From Form Factors in N=4 SYM to Higgs Amplitudes

Andi Brandhuber

Queen Mary, University of London

work done over the last years in collaboration with:

Gurdogan, Korres, Mooney, Spence, Travaglini, Yang
arXiv:1201.4170, 1107.5067, 1011.1899

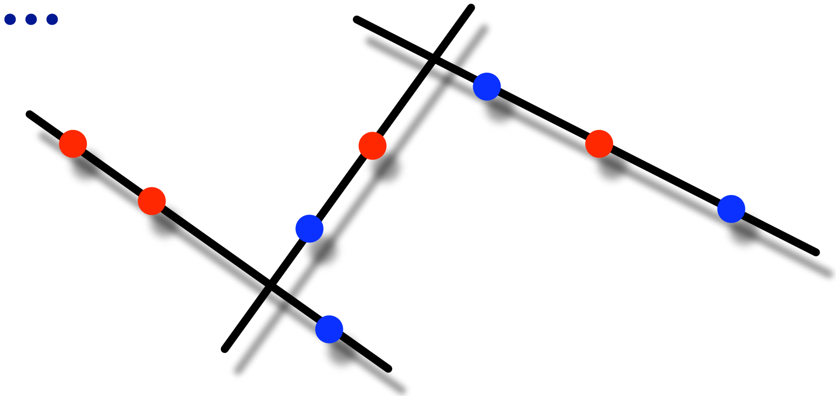
related work by: van Neerven (1986!),   Bork-Kazakov-Vartanov,   Alday-Maldacena-Zhiboedov,   Gehrmann-Henn-Huber

PARMA INT'L SCHOOL OF THEORETICAL PHYSICS   3-8/9/2012

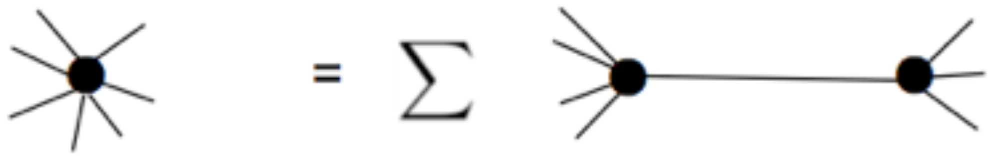
Why calculate scattering amplitudes?

- Strong Physics motivations
  - LHC physics
  - precision perturbative QCD
- Mathematical physics motivations
  - AdS/CFT
  - Dualities
  - Hidden symmetries
  - Integrability

Hidden Simplicity ...



- Why are amplitudes so simple?
  Can we make use of this fact?
  - Geometry in twistor space (Witten 2003)
  - Iterative structures of S-matrix of gauge theory & gravity



- Avoid problems of standard Feynman rules
  - gauge dependence, ghosts
  - off-shell
  - large number of diagrams

... Inspires New Methods

- Novel Methods

  e.g. one-loop amplitudes

  - MHV Diagrams
    (Cachazo-Svrcek-Witten)
  - Modern Unitarity algebraic; no phase space/dispersion integrals!
    (Bern, Dixon, Dunbar, Kosower,... Britto, Cachazo, Feng,...)
  - On-shell Recursion Relations (Britto-Cachazo-Feng-Witten)

- Important common features

  - Only on-shell quantities needed
    e.g. MHV rules need only

$$\mathcal{A}_{n,\mathrm{MHV}} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

$$\langle ij\rangle \equiv \lambda_a^i \lambda_b^j \varepsilon^{ab} \;,\; [ij] \equiv \tilde{\lambda}_{\dot{a}}^i \tilde{\lambda}_{\dot{b}}^j \varepsilon^{\dot{a}\dot{b}}$$

  - use analytic structure of amplitudes (poles, branch cuts) in an essential way (Realisation of old S-matrix programme)
  - complex momenta (natural arena for twistors)

Modern Methods

- Remarkably none of these new insights comes from staring at the Lagrangian but rather...
  - explicit amplitudes ⇒ new, hidden structures ⇒ new methods ⇒ more amplitudes ⇒ ...
- Also, these new tools are very general, apply to gravity and even to QCD, hence are important for LHC and are used for numerical calculations
- overarching goal: find reformulation of QFT that makes these structures and symmetries manifest

Outline of Lectures

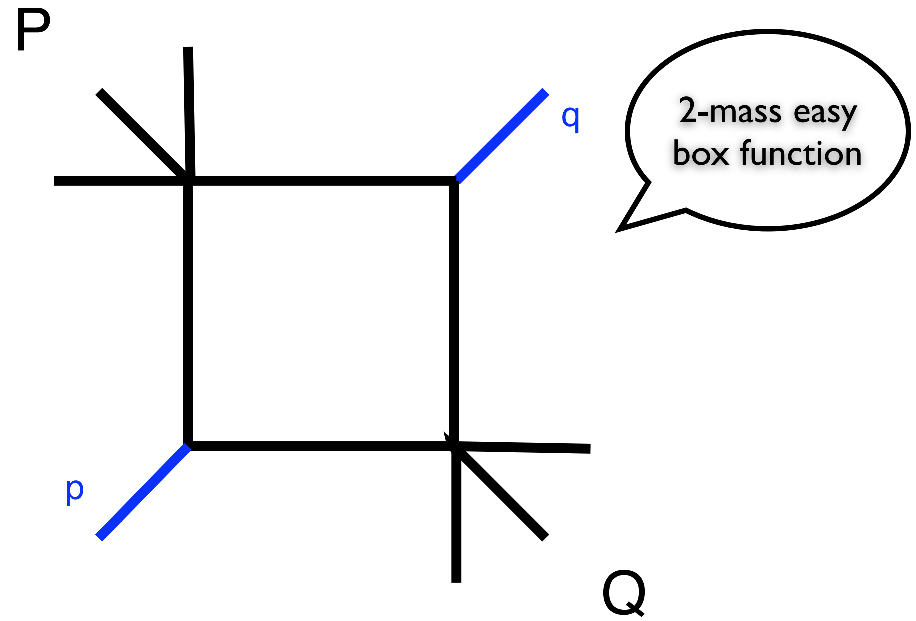
- Brief Review of some recent key developments in N=4 SYM
- Motivation and short review of the applications of form factors in N=4 and QCD
- Form Factors in N=4 SYM
  - Form Factors at tree and one loop level
  - Super Form Factors
  - 2-loop Sudakov Form Factor in N=4
  - 2-loop n=3 FF in N=4 from Unitarity and Symbology
  - Unexpected relation to Higgs amplitudes in QCD

Brief Review of some recent key developments in N=4 SYM

Iterative Structures at Weak Coupling

- Simplest one-loop amplitude is the n-point MHV amplitude in N=4 SYM (colour-structure stripped off):

$$\mathcal{A}_{MHV}^{1-loop} = \mathcal{A}_{MHV}^{tree} \sum_{p,q} 1 \times$$



- Calculated using unitarity in 1994 (Bern-Dixon-Dunbar-Kosower)
- From MHV diagrams in 2004 (AB-Spence-Travaglini)

Surprising iterative structure at two loops...

helicity blind

- n-point MHV amplitude in N=4: $\mathcal{A}_n^{(L)} = \mathcal{A}_n^{tree} \mathcal{M}_n^{(L)}$

- First observed for 4 gluon scattering in planar N=4 SYM at 2 loops (Anastasiou-Bern-Dixon-Kosower)

$$\mathcal{M}_n^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 = f^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)$$

IR divergent

$D = 4 - 2\epsilon$

- Four universal constants

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2 \qquad C^{(2)} = -\frac{1}{2}\zeta_2^2$$

2-loop cusp & collinear anomalous dimensions

BDS-Ansatz/Exponentiation

- Based on earlier work (Anastasiou-Bern-Dixon-Kosower) Bern-Dixon-Smirnov (BDS) found iterative structures for n=4 at 3 loops and proposed an all-loop order formula for n-point MHV amplitudes in planar N=4 SYM.

helicity blind

$$a \sim g_{\mathrm{YM}}^2 N/(8\pi^2)$$

$$\mathcal{A}_n^{(L)} = \mathcal{A}_n^{tree} \mathcal{M}_n^{(L)}$$

$$\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}_n^{(L)} \overset{BDS}{\equiv} \exp\left[\mathrm{Div} + \gamma_K \mathrm{Finite}^{(1)}(p_1,\ldots,p_n)\right]$$

- $\mathrm{Finite}^{(1)}(p_1,\ldots,p_n)$ finite part of one-loop amplitude
- γ_K cusp anomalous dimension: BES equation $\Rightarrow$ integrability
- leading IR divergence in Div is proportional to γ_K

Comments

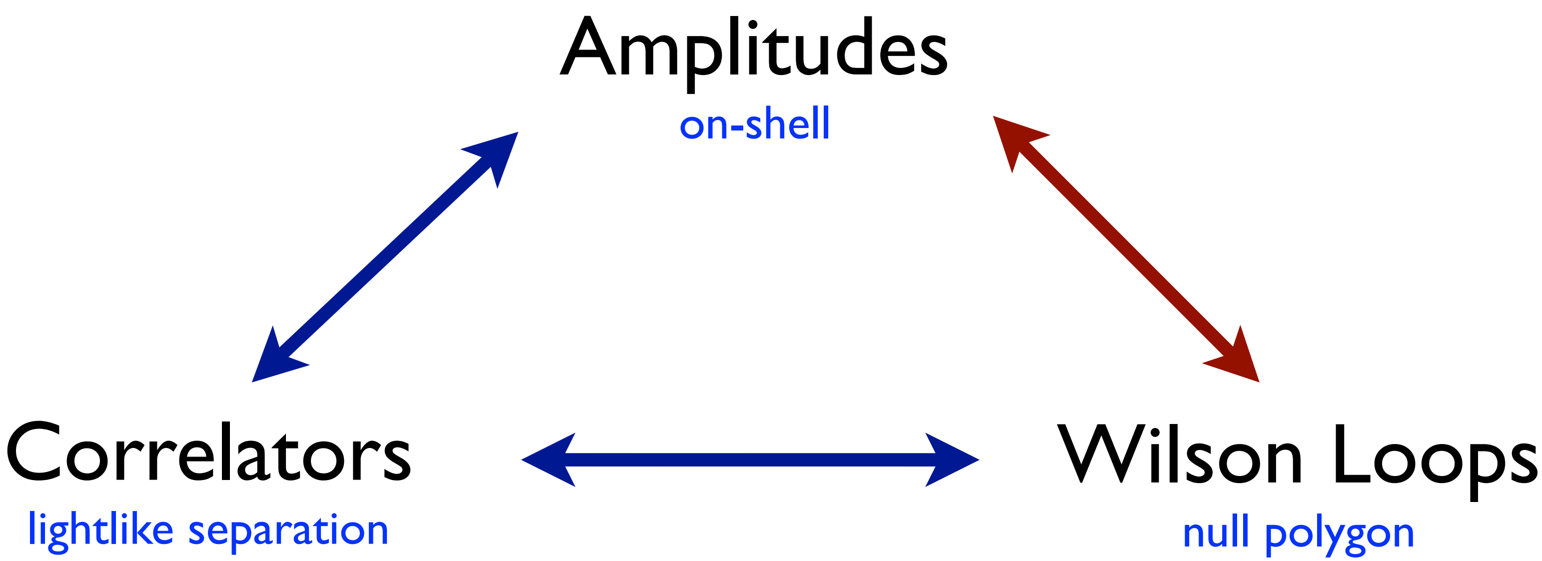
- Motivated by universal factorisation & exponentiation of IR divergences=Form Factors (true in any gauge theory)
- Miracle in N=4: exponentiation of finite parts of the amplitude
- Confirmed by strong coupling calculation using AdS/CFT by Alday-Maldacena and at weak coupling for n=4, 5
- However for n > 5 BDS ansatz needs modification ⇒ remainder function $\mathcal{R}_n$

$$\mathcal{M} = e^{\mathrm{BDS}+\mathcal{R}}$$

- hard to calculate directly, luckily there is much better way to calculate $\mathcal{R}_n$ which involves Wilson loops...

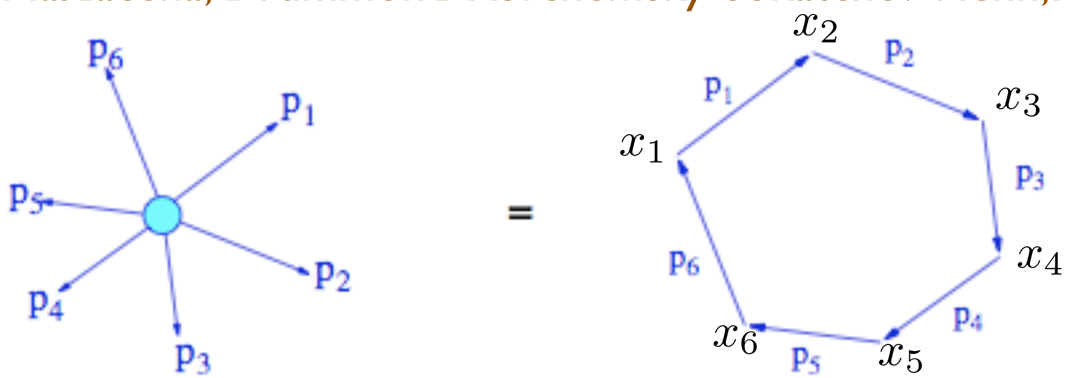
New dualities/symmetries in N=4

- N=4 SYM: three seemingly unrelated objects of interest
  Correlators // Amplitudes (S-Matrix) // Wilson Loops
- AdS/CFT: know how to deal with them at strong coupling
- Strong evidence that all three are actually related => Triality
  (Eden, Maldacena, Korchemsky, Sokatchev, Heslop,...)



Amplitude/Wilson Loop duality

Alday-Maldacena, Drummond-Korchemsky-Sokatchev-Henn, AB-Heslop-Travaglini



$$p_i = x_i - x_{i-1}$$

$$\sum_{i=1}^{n} p_i = 0$$

- Planar MHV n-point amplitude "=" <W[*C*]> where *C* is lightlike n-gon in (T-)dual momentum space

- Subtract wellknown universal IR/UV divergences
  ⇒ finite remainder function

$$\mathcal{R}_n^A = \mathcal{R}_n^W$$

- Strong coupling: minimal area with boundary *C* in T-dual AdS (Alday, Maldacena, Gaiotto, Sever, Viera): integrability, Y-system, ...

