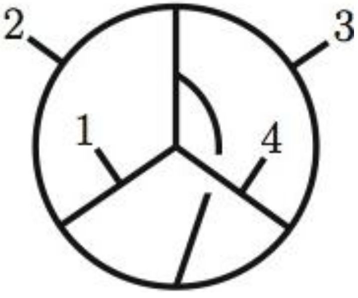
Some Surprising Properties of Gauge and Gravity Amplitudes

Parma Summer School

Sept 6, 2011

Zvi Bern, UCLA

Lecture 2,3: Application of duality between color and kinematics to UV properties of supergravity.



Outline

In today's lecture I will show you how the duality between color and kinematics helps us understand UV properties of gravity.

Review of Previous Lecture

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

flat metric

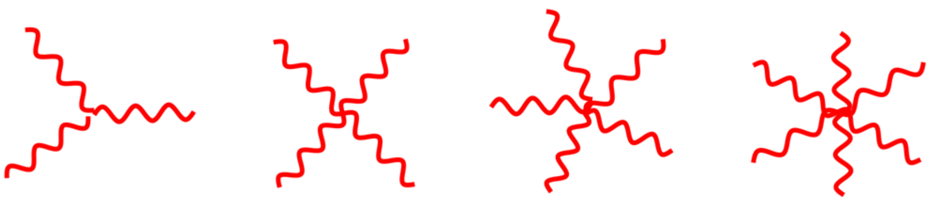
graviton field

metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions



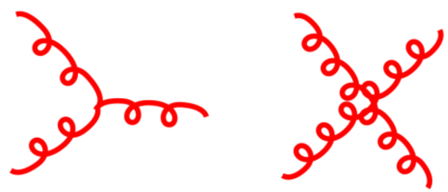
$+ \cdots$

terrible mess

Non-renormalizable

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

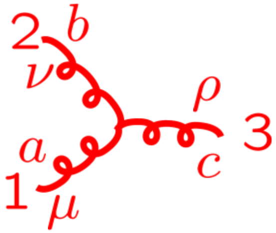


Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Three Vertices

Standard Feynman diagram approach.



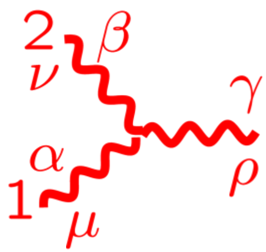
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$\begin{aligned} &G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\ &\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ &\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ &\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ &\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

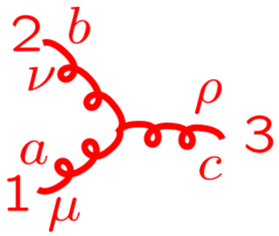
Definitely not a good approach.

Simplicity of Gravity Amplitudes

On-shell viewpoint much more powerful.

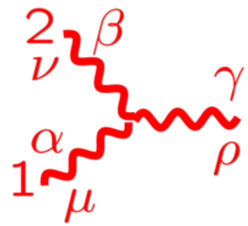
***On-shell* three vertices contains all information:** $k_i^2 = 0$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$
$$\times\, (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

double copy of Yang-Mills vertex.

- **Using modern on-shell methods, any gravity scattering amplitude constructible solely from *on-shell* 3 vertices. BCFW recursion for trees, BDDK unitarity method for loops.**
- **Higher-point vertices irrelevant!**

Gravity vs Gauge Theory

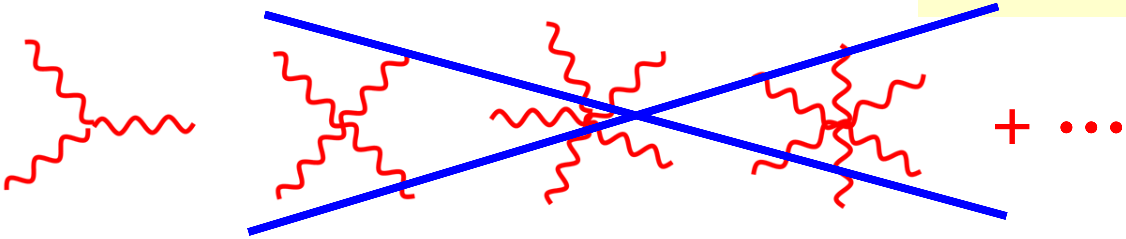
Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric — flat metric — graviton field

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

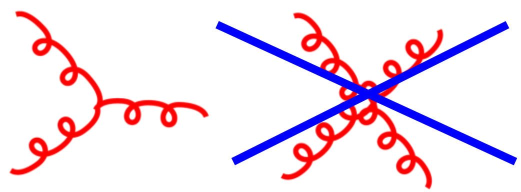
Infinite number of irrelevant interactions!



Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



Only three-point Interactions needed

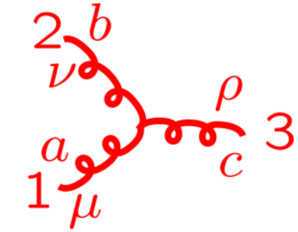
Gravity seems ~~so much~~ no more complicated than gauge theory.

