# Symmetries and Dualities of Scattering amplitudes

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Lecture 5

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# $\mathcal{N}=4$ super Yang Mills: The simplest interacting 4d QFT

- Field content: All fields in adjoint of $SU(N)$, $N \times N$ matrices
  - Gluons: $A_\mu$, $\mu = 0,1,2,3$, $\Delta = 1$
  - 6 real scalars: $\Phi_I$, $I = 1,\ldots,6$, $\Delta = 1$
  - $4 \times 4$ real fermions: $\Psi_{\alpha\,A}$, $\bar{\Psi}^{\dot{\alpha}}_{A}$ ,$\alpha,\dot{\alpha} = 1,2$. $A = 1,2,3,4$, $\Delta = 3/2$
  - Covariant derivative: $\mathcal{D}_\mu = \partial_\mu - i[A_\mu, *]$, $\Delta = 1$
- Action: Unique model completely fixed by SUSY

$$S = \frac{1}{g_{\mathrm{YM}}{}^2} \int d^4x \, \mathrm{Tr}\Big[\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu \Phi_I)^2 - \tfrac{1}{4}[\Phi_I,\Phi_J][\Phi_I,\Phi_J] + \bar{\Psi}^A_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_{\mu}\mathcal{D}^\mu \Psi_{\beta\,A} - \tfrac{i}{2}\Psi_{\alpha\,A}\sigma^{AB}_{I}\epsilon^{\alpha\beta}\,[\Phi^I,\Psi_{\beta\,B}] - \tfrac{i}{2}\bar{\Psi}_{\dot{\alpha}\,A}\sigma^{AB}_{I}\epsilon^{\dot{\alpha}\dot{\beta}}\,[\Phi^I,\bar{\Psi}_{\dot{\beta}\,B}]\Big]$$

- $\boxed{\beta_{g_{\mathrm{YM}}} = 0}$: Quantum Conformal Field Theory, 2 parameters: $N$ & $\lambda = g_{\mathrm{YM}}{}^2 N$
- Shall consider 't Hooft planar limit: $N \to \infty$ with $\lambda$ fixed.
- Is the 4d interacting QFT with highest degree of symmetry!

$\Rightarrow$ **"H-atom of gauge theories"**

# Superconformal symmetry

- Symmetry: $\mathfrak{so}(2,4) \otimes \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$

Poincaré: $p^{\alpha\dot\alpha} = p_\mu \, (\sigma^\mu)^{\dot\alpha\beta}, \quad m_{\alpha\beta}, \quad \bar m_{\dot\alpha\dot\beta}$

Conformal: $k_{\alpha\dot\alpha}, \quad d$ ($c$ : central charge)

R-symmetry: $r_{AB}$

Poncaré Susy: $q^{\alpha\, A}, \bar q^{\dot\alpha}_A$ Conformal Susy: $s_{\alpha\, A}, \bar s^A_{\dot\alpha}$

- $4+4$ Supermatrix notation $\bar A = (\alpha, \dot\alpha | A)$

$$J^{\bar A}{}_{\bar B} = \begin{pmatrix} m^\alpha{}_\beta - \frac{1}{2}\,\delta^\alpha_\beta\,(d + \frac{1}{2}c) & k^\alpha{}_{\dot\beta} & s^\alpha{}_B \\ p^{\dot\alpha}{}_\beta & \overline{m}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\,(d - \frac{1}{2}c) & \bar q^{\dot\alpha}{}_B \\ q^A{}_\beta & \bar s^A{}_{\dot\beta} & -r^A{}_B - \frac{1}{4}\delta^A_B\, c \end{pmatrix}$$

- Algebra:

$$[J^{\bar A}{}_{\bar B}\,,\, J^{\bar C}{}_{\bar D}\} = \delta^{\bar C}_{\bar B}\, J^{\bar A}{}_{\bar D} - (-1)^{(|\bar A|+|\bar B|)(|\bar C|+|\bar D|)}\delta^{\bar A}_{\bar D}\, J^{\bar C}{}_{\bar B}$$

# Gluon Amplitudes and Helicity Classification

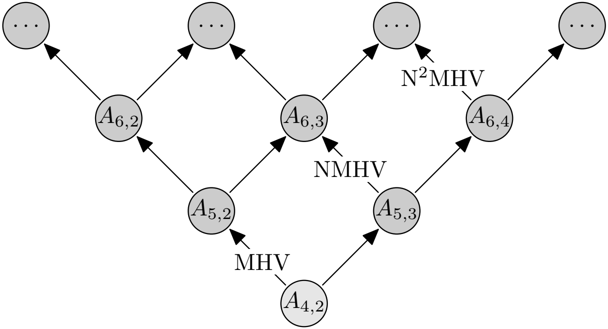
Classify gluon amplitudes by # of helicity flips