Applications & Implications

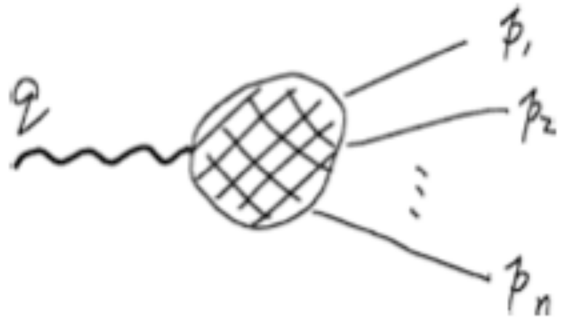
- New Hidden Symmetries
- WL conformally invariant => dual conformal symmetry of amplitudes (part of Yangian) (Drummond, Henn, Korchemsky, Sokatchev;AB, Heslop, Travaglini; Plefka, Beisert, McLoughlin, Loebbert, Bargheer, Galleas)
  - many new developments: momentum twistors, new formulation of loop integrands, duality with correlators... (Arkani-Hamed, Cachazo, Trnka, Kaplan, Bourjailly, Mason, Skinner, Bullimore, Hodges, Alday, Maldacena, Korechemsky, Sokatchev, Eden, Heslop...)
- New Results for Amplitudes
  - dual conformal symmetry fixes 4 & 5 point amplitudes to all orders (non-perturbative result!)(Drummond, Henn, Korchemsky, Sokatchev)
  - WL much simpler than amplitude integrals
    - ALL n-point, 2-loop MHV amplitudes numerically (Anastasiou-AB-Heslop-Khoze-Spence)
    - Analytic (17 page !) formula for n=6 (Del-Duca, Duhr, Smirnov) expressible in 2 lines!! (see later)(Goncharov,Spradlin,Vergu,Volovich)

Motivation and short review of applications of form factors in N=4 super Yang-Mills (SYM) and QCD

Form Factors in N=4

- more general objects than correlation functions, Wilson loops, amplitudes: e.g. Wilson loops with operator insertions, correlators of Wilson loops ...
- Form Factors: interpolate between correlators and amplitudes, partially off-shell

$$\int d^4x \, e^{-iqx} \, \langle 1 \cdots n | \mathcal{O}(x) | 0 \rangle \; = \; \delta^{(4)}(q - \sum_{i=1}^{n} p_i) \, \langle 1 \cdots n | \mathcal{O}(0) | 0 \rangle$$



- Function of: choice of operator, and the momenta & helicities of the on-shell states

Sudakov Form Factors

- Simplest example: Sudakov FF (n=2):
- governed by an evolution equation => exponentiation

$$F_2(q^2) = \exp \sum_{l=1}^{\infty} a^l \left(-\frac{q^2}{\mu^2}\right)^{-l\epsilon} \left[-\frac{\gamma_K^{(l)}}{4(l\epsilon)^2} + \dots \right] \qquad q^2 = (p_1 + p_2)^2$$

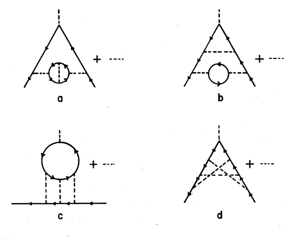
  - dim'l regularisation $D = 4 - 2\epsilon$,'t Hooft coupling $a \sim N g_{YM}^2$
  - γ_K = "cusp anomalous dimension",
- In N=4 2-loop Sudakov FF first studied by Van Neerven (will rederive that later)
- planar amplitudes in N=4 factorise into finite hard part and product of Sudakov FFs (capturing universal IR divs)

$$A_n \sim \left[\prod_{i=1}^{n} F_2((p_i + p_{i+1})^2) \right]^{1/2} H_n(p_1, \dots, p_n)$$

Appears in many physical contexts

- Three-loop correction to electron g–2

72 diagrams like



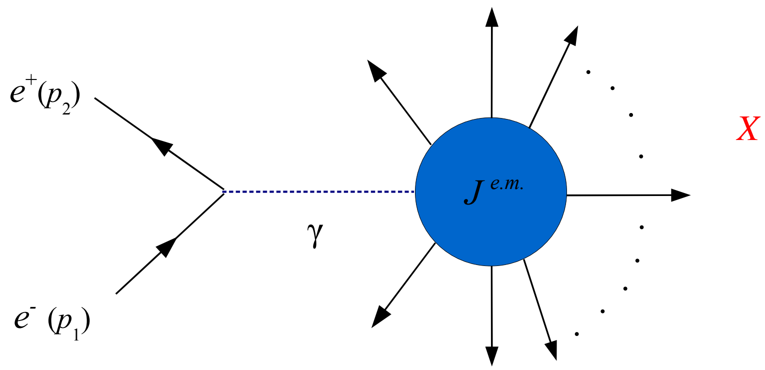
$$= (1.181241456...)\,(\alpha_{\text{e.m.}}/\pi)^3$$

(Cvitanovic & Kinoshita '74)

(Laporta & Remiddi '96)

- wild oscillations between individual diagram
- result is O(1) $\Rightarrow$ mysterious cancellations
- explained by grouping Feynman diagrams into gauge invariant combinations (Cvitanovic)

- $e^+ e^- \rightarrow$ hadrons (LEP):



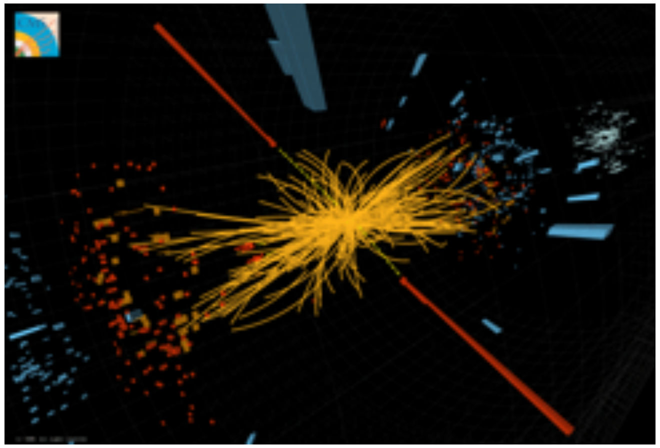
$$e\,\bar{v}(p_2)\gamma_\mu u(p_1)\;\frac{\eta^{\mu\nu}}{(p_1+p_2)^2}\;(-e)\langle X|\,J_\nu^{e.m.}(0)\,|0\rangle$$

$e^+ e^- \rightarrow$ hadrons (X)

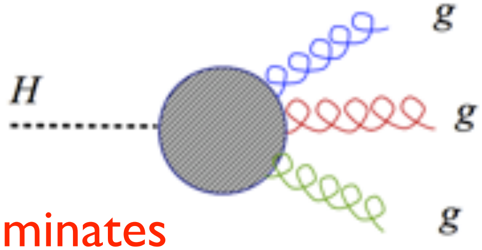
all orders in α_{strong}, first order in $\alpha_{\text{e.m.}}$

Form Factor of
hadronic electromagnetic current

- NNLO calculation for X=3 partons used for precision determination of strong coupling constant (Gehrmann, Glover, Heinrich,...)
- Relevant master integrals worked out by (Gehrmann, Remiddi)
- Can think of Form Factor as (good approximation of) amplitudes in theories with couplings of different strength!



- Higgs + multi-gluon amplitudes
  - dominant Higgs production channel at the LHC through gluon fusion
  - coupling to gluons through a fermion loop
    - proportional to the mass of the quark ⇒ top quark dominates



  - for $M_H < 2\,m_t$ integrate out the top quark

- Effective Lagrangian description $\mathcal{L}_{\rm eff} \sim H\,{\rm Tr}\,F^2$
  - coupling $\frac{\alpha_S}{12\pi v}\,, \; v = 246 GeV$ is independent of m_t

- In N=4 different operators related by SUSY:
  - form factor of $\mathrm{Tr}\,(F_{\mathrm{SD}})^2$ (= amplitude of a different theory!)

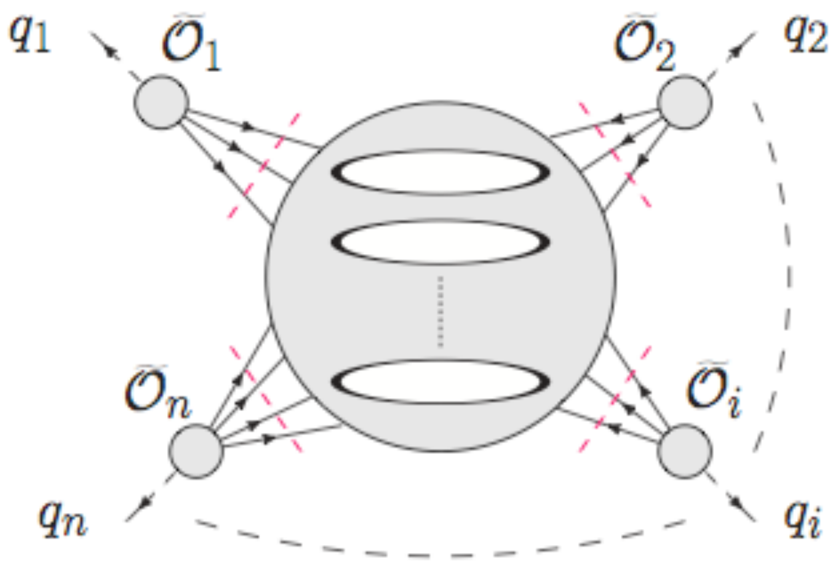
$$F_{\mathrm{Tr}\,F^2_{\mathrm{SD}}}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\;\; \langle state|\, \mathrm{Tr}\, F^2_{\mathrm{SD}}(x)\,|0\rangle$$

  - in N=4 SYM, this is related to the form factor of $\mathrm{Tr}\,(\phi_{12})^2$

$$F_{\mathrm{Tr}\phi^2_{12}}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\;\; \langle state'|\, \mathrm{Tr}\, \phi^2_{12}(x)\,|0\rangle$$

    - $\mathrm{Tr}\;\phi^2{}_{12}$ and $\mathrm{Tr}\;F_{\mathrm{SD}}{}^2$ part of the same 1/2 BPS supermultiplet
    - supersymmetric form factor of the chiral part of the stress tensor multiplet (Brandhuber, Gurdogan, Mooney, Yang, GT)

- form factors also appear in unitarity cuts of momentum space correlation functions: see e.g. Engelund-Roiban for a recent application



Tree-Level Form Factors in N=4 SYM

Tree level Form Factors in N=4 SYM

- Simplest form factors: consider scalar 1/2 BPS operators
  - e.g. $O(x)$ = Tr $(\phi_{12}\ \phi_{12})(x)$ where $\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{\phi}^{CD}$
  - Sudakov form factor: $<0 \mid O(0) \mid \phi_{12}(p_1)\ \phi_{12}(p_2)>$
    Important note: O is a colour singlet
  - equal to 1 at tree level
- Supersymmetry
  $$Q^{\alpha}_{A}\phi_{12} = 0 \quad , A = 1,2$$
  $$\bar{Q}^{\dot{\alpha}A}\phi_{12} = 0 \quad , A = 3,4$$
  - acting with $Q^{\alpha}_{A=3,4}$ creates new operators which are part of chiral part of stress-tensor multiplet (schematically): e.g.

    $$Tr(\phi\lambda) \qquad\qquad Tr(\lambda\lambda)$$

  - up to (we will encounter this at several places)
    $$Tr(F^2_{SD} + g\psi[\phi,\psi] + g^2[\phi,\phi]^2)$$ on-shell Lagrangian:

MHV Form Factor

- Simplest generalisation of Sudakov FF with n-particle state
- "MHV" family: add positive-helicity gluons

$$\int d^4x \; e^{iqx} \; \langle 0| \,\mathrm{Tr}(\phi_{12}\phi_{12})(x)\; |g^+(p_1)\cdots\phi_{12}(p_i)\cdots\phi_{12}(p_j)\cdots g^+(p_n)\rangle$$

$$= \frac{\langle i\, j\rangle^2}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle} \quad \delta^{(4)}(q - \sum_i p_i) \quad \underline{\textbf{tree}}$$

Recall

$$p_{\alpha\dot{\alpha}} \;:=\; \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$

$$\langle l\, m\rangle \;:=\; \epsilon_{\alpha\beta}\, \lambda_l^\alpha\, \lambda_m^\beta$$

- $F_{\text{MHV}}(1,\ldots,i\ldots,j,\ldots,n) \;=\; \dfrac{\langle i\, j\rangle^2}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle}$
- Calculation: Feynman diagrams or BCFW recursion:
  - result can be guessed from spinor weights, dimensionality (and factorisation)

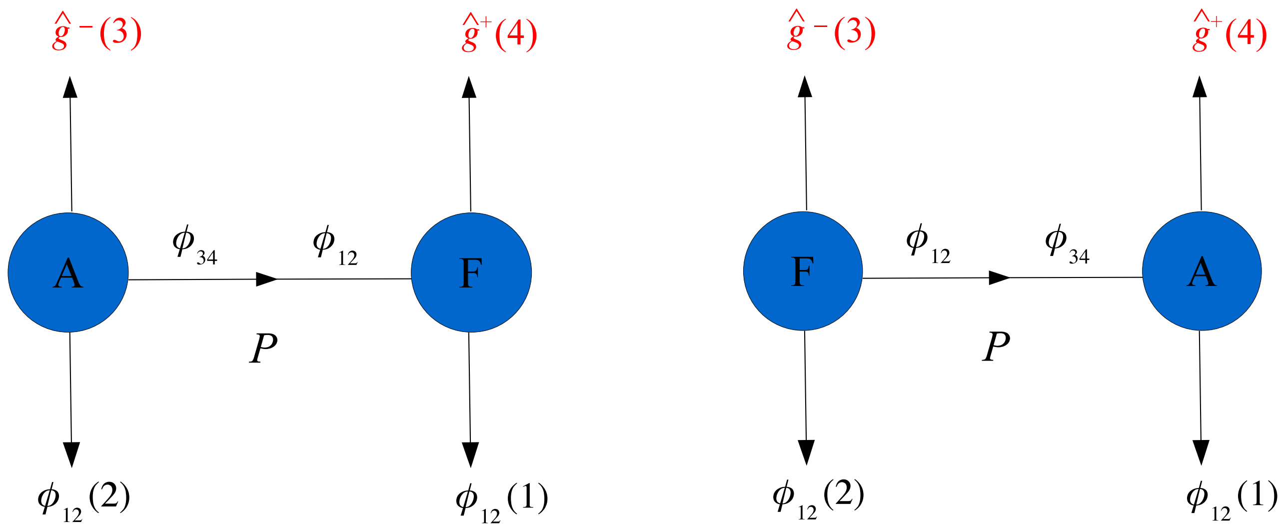
- structure very similar to that of MHV amplitudes in N=4
  - Simplicity (simplest of all form factors)
  - holomorphic function of spinor variables
  - localises on a line in Penrose's twistor space, as MHV amplitudes
  - numerator can be derived from a supersymmetric δ-function (later!)