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)

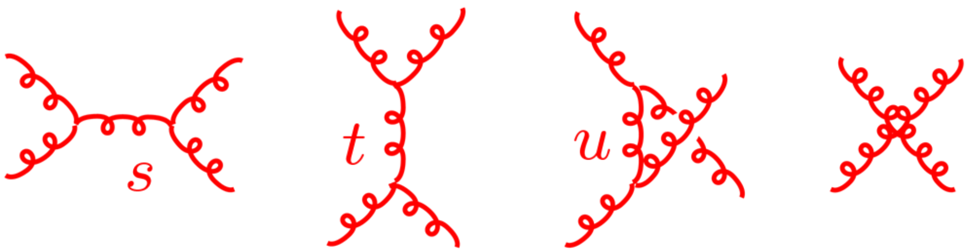
coupling constant → color factor ← momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1a_2b}f^{ba_4a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1+k_2)^2$ $u = (k_1+k_3)^2$
$t = (k_1+k_4)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

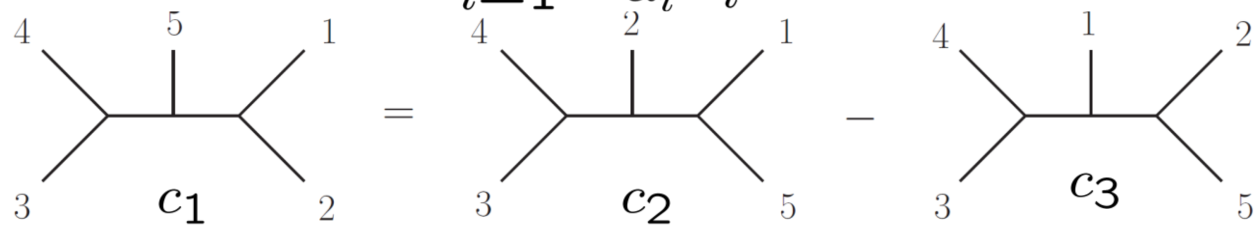
Consider five-point tree amplitude: ZB, Carrasco, Johansson (BCJ)

color factor

kinematic numerator factor

Feynman propagators

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p_i^2}$$



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

Claim: At n-points we can always find a rearrangement so color and kinematics satisfy the same algebraic constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng,Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

Gravity and Gauge Theory

kinematic numerator → n_i; color factor → c_i

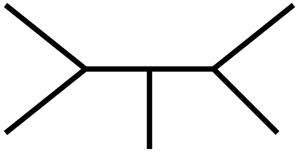
gauge theory:
$$\frac{1}{g^{n-2}} \mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

sum over diagrams with only 3 vertices

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity:
$$-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

Gravity numerators are a double copy of gauge-theory ones.

This works for ordinary Einstein gravity and susy versions.

Cries out for a unified description of the sort given by string theory!

Duality for BLG Theory

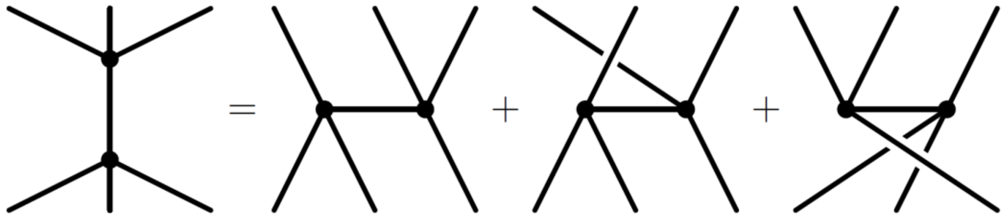
BLG based on a 3 algebra

Bagger, Lambert, Gustavsson (BLG)

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

***D* = 3 Chern-Simons gauge theory**

Four-term color identity:



$$c_s = c_t + c_u + c_v \Rightarrow n_s = n_t + n_u + n_v$$

Such numerators explicitly found at 6 points.

Bargheer, He, and McLoughlin

What is the double copy?

Explicit check at 4 and 6 points shows it is the $E_{8(8)}$ N = 16 supergravity of Marcus and Schwarz. Very non-trivial!

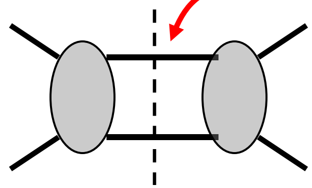
A hidden 3 algebra structure exists in this supergravity.

Bern, Dixon, Dunbar and Kosower (1998)

Unitarity Method: Rewrite of QFT

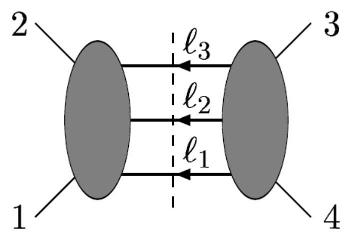
on-shell

Two-particle cut:

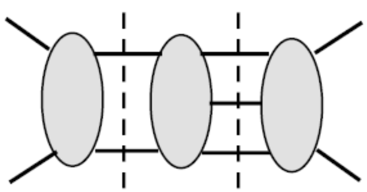


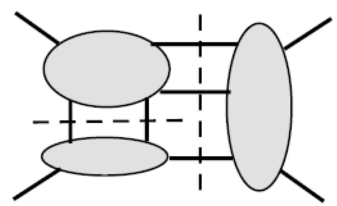
Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Three-particle cut:



Generalized unitarity as a practical tool:





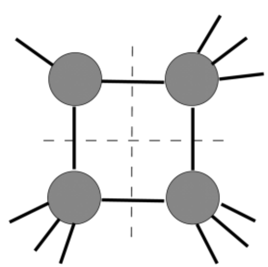
Different cuts merged to give an expression with correct cuts in all channels.