- By SUSY Ward identities: $\mathcal{A}_n(1^+, 2^+, \ldots, n^+) = 0 = \mathcal{A}_n(1^-, 2^+, \ldots, n^+)$ true to all loops
- Maximally helicity violating (MHV) amplitudes

$$\mathcal{A}_n(1^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \delta^{(4)}\Big(\sum_i p_i\Big) \frac{\langle i, j\rangle^4}{\langle 1, 2\rangle \langle 2, 3\rangle \ldots \langle n, 1\rangle}$$

[Parke, Taylor]

- Next-to-maximally helicity amplitudes ($\mathrm{N}^k$MHV) have more involved structure!




$$A_{n,m} : g_+^{n-m} g_-^m$$

[Picture from T. McLoughlin]

# On-shell superspace

- Augment $\lambda_i^\alpha$ and $\tilde{\lambda}_i^{\dot{\alpha}}$ by Grassmann variables $\eta_i^A \quad A = 1, 2, 3, 4$ [Nair]
- On-shell superspace $(\lambda_i^\alpha, \tilde{\lambda}^{\dot{\alpha}}, \eta_i^A)$ with on-shell superfield:

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)$$

- Superamplitudes: $\left\langle \Phi(\lambda_1, \tilde{\lambda}_1, \eta_1)\, \Phi(\lambda_2, \tilde{\lambda}_2, \eta_2) \ldots \Phi(\lambda_n, \tilde{\lambda}_n, \eta_n) \right\rangle$

  Packages all $n$-parton gluon$^\pm$-gluino$^{\pm 1/2}$-scalar amplitudes

- General form of **tree superamplitudes**:

$$\mathbb{A}_n = \frac{\delta^{(4)}(\sum_i \lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\sum_i \lambda_i \eta_i)}{\langle 1,2\rangle \langle 2,3\rangle \ldots \langle n,1\rangle} \mathcal{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

  Conservation of super-momentum: $\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^A) = (\sum_i \lambda^\alpha \eta_i^A)^8$

- $\eta$-expansion of $\mathcal{P}_n$ yields N$^k$MHV-classification of superamps as $h(\eta) = -1/2$

$$\mathcal{P}_n = \mathcal{P}_n^{\text{MHV}} + \eta^4\, \mathcal{P}_n^{\text{NMHV}} + \eta^8\, \mathcal{P}_n^{\text{NNMHV}} + \ldots + \eta^{4n-16}\, \mathcal{P}_n^{\overline{\text{MHV}}}$$

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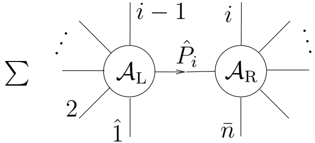
# Super BCFW-recursion

- Efficient way of constructing tree-level amplitudes via BCFW recursion using an on-shell superspace via shift in $(\lambda_i, \tilde{\lambda})$ and $\eta_i$ [Elvang et al, Arkani-Hamed et al, Brandhuber et al]

$$\mathbb{A}_n = \sum_i \int d^{4\eta_P}\, \mathbb{A}^L_{i+1}\, \frac{1}{P_i^2}\, \mathbb{A}^R_{n-i+1}$$




- Reformulation of recursion relations in terms of functions $\mathcal{P}_n(1,2,\ldots,n)$:

$$\mathcal{P}_n = \mathcal{P}_{n-1}(\hat{P},3,\ldots,\hat{n}) + \sum_{i=4}^{n-1} R_{n;2,i}\, \mathcal{P}_i(\hat{1},2,\ldots,-\hat{P}_i)\, \mathcal{P}_{n-i+2}(\hat{P}_i,i,\ldots,\hat{n})$$

- Is much simpler and can be solved analytically!

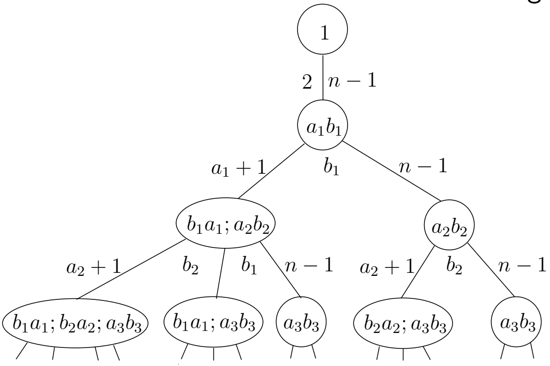
$\Rightarrow$ $\mathcal{P}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})$ known in closed analytical form at tree-level [Drummond,Henn]

# The Drummond-Henn solution

$\mathcal{P}_n$ expressed as sums over $R$-invariants determined by paths on rooted tree

$$\mathcal{P}_n^{\text{N}^k\text{MHV}} = \sum_{\substack{\text{all paths}\\ \text{of length } k}} 1 \cdot R_{n,a_1b_1} \cdot R_{n,\{I_2\},a_2b_2}^{\{L_2\};\{U_2\}} \cdot \ldots \cdot R_{n,\{I_p\},a_pb_p}^{\{L_p\};\{U_p\}}$$

E.g.




$$\mathcal{P}^{\text{NMHV}} = \sum_{1<a_1,b_1<n} R_{n,a_1b_1}$$

$$\mathcal{P}_n^{\text{N}^2\text{MHV}} = \sum_{1<a_1,b_1<n} R_{n;a_1b_1} \times \left[ \sum_{a_1<a_2,b_2\leq b_1} R_{n;b_1a_1;a_2b_2}^{0;a_1b_1} + \sum_{b_1\leq a_2,b_2<n} R_{n;a_2b_2}^{a_1b_1;0} \right]$$

$$R_{n;b_1a_1;b_2a_2;\ldots;b_ra_r;ab} = \frac{\langle a\ a-1\rangle\langle b\ b-1\rangle\delta^{(4)}(\langle\xi|x_{a_ra}x_{ab}|\theta_{ba_r}\rangle + \langle\xi|x_{a_rb}x_{ba}|\theta_{aa_r}\rangle)}{x_{ab}^2\langle\xi|x_{a_ra}x_{ab}|b\rangle\langle\xi|x_{a_ra}x_{ab}|b-1\rangle\langle\xi|x_{a_rb}x_{ba}|a\rangle\langle\xi|x_{a_rb}x_{ba}|a-1\rangle},$$

with

$$\langle\xi| = \langle n|x_{nb_1}x_{b_1a_1}x_{a_1b_2}x_{b_2a_2}\ldots x_{b_ra_r}\,.$$

# Dual Superconformal symmetry

- Introduce dual on-shell superspace [Drummond, Henn, Korchemsky, Sokatchev]

$$(x_i - x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^\alpha \, \tilde{\lambda}_i^{\dot{\alpha}} \qquad (\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda_i^\alpha \, \eta_i^A$$

- Transformation properties under inversions $I[\ldots]$ in dual $x$-space

$$I[\langle i\, i+1\rangle] = \frac{\langle i\, i+1\rangle}{x_i^2} \quad I[\delta^4(p)\delta^8(q)] = \delta^4(p)\delta^8(q)$$

$$I[\langle n|x_{na}x_{ab}|b\rangle] = \frac{\langle n|x_{na}x_{ab}|b\rangle}{x_n^2 x_a^2 x_b^2}\,, \quad I[\langle n|x_{na}x_{ab}|b-1\rangle] = \frac{\langle n|x_{na}x_{ab}|b-1\rangle}{x_n^2 x_a^2 x_{b-1}^2}$$

- One shows that $I[R_{n;b_1a_1;\ldots;b_ra_r;ab}] = R_{n;b_1a_1;\ldots;b_ra_r;ab}$ as all weights cancel!
- Simple proof of dual conformal symmetry: $R_{n,st}$ is I-invariant, assume $\mathcal{P}_{k<n}$ are I-invariant. Then RHS of recursion relation is invariant too, thus $\mathcal{P}_n$ also I-invariant.
- Hence:

$$I[\mathbb{A}_n] = x_1^2\, x_2^2 \ldots x_n^2\, \mathbb{A}_n$$

# Infinitesimal form of dual superconformal symmetry

- Infinitesimally one has: $K^{\alpha\dot\alpha} = \sum_i x_i^{\alpha\dot\beta}\, x_i^{\dot\alpha\beta}\, \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta}\, \theta_i^{\alpha\, B}\, \frac{\partial}{\partial \theta_i^{\beta\, B}}$.
  Bosonic part derives from $K_\mu = x^2\, \partial_\mu - 2x_\mu\, x\cdot\partial$.
- Indeed: Trees are dual superconformal covariant:

$$K^{\alpha\dot\alpha}\, \mathbb{A}_n^{\text{tree}} = -\sum_{i=1}^{n} x_i^{\alpha\dot\alpha}\, \mathbb{A}_n^{\text{tree}} \qquad S^{\alpha A}\, \mathbb{A}_n^{\text{tree}} = -\sum_{i=1}^{n} \theta_i^{\alpha A}\, \mathbb{A}_n^{\text{tree}}$$

$\Rightarrow$ $\boxed{\tilde K = K + \sum_i x_i \text{ and } \tilde S = S + \sum_i \theta_i}$ annihilate the amplitude.

- Extend dual superconformal generators so that they commute with constraints

$$(x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha} \qquad (\theta_i - \theta_{i+1})^{\alpha\, A} = \lambda_i^\alpha\, \eta_i^A$$

leads to expression for $K^{\alpha\dot\alpha}$ acting in joint super-space $\{\lambda_i, \tilde\lambda_i, \eta_i; x_i, \theta_i\}$

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$$+ x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^B_{i+1\,\alpha}\partial_{iB}$$

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$$+ x_{i\dot\alpha}{}^{\beta} \lambda_{i\alpha} \partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta} \tilde\lambda_{i\dot\alpha} \partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha} \theta^B_{i+1\,\alpha} \partial_{iB}$$

# The natural question

**Q**: What algebraic structure emerges when one commutes conformal with dual conformal generators? [Drummond,Henn,Plefka]

**First Task:** Tranform dual superconformal generators expressed in dual space $(x_i, \theta_i)$ into original on-shell superspace $(\lambda_i, \tilde{\lambda}_i, \eta_i)$!

1. Open chain by droping $x_{n+1} = x_1$ and $\theta_{n+1} = \theta_1$ conditions, implemented via $\delta$-fcts: $\delta^{(4)}(p)\,\delta^{(8)}(q) = \delta^{(4)}(x_1 - x_{n+1})\,\delta^{(8)}(\theta_1 - \theta_{n+1})$
2. Express dual variables via "non-local' relations:

$$x_i^{\alpha\dot{\alpha}} = x_1^{\alpha\dot{\alpha}} + \sum_{j<i} \lambda_j^\alpha\,\tilde{\lambda}_j^{\dot{\alpha}} \qquad \theta_i^{\alpha A} = \theta_1^{\alpha A} + \sum_{j<i} \lambda_j^\alpha\,\eta_j^A$$

   Now set $x_1 = \theta_1 = 0$ by dual translation $P$ and Poincare Susy $Q$.

3. Can now drop all $x_1$ and $\theta_i$ derivatives in dual superconformal generators.

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3. Can now drop all $x_1$ and $\theta_i$ derivatives in dual superconformal generators.

# Dual $\mathfrak{psu}(2,2|4)$ generators

- Dual superconformal generators acting in standard on-shell superspace $(\lambda, \tilde{\lambda}, \eta)$:

$$P_{\alpha\dot{\alpha}} = 0\,, \qquad Q_{\alpha A} = 0\,, \qquad \overline{Q}^{A}_{\dot{\alpha}} = \sum_i \eta_i^A \partial_{i\dot{\alpha}} = \bar{s}^A_{\dot{\alpha}}$$

$$M_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)} = \bar{m}_{\dot{\alpha}\dot{\beta}}\,, \qquad \overline{M}_{\dot{\alpha}\dot{\beta}} = \sum_i \tilde{\lambda}_{i(\dot{\alpha}}\partial_{i\dot{\beta})} = m_{\alpha\beta}\,,$$

$$R^A{}_B = \sum_i \eta_i^A\,\partial_{iB} - \tfrac{1}{4}\delta^A_B \eta_i^C \partial_{iC} = -r^A{}_B\,,$$

$$D = \sum_i -\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} - \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} = -d\,,$$

$$C = \sum_i -\tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} + \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} + \tfrac{1}{2}\eta_i^A \partial_{iA} = 1 - c\,,$$

$$S^A_\alpha = \sum_i \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot{\beta}}\eta_i^A \partial_{i\dot{\beta}} - \theta^B_{i+1\,\alpha}\eta_i^A \partial_{iB}\,,$$

$$\overline{S}_{\dot{\alpha}A} = \sum_i \tilde{\lambda}_{i\dot{\alpha}}\partial_{iA} = \bar{q}_{\dot{\alpha}A}\,,$$

$$K_{\alpha\dot{\alpha}} = \sum_i x_{i\dot{\alpha}}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot{\beta}}\tilde{\lambda}_{i\dot{\alpha}}\partial_{i\dot{\beta}} + \tilde{\lambda}_{i\dot{\alpha}}\theta^B_{i+1\,\alpha}\partial_{iB}$$

# Nonlocal structure of dual $K$ and $S$

- We are left with the dual generators $K$ and $S$, all others trivially related to standard superconformal generators.

$$\tilde{K}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} x_i^{\dot{\alpha}\beta}\,\lambda_i^{\alpha}\,\frac{\partial}{\partial\lambda_i^{\beta}} + x_{i+1}^{\alpha\dot{\beta}}\,\tilde{\lambda}_i^{\dot{\alpha}}\,\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\alpha}}\,\theta_{i+1}^{\alpha B}\,\frac{\partial}{\partial\eta_i^{B}} + x_i^{\alpha\dot{\alpha}}$$

$$x_i^{\alpha\dot{\alpha}} = \sum_{j=1}^{i-1} \lambda_j^{\alpha}\,\tilde{\lambda}_j^{\dot{\alpha}} \qquad \theta_{i+1}^{\alpha A} = \sum_{j=1}^{i} \lambda_j^{\alpha}\,\eta_j^{A}$$

Nonlocal structure!

# Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

- Can show that dual superconformal generators $K$ and $S$ may be lifted to level 1 generators of a Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$:

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(0)} \qquad \text{conventional superconformal symmetry}$$

$$[J_a^{(1)}, J_b^{(0)}\} = f_{ab}{}^c\, J_c^{(1)} \qquad \text{from dual conformal symmetry}$$

with nonlocal generators

$$\boxed{J_a^{(1)} = f^{cb}{}_a \sum_{1<j<i<n} J_{i,b}^{(0)}\, J_{j,c}^{(0)}}$$

and super Serre relations (representation dependent). [Dolan,Nappi,Witten]

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}\}\} + (-1)^{|a|(|b|+|c|)}[J_b^{(1)}, [J_c^{(1)}, J_a^{(0)}\}\} + (-1)^{|c|(|a|+|b|)}[J_c^{(1)}, [J_a^{(1)}, J_b^{(0)}\}\}$$
$$= h(-1)^{|r||m|+|t||n|}\{J_l^{(0)}, J_m^{(0)}, J_n^{(0)}]f_{ar}{}^l f_{bs}{}^m f_{ct}{}^n f^{rst}.$$

# Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ SYM

- Bosonic invariance $\boxed{p^{(1)}_{\alpha\dot\alpha}\,\mathbb{A}_n = 0}$ with

$$p^{(1)}_{\alpha\dot\alpha} = \tilde{K}_{\alpha\dot\alpha} + \Delta K_{\alpha\dot\alpha} = \frac{1}{2}\sum_{i<j}(m_{i,\alpha}{}^{\gamma}\delta^{\dot\gamma}_{\dot\alpha} + \bar{m}_{i,\dot\alpha}{}^{\dot\gamma}\delta^{\gamma}_{\alpha} - d_i\,\delta^{\gamma}_{\alpha}\delta^{\dot\gamma}_{\dot\alpha})\,p_{j,\gamma\dot\gamma} + \bar{q}_{i,\dot\alpha C}\,q^{C}_{j,\alpha} - (i \leftrightarrow j)$$

- In supermatrix notation: $\bar{A} = (\alpha, \dot\alpha | A)$

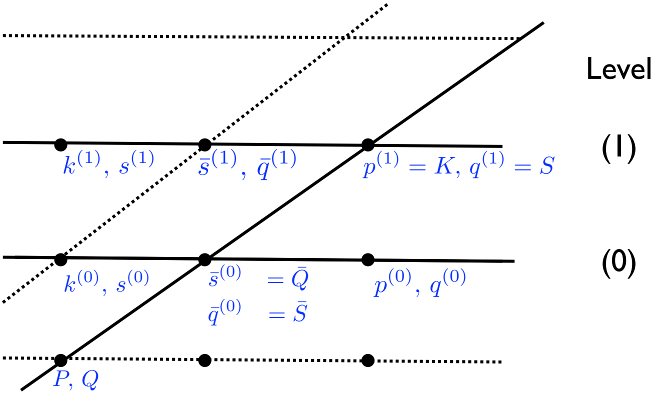
$$J^{\bar{A}}{}_{\bar{B}} = \begin{pmatrix} m^{\alpha}{}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\,(d + \frac{1}{2}c) & k^{\alpha}{}_{\dot\beta} & s^{\alpha}{}_{B} \\ p^{\dot\alpha}{}_{\beta} & \overline{m}^{\dot\alpha}{}_{\dot\beta} + \frac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\,(d - \frac{1}{2}c) & \bar{q}^{\dot\alpha}{}_{B} \\ q^{A}{}_{\beta} & \bar{s}^{A}{}_{\dot\beta} & -r^{A}{}_{B} - \frac{1}{4}\delta^{A}_{B}\,c \end{pmatrix}$$

and $\boxed{J^{(1)\,\bar{A}}{}_{\bar{B}} := -\sum_{i>j}(-1)^{|\bar{C}|}(J^{\bar{A}}_{i}{}_{\bar{C}}\,J^{\bar{C}}_{j}{}_{\bar{B}} - J^{\bar{A}}_{j}{}_{\bar{C}}\,J^{\bar{C}}_{i}{}_{\bar{B}})}$

- Integrable spin chain picture also for colour ordered scattering amplitudes!
- Implies an infinite-dimensional symmetry algebra for $\mathcal{N}=4$ SYM scattering amplitudes!

# Summary of Yangian Structure

- Combination of standard and dual superconformal symmetry lifts to Yangian $Y[\mathfrak{psu}(2,2|4)]$ [Picture: Beisert]




- Tree level superamplitudes invariant: $\boxed{\mathcal{J} \circ \mathbb{A}_n^{\text{tree}} = 0}$ for $\mathcal{J} \in Y[\mathfrak{psu}(2,2|4)]$.

# Dual conformal symmetry at loop level

- 4-point MHV-amplitude at 1-loop: $(a = \lambda/8\pi^2)$

$$\mathbb{A}_4^{\text{MHV, 1-loop}} = \mathbb{A}_4^{\text{MHV, tree}} \cdot \frac{a}{2}\, st \cdot I(s,t)$$

Scalar box integral: $I(s,t) = \int \frac{d^4k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$

No bubbles or triangles!

- Transform to dual coordinates: $x_{ij} = x_i - x_j$

$$p_1 = x_{12} \quad p_2 = x_{23} \quad p_3 = x_{34} \quad p_4 = x_{41} \quad k = x_1 - x_5$$

then $\boxed{I(s,t) = \int \frac{d^4x_5}{x_{15}^2\, x_{25}^2\, x_{35}^2\, x_{45}^2}}$ which is (naively) dual conformal invariant

$$I[\frac{d^4x_5}{x_{15}^2\, x_{25}^2\, x_{35}^2\, x_{45}^2}] = x_1^2 x_2^2 x_3^2 x_4^2 \, \frac{d^4x_5}{x_{15}^2\, x_{25}^2\, x_{35}^2\, x_{45}^2}$$

- Note $st = (2p_1 \cdot p_2)(2p_1 \cdot p_3) = x_{13}^2\, x_{24}^2$, hence $st\, I(s,t)$ is dual conformal inv.

# Pseudo conformal invariance at loop level

- One-loop box is only "pseudo-conformal" invariant as $I(s,t)$ is IR-divergent and needs to be regularized: $\boxed{d^4x_5 \to d^{4-2\epsilon}x_5}$. This breaks dual conformal invariance.
- Indeedexact dual conformal invariance would imply $st\, I(s,t) = 0$ as there are no conformal invariant cross-ratios for 4 light-like seperated points:

$$\text{Dual conformal cross-ratios:} \quad R(i,j,k,l) = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

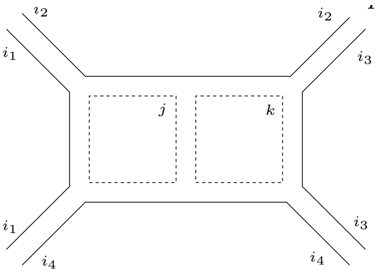
- Indeed one finds a non-vanishing result

$$\mu^{2\epsilon} e^{-\epsilon\gamma_E}\, st\, I(s,t) = \frac{2}{\epsilon^2}\left[ (\frac{\mu^2}{s})^\epsilon + (\frac{\mu^2}{t})^\epsilon \right] - \log^2(s/t) - \frac{4\pi^2}{3}$$

$\Rightarrow$ dual conformal anomaly

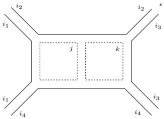
- "Pseudo" dual conformal invariance still a very useful concept as it constrains the possible scalar-integrals appearing at higher loops.

- E.g. at 2 loops: Only one integral is allowed by dual conformal symmetry:




Similar restrictions at higher loops.

- One observes exponentiation: [Bern,Dixon,Smirnov]



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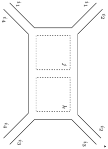
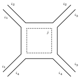


$= \exp\Big[\Gamma_{\mathsf{cusp}}(\lambda)$



$\Big]\Big|_{\lambda^2}$

# What about higher loops?

- Spezialize to MHV for simplicity: $\boxed{\mathcal{A}_n^{\text{MHV}} = \mathcal{A}_{n,0}^{\text{MHV}}\,\mathcal{M}_n^{\text{MHV}}(p_i\cdot p_j;\lambda)}$
- All loop planar amplitudes can be split into IR divergent and finite parts:

$$\ln \mathcal{M}_n^{\text{MHV}} = D_n + F_n + \mathcal{O}(\epsilon)$$

IR divergencies exponentiate in any gauge theory $(a = \lambda/8\pi^2)$ [Mueller,Collins,Sterman,. . . ]

$$D_n = -\frac{1}{2}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{G^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}(2p_i\cdot p_j)^{l\epsilon}$$

$$\Gamma_{\text{cusp}}(a) = \sum_l a^l \Gamma_{\text{cusp}}^{(l)}\,, \quad \text{cusp anomalous dimension}$$

$$G(a) = \sum_l a^l G^{(l)}\,, \quad \text{colinear anomalous dimension}$$

- IR divergencies break $\{s,\bar{s},k,K,S,\bar{Q}\}$ but leave $\{p,q,\bar{q},P,Q,\bar{S}\}$ intact.

[Korchemsky,Sokatchev]

# Dual conformal anomaly

- Breaking of $K_\mu$ is under control and proportional to $\Gamma_{\text{cusp}}(g)$ for MHV amplitudes. From dual Wilson loop picture: UV anomaly due to cusps for finite piece $F_n$

$$K_\mu F_n = \sum_{i=1}^{n}\left[2x_{i\mu}x_i^\nu \frac{\partial}{\partial x_i^\nu} - x_i^2 \frac{\partial}{\partial x_i^\mu}\right] F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{n}\left[x_{i,i+1}^\mu \ \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right] F_n$$

- Conjecture: Dual superconformal 'anomaly' is the same for MHV and non-MHV amplitudes [Drummond,Henn,Korchemsky,Sokatchev '08]
- 'Anomaly' fixes the MHV 4 & 5 gluon amplitudes completely ⇔ BDS-ansatz. Nontrivial structure starts with $n = 6$.
- ⇒ Remainder function, non-trivial function of dual conformal invariants
- **Q:** Can the other broken Yangian symmetry be repaired at loop level?

  ⇒ Does this constrain the answers?

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# From $\mathcal{N}=4$ SYM trees to massless QCD

**Goal:** Project onto component field amplitudes [Dixon, Henn, Plefka, Schuster]

$$x_i - x_{i+1} = p_i \qquad x_{ij} := x_i - x_j \overset{i<j}{=} p_i + p_{i+1} + \cdots + p_{j-1}$$

- All amplitudes expressed via momentum invariants $x_{ij}^2$ and the scalar quantities:

$$\begin{aligned}\langle n a_1 a_2 \ldots a_k | a\rangle &:= \langle n | x_{na_1} x_{a_1a_2} \ldots x_{a_{k-1}a_k} | a \rangle \\ &= \lambda_n^{\alpha} (x_{na_1})_{\alpha\dot{\beta}} (x_{a_1a_2})^{\dot{\beta}\gamma} \ldots (x_{a_{k-1}a_k})^{\dot{\delta}\rho} \lambda_{a\,\rho}\end{aligned}$$

- Building blocks for amps: $\tilde{R}$ invariants and path matrix $\Xi_n^{\text{path}}$

$$\tilde{R}_{n;\{I\};ab} := \frac{1}{x_{ab}^2} \frac{\langle a(a-1)\rangle}{\langle n\,\{I\}\,ba|a\rangle\,\langle n\,\{I\}\,ba|a-1\rangle} \frac{\langle b(b-1)\rangle}{\langle n\,\{I\}\,ab|b\rangle\,\langle n\,\{I\}\,ab|b-1\rangle};$$

$$\Xi_n^{\text{path}} := \begin{pmatrix} \langle nc_0\rangle & \langle nc_1\rangle & \ldots & \langle nc_p\rangle \\ (\Xi_n)^{c_0}_{a_1b_1} & (\Xi_n)^{c_1}_{a_1b_1} & \ldots & (\Xi_n)^{c_p}_{a_1b_1} \\ (\Xi_n)^{c_0}_{\{I_2\};a_2b_2} & (\Xi_n)^{c_1}_{\{I_2\};a_2b_2} & \ldots & (\Xi_n)^{c_p}_{\{I_2\};a_2b_2} \\ \vdots & \vdots & & \vdots \\ (\Xi_n)^{c_0}_{\{I_r\};a_rb_r} & (\Xi_n)^{c_1}_{\{I_r\};a_rb_r} & \ldots & (\Xi_n)^{c_p}_{\{I_r\};a_rb_r} \end{pmatrix}$$

# All gluon-gluino trees in $\mathcal{N} = 4$ SYM [Dixon, Henn, Plefka, Schuster]

- MHV gluon amplitudes [Parke, Taylor]

$$A_n^{\text{MHV}}(c_0^-, c_1^-) = \delta^{(4)}(p) \frac{\langle c_0\, c_1\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle}$$

- N$^{\text{p}}$MHV gluon amplitudes:

$$A_n^{\text{N}^{\text{p}}\text{MHV}}(c_0^-, \ldots, c_{p+1}^-) = \frac{\delta^{(4)}(p)}{\langle 1\, 2\rangle \ldots \langle n\, 1\rangle} \sum_{\substack{\text{all paths} \\ \text{of length } p}} \left( \prod_{i=1}^{p} \tilde{R}_{n;\{I_i\};a_i b_i}^{L_i;R_i} \right) (\det \Xi)^4$$

- MHV gluon-gluino amplitudes (single flavor)

$$A_n^{\text{MHV}}(a^-, b_q, c_{\bar{q}}) = \delta^{(4)}(p) \frac{\langle a\, c\rangle^3 \langle a\, b\rangle}{\langle 1\, 2\rangle \ldots \langle n\, 1\rangle}$$

- N$^{\text{p}}$MHV gluon-gluino amplitudes:

$$A_{(q\bar{q})^k,n}^{\text{N}^{\text{p}}\text{MHV}}(\ldots, c_k^-, \ldots, (c_{\alpha_i})_q, \ldots, (c_{\bar{\beta}_j})_{\bar{q}}, \ldots) =$$

$$\frac{\delta^{(4)}(p)\text{sign}(\tau)}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle} \times \sum_{\substack{\text{all paths} \\ \text{of length } p}} \left( \prod_{i=1}^{p} \tilde{R}_{n;\{I_i\};a_i b_i}^{L_i;R_i} \right) \left( \det \Xi\big|_q \right)^3 \det \Xi(q \leftrightarrow \bar{q})\big|_{\bar{q}}$$

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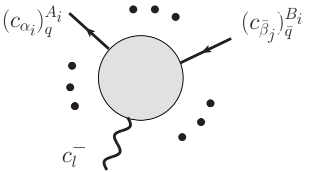



- N$^{\text{p}}$MHV gluon-gluino amplitudes:

$$A^{\text{N}^{\text{p}}\text{MHV}}_{(q\bar{q})^k,n}(\ldots, c_k^-, \ldots, (c_{\alpha_i})_q, \ldots, (c_{\bar{\beta}_j})_{\bar{q}}, \ldots) =$$

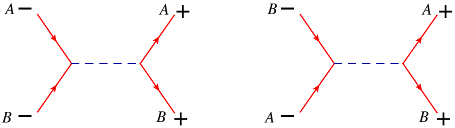
$$\frac{\delta^{(4)}(p)\text{sign}(\tau)}{\langle 1\ 2\rangle\langle 2\ 3\rangle\ldots\langle n\ 1\rangle}\times\sum_{\substack{\text{all paths}\\ \text{of length } p}}\left(\prod_{i=1}^{p}\tilde{R}^{L_i;R_i}_{n;\{I_i\};a_ib_i}\right)\left(\det\Xi\big|_q\right)^3\det\Xi(q\leftrightarrow\bar{q})\big|_{\bar{q}}$$

# From $\mathcal{N} = 4$ to massless QCD trees

- Differences in color: SU(N) vs. SU(3); Fermions: adjoint vs. fundamental
  Irrelevant for color ordered amplitudes, as color d.o.f. stripped off anyway. E.g. single quark-anti-quark pair

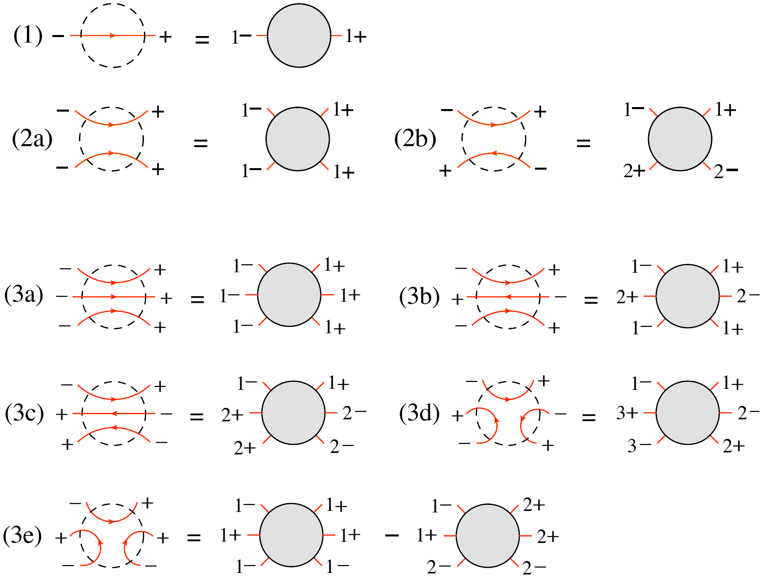
$$\mathcal{A}_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \ldots, n) = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}})_{i_2}^{\bar{i}_1} \, A_n^{\text{tree}}(1_{\bar{q}}, 2_q, \sigma(3), \ldots, \sigma(n))$$

  Color ordered $A_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \ldots, n)$ from two-gluino-$(n-2)$-gluon amplitude.
- For more than one quark-anti-quark pair needs to accomplish:
  (1) Avoid internal scalar exchanges (due to Yukawa coupling)



# From $\mathcal{N} = 4$ to massless QCD trees

(2) Allow all fermion lines present to be of different flavor



# From $\mathcal{N} = 4$ to massless QCD trees

- Also worked out explicitly for 4 quark-anti-quark pairs.
- Conclusion: Obtained all (massless) QCD trees from the $\mathcal{N} = 4$ SYM trees
- Comparison of numerical efficiency to Berends-Giele recursion: Analytical formulae faster for MHV and NMHV case, competitive for NNMHV

  [Biedermann, Uwer, Schuster, Plefka, Hackl]