- Non-MHV form factors: add g^-'s in the external state

$$F_{\rm NMHV}(1,\ldots,4) \;=\; \langle 0|\,{\rm Tr}(\phi_{12}\phi_{12})(0)\;|\;\phi_{12}(p_1)\,\phi_{12}(p_2)\,g^-(p_3)\,g^+(p_4)\rangle$$

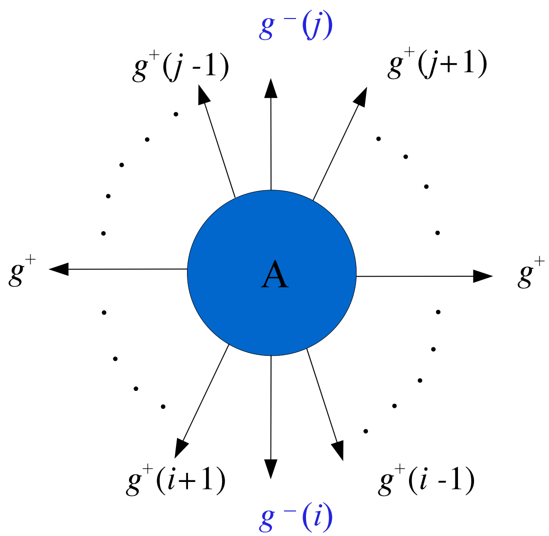
- Tree level: BCFW recursion relations

2 recursive diagrams



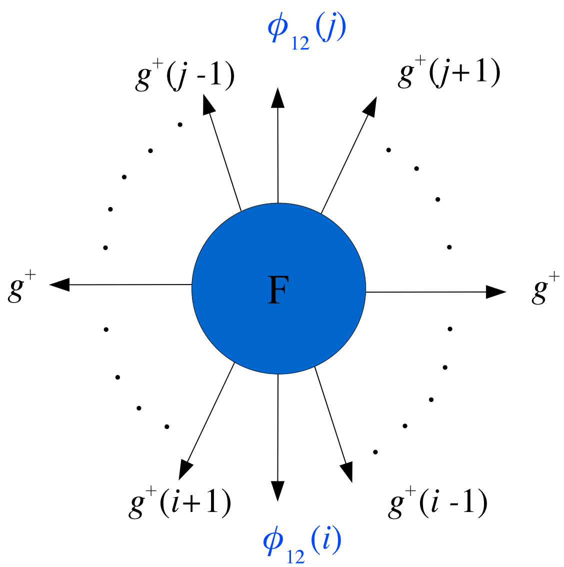
$$F_{\rm NMHV}(1,\ldots,4) \;=\; \frac{[24]^2}{[23][34]}\,\frac{1}{s_{234}}\,\frac{\langle 1\,|q|\,4]}{\langle 1\,|q|\,2]} \;+\; \frac{\langle 13\rangle^2}{\langle 34\rangle\langle 41\rangle}\,\frac{1}{s_{341}}\,\frac{\langle 3\,|q|\,2]}{\langle 1\,|q|\,2]}$$

- Tree level: MHV diagram expansion
  - in addition to usual MHV amplitudes, continued off shell to vertices...



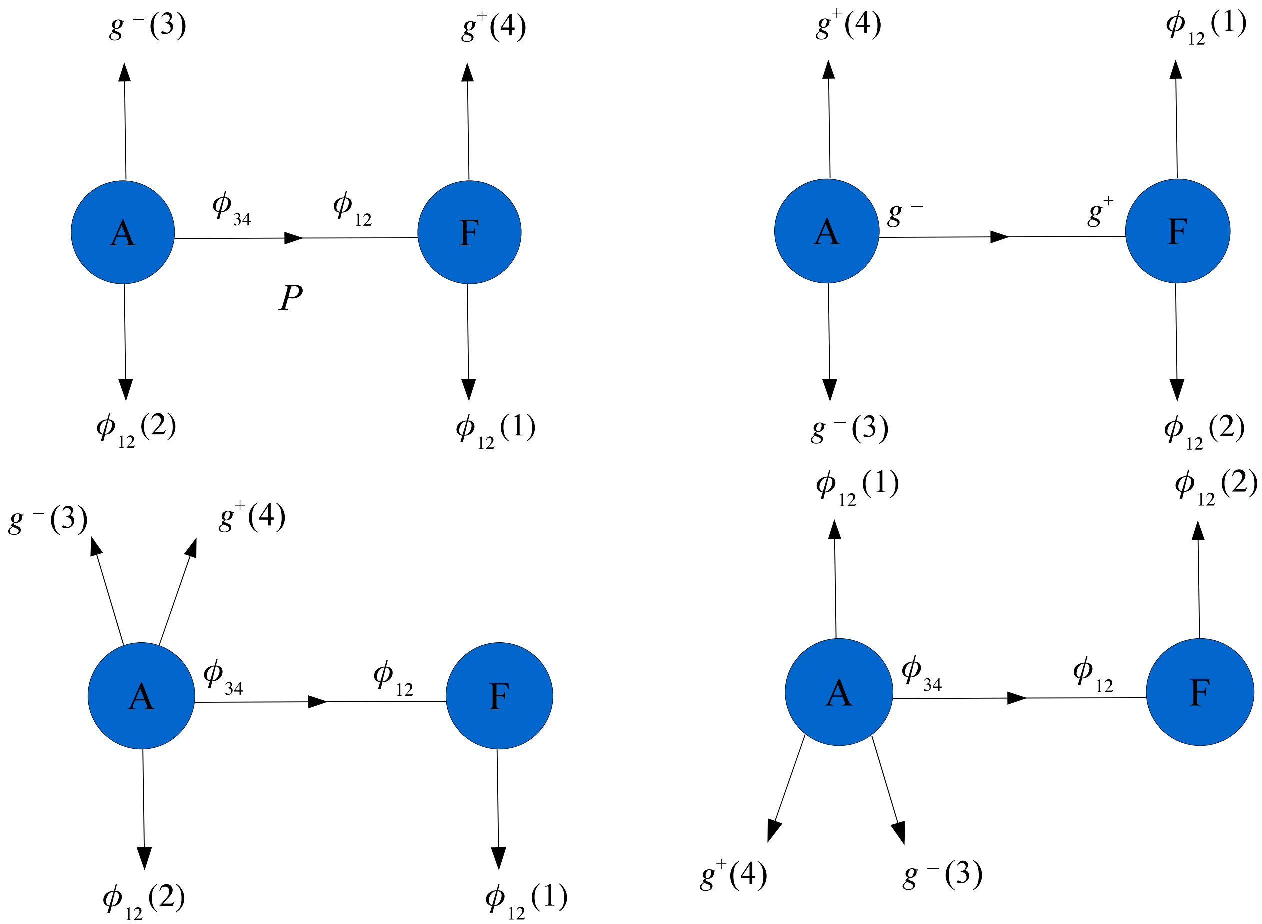
$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle}$$

  - ...add the MHV form factor as a new vertex



$$= \frac{\langle i\, j\rangle^2}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle}$$

- 4 MHV diagrams (can be reduced to 3)



- result agrees with recursion relation
- independent of choice of reference spinor

Comments

- large classes of tree level form factors have been calculated using MHV diagrams and BCFW recursion relations
- in N=4 form factors with different operator insertions (and slightly modified on-shell states) are related by SUSY Ward identities => see Super Form Factors later
- tree form factors are important input in the calculation of loop-level form factors using unitarity!

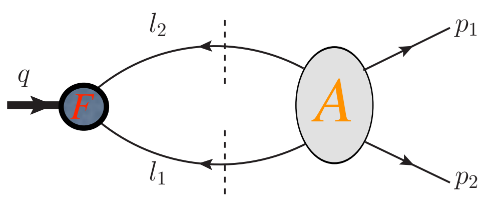
One-Loop Form Factors in N=4 SYM

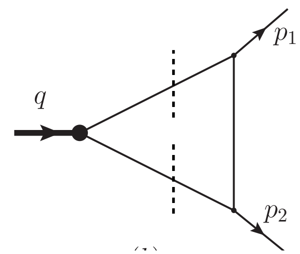
One loop Form Factors in N=4

(Brandhuber, Spence, GT, Yang; + Gurdogan & Mooney)

- Warm-up: Sudakov form factor from unitarity

$$F(q^2) \ := \ \langle 0|\, \mathrm{Tr}(\phi_{12}\phi_{12})(0)\ |\phi_{12}(p_1)\phi_{12}(p_2)\rangle \qquad q := p_1 + p_2$$





$$\left[F(q^2)\right]^{1\,\mathrm{loop}} = 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}+\mathcal{O}(\epsilon)\Big]$$

$D = 4 - 2\epsilon$ ☜

regulates infrared divergences

- agrees with calculation of van Neerven

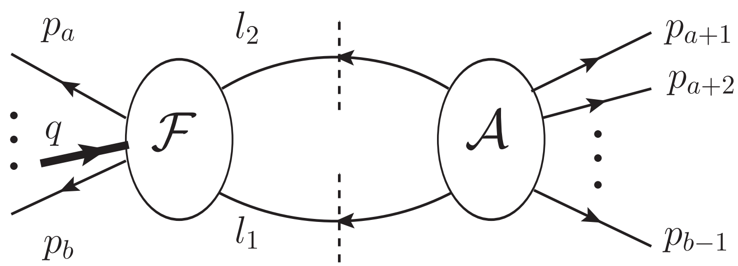
Multi-leg One-Loop MHV Form Factors

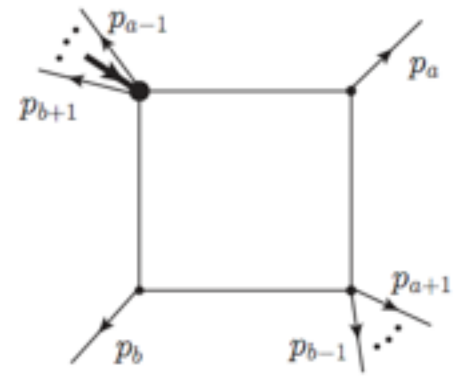
$$F_{\rm MHV} \;=\; \langle 0|\,{\rm Tr}(\phi_{12}\phi_{12})(0)\;|g^+(p_1)\cdots\phi_{12}(p_i)\cdots\phi_{12}(p_j)\cdots g^+(p_n)\rangle$$

- MHV:
  - In N=4 result proportional to tree level form factor $F^{(0)}$
  - sum of two-mass easy box functions and triangle functions
  - result very similar to the MHV amplitude...
  - ...except that q can be inserted in all possible ways ("nonplanarity" of momentum flow)
  - and MHV amplitude contains only box functions

One-Loop MHV Form Factors

- Calculation
- use generic 2-particle cuts





a "two-mass easy" box function:
two opposite legs, p_a and p_b, are massless

- n-point MHV one-loop FF

- $$F^{(1)}_{\rm MHV}(1,\ldots,n) \;=\; F^{\rm tree}_{\rm MHV}(1,\ldots,n)\Big[-\sum_{l=1}^{n}\frac{(-s_{ll+1})^{-\epsilon}}{\epsilon^2} + \sum_{a,b}{\rm Fin}^{\rm 2me}(p_a,p_b,P,Q)\Big]$$

Super Form Factors in N=4 SYM

Supersymmetric form factors

(Brandhuber, GT, Yang; + Gurdogan & Mooney; Bork, Kazakov, Vartanov)

- In N=4 SYM: re-package component amplitudes into superamplitudes

- From form factors to super form factors:
  - supersymmetrise the state (Nair)
  - we can also supersymmetrise the operator!
  - supersymmetry relates e.g. $\mathrm{Tr}\,\phi^2_{12}$ and $\mathrm{Tr}\,F_{\mathrm{SD}}^2$ form factors

- We could do this using harmonic superspace but will use here a more pedestrian approach

N=4 superamplitudes -- recap:

momentum

$$\mathcal{A}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \delta^{(4)}\Big(\sum_i \lambda_i \tilde{\lambda}_i\Big)\, \delta^{(8)}\Big(\sum_i \eta_i \lambda_i\Big)\, A(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$$

(Nair)

supermomentum

- η_A fermionic variables, A =1,...,4 is an SU(4) index
- expansion in η generates all component amplitudes
  - p_i powers of η_i corresponds to helicity $h_i = 1 - p_i/2$
- MHV: $A = \dfrac{1}{\langle 1\,2\rangle\langle 2\,3\rangle \cdots \langle n\,1\rangle}$
- gluons i^-, j^- : pick coefficient of $(\eta_i)^4(\eta_j)^4 \langle i\,j\rangle^4$ $\Rightarrow$

$$A(1_{g+}, \ldots, i_{g-}, \ldots, j_{g-}, \ldots, n_{g+}) = \frac{\langle ij\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}$$

(Parke & Taylor)

- $(\lambda, \tilde{\lambda}, \eta)$ $\Rightarrow$ Nair's on-shell, chiral superspace

- Chiral part of the stress-tensor multiplet operator $\mathcal{T}(x, \theta^{3,4})$
  - recent analysis of correlators (Eden, Heslop, Korchemsky, Sokatchev)
  - $\mathrm{Tr}\,(\phi_{12}\ \phi_{12})(x)$ is the lowest component; contains also the (on-shell) Lagrangian
- (Full) stress-tensor multiplet $\mathcal{T}(x, \theta^{3,4}, \bar{\theta}_{1,2})$
  - 1/2 BPS condition expressed very nicely:
    - $\theta^{1,2}, \bar{\theta}_{3,4}$ (conjugate variables) don't appear = (Grassmann analyticity condition) (Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev; Howe & West)
  - Chiral part is simply $\mathcal{T}(x, \theta^{3,4}, \bar{\theta}_{1,2} = 0)$

- More explicitly:

$$\mathcal{T}(x,\theta^{3,4},\bar\theta_{1,2}) = Tr(W_{12}W_{12})$$
$$= Tr(\phi_{12}\phi_{12}) + \ldots + (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)T_{\mu\nu} + \ldots$$

  - still a non-chiral object !

- Chiral part of $\mathcal{T}$: (Eden, Heslop, Korchemsky, Sokatchev)

$$\mathcal{T}(x,\theta^{3,4}) := \mathcal{T}(x,\theta^{3,4},\bar\theta_{1,2}=0)$$
$$= Tr(\phi_{12}\phi_{12}) + \ldots + \frac{1}{3}(\theta)^4\mathcal{L}$$

  - natural choice to match to Nair chiral superspace
  - $\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + \sqrt{2}g\lambda^{\alpha A}[\phi_{AB},\lambda^B_\alpha] - \frac{1}{8}g^2[\phi^{AB},\phi^{CD}][\phi_{AB},\phi_{CD}]\Big]$

- Super form factor is defined as (super) Fourier transform:

$$\mathcal{F} = \int d^4x d^4\theta_\alpha^{3,4} e^{-iq\cdot x - i\gamma_{3,4}^\alpha \theta_\alpha^{3,4}} \langle 1 \ldots n | \mathcal{T}(x, \theta^{3,4}) | 0 \rangle$$

  - depends on q and $\gamma_{3,4}^\alpha$ (conjugate to x and $\theta_\alpha^{3,4}$)
  - $<1 \ldots n| = <0| \Phi(p_1,\eta_1) \ldots \Phi(p_n,\eta_n)$ Nair superstate

$$\Phi(p,\eta) = g^+(p) + \eta_A \psi^A(p) + \frac{\eta_A \eta_B}{2!}\phi^{AB}(p) + \epsilon^{ABCD}\frac{\eta_A\eta_B\eta_C}{3!}\tilde{\psi}_D(p) + \eta_1\eta_2\eta_3\eta_4 g^-(p)$$

- Next: action of supersymmetry
  - charges $Q_{3,4}^\alpha$ and $Q_{1,2}^\alpha$
  - from (super) translation invariance and doing Fourier transf.:

$$\mathcal{F} = \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\delta^{(4)}(\gamma_{3,4} - \sum_i \lambda_i \eta_{i;3,4})\delta^{(4)}(\sum_i \lambda_i \eta_{i;1,2}) R$$

- **Standard derivation of Ward identities:**
  (Grisaru, Pendleton, van Nieuwenhuizen; Mangano & Parke; Elvang, Freedman & Kiermaier)

  - denote $\mathcal{F} \;:=\; \langle 1\cdots n|\,\mathcal{O}\;|0\rangle \;=\; \langle 0|\Phi(1)\cdots\Phi(n)\,\mathcal{O}\;|0\rangle$
  - expand $0 \;=\; \langle 0|[s\,,\Phi(1)\cdots\Phi(n)\,\mathcal{O}\,]|0\rangle$
    (s is a symmetry generator):

$$0 \;=\; \langle 0|\Phi(1)\cdots\Phi(n)\,[s\,,\,\mathcal{O}]\,|0\rangle \;+\; \sum_{i=1}^{n}\langle 0|\,\Phi(1)\cdots[s\,,\,\Phi(i)]\cdots\Phi(n)\,\mathcal{O}|0\rangle$$

- **Realisation of the supersymmetry generators:**

  - the algebra of the Q-generators closes off-shell on the chiral part of the stress-tensor multiplet (Eden, Heslop, Korchemsky, Sokatchev)

$$\left[Q_{1,2},\mathcal{T}(x,\theta^{3,4})\right]=0\;,\quad \left[Q_{3,4},\mathcal{T}(x,\theta^{3,4})\right]=i\frac{\partial}{\partial\theta^{3,4}}\mathcal{T}(x,\theta^{3,4})$$

  - we also have

$$\bar{Q}^{3,4}_{\dot\alpha}=-\theta^{\alpha;3,4}\frac{\partial}{\partial x^{\dot\alpha\alpha}}$$

  - so that $Q_{3,4}$, $\bar{Q}^{3,4}$ close on translations!

- standard action of supersymmetry charges on Nair superstate: $\langle i|\, s \;=\; \langle 0|\Phi(i)\, s \;=\; \langle 0|\, [\Phi(i)\,,\, s]$

  - $P:$ $$\langle i|P \;=\; \langle i|p_i$$

  - $Q:$ $$\langle i|Q \;=\; \langle i|\lambda_i\eta_i$$

  - $\bar{Q}\;:$ $$\langle i|\,\bar{Q} \;=\; \langle i|\,\frac{\partial}{\partial\eta_i}\tilde{\lambda}_i$$

- we obtain three Ward identities:

$$Q_{1,2},\ Q_{3,4}: (\sum_i \lambda_i \eta_{i;1,2})\mathcal{F}(q,\gamma_{3,4};1,\dots,n) = 0$$

$$(\sum_i \lambda_i \eta_{i;3,4} - \gamma_{3,4})\mathcal{F}(q,\gamma_{3,4};1,\dots,n) = 0$$

$$\bar{Q}^{3,4}: \qquad (\sum_i \tilde{\lambda}_i \frac{\partial}{\partial \eta_{i;3,4}} - q\frac{\partial}{\partial \gamma_{3,4}})\mathcal{F}(q,\gamma_{3,4};1,\dots,n) = 0$$

- solution is

$$\mathcal{F} = \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\delta^{(4)}(\gamma_{3,4} - \sum_i \lambda_i \eta_{i;3,4})\delta^{(4)}(\sum_i \lambda_i \eta_{i;1,2})R$$

- constraint on *R*: $(\sum_i \tilde{\lambda}_i \frac{\partial}{\partial \eta_{i;3,4}} - q\frac{\partial}{\partial \gamma_{3,4}})R = 0$ on δ-function support

- Note action of supercharges after Fourier transform: $x \longrightarrow q$

$$\theta \longrightarrow \gamma$$

A few explicit examples

- Super MHV: $$R^{\mathrm{MHV}} = \frac{1}{\langle 12\rangle \cdots \langle n1\rangle}$$

- Hence:

$$\mathcal{F}_{MHV} = \frac{\delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\delta^{(4)}(\gamma_{A=3,4} - \sum_i \lambda_i \eta_{i;A=3,4})\delta^{(4)}(\sum_i \lambda_i \eta_{i;A=1,2})}{\langle 12\rangle \ldots \langle n1\rangle}$$

- Form factor of Tr $(\phi_{12}\ \phi_{12})$

$$\mathcal{F} = \int d^4x d^4\theta^{3,4} e^{-iq\cdot x - i\gamma_{3,4}\theta^{3,4}} \langle 1 \ldots n | \mathcal{T}(x, \theta^{3,4}) | 0 \rangle$$

$$= \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(4)}(\gamma_{A=3,4} - \sum_i \lambda_i \eta_{i;A=3,4}) \delta^{(4)}(\sum_i \lambda_i \eta_{i;A=1,2}) R_{MHV}$$

$$\mathcal{T}(x, \theta^{3,4}) = Tr(\phi_{12}\phi_{12}) + \ldots + (\theta^{3,4})^4 \mathcal{L}$$

- need $(\theta)^0$ component of $\mathcal{T} \Rightarrow (\gamma)^4$ term

- result: $\mathcal{F}_{Tr\phi_{12}^2} = \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(4)}(\sum_i \lambda_i \eta_{i;A=1,2}) R_{MHV}$

- see earlier result $\dfrac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle}$
  (i,j scalars, remaining particles are g^+'s)

- $< i\ j >^2$ from expansion of fermionic δ-function

- Form factor of on-shell Lagrangian

$$\mathcal{F} = \int d^4x d^4\theta^{3,4} e^{-iq\cdot x - i\gamma_{3,4}\theta^{3,4}} \langle 1 \ldots n | \mathcal{T}(x, \theta^{3,4}) | 0 \rangle$$

$$= \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(4)}(\gamma_{A=3,4} - \sum_i \lambda_i \eta_{i;A=3,4}) \delta^{(4)}(\sum_i \lambda_i \eta_{i;A=1,2}) R_{MHV}$$

- NOW: need $(\theta)^4$ component of $\mathcal{T} \Rightarrow (\gamma)^0$ term
- result: $\mathcal{F}_{\mathcal{L}} = \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \; \delta^{(8)}(\sum_i \lambda_i \eta_i) \; R$
- e.g. MHV: $\delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \, \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$
- $<i\, j>^4$ from δ-function
- same as Higgs + multi-gluon amplitude (Dixon, Glover, Khoze)
- same as gluon MHV amplitude for $q = 0$: $\mathcal{F}_{\mathcal{L}}|_{q=0} \sim \frac{\partial}{\partial(1/g^2)} \mathcal{A}$
  (Lagrangian insertion trick) (Intriligator; Eden, Howe, Schubert, Sokatchev, West)

- Maximally non-MHV form factor
    - there are other very simple form factors besides MHV:

$$\langle 1^- \cdots n^- |\, \mathrm{Tr}\, F_{\mathrm{SD}}^2(0)\, |0\rangle \;=\; \frac{q^4}{[1\,2]\,[2\,3]\,\cdots[n\,1]}$$

    - This is the same as the Higgs plus "minus only" gluons amplitude (Dixon, Glover, Khoze)

$$A_n(H,\, g_1^-, \cdots, g_n^-)$$

- This can be derived from our supersymmetric form factor, starting from the MHV super form factor!

2-loop Sudakov FF in N=4 from Unitarity

Higher Loop Form Factors in N=4

- Sudakov form factor for $\mathcal{O} = Tr\Phi_{12}^2$ in N=4 first studied by van Neerven using Feynman diagrams up to 2 loops
  - depends only on $q^2 = (p_1 + p_2)^2$
- Recent rederivations using unitarity, SUSY Feynman rules or different regulators (AB-Travaglini-Yang, Bork-Kazakov-Vartanov, Henn-Moch-Naculich)
  - Recent 3 loop calculation (Gehrmann-Henn-Huber) reproduces IR divergences of know amplitudes
- Also $Tr\Phi_{12}^k \ , \ k > 2$ at 2 loops (Bork-Kazakov-Vartanov)

Exponentiation of Sudakov FF

$$\log[\sum_{l=0}^{\infty} a^l F^{(l)}(q^2,\epsilon)] = \sum_{l=1}^{\infty} a^l [f^{(l)}(\epsilon) F^{(1)}(q^2, l\epsilon) + C^{(l)}]$$

- Consider log of full form factor

$$\log[\sum_{l=0}^{\infty} \lambda^l F^{(l)}(q^2,\epsilon)] = \sum_{l=1}^{\infty} \lambda^l [f^{(l)}(\epsilon) F^{(1)}(q^2, l\epsilon) + C^{(l)}]$$

- expressed in terms of one-loop form factor and universal constants (1-loop cusp anomalous dimensions, ...):

$$f(\epsilon)^{(l)} = f_0^{(l)} + f_1^{(l)}\epsilon + f_2^{(l)}\epsilon^2 \quad , \quad C^{(l)}$$

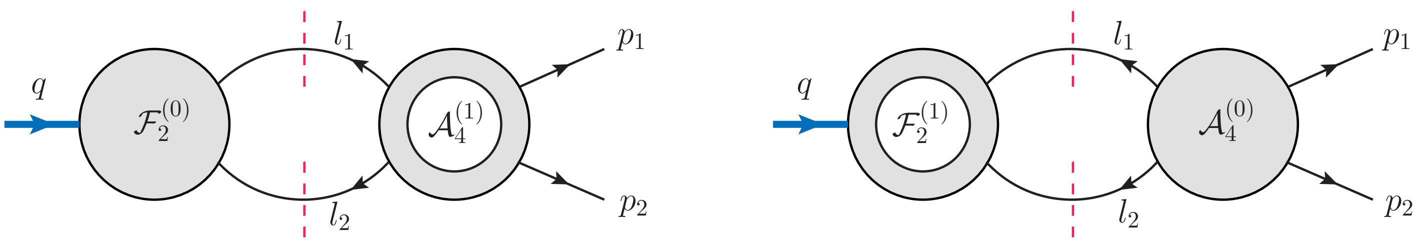
- At two loops this implies a recursive relation
- $\log F|_{2-loop} = F^{(2)} - \frac{1}{2}(F^{(1)})^2 = f^{(2)}(\epsilon) F^{(1)}(q^2,\epsilon) + C^{(2)}$

Two loop calculation

- Sudakov:

$$F(q^2) \ := \ \langle 0 |\, \mathrm{Tr}(\phi_{12}\phi_{12})(0) \ |\phi_{12}(p_1)\phi_{12}(p_2)\rangle$$

  - simple illustration of the technique



  - F proportional to δ^{ab}
  - non-planar one-loop amplitude are also relevant in the cuts!

- One-loop complete amplitude (planar + non-planar)

Complete: $\mathcal{A}^{(1)} = A_{\rm P}^{(1)} + A_{\rm NP}^{(1)}$ where

P: $$A_{\rm P}^{(1)} = N \sum_{\sigma \in S_n/\mathbb{Z}_n} {\rm Tr}(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}})\, A_{n;1}^{[1]}(\sigma_1, \ldots, \sigma_n)$$

NP: $$A_{\rm NP}^{(1)} = \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_{n;c}} {\rm Tr}(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_{c-1}}})\, {\rm Tr}(T^{a_{\sigma_c}} \cdots T^{a_{\sigma_n}})\, A_{n;c}^{[1]}(\sigma_1, \ldots, \sigma_n)$$

- $A^{[1]}{}_{n;c}$ linear combinations of colour-ordered amplitudes $A^{[1]}{}_{n;1}$
  (Bern, Dixon, Dunbar, Kosower)

- contracting with tree form factor ~ $\delta^{a_1 a_2}$ we get:

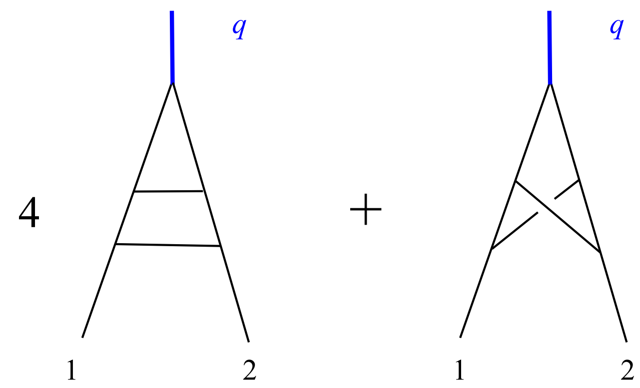
P: $$N\, \delta^{a_1 a_2} {\rm Tr}(T^{a_1} T^{a_2} T^X T^Y) = N^2\, {\rm Tr}(T^X T^Y) = N^2\, \delta^{XY}$$

NP: $$\delta^{a_1 a_2}\, {\rm Tr}(T^{a_1} T^{a_2})\, {\rm Tr}(T^X T^Y) = N^2\, {\rm Tr}(T^X T^Y) = N^2\, \delta^{XY}$$

both leading in colour!

- Final result obtained very easily:

$$F^{(2)}(q^2) = \quad 4 \; [\text{planar ladder}] \; + \; [\text{crossed ladder}]$$



- agrees with van Neerven
- two-loop result exponentiates as expected: (more on this later)

$$\left[F(q^2)\right]^{1\,\text{loop}} = 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}+\mathcal{O}(\epsilon)\Big]$$

$$\left[\text{Log}\;F(q^2)\right]^{2\,\text{loop}} = (-q^2)^{-2\epsilon}\Big[\frac{\zeta_2}{\epsilon^2}+\frac{\zeta_3}{\epsilon}+\mathcal{O}(\epsilon)\Big]$$

- result is transcendental (non-planar integral topology)

Next: Iterative structure beyond 2 points ?

(Brandhuber, Yang, GT)

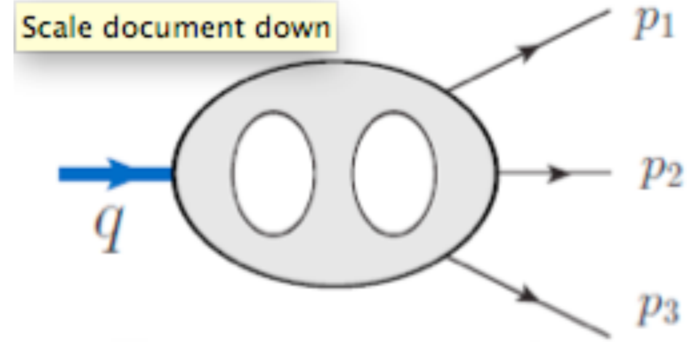
- Goal: test exponentiation beyond the Sudakov
  - our present data: three-point MHV form factors at two loops
  - construct the Log of the form factor at two loops

$$\left[\log \mathcal{F}_n\right]^{(2)} := \mathcal{F}_n^{(2)} - \frac{1}{2}\left(\mathcal{F}_n^{(1)}(\epsilon)\right)^2$$

  - recast it in terms of one-loop form factor
  - Ingredients:
    - one-loop form factor to higher orders in ϵ
    - two-loop form factor
    - (generalised) unitarity cuts

2-loop, 3-point FF in N=4 from Unitarity and Symbology

3-point 2-loop MHV FF in N=4



- Will focus on 3-point FF at 2-loops

$$F_3^{\mathrm{MHV},(2)}(1,2,3) := \langle \phi_{12}(p_1)\phi_{12}(p_2)g^+(p_3)|Tr(\phi_{12}^2)(0)|0\rangle$$

- nothing special about helicities of external particles: tree level FF (and color factor) can be stripped off

$$F_3^{\mathrm{MHV},(2)}(1,2,3) := F_3^{\mathrm{MHV},(0)}(1,2,3)\mathcal{G}_3^{(2)}$$

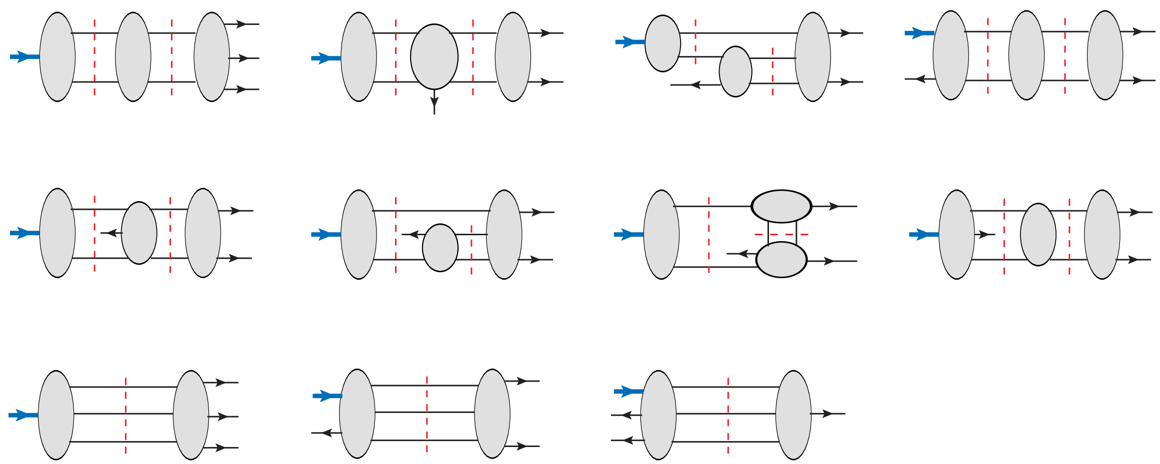
$$F_3^{\mathrm{MHV},(0)}(1,2,3) = \frac{\langle 12\rangle}{\langle 23\rangle\langle 31\rangle} \qquad \langle ij\rangle \equiv \lambda_a^i \lambda_b^j \varepsilon^{ab} \ , \ [ij] \equiv \tilde{\lambda}_{\dot{a}}^i \tilde{\lambda}_{\dot{b}}^j \varepsilon^{\dot{a}\dot{b}}$$

- $\mathcal{G}_3^{(2)}$ is a helicity-blind, scalar function of kinematic variables and is symmetric under permutations of external legs; depends on: $s_{12} = (p_1+p_2)^2, \ s_{23} = (p_2+p_3)^2, \ s_{31} = (p_1+p_3)^2$

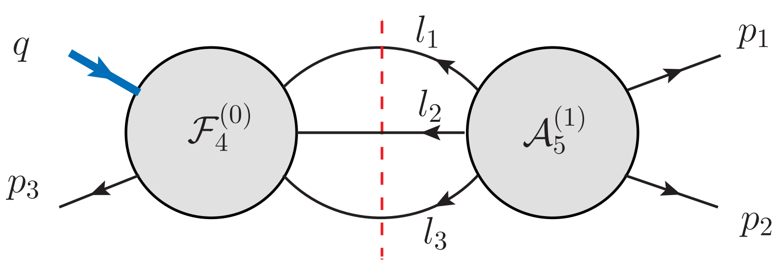
$$q^2 = s_{12} + s_{23} + s_{31}$$

Generalised unitarity

- Strategy:
  - detect all possible integrals and coefficients with iterated 2-particle cuts (some numerator ambiguities)



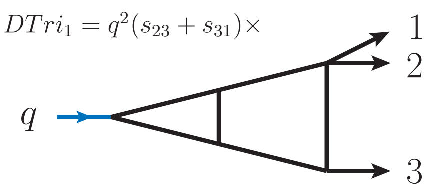
  - remove remaining ambiguities with triple-cuts such as



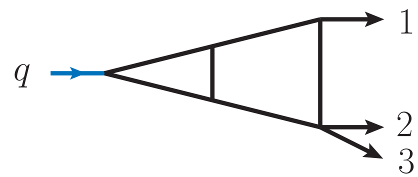
- # Final result:

$$\frac{F_3^{(2)}}{F_3^{\text{tree}}} = \sum_{i=1}^{2} (DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}$$

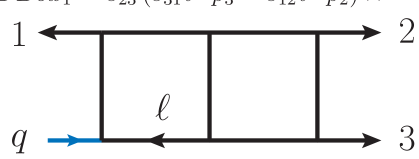
$DTri_1 = q^2(s_{23}+s_{31})\times$



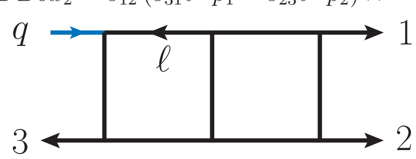
$DTri_2 = q^2(s_{12}+s_{31})\times$



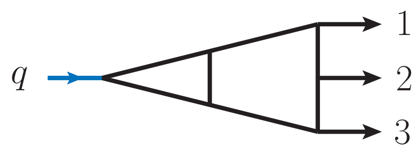
$DBox_1 = s_{23}\left(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2\right)\times$



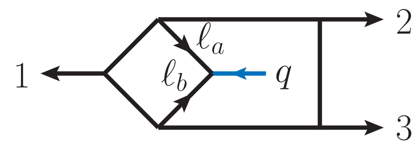
$DBox_2 = s_{12}\left(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2\right)\times$



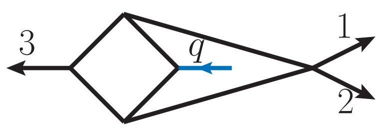
$TriPent = q^2 s_{12}s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23}+s_{31})\times$



* result expressed in terms of two-loop planar and non-planar integrals

Numerical results

- dimensional regularisation
- some integrals are known analytically in terms of (several pages of) Goncharov multiple polylogs (Gehrmann, Remiddi)
  - fixed degree of transcendentality: 2-loop, degree = 4
  - will see later that answer contains only classical polylogs!
- We used sophisticated numerical tools for evaluation, Mellin-Barnes representation:
  MB.m (Czakon)   AMBRE (Gluza, Kajda, Riemann, Yundin)
- reproduce expected IR divergences (sanity check)

Finite Remainder (Exponentiation?)

- Define n-point FF remainder a la ABDK/BDS for amplitudes

$$\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{G}_n^{(1)}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\,\mathcal{G}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

- Signature of exponentiation at 2-loops: $\mathcal{R}_n^{(2)} = 0!$ true for n=2 (Sudakov) but what about n>2?

- with 2 universal n- and kinematic independent quantities!

- both fixed by Sudakov FF (n=2) and collinear factorisation:

$$f(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2 \quad , \quad C^{(2)} = 4\zeta_4$$

- $f(\epsilon)$ contains cusp anomalous and collinear anomalous dimensions; cusp anomalous dim. known to all orders in coupling (integrability) (Beisert-Eden-Staudacher)

Finite remainder $\mathcal{R}_n^{(2)}$

- Properties:
- Finite, regulator independent
- Subtraction removes universal IR divergences and trivializes collinear limits

$$p_i || p_{i+1} : \quad s_{i,i+1} \to 0 : \quad \mathcal{R}_{n+1}^{(2)} \to \mathcal{R}_n^{(2)} \quad \text{and} \quad \mathcal{R}_3^{(2)} \to 0$$

- function of scaling invariant ratios; not dual conformal invariant; in general remainder depends on 3 n - 7 ratios
  - here n=3: two independent ratios

$$(u,v,w) := (\frac{s_{12}}{q^2}, \frac{s_{23}}{q^2}, \frac{s_{31}}{q^2}) \quad , \quad q^2 = s_{12} + s_{23} + s_{31} = (p_1 + p_2 + p_3)^2$$

$$\to u + v + w = 1$$

Finite remainder numerical results

(u, v, w)	numerical $\mathcal{R}_3^{(2)}$	est. error
(1/3, 1/3, 1/3)	-0.1519	0.02
(1/4, 1/4, 1/2)	-0.1203	0.02
(1/5, 2/5, 2/5)	-0.1301	0.02
(1/2, 1/3, 1/6)	-0.1080	0.03

- IR divergences cancel as expected
- important cross check for unitarity based calculation
- non-trivial remainder: varies between zero (collinear limit) and about -0.15 (symmetric kinematic point)
- Q: is there a more direct way to get the analytic answer?

Analytic Answer from Symbology

(Goncharov 2009; Goncharov-Spradlin-Vergu-Volovich 2010)

- Ultimate Goal: obtain results algebraically without ever touching an integral just from symmetries, physical constraints
- results of loop calculations given in terms of Goncharov polylogarithms: logs, classical polylogs, harmonic polylogs...
  - degree-k Goncharov polylog = k-fold iterated integral

$$G(a_k, a_{k-1}, \ldots, a_1; z) = \int_0^z G(a_{k-1}, \ldots, a_1; t) \frac{dt}{t - a_k}, \quad G(z) = 1$$

  - Principle of maximal transcendentality:
    N=4 SYM: k=2 x L(oops)
- the symbol S(f) captures important analytic structure of f while "forgetting" lower degree pieces and locations of branch cuts

- Important property of the symbol is that it reduces complicated polylog identities to linear algebra!!
  - First application of this to 2-loop, 6-point amplitude in N=4 by Goncharov-Spradlin-Vergu-Volovich
    Analytic 17 page formula (Del Duca-Duhr-Smirnov) =>
    1-2 lines of classical polylogs (GSVV)

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}.$$

- More recently application of this in QCD for 2-loop Higgs + 3 parton amplitudes (Duhr)

Crash Course in Symbology

- consider degree k = 2 L *pure functions:*
  recursive definition as k-fold iterated integrals =>

$$df^{(k)} := \sum_i f_i^{(k-1)} d\log R_i$$

- The Symbol is then defined recursively:
  element of *k-fold tensor product of algebraic functions R*

$$\mathcal{S}(f^{(k)}) := \sum_i \mathcal{S}(f_i^{(k-1)}) \otimes R_i = \ldots = \sum_{\vec{r}} R_{r_1} \otimes \ldots \otimes R_{r_k}$$

- integrability: $d^2 f^{(k)} = 0$

- further properties

$$\ldots \otimes R_i R_j \otimes \ldots = \ldots \otimes R_i \otimes \ldots + \ldots \otimes R_j \otimes \ldots$$

$$\ldots \otimes c \otimes \ldots = 0$$

Symbology

- Examples: $\mathcal{S}(\log x \log y) = x \otimes y + y \otimes x$

$$Li_2(x) = -\int \log(1-x) d\log(x) \rightarrow$$

$$\mathcal{S}(Li_2(x)) = -(1-x) \otimes x$$

$$\mathcal{S}(Li_3(x)) = -(1-x) \otimes x \otimes x$$

the symbol forgets this guy

$$Li_2(x) \quad + \quad Li_2(1-x) = -\log(x)\log(1-x) + \zeta_2 \rightarrow$$

$$-(1-x) \otimes x - x \otimes (1-x) = \mathcal{S}(-\log(x)\log(1-x))$$

- If f has a discontinuity starting at t=0 then the symbol takes the form (first entry condition): (Maldacena-Sever-Viera)

$$\mathcal{S}(f) = \sum_t t \otimes \mathcal{S}(\Delta_t f) \qquad t = (p_i + \ldots + p_j)^2 = 0$$

  - important: unitarity tells us where discontinuities start

- integrability: not every k-tensor is the symbol of a function!

$$\sum_r dR_i^r \wedge dR_{i+1}^r R_1^r \otimes \ldots R_{i-1}^r \otimes R_{i+2}^r \otimes \ldots R_k^r = 0$$

  - Example $x \otimes y$ is not the symbol of a function since $dx \wedge dy \neq 0$
  - But $x \otimes y + y \otimes x = S(\log x \log y)$

- physical constraints can restrict the symbol further
  - Wilson loop OPE (Gaiotto-Maldacena-Sever-Vieira)
  - kinematic limits (Regge limits) (Dixon-Drummond-Henn)

- Symbols have been applied successfully in several examples
  - 2&3-loop amplitudes in 1+1 dim'l kinematics (Goddard, Heslop, Khoze)
  - MHV 2&3-loop, NMHV 2-loop amplitudes, symbol of all MHV 2-loop amps. (Caron-Huot, Dixon-Drummond-Heslop)
  - 2-loop,3-point Form Factor (AB-Gang-Travaglini)

2-Loop Form Factor from Symbology

- Strategy: construct most general symbol assuming all entries are taken from $\{u, v, w, 1-u, 1-v, 1-w\}$ (degree 4 tensor as we are interested in 2-loops $\Rightarrow$

  1296 terms) and impose:

  - integrability: S is symbol of a function
  - symmetries (permutation of legs, scaling invariance),
  - first entry condition (branch cuts start at correct location)
  - further constraints on 2nd and last entry (Gaiotto Maldacena Sever Vieira, Caron-Huot, Dixon Drummond Henn)
  - trivial collinear limits

- $\Rightarrow$ Unique symbol!

$$
\begin{aligned}
\mathcal{S}^{(2)} \quad = \quad & -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\
& -u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\
& -u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\
& +u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\
& +u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\
& +u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\
& +u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \\
& + \text{cyclic permutations}
\end{aligned}
$$

Comments on entry conditions:

(I) look at the final entries

(II) if the first entry is e.g. u then the 2nd entry is taken from the list {v,w,1-u}

Also easy to check collinear limit i.e. S = 0 if u→0:

for this you need to use that the symbol vanishes if an entry is 1 e.g. (1-u) → 1 and that for u→ 1 you also have w= 1-v due to u+v+w= 1.

- Final task: find a function that has this symbol
- Luckily our symbol has a particular symmetry property (Goncharov)

$$\mathcal{S}^{(2)}_{abcd} - \mathcal{S}^{(2)}_{bacd} - \mathcal{S}^{(2)}_{abdc} + \mathcal{S}^{(2)}_{badc} - (a \leftrightarrow c\,,\, b \leftrightarrow d) \;=\; 0$$

  - ...which implies only classical (poly)logs may appear

$$\log x_1 \log x_2 \log x_3 \log x_4 \;,\; \mathrm{Li}_2(x_1) \log x_2 \log x_3 \;,\; \mathrm{Li}_2(x_1) \mathrm{Li}_2(x_2) \;,\; \mathrm{Li}_3(x_1) \log x_2 \text{ and } \mathrm{Li}_4(x_i)$$

- but what are the allowed arguments? Nobody knows so need to guess, for us the following list was sufficient

$$\left(u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v}\right)$$

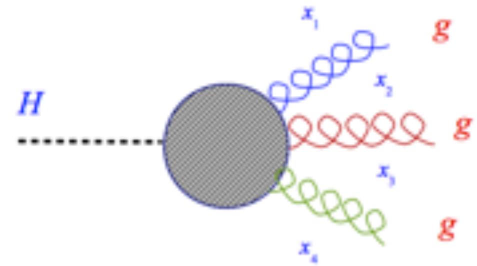
- $\Rightarrow$ One line answer for appropriately chosen font size

Final Answer

$$\mathcal{R}_3^{(2)} = -2\left[J_4\left(-\frac{uv}{w}\right) + J_4\left(-\frac{vw}{u}\right) + J_4\left(-\frac{wu}{v}\right)\right] - 8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right) + \frac{\log^4 u_i}{4!}\right]$$
$$-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i) + \frac{\log^2 u_i}{2!}\right]^2 + \frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2 - \frac{\log^4(uvw)}{4!} - \frac{23}{2}\zeta_4\,,$$
$$J_4(z) := \mathrm{Li}_4(z) - \log(-z)\mathrm{Li}_3(z) + \frac{\log^2(-z)}{2!}\mathrm{Li}_2(z) - \frac{\log^3(-z)}{3!}\mathrm{Li}_1(z) - \frac{\log^4(-z)}{48}$$

- In this answer remaining ambiguities of the symbol have been removed using collinear limits and symmetries
- This answer combines pages and pages of meaningless lists of Goncharov polylogarithms: only classical polylogs appear!
- Next: mysterious connection to QCD amplitude g g → g H
  (Gehrmann-Glover-Jaquier-Koukoutsakis)

Higgs + parton amplitudes in QCD



- Higgs + 3 partons (Koukoutsakis 2003; Gehrmann, Glover, Jaquier & Koukoutsakis 2011)
  - $H \rightarrow g^+ \, g^- \, g^-$ MHV
  - $H \rightarrow g^+ \, g^+ \, g^+$ maximally non-MHV
  - $H \rightarrow q \;\; \bar{q} \;\; g$ fundamental quarks

$$F^{\text{tree}}(H, g_1^-, g_2^-, g_3^+) = \frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

$$F^{\text{tree}}(H, g_1^+, g_2^+, g_3^+) = \frac{q^4}{[1\,2]\,[2\,3]\,[3\,1]}$$

$$q^2 = M_H^2$$

- In N=4 SYM:
  - $(H \, g^+ \, g^- \, g^-)$ and $(H \, g^+ \, g^+ \, g^+)$ both derived from super form factor
  - from supersymmetric Ward identities:

$$\frac{F^{(L)}(g_1^-, g_2^-, g_3^+)}{F^{\text{tree}}(g_1^-, g_2^-, g_3^+)} = \frac{F^{(L)}(g_1^+, g_2^+, g_3^+)}{F^{\text{tree}}(g_1^+, g_2^+, g_3^+)} = \mathcal{G}^{(L)}(u, v, w) \quad \leftarrow \text{what we computed}$$

• Feynman diagram based calculation of $H \rightarrow g^{+}\, g^{-}\, g^{-}$ amplitude in QCD gives this!

$$
\begin{aligned}
B_\alpha^{(1)} &= 0\,, && \text{(A.2)}\\
C_\alpha^{(1)} &= \frac{1}{6}\Big[G(1-z,y) - H(1,z) + H(0,z) + G(0,y)\Big]\\
&\quad - \frac{1}{6}\Big((y+z)(1-y) - z^2\Big) - i\frac{\pi}{2}\,, && \text{(A.3)}\\
A_\beta^{(1)} &= \Bigg[\frac{1}{2}\Big(- G(1-z,0,y) - H(1,0,z) - G(0,1-z,y) - H(0,1,z)\\
&\quad - H(0,z)G(1-z,y) + G(0,y)H(1,z) - G(0,y)H(0,z)\Big) + G(-z,1-z,y)\\
&\quad + G(1,0,y) - H(1,z)G(-z,y)\\
&\qquad + \frac{11}{12}\Big(- G(1-z,y) + H(1,z) - H(0,z) - G(0,y)\Big)\Bigg]\\
&\quad - \frac{z}{6(1-y-z)}\Big(1 - \frac{1}{1-y-z} + \frac{z}{1-y-z}\Big) - \frac{\pi^2}{12} + i\frac{11\pi}{4}\,, && \text{(A.4)}\\
B_\beta^{(1)} &= 0\,, && \text{(A.5)}\\
C_\beta^{(1)} &= \frac{1}{6}\Bigg[G(1-z,y) - H(1,z) + H(0,z) + G(0,y)\\
&\quad - \frac{z}{1-y-z}\Big(- 1 + \frac{1}{1-y-z} - \frac{z}{1-y-z}\Big)\Bigg] - i\frac{\pi}{2}\,, && \text{(A.6)}\\
A_\gamma^{(1)} &= \frac{1}{6}\Bigg[3\Big(- G(0,y)H(1,z) + H(0,z)G(1-z,y) + H(0,1,z) + G(0,1-z,y)\\
&\quad - G(1,0,y) + G(1-z,0,y)\Big) + 6\Big(H(1,z)G(-z,y) - G(-z,1-z,y)\Big)\\
&\quad + 5\Big(G(0,y) + H(0,z)\Big) + \frac{13}{2}\Big(- H(1,z) + G(1-z,y)\Big) - \frac{89}{6} - \frac{3(z-1)}{2y}\Bigg]\\
&\quad - i\frac{11\pi}{4}\,, && \text{(A.7)}\\
B_\gamma^{(1)} &= \frac{1}{6}\Bigg[3\Big(- G(0,y)H(0,z) + G(1,0,y) - H(1,0,z)\Big) - \frac{\pi^2}{2} - \frac{27}{2} - \frac{3(z-1)}{2y}\Bigg]\,, && \text{(A.8)}\\
C_\gamma^{(1)} &= \frac{1}{6}\Bigg[\frac{1}{2}\Big(- G(0,y) - H(0,z)\Big) + 2\Big(H(1,z) - G(1-z,y)\Big) + \frac{10}{3}\Bigg] + i\frac{\pi}{2}\,. && \text{(A.9)}
\end{aligned}
$$

B. Two-loop helicity coefficients

The finite contributions to the renormalized two-loop helicity coefficients, decomposed in colour factors according to (5.20) are:

$$
\begin{aligned}
A_\alpha^{(2)} = \Bigg[\frac{1}{2}\Big(&- G(1-z,-z,1-z,0,y) - G(1-z,-z,0,1-z,y) + G(1-z,1-z,0,0,y)\\
&+ G(1-z,0,-z,1-z,y) + G(1-z,0,1-z,0,y) - G(1-z,0,1,0,y)\\
&+ G(1-z,0,0,1-z,y) + H(1,1,0,0,z) + H(1,0,1,0,z) + H(1,0,0,1,z)\\
&+ H(1,0,0,z)G(1-z,y) - H(1,0,z)G(-z,0,y) + H(1,0,z)G(1-z,-z,y)\\
&- G(1,0,y)H(1,0,z) + H(1,z)G(1-z,-z,0,y) + H(1,z)G(1-z,0,-z,y)
\end{aligned}
$$

$$
\begin{aligned}
&- H(1,z)G(1-z,0,0,y) - G(0,-z,1-z,y) + G(0,-z,y)H(1,0,z)\\
&- G(0,1-z,-z,1-z,y) + G(0,1-z,1-z,0,y) + G(0,1-z,0,1-z,y)\\
&+ G(0,1-z,-z,y)H(1,z) - G(0,-z,1-z,y) + G(0,1-z,1-z,y)H(0,z)\\
&- G(0,1-z,1,0,y) - G(0,1-z,0,y)H(1,z) - G(0,1-z,0,y)H(0,z)\\
&- H(0,z)G(1-z,0,1-z,y) - H(0,z)G(1-z,0,0,y) - G(0,y)H(1,1,0,z)\\
&- G(0,y)H(1,0,1,z) + G(0,y)H(1,0,0,z) - G(0,1,1-z,0,y) + H(0,1,1,0,z)\\
&- H(0,1,1,z)G(0,y) - G(0,1,0,1-z,y) + H(0,1,0,1,z)\\
&+ H(0,1,0,z)G(1-z,y) + H(0,1,0,z)G(0,y) + G(0,1,0,y)H(1,z)\\
&- G(0,1,0,y)H(0,z) - H(0,1,z)G(-z,0,y) - H(0,1,z)G(1,0,y)\\
&+ H(0,1,z)G(0,-z,y) + G(0,0,1-z,1-z,y) - G(0,0,1-z,y)H(1,z)\\
&- G(0,0,1-z,y)H(0,z) + H(0,0,1,1,z) + H(0,0,1,z)G(0,y)\\
&+ H(0,0,z)G(1-z,1-z,y) + H(0,0,z)G(1-z,0,y) + H(0,0,z)G(0,1-z,y)\\
&+ G(0,0,y)H(1,1,z) - G(0,0,y)H(1,0,z) - G(0,0,y)H(0,1,z)\\
&+ G(0,0,y)H(0,0,z) + H(0,z)G(-z,1-z,0,y) + H(0,z)G(-z,0,1-z,y)\\
&- H(0,z)G(1-z,-z,1-z,y) - H(0,z)G(1-z,1-z,0,y)\\
&- H(0,z)G(1-z,1,0,y) + H(0,z)G(1,1-z,0,y) + H(0,z)G(1,0,1-z,y)\Big)\\
&\qquad + \Big(- G(-z,1-z,1-z,0,y) - G(-z,1-z,0,1-z,y)\\
&- G(-z,0,1-z,1-z,y) - G(1-z,1-z,-z,1-z,y) + G(1-z,1-z,1,0,y)\\
&+ G(1-z,1,1-z,0,y) + G(1-z,1,0,1-z,y) - G(1-z,1,0,0,y)\\
&- G(1-z,0,-z,1-z,y) - G(1,1-z,0,0,y) - H(1,1,0,z)G(-z,y)\\
&- H(1,1,z)G(-z,0,y) - G(1,0,1-z,0,y) + H(1,0,1,z)G(-z,y)\\
&- G(1,0,0,1-z,y) + G(1,0,0,y)H(1,z) - H(1,0,z) + H(1,0,z)G(-z,1-z,y)\\
&- H(1,0,z)G(1-z,1-z,y) + H(1,0,z)G(1-z,0,y) + H(1,z)G(-z,1-z,0,y)\\
&+ H(1,z)G(-z,0,1-z,y) + H(1,z)G(1-z,1-z,-z,y)\\
&- H(1,z)G(1-z,1,0,y) - G(0,-z,1-z,1-z,y) + G(0,-z,1-z,0,y)\\
&+ G(0,-z,0,1-z,y) + G(0,-z,1-z,y) - G(0,-z,y)H(1,1,z)\\
&+ G(0,-z,1-z,y)H(0,z) - G(0,-z,0,y)H(1,z) + G(0,-z,1-z,y)H(1,z)\\
&+ H(0,1,1,z)G(-z,y) - H(0,1,z)G(-z,1-z,y) + H(0,1,z)G(1-z,1-z,y)\\
&+ H(0,1,z)G(1-z,0,y) + G(0,0,-z,1-z,y) - G(0,0,-z,y)H(1,z)\\
&- G(0,0,1,0,y) - H(0,z)G(-z,1-z,1-z,y) + H(0,z)G(1-z,0,0,y)\\
&- H(0,z)G(1,0,0,y)\Big)\\
&\qquad + \frac{3}{2}\Big(- G(0,-z,1-z,y)H(0,z) - H(0,1,z)G(1-z,-z,y)\\
&- H(0,0,1,z)G(1-z,y)\Big)\\
&\qquad + 2\Big(- G(-z,-z,-z,1-z,y) + G(-z,-z,1-z,1-z,y)
\end{aligned}
$$

$$+ G(-z,1-z,-z,1-z,y) + G(1-z,-z,-z,1-z,y) - G(1,1,1,0,y)$$
$$+ G(1,1,0,0,y) + H(1,1,0,z)G(1-z,y) + H(1,1,z)G(-z,-z,y) + G(1,0,1,0,y)$$
$$+ H(1,z)G(-z,-z,-z,y) - H(1,z)G(-z,-z,1-z,y)$$
$$- H(1,z)G(-z,1-z,-z,y) - H(1,z)G(1-z,-z,-z,y) + G(0,1,1,0,y)$$
$$+ H(0,1,z)G(-z,-z,y) + H(0,0,1,z)G(-z,y)\Big)$$
$$+\frac{11}{24}\Big(- G(0,1-z,y)H(0,z) + H(0,1,z)G(0,y) - H(0,z)G(1-z,0,y)$$
$$+ G(0,y)H(1,0,z)\Big)$$
$$+\frac{11}{6}\Big(- G(-z,-z,1-z,y) + G(1-z,1-z,0,y) + G(1-z,0,1-z,y)$$
$$+ G(1-z,0,0,y) - G(1,1,0,y) + H(1,0,0,z) - H(1,0,z)G(1-z,y)$$
$$+ H(1,z)G(-z,-z,y) - H(1,z)G(1-z,0,y) + G(0,1-z,1-z,y)$$
$$- G(0,1-z,y)H(1,z) + G(0,1-z,0,y) - H(0,1,1,z) + H(0,1,z)G(-z,y)$$
$$+ G(0,0,1-z,y) + H(0,0,z)G(1-z,y) + H(0,0,z)G(0,y) - G(0,0,y)H(1,z)$$
$$+ G(0,0,y)H(0,z) + H(0,z)G(1-z,1-z,y) + G(0,y)H(1,1,z)\Big)$$
$$+\frac{11}{4}\Big(- H(1,0,1,z) + H(0,1,z)G(1-z,y) + H(0,1,0,z)\Big)$$
$$+\frac{11}{3}\Big(- G(1,0,0,y) - G(-z,1-z,1-z,y) - H(1,1,z)G(-z,y)$$
$$+ H(1,z)G(-z,1-z,y)\Big)$$
$$+\frac{55}{12}\Big(- G(1-z,-z,1-z,y) + H(1,z)G(1-z,-z,y) - G(0,1,0,y)\Big)$$
$$+\frac{\pi^2}{3}\Big(G(1,1,y) - G(0,1,y) + H(1,1,z) - G(1,0,y) + H(0,0,z) + G(0,0,y)\Big)$$
$$+\Big(\frac{\zeta_3}{4} - \frac{33\pi^2}{8}\Big)\Big(- G(1-z,y) + H(1,z) - H(0,z) - G(0,y)\Big)$$
$$+\Big(- \frac{121}{48} + \frac{\pi^2}{3}\Big)\Big(- G(1-z,1-z,y) - H(1,1,z) + H(1,z)G(1-z,y)$$
$$- H(0,0,z) - G(0,0,y)\Big)$$
$$+\Big(\frac{49}{48} - \frac{\pi^2}{8}\Big)\Big(- G(1-z,0,y) - G(0,1-z,y) - H(0,z)G(1-z,y)$$
$$+ G(0,y)H(1,z) - G(0,y)H(0,z)\Big)$$
$$+\Big(\frac{67}{18} + \frac{\pi^2}{12}\Big)\Big(G(-z,1-z,y) + G(1,0,y) - H(1,z)G(-z,y)\Big)$$
$$- \Big(\frac{245}{144} - \frac{\pi^2}{8}\Big)H(1,0,z) - \Big(\frac{389}{144} - \frac{\pi^2}{8}\Big)H(0,1,z)$$
$$- \Big(\frac{13}{8} + \frac{451\pi^2}{96}\Big)G(1-z,y) + \Big(\frac{13}{8} + \frac{1265\pi^2}{288}\Big)H(1,z)$$
$$+ \Big(\frac{13}{8} + \frac{1133\pi^2}{288}\Big)\Big(- H(0,z) - G(0,y)\Big) + \frac{11\pi^2 G(1,y)}{36}$$

$$-\frac{1}{36}\Big(\frac{5029\pi^2}{24} - 72\zeta_4 + \frac{99\zeta_3}{4} + \frac{3\pi^4}{16} - \frac{1321}{6}\Big)\Big]$$
$$+ \frac{1}{6}\Big((y+z)(1-y) - z^2\Big)\Big[+ G(1,0,y) + G(-z,1-z,y) - H(1,z)G(-z,y)$$
$$+ G(0,y)H(1,z) - G(0,y)H(0,z)$$
$$+\frac{1}{2}\Big(- G(1-z,0,y) - H(1,0,z) - G(0,1-z,y) - H(0,1,z)$$
$$- H(0,z)G(1-z,y) - G(0,y)H(1,z) + G(0,y)H(0,z)\Big)$$
$$- \frac{41}{12}\Big(G(1-z,y) - H(1,z)\Big) + \frac{19}{12}H(0,z) - \frac{3\pi^2}{2} + \frac{247}{18}\Big]$$
$$+ \Big(\frac{25z}{12}(- 1 + \frac{1}{1-y-z}) - \frac{15z^2}{4(1-y-z)} - \frac{yz}{6} + \frac{5z^2}{6}(1 + \frac{2z}{1-y-z}$$
$$+ \frac{1}{(1-y-z)^2}(1 - 2z + z^2))\Big)\Big[G(0,y)H(0,z) + H(1,0,z) - G(1,0,y)\Big]$$
$$+ \Big(\frac{25y}{12z} - \frac{9y}{4} + \frac{yz}{6} - \frac{15y^2}{4z} + y^2 + \frac{5y^2}{6}(\frac{2y}{z} + \frac{1}{z^2}(1 - 2y + y^2))\Big) \times$$
$$\Big[G(1-z,0,y) - G(1,0,y) - G(-z,1-z,y) + H(1,z)G(-z,y) + G(0,1-z,y)$$
$$+ H(0,1,z) - G(0,y)H(1,z)\Big]$$
$$+ \Big(\frac{25z}{12y} - \frac{9z}{4} + \frac{yz}{6} - \frac{15z^2}{4y} + z^2 + \frac{5z^2}{6}(\frac{2z}{y} + \frac{1}{y^2}(1 - 2z + z^2))\Big) \times$$
$$\Big[H(0,z)G(1-z,y) - G(-z,1-z,y) + H(1,z)G(-z,y)\Big]$$
$$+ \frac{1}{36}\Big(63 - 93(y+z) + 4yz + \frac{30z}{y}(1 - 2z + z^2) + \frac{30y}{z}(1 - 2y + y^2)$$
$$+ 30(y^2 + z^2)\Big)\Big[G(1-z,y) - H(1,z)\Big]$$
$$- \frac{1}{36}\Big(- 63z + 60z^2 - 30z^2(1-z)(\frac{1}{y} + \frac{1}{1-y-z}) + 26y(1-y-z)\Big)H(0,z)$$
$$- \frac{1}{36}\Big(\frac{93z(1-z)}{2} - \frac{145y}{2} + \frac{27yz}{2} + \frac{79y^2}{2} + \frac{30z}{1-y-z}(- 1 + 2z - z^2)$$
$$- \frac{30y^2(1-y)}{z}\Big)G(0,y)$$
$$+ \frac{\pi^2}{2}\Big(\frac{25z}{36(1-y-z)} - \frac{2z}{9} + \frac{5z^2}{18(1-y-z)}(2z - \frac{9}{2} + \frac{1}{1-y-z}(1 - 2z + z^2))$$
$$- \frac{7z^2}{36} - \frac{19yz}{36} + \frac{17y(1-y)}{36}\Big)$$
$$+i\pi\Big[\frac{55}{24}\Big(- H(0,z)G(1-z,y) - H(0,z)G(0,y) - H(0,1,z)$$
$$- 2H(1,z)G(-z,y) + H(1,z)G(0,y) - H(1,0,z) - G(1-z,0,y)$$
$$+ 2G(-z,1-z,y) - G(0,1-z,y) + 2G(1,0,y) + \frac{11}{6}(- H(0,z) + H(1,z)$$

$$- G(1-z,y) - G(0,y)) + \frac{1}{3}(y(1-y-z)+z(1-z))\Big) - \frac{77\pi^2}{288} + \frac{3\zeta_3}{4} + \frac{185}{24}\Big] \qquad \text{(B.1)}$$

$$B_\alpha^{(2)} = 0\,, \qquad \text{(B.2)}$$

$$C_\alpha^{(2)} = 0\,, \qquad \text{(B.3)}$$

$$
\begin{aligned}
D_\alpha^{(2)} = {} & \frac{y}{12}\Big(\frac{y}{z^2}(-1+2y-y^2) + \frac{2}{z}(1-y^2) - 4 + 2z + y\Big)\Big[+ G(1-z,0,y) \\
& + G(0,1-z,y) + H(0,1,z) - G(0,y)H(1,z) - G(1,0,y) + H(1,z)G(-z,y) \\
& - G(-z,1-z,y)\Big] \\
& + \frac{z}{12}\Big(\frac{z}{(1-y-z)^2}(-1+2z-z^2) + \frac{2}{1-y-z}(1-z^2) - 2 - z - 2y\Big) \times \\
& \Big[H(1,0,z) + G(0,y)H(0,z) - G(1,0,y)\Big] \\
& + \frac{z}{12}\Big(\frac{z}{y^2}(-1+2z-z^2) + \frac{2}{y}(1-z^2) - 4 + z + 2y\Big)\Big[H(0,z)G(1-z,y) \\
& + H(1,z)G(-z,y) - G(-z,1-z,y)\Big] \\
& + \frac{1}{36}\Big(\frac{15}{2} + \frac{3z}{y}(-1+2z-z^2) + \frac{3y}{z}(-1+2y-y^2) - \frac{9}{2}(y^2+z^2) \\
& - 3(y+z+yz)\Big)\Big[G(1-z,y) - H(1,z)\Big] \\
& - \frac{1}{36}\Big(3z^2(1-z)(\frac{1}{y} + \frac{1}{1-y-z}) - 6z - \frac{15z^2}{2}\Big)H(0,z) \\
& - \frac{1}{36}\Big(\frac{3z}{1-y-z}(1-2z+z^2) - 3z(1-z) + \frac{3y^2(1-y)}{z} \\
& - 6y - 3yz - \frac{9y^2}{2}\Big)G(0,y) - \frac{1}{18}\Big[-\frac{201}{8} + 18\zeta_3\Big] - \frac{1}{6}\Big((y+z)(1-y) - z^2\Big) \\
& + \frac{\pi^2}{72}\Big(\frac{z}{1-y-z}(2 - \frac{z}{1-y-z} + \frac{2z^2}{1-y-z} - 2z^2 - \frac{z^3}{1-y-z}) - z(2+z+2y)\Big) \\
& + i\frac{\pi}{4}\,, \qquad \text{(B.4)}
\end{aligned}
$$

$$
\begin{aligned}
E_\alpha^{(2)} = {} & \frac{1}{6}\Big[\frac{1}{2}\Big(+ G(0,1-z,y)H(0,z) - H(0,1,z)G(0,y) + H(0,z)G(1-z,0,y) \\
& - G(0,y)H(1,0,z)\Big) \\
& + 2\Big(G(-z,-z,1-z,y) - G(1-z,1-z,0,y) - G(1-z,0,1-z,y) \\
& - G(1-z,0,0,y) - G(0,1-z,0,y) + G(1,1,0,y) - H(1,0,0,z) \\
& + H(1,0,z)G(1-z,y) - H(1,z)G(-z,-z,y) + H(1,z)G(1-z,0,y) \\
& - G(0,1-z,1-z,y) + G(0,1-z,y)H(1,z) + H(0,1,1,z) - H(0,1,z)G(0,y) \\
& - H(0,1,z)G(-z,y) + H(0,1,z)G(0,y) - G(0,0,1-z,y) \\
& - H(0,0,z)G(1-z,y) - H(0,0,z)G(0,y) + G(0,0,y)H(1,z) - G(0,0,y)H(0,z)
\end{aligned}
$$

$$
\begin{aligned}
& - H(0,z)G(1-z,1-z,y) - G(0,y)H(1,1,z)\Big) \\
& \quad + 3\Big(H(1,0,1,z) + G(0,1-z,y) - H(0,1,0,z) - H(0,1,z)G(1-z,y)\Big) \\
& \quad + 4\Big(G(-z,1-z,1-z,y) + H(1,1,z)G(-z,y) + G(1,0,0,y) \\
& - H(1,z)G(-z,1-z,y)\Big) \\
& \quad + 5\Big(G(1-z,-z,1-z,y) - H(1,z)G(1-z,-z,y) + G(0,1,0,y)\Big) \\
& \quad - \frac{10}{3}\Big(G(-z,1-z,y) + G(1,0,y) - H(1,z)G(-z,y)\Big) \\
& \quad + \frac{11}{2}\Big(- G(1-z,1-z,y) - H(1,1,z) + H(1,z)G(1-z,y) - H(0,0,z) \\
& - G(0,0,y)\Big) - \frac{7}{2}\Big(- H(1,0,z) - H(0,1,z)\Big) - \frac{19G(0,1-z,y)}{6} \\
& \quad + \frac{1}{6}\Big(- G(1-z,0,y) - H(0,z)G(1-z,y) + G(0,y)H(1,z) - G(0,y)H(0,z)\Big) \\
& \quad + \Big(\frac{103}{18} + \frac{41\pi^2}{8}\Big)G(1-z,y) - \Big(\frac{103}{18} + \frac{7\pi^2}{24}\Big)H(1,z) + \Big(\frac{35}{36} - \frac{5\pi^2}{24}\Big)H(0,z) \\
& + \Big(\frac{103}{18} - \frac{5\pi^2}{24}\Big)G(0,y) - \frac{\pi^2}{3}\Big(G(1,y) + \frac{27G(1-z,y)}{2}\Big) - \frac{1}{6}\Big(\frac{1781}{12} + \frac{63\zeta_3}{2} \\
& - \frac{1879\pi^2}{24}\Big)\Big] \\
& \quad + \frac{1}{36}\Big((y+z)(1-y) - z^2\Big)\Big[6\Big(- G(-z,1-z,y) - G(1,0,y) \\
& + H(1,z)G(-z,y)\Big) \\
& \quad + 3\Big(- G(0,y)H(1,z) + G(1-z,0,y) + H(1,0,z) + G(0,1-z,y) \\
& + H(0,1,z) + H(0,z)G(1-z,y)\Big) + G(0,y)H(0,z) + 2G(0,y)H(0,z) \\
& + 23G(1-z,y) - 23H(1,z) - 19H(0,z) + \frac{\pi^2}{2} - \frac{275}{3}\Big] \\
& \quad + \Big(\frac{-7y}{6}(1-\frac{1}{z}) + \frac{11y^3}{12z^2}(\frac{z^2}{y} + \frac{1}{y} - 2 + 2z + y) - \frac{3y^2}{z}\Big)\Big[- G(1-z,0,y) \\
& + G(-z,1-z,y) + G(1,0,y) - H(1,z)G(-z,y) - G(0,1-z,y) \\
& - H(0,1,z) + G(0,y)H(1,z)\Big] \\
& \quad + \Big(\frac{-7z}{6}(1-\frac{1}{y}) + \frac{11z^3}{12y^2}(\frac{y^2}{z} + \frac{1}{z} - 2 + 2y + z) - \frac{3z^2}{y}\Big)\Big[G(-z,1-z,y) \\
& - H(1,z)G(-z,y) - H(0,z)G(1-z,y)\Big] \\
& \quad + \Big(\frac{-7z}{6}(1 - \frac{1}{1-y-z}) + \frac{11z^3}{12(1-y-z)^2}(\frac{(1-y-z)^2}{z} + \frac{1}{z} - 2 \\
& + 2(1-y-z) + z) - \frac{3z^2}{1-y-z}\Big)\Big[- H(1,0,z) + G(1,0,y) - G(0,y)H(0,z)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{36}\Big(- \frac{57}{2} + \frac{33z}{y}(- 1 + 2z - z^2 - yz) + \frac{33y}{z}(- 1 + 2y - y^2 - yz) \\
&+ \frac{123}{2}(y+z) - \frac{7yz}{2}\Big)\Big[G(1-z,y) - H(1,z)\Big] \\
&\quad - \frac{1}{36}\Big(- \frac{57}{2}(1+2z^2) + \frac{39z}{2} + 33z^2(1-z)(\frac{1}{y} + \frac{1}{1-y-z}) \\
&+ \frac{77y}{2}(- 1 + y + z)\Big)H(0,z) \\
&\quad - \frac{1}{36}\Big(\frac{33z}{1-y-z}(1-2z+z^2) + \frac{105z^2}{2}(1-\frac{1}{z}) + y(\frac{77}{2} - \frac{27z}{2} + \frac{33y}{z} \\
&- 43y - \frac{33y^2}{z})\Big)G(0,y) \\
&\quad + \frac{\pi^2}{2}\Big(\frac{7z}{18}(1 - \frac{1}{1-y-z}) + \frac{11z^2}{36(1-y-z)^2}(- 1 - (1-y-z)^2 + 2z \\
&- 2z(1-y-z) - z^2) + \frac{z^2}{1-y-z}\Big) \\
&\quad +i\pi\Big[\frac{5}{12}\Big(H(0,z)G(1-z,y) + H(0,z)G(0,y) + H(0,1,z) \\
&+ 2H(1,z)G(-z,y) - H(1,z) * G(0,y) + H(1,0,z) + G(1-z,0,y) \\
&- 2G(-z,1-z,y) + G(0,1-z,y) - 2G(1,0,y) + \frac{11}{3}(+ H(0,z) - H(1,z) \\
&+ G(1-z,y) + G(0,y))\Big) + \frac{65}{72}\Big(z(z-1) - y(1-y-z)\Big) + \frac{7\pi^2}{144} - \frac{71}{18}\Big] \,, \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
F^{(2)}_\alpha = &\quad \frac{1}{36}\Big[3\Big(G(1-z,1-z,y) + H(1,1,z) - H(1,z)G(1-z,y) + H(0,0,z) \\
&+ G(0,0,y)\Big) + G(1-z,0,y) - H(1,0,z) + G(0,1-z,y) - H(0,1,z) \\
&+ H(0,z)G(1-z,y) - G(0,y)H(1,z) + G(0,y)H(0,z) \\
&+ \frac{10}{3}\Big(- G(0,y) - H(0,z) + H(1,z) - G(1-z,y)\Big) - \frac{29\pi^2}{4}\Big] \\
&\quad + \frac{1}{36}\Big((y+z)(1-y) - z^2\Big)\Big[- G(0,y) - H(0,z) + H(1,z) \\
&- G(1-z,y) + \frac{10}{3}\Big] - \frac{z(1-y-z)}{18}G(0,y) - \frac{y(1-y-z)}{18}H(0,z) \\
&- \frac{yz}{18}\Big(G(1-z,y) - H(1,z)\Big) \\
&\quad +i\frac{5\pi}{36}\Big[- H(0,z) + H(1,z) - G(1-z,y) - G(0,y) \\
&+ \Big(y(1-z-y) + 2 + z - z^2\Big)\Big] \,, \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
A^{(2)}_\beta = &\ \frac{1}{2}\Big[- G(1-z,-z,1-z,0,y) - G(1-z,-z,0,1-z,y) + G(1-z,1-z,0,0,y) \\
&- G(1-z,0,-z,1-z,y) + G(1-z,0,1-z,0,y) - G(1-z,0,1,0,y) \\
&+ G(1-z,0,0,1-z,y) + H(1,1,0,0,z) + H(1,0,1,0,z) + H(1,0,0,1,z)
\end{aligned}
$$

$$
\begin{aligned}
&+ H(1,0,0,z)G(1-z,y) - H(1,0,z)G(-z,0,y) + H(1,0,z)G(1-z,-z,y) \\
&- G(1,0,y)H(1,0,z) + H(1,z)G(1-z,-z,0,y) + H(1,z)G(1-z,0,-z,y) \\
&- H(1,z)G(1-z,0,0,y) - G(0,-z,1-z,y)H(0,z) + G(0,-z,y)H(1,0,z) \\
&- G(0,1-z,-z,1-z,y) + G(0,1-z,-z,y)H(1,z) + G(0,1-z,1-z,y)H(0,z) \\
&+ G(0,1-z,0,1-z,y) - G(0,1-z,1,0,y) + G(0,1-z,1-z,0,y) \\
&- G(0,1-z,0,y)H(1,z) - G(0,1-z,0,y)H(0,z) - G(0,1,1-z,0,y) \\
&+ H(0,1,1,0,z) - H(0,1,1,z)G(0,y) - G(0,1,0,1-z,y) + H(0,1,0,1,z) \\
&+ H(0,1,0,z)G(1-z,y) + H(0,1,0,z)G(0,y) + G(0,1,0,y)H(1,z) \\
&- G(0,1,0,y)H(0,z) - H(0,1,z)G(-z,0,y) - H(0,1,z)G(1,0,y) \\
&+ H(0,1,z)G(0,-z,y) + H(0,z)G(-z,1-z,0,y) + H(0,z)G(-z,0,1-z,y) \\
&- H(0,z)G(1-z,-z,1-z,y) - H(0,z)G(1-z,1,0,y) + H(0,z)G(1,1-z,0,y) \\
&+ H(0,z)G(1,0,1-z,y) - H(0,z)G(1-z,1-z,0,y) \\
&- H(0,z)G(1-z,0,1-z,y) + H(0,z)G(1-z,0,0,y) - G(0,y)H(1,1,0,z) \\
&- G(0,y)H(1,0,1,z) + G(0,y)H(1,0,0,z) + G(0,0,1-z,1-z,y) \\
&- G(0,0,1-z,y)H(1,z) - G(0,0,1-z,y)H(0,z) + H(0,0,1,1,z) \\
&+ H(0,0,1,z)G(0,y) + H(0,0,z)G(1-z,1-z,y) + H(0,0,z)G(1-z,0,y) \\
&+ H(0,0,z)G(0,1-z,y) - G(0,0,y)H(1,0,z) + G(0,0,y)H(1,1,z) \\
&- G(0,0,y)H(0,1,z) + G(0,0,y)H(0,0,z) \\
&\quad + 2\Big(- G(-z,1-z,1-z,0,y) - G(-z,1-z,0,1-z,y) \\
&- G(-z,0,1-z,1-z,y) - G(1-z,1-z,-z,1-z,y) + G(1-z,1-z,1,0,y) \\
&+ G(1-z,1,1-z,0,y) + G(1-z,1,0,1-z,y) - G(1-z,1,0,0,y) \\
&- G(1,1-z,0,0,y) - H(1,1,0,z)G(-z,y) - H(1,1,z)G(-z,0,y) \\
&- G(1,0,1-z,0,y) + H(1,0,1,z)G(-z,y) - H(1,0,z)G(1-z,1-z,y) \\
&+ H(1,0,z)G(1-z,0,y) - G(1,0,0,1-z,y) + G(1,0,0,y)H(1,z) \\
&+ H(1,0,z)G(-z,1-z,y) + H(1,z)G(-z,1-z,0,y) + H(1,z)G(-z,0,1-z,y) \\
&+ H(1,z)G(1-z,1-z,-z,y) - H(1,z)G(1-z,1,0,y) - G(0,-z,1-z,1-z,y) \\
&+ G(0,-z,1-z,0,y) + G(0,-z,0,1-z,y) - G(0,-z,0,y)H(1,z) \\
&+ G(0,-z,1-z,y)H(1,z) - G(0,-z,y)H(1,1,z) + H(0,1,1,z)G(-z,y) \\
&- H(0,1,z)G(-z,1-z,y) + H(0,1,z)G(1-z,1-z,y) + H(0,1,z)G(1-z,0,y) \\
&+ G(0,0,-z,1-z,y) - G(0,0,-z,y)H(1,z) - H(0,z)G(-z,1-z,1-z,y) \\
&- H(0,z)G(1,0,0,y) - G(0,0,1,0,y)\Big) \\
&\quad + 3\Big(- H(0,1,z)G(1-z,-z,y) - H(0,0,1,z)G(1-z,y)\Big) \\
&\quad + 4\Big(- G(-z,-z,-z,1-z,y) + G(-z,-z,1-z,1-z,y) \\
&+ G(-z,1-z,-z,1-z,y) + G(1-z,-z,-z,1-z,y) - G(1,1,1,0,y) \\
&+ G(1,1,0,0,y) + H(1,1,0,z)G(1-z,y) + H(1,1,z)G(-z,-z,y) + G(1,0,1,0,y)
\end{aligned}
$$

- QCD answer from Gehrmann, Glover, Jaquier & Koukoutsakis :

  - expressed in terms of (several pages of) Goncharov polylogarithms of degree = 0 to 4

- Next, relate N=4 SYM and QCD form factors:

  - take maximally transcendental piece of $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$

  - convert the QCD remainder (Catani) into our ABDK/BDS-type remainder

in practice: $$\mathcal{R}^{(2)} = F^{(2)}_{\rm GGJK} - \frac{1}{2}\,(F^{(1)}_{\rm GGJK})^2$$

- We find a surprising relation...

$$\mathcal{R}^{(2)}_{H\,g^-g^-g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{H\,g^+g^+g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{\mathcal{N}=4\,\text{SYM}}$$

!

- N=4 result = maximally transcendental piece of QCD result; (several pages!) can be reduced to a few lines of classical polylogarithms
- this phenomenon has been seen previously in anomalous dimension of operators (Kotikov-Lipatov-Onishchenko-Velizhanin)
- first example with non-trivial kinematic dependence, but likely only true for small n
- remaining parts of the QCD amplitudes also expressible in terms of simpler functions (Duhr)
- Q: can QCD amplitudes be calculated directly with Symbols ?

Summary

- Hidden structures and simplicity of (amplitudes &) form factors

- Form factors in N=4 super Yang-Mills
  - tree, one and two loops (on-shell recursion relations, unitarity)
  - exponentiation of Sudakov form factor
- Three-point form factor in N=4 super Yang-Mills & QCD
  - Analytic remainder function from symbols and explicit calculations
  - relation to Higgs + multi-gluon QCD remainder...

Open questions

- More loops, more legs: 3-loops under way
- Further applications of symbol to QCD and other theories? e.g. ABJM
- Connection to correlation functions?
- Representation in terms of Wilson lines?
- Recursion relations for form factors integrands?
- Symmetries?
- BCJ? ...