Bern, Dixon and Kosower
Britto, Cachazo and Feng; Forde;
Ossala, Pittau, Papadopolous, and many others

Now a standard tool

complex momenta to solve cuts

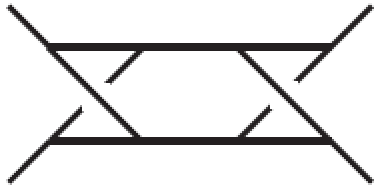
Britto, Cachazo and Feng

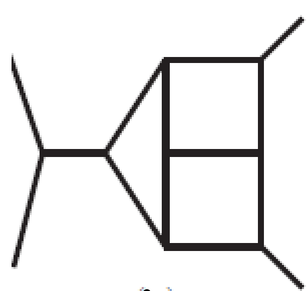


Discussed by Zoltan

Loop-Level Generalization

ZB, Carrasco, Johansson (2010)





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gravity

symmetry factor

Loop-level conjecture is identical to tree-level one except for symmetry factors and loop integration.
Double copy works if numerator satisfies duality.

Generalized Gauge Invariance

BCJ
ZB, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{m-2+2L}}\mathcal{A}_m^{\text{loop}} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}} = 0$$

$$(c_\alpha + c_\beta + c_\gamma) f(p_i) = 0$$

Above is just a definition of generalized gauge invariance

gravity

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}} = 0$$

- **Gravity inherits generalized gauge invariance from gauge theory!**
- **Double copy works even if only one of the two copies has duality manifest!**

Generalized Gauge Invariance

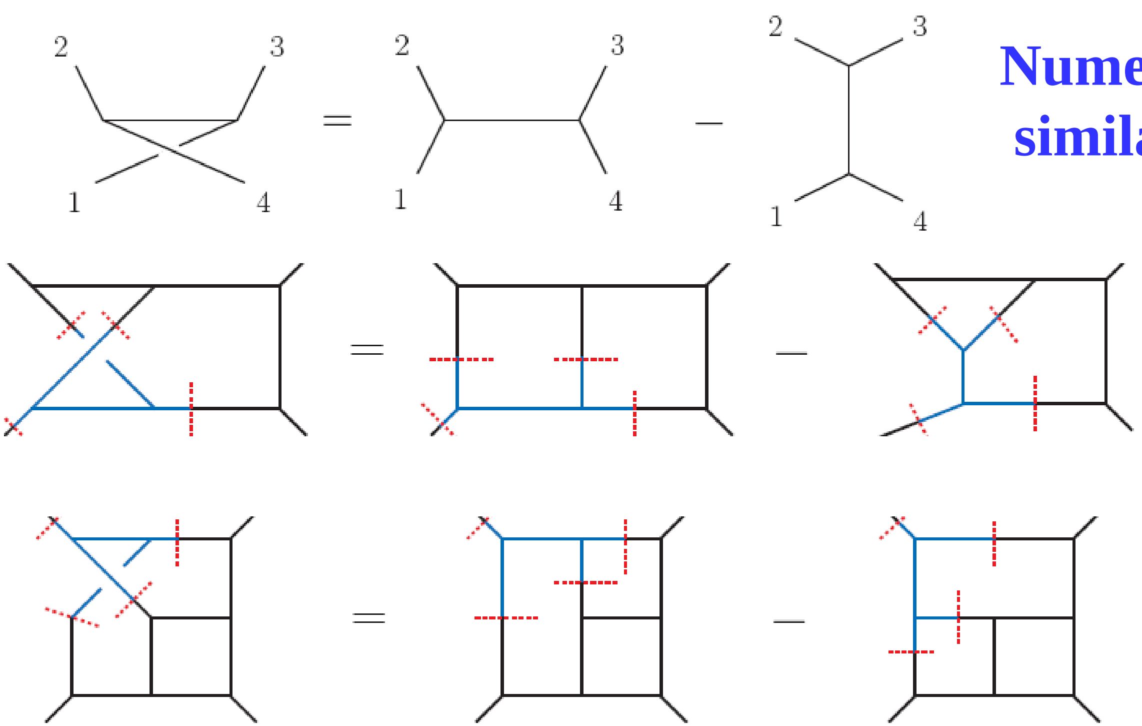
• Generalized gauge invariance means symmetries of gauge theory inherited by gravity.

• If we see a UV cancellation in a gauge theory we should expect a corresponding cancellation in gravity.

Nonplanar is Locked to Planar

ZB, Carrasco, Johansson

Generally, planar is simpler than non-planar. Can we obtain non-planar from planar? The answer is yes!



Numerators satisfy identities similar to color Jacobi identities.

Numerator relations

Planar gives nonplanar.

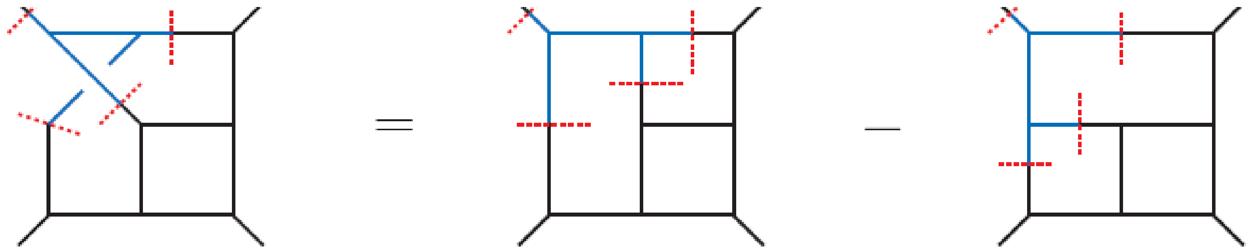
Interlocking set of equations restrict numerators

Gives us hope that once we solve planar *N*=4 sYM we will be able to do the same for non-planar!

BCJ

Gravity integrands are free!

Ideas generalize to loops:



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

If you have a set of duality satisfying numerators. To get:

gauge theory → gravity theory

simply take

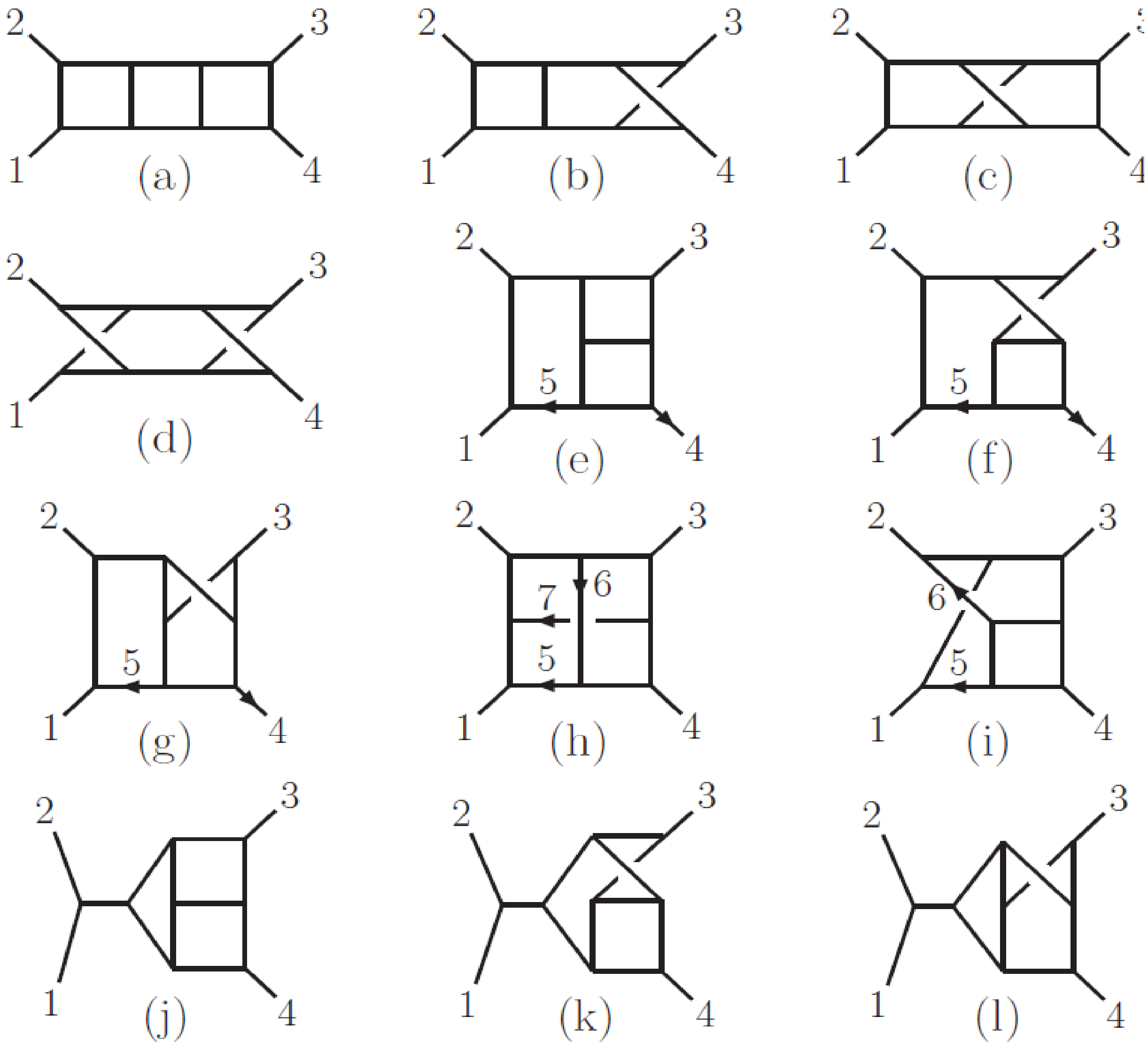
color factor → kinematic numerator

Gravity loop integrands are trivial to obtain!

Three loop $N = 4$ sYM

ZB, Carrasco, Johansson (2010)

$N = 4$ super-Yang-Mills integrand



One diagram to rule them all

- **These are numerator Jacobi relations.**
- **One-loop triangle subdiagrams vanish in *N* = 4 sYM so many numerators simply vanish**

$$N^{(a)} = N^{(b)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,$$
$$N^{(b)} = N^{(d)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,$$
$$N^{(c)} = N^{(a)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,$$
$$N^{(d)} = N^{(h)}(k_3, k_1, k_2, l_7, l_6, k_{1,3} - l_5 + l_6 - l_7) + N^{(h)}(k_3, k_2, k_1, l_7, l_6, k_{2,3} + l_5 - l_7)\,,$$
$$N^{(f)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,$$
$$N^{(g)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7)\,,$$
$$N^{(h)} = -N^{(g)}(k_1, k_2, k_3, l_5, l_6, k_{1,2} - l_5 - l_7) - N^{(i)}(k_4, k_3, k_2, l_6 - l_5, l_5 - l_6 + l_7 - k_{1,2}, l_6)\,,$$
$$N^{(i)} = N^{(e)}(k_1, k_2, k_3, l_5, l_7, l_6) - N^{(e)}(k_3, k_2, k_1, -k_4 - l_5 - l_6, -l_6 - l_7, l_6)\,,$$
$$N^{(j)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(e)}(k_2, k_1, k_3, l_5, l_6, l_7)\,,$$
$$N^{(k)} = N^{(f)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(f)}(k_2, k_1, k_3, l_5, l_6, l_7)\,,$$
$$N^{(l)} = N^{(g)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(g)}(k_2, k_1, k_3, l_5, l_6, l_7)\,,$$

All numerators solved in terms of numerator (e)

One diagram to rule them all

ZB, Carrasco, Johansson (2010)

***N* = 4 super-Yang-Mills integrand**

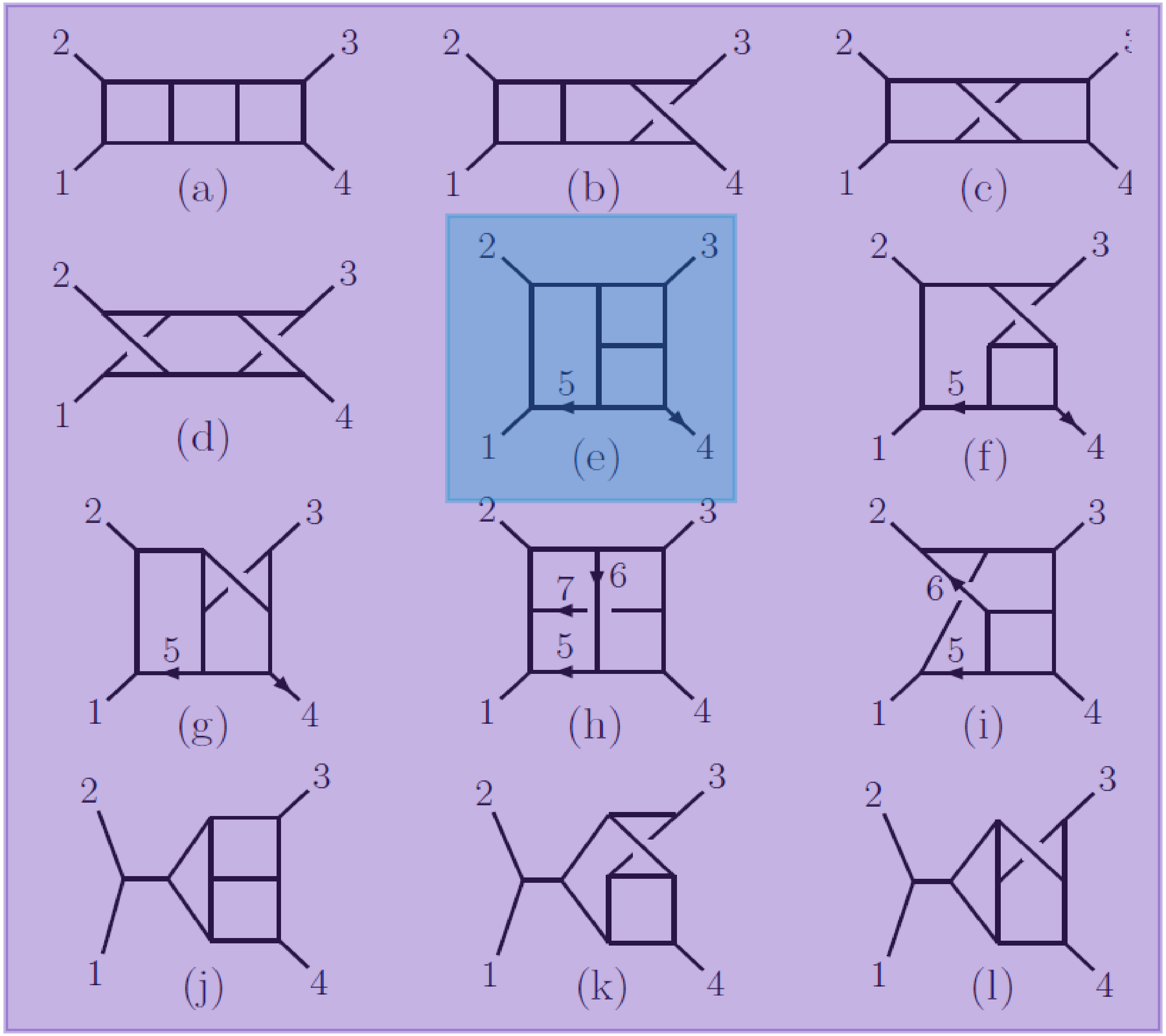


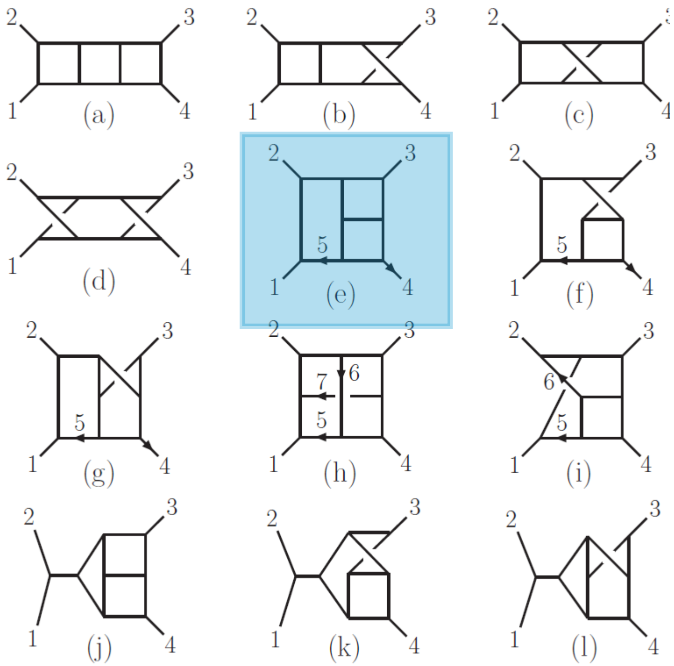
Diagram (e) is the master diagram.

Determine the master numerator in proper form and duality gives all others.

***N* = 8 sugra given by double copy.**

Explicit Three-Loop Check for Maximal Susy

ZB, Carrasco, Johansson (2010)



$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k$$

***N*=4 sYM theory.**

Only 12 diagrams contribute.

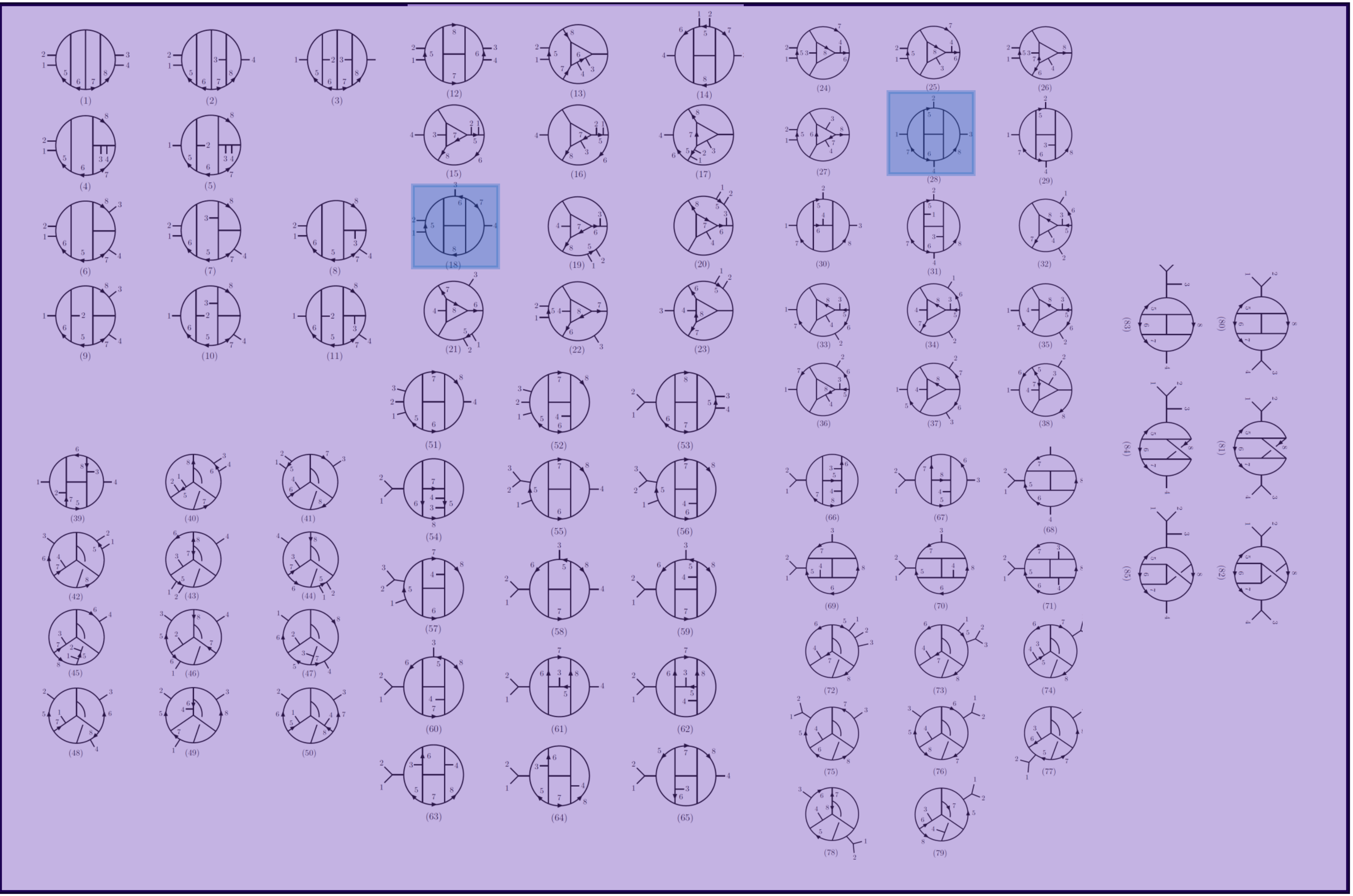
$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$
$$\tau_{ij} = 2k_i \cdot l_j$$

Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35} + \tau_{45} + t\right) - t\left(\tau_{25} + \tau_{45}\right) + u\left(\tau_{25} + \tau_{35}\right) - s^2\right)/3$
(h)	$\left(s\left(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u\right)\right.$ $\left. + t\left(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}\right) + s^2\right)/3$
(i)	$\left(s\left(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t\right)\right.$ $\left. + t\left(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}\right) + u\,\tau_{25} + s^2\right)/3$
(j)-(l)	$s(t-u)/3$

- **Duality works!**
- **Double copy works!**
- ***N* = 8 supergravity is free.**

$\mathcal{N}$ = 4 sYM Four Loops

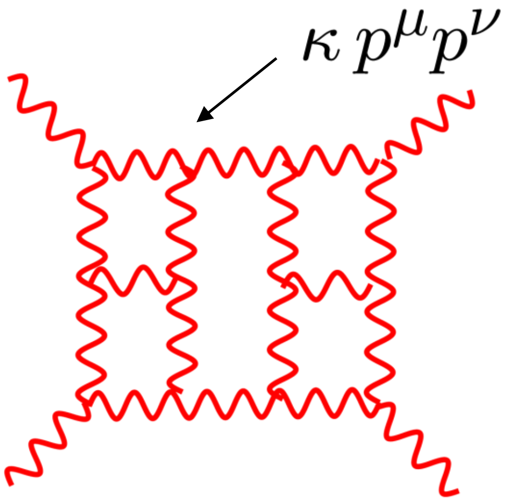
ZB, Carrasco, Dixon, Johansson, Roiban (to appear)



Application: UV Properties of Gravity

Power Counting at High Loop Orders

$$\kappa = \sqrt{32\pi G_N}$$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Reasons to focus on extended supergravity:

- **With more susy expect better UV properties.**
- **High symmetry implies technical simplicity.**

Finiteness of *N* = 8 Supergravity?

We are interested in UV finiteness of *N* = 8 supergravity because it would imply a new symmetry or non-trivial dynamical mechanism. No known symmetry can render a UV theory finite.

The discovery of either would have a fundamental impact on our understanding of gravity.

Note: Perturbative finiteness is not the only issue for consistent gravity: Nonperturbative completions? High energy behavior of theory? Realistic models?

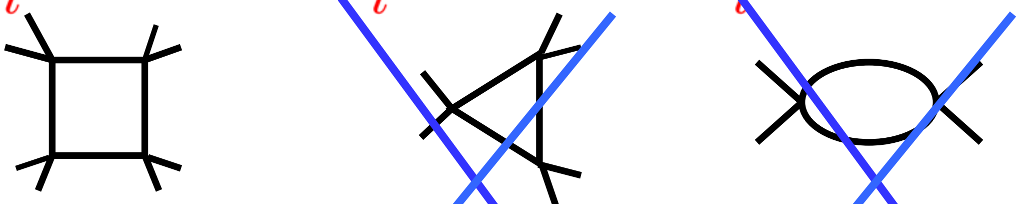
$\mathcal{N} = 8$ Supergravity No-Triangle Property

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Proofs by Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

One-loop $D = 4$ theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

Brown, Feynman; Passarino and Veltman, etc

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$



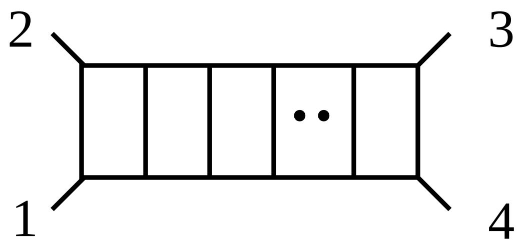
$$\int \frac{d^4 p}{(p^2)^4} \qquad \int \frac{d^4 p}{(p^2)^3} \qquad \int \frac{d^4 p}{(p^2)^2}$$

- **In $\mathcal{N} = 4$ Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.**
- **The "no-triangle property" is the statement that same holds in $\mathcal{N} = 8$ supergravity. Non-trivial constraint on analytic form of amplitudes.**
- **Unordered nature of gravity is important for this property**

Bjerrum-Bohr and Vanhove

$N = 8$ L-Loop UV Cancellations

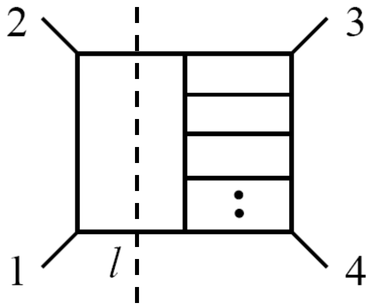
ZB, Dixon, Roiban



$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor

From 2 particle cut:

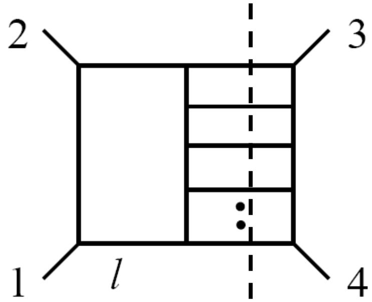


1 in $N = 4$ YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

L-particle cut



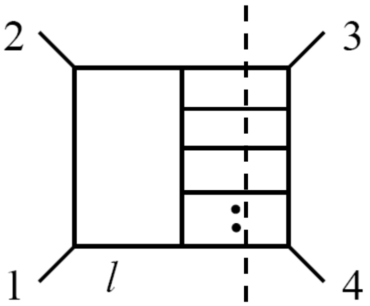
- **Numerator violates one-loop "no-triangle" property.**
- **Too many powers of loop momentum in one-loop subamplitude.**
- **After cancellations behavior is same as in $N = 4$ Yang-Mills!**

- **UV cancellation exist to *all* loop orders! (not a proof of finiteness)**
- **These all-loop cancellations *not* explained by supersymmetry alone.**
- **Existence of these cancellations drive our calculations!**

Origin of Cancellations?

There does not appear to be a supersymmetry explanation for observed cancellations, especially as the loop order increases.

If it is *not* supersymmetry what might it be?



Where is First Potential UV Divergence in $D= 4$ $N = 8$ Sugra?

Various opinions, pointing to divergences over the years:

3 loops	Superspace power counting	Deser, Kay (1978) Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985), etc
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N} = 6$ harmonic superspace exists; algebraic renormalisation argument	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N} = 7$ harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* offshell $\mathcal{N} = 8$ superspace exists; lightcone gauge locality arguments; Algebraic renormalization arguments; $E_{7(7)}$ symmetry.	Grisaru and Siegel (1982); Howe, Stelle and Bossard (2009) Vanhove; Bjornsson, Green (2010) Kiermaier, Elvang, Freedman(2010) Ramond, Kallosh (2010); Biesert, et al (2010)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy and duality.	Kallosh; Howe and Lindström (1981)
9 loops	Assumes Berkovits' superstring non-renormalization theorems carries over to D=4 $\mathcal{N} = 8$ supergravity and extrapolates to 9 loops.	Green, Russo, Vanhove (2006) (retracted)

No divergences demonstrated above. Arguments based on lack of symmetry protection. An unaccounted symmetry can make the theory finite.

To end debate we need solid calculations.

Opinions from the 80's

Supergravity well studied in the late 70's and 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that *all* supergravity theories diverge (at three loops) has been widely accepted for over 25 years

$\frac{1}{\epsilon}R^4$ **was expected counterterm**

Today's opinions

Go around the room and ask the professors if *N* = 8 sugra can be UV finite.

You will get responses along the lines of:

- **"That's crazy" (Ask Tom Banks)**
- **"These people are wasting their time"**
- **"I'll believe it when I see a proof I trust"**
- **"It may be interesting but it can't be finite"**

Question: How should one argue with a roomful of people who think you are wrong?

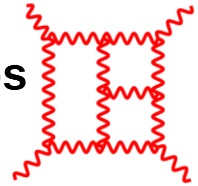
Answer: “Shut Up and Calculate!”

Here I will show you how the duality and double copy allows us to do the seemingly impossible calculations to settle the question of UV properties in quantum gravity.

Feynman Diagrams for Gravity

Suppose we want to check if opinions are true using Feynman digrams

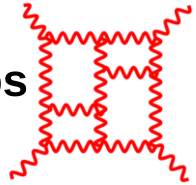
3 loops



~10^{20} TERMS

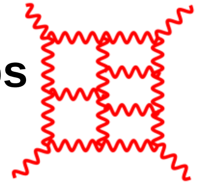
Has never been calculated via Feynman diagrams.

4 loops



~10^{26} TERMS

5 loops



~10^{31} TERMS

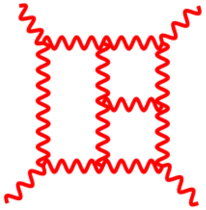
Five loops is currently the crucially needed computation

- Calculations to settle this seemed utterly hopeless!
- Seemed destined for dustbin of undecidable questions.

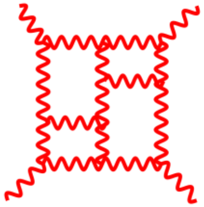
Unitarity Method + Double Copy

Z B, John Joseph Carrasco, David Kosower, Lance Dixon, Henrik Johansson, Radu Roiban

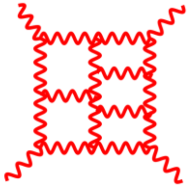
For N = 8 supergravity.

3 loops  ~10 TERMS

Can do 3 loops on the blackboard!

4 loops  $\sim 10^2$ TERMS

Much more manageable!

5 loops  $\sim 10^3$ TERMