusion: The AdS/CFT duality appears to be exactly true!

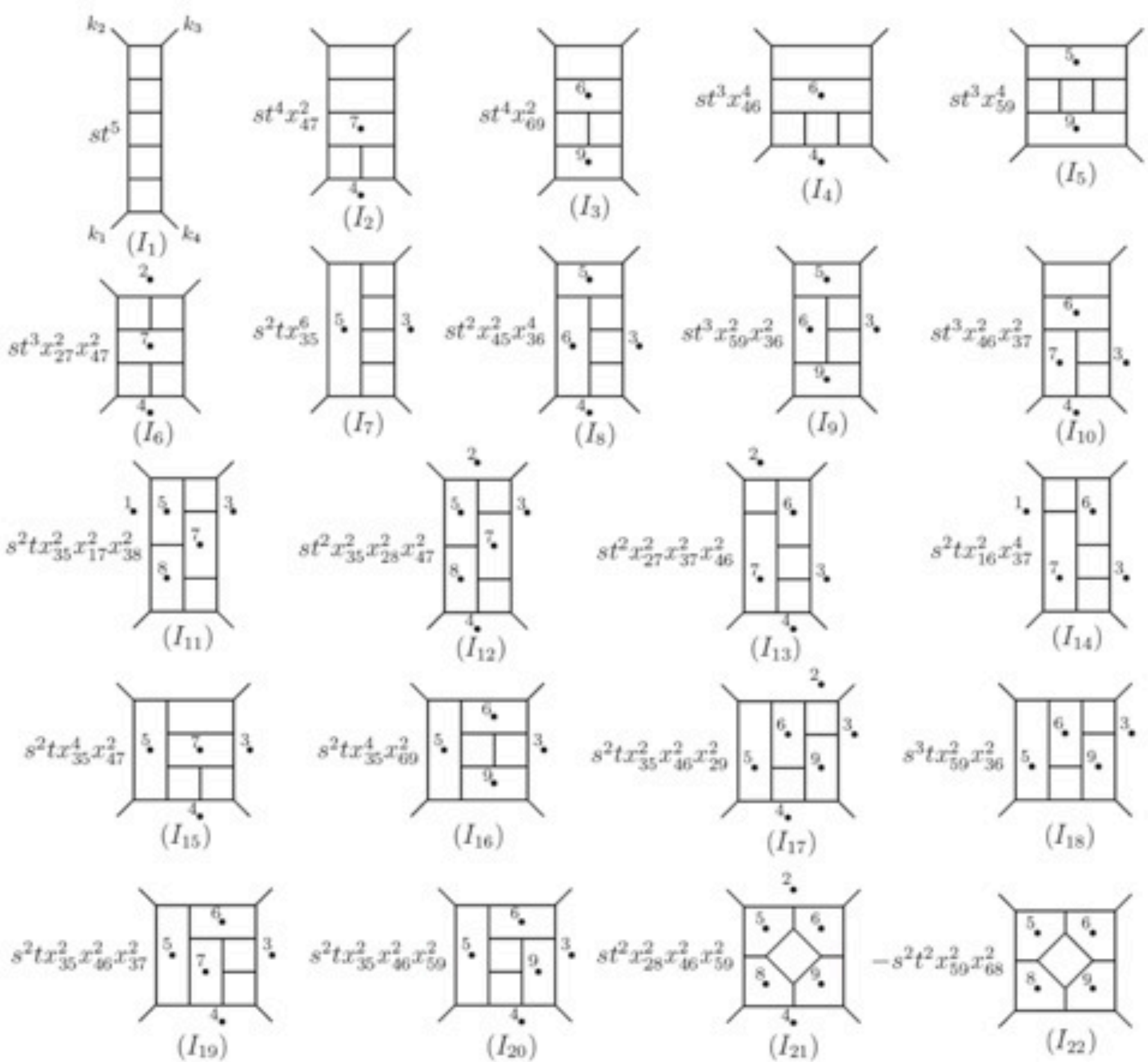
Hint: The scaling function also appears in scattering amplitudes!

Analytic Structure in the Coupling Constant Plane



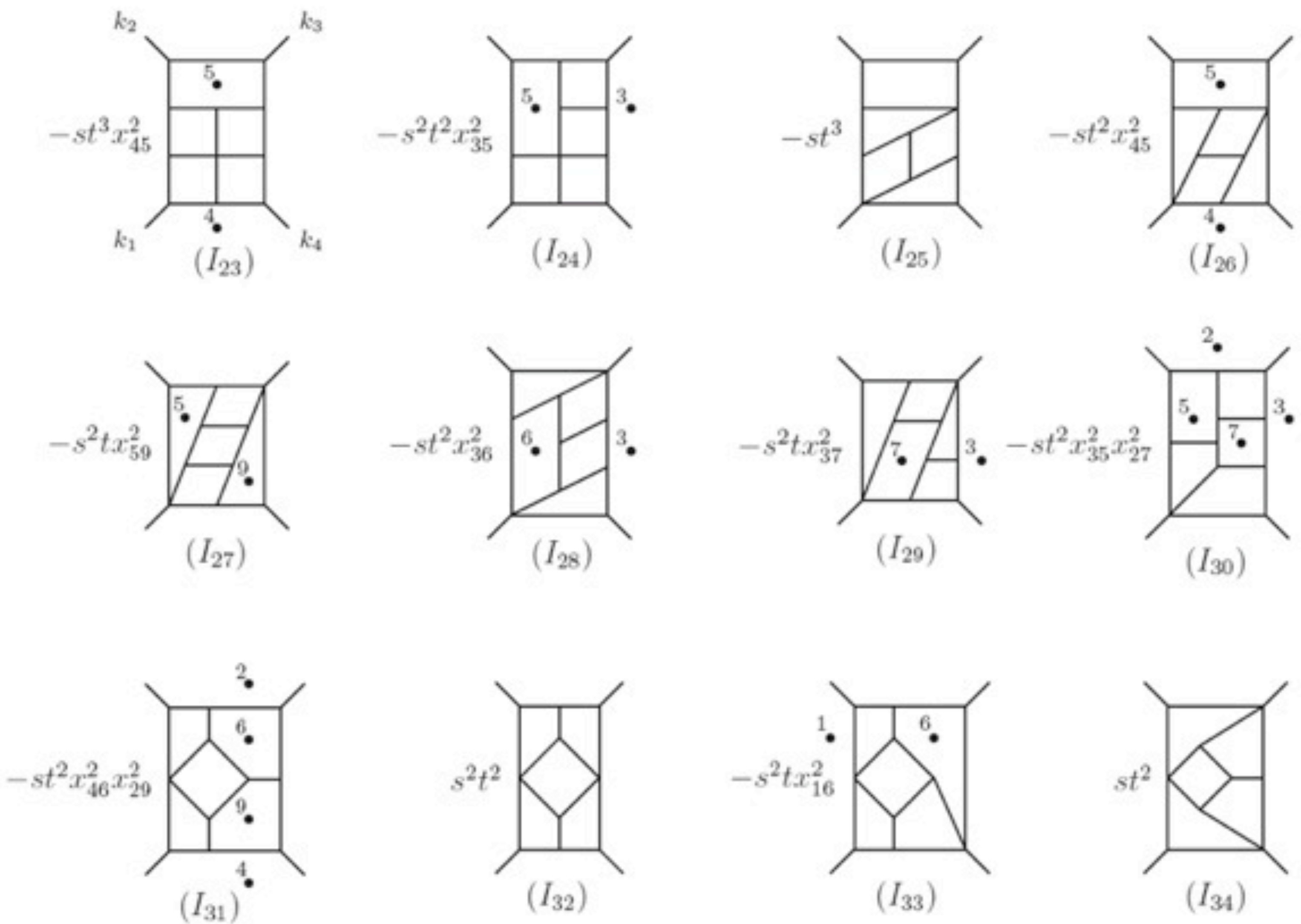
Challenge I: Compute 5-Loop diagrams ...

[Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower ‘07]



... and ...

[Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower ‘07]



… and compare to the 5-Loop Prediction

[Beisert, Eden, MS '06]

$$
\begin{aligned}
f(g) = \quad & 8\,g^2 - \tfrac{8}{3}\,\pi^2 g^4 + \tfrac{88}{45}\,\pi^4 g^6 - 16\left(\tfrac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \\
& +32\left(\tfrac{887}{14175}\,\pi^8 + \tfrac{4}{3}\,\pi^2 \zeta(3)^2 + 40\,\zeta(3)\,\zeta(5)\right) g^{10} \mp \ldots\,.
\end{aligned}
$$

Challenge II: Compute 3-Loop String Corrections ...

... and compare to the 3-loop prediction

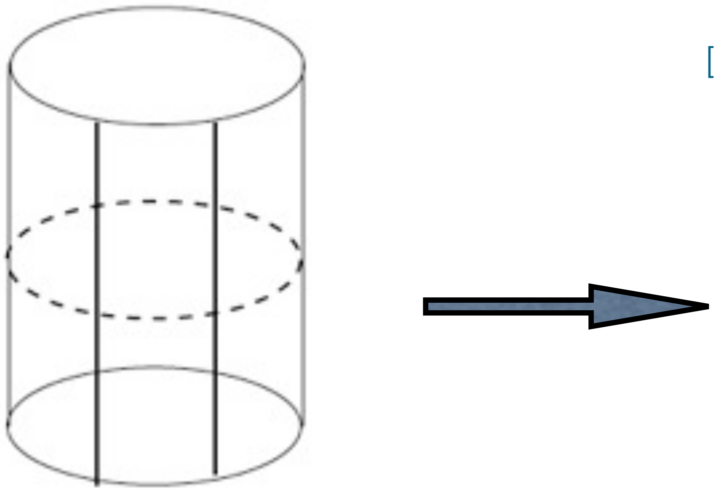
[Beisert, Eden, MS '06, Basso, Korchemsky, Kotański '07]

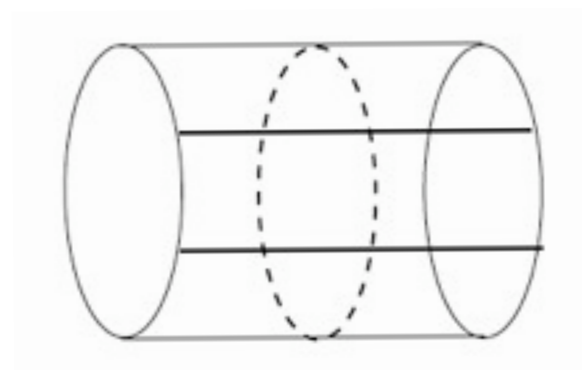
$$\boxed{f\Big(g + \frac{3\log 2}{4\,\pi}\Big) = 4\,g\; - \frac{\mathrm{K}}{4\pi^2}\,\frac{1}{g} - \frac{27\,\zeta(3)}{2^9\,\pi^3}\,\frac{1}{g^2} - \ldots}$$

Lüscher Corrections and Thermodynamic Bethe Ansatz

[Lüscher '86; A.B. Zamolodchikov '90]

[Ambjørn, Janik, Kristjansen '06; Arutyunov, Frolov, '07, '08; Bajnok, Janik '08]





In the TBA, one "turns around" the world sheet cylinder of the string σ-model, and considers scattering in the cross channel.

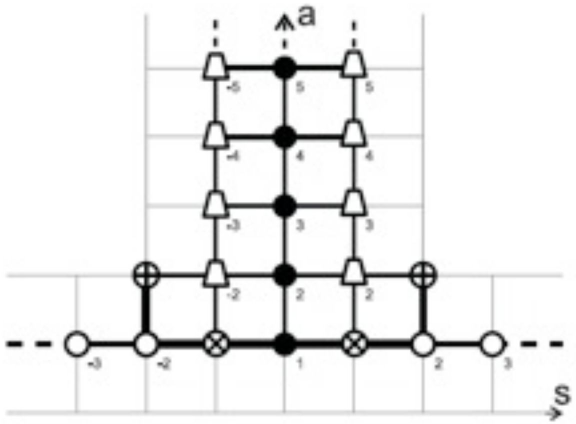
The virtual field-theoretic corrections solve the wrapping problem.

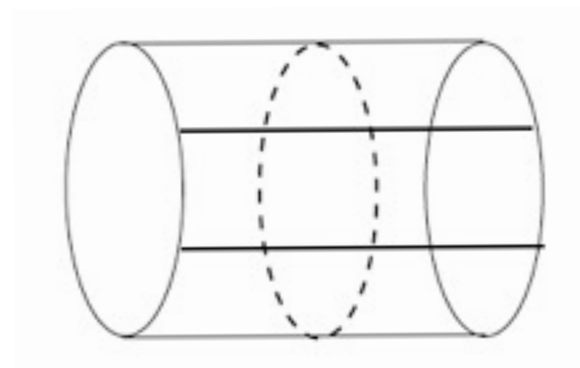
All-loop TBA equations, Y-System

Ground State TBA: [Bombardelli, Fioravanti, Tateo '09, Arutyunov, Frolov '09, Gromov, Kazakov, Kozak, Vieira '09]

Claim: Exact spectrum of planar $\mathcal{N} = 4$. [Gromov, Kazakov, Kozak, Vieira '09]

Infinite system of integral equations for "Y-functions" living on a lattice:

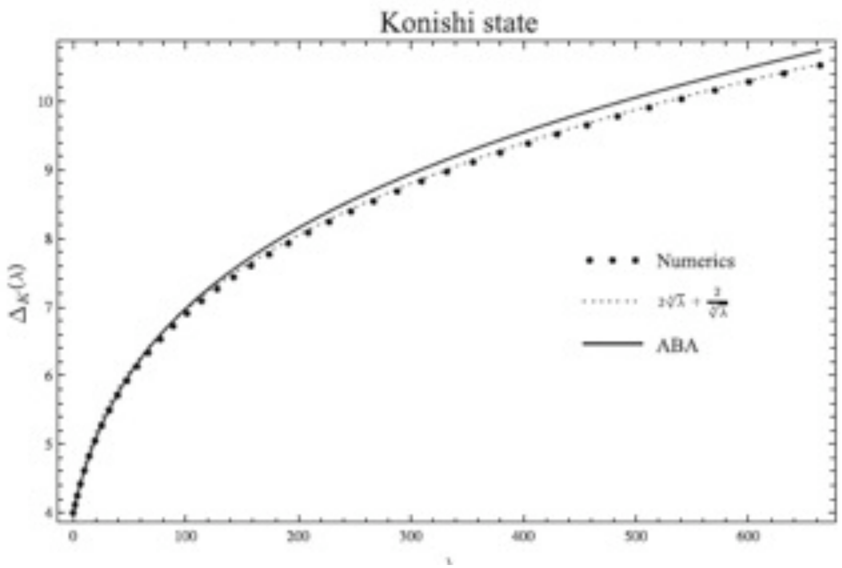




How to derive (as opposed to guess) the Y-system?

What is the physics of it?

Predictions from the All-Loop TBA and the Y-System



[Gromov, Kazakov, Vieira '09]

- A numerical plot for Konishi was obtained and confirmed in [Frolov '10].

- It fits very well weak coupling. Analytical results up to five loops.

- It also fits well the known strong coupling behavior $2\,\lambda^{\frac{1}{4}} + 2\,\lambda^{-\frac{1}{4}} + \ldots$.

- Analytic Structure of the expansion not quite clear though.

Weak Coupling Challenges for AdS/CFT Integrability

- It is very important to explore the predictions of integrability for short operators in $\mathcal{N} = 4$ gauge theory to much higher orders.

- Analytic 5-loop Konishi [Bajnok, Hegedus, Janik, Łukowski '09] confirmed in [Eden et. al. '12]

$$\gamma = 12\,g^2 - 48\,g^4 + 336\,g^6 - 96\,(26 - 6\,\zeta(3) + 15\,\zeta(5))\,g^8 + +96\,\Big(158 + 72\,\zeta(3) - 54\,\zeta(3)^2 - 90\,\zeta(5) + 315\,\zeta(7)\Big) + \ldots$$

- The dots are not boring!

- Results from mathematical Feynman graph theory and algebraic geometry indicate that at some order multiple zeta functions are expected. 6 loops in ϕ^4-theory. [Bloch, Broadhurst, Brown, Kreimer] 8 loops? Double-Wrapping?

- A non-Tate graph appears in planar ϕ^4-theory at 9 loops. [Brown, Schnetz '10] Integrals not expressible as zeta functions! 12 loops? Triple-Wrapping?

Solvable Structures in the Planar AdS/CFT System

- Spectral Problem
- Gluon Amplitudes
- Wilson Loops
- High Energy Scattering (BFKL)

These are all related, and all show (unproven) hints of integrability!

- Recently much progress with $\mathcal{N} = 4$ amplitudes and Wilson loops.
- Exciting hints at Yangian structures in planar $\mathcal{N} = 4$ amplitudes.

[Drummond, Henn, Plefka '09; Drummond, Ferro '10; Beisert, Henn, McLoughlin, Plefka '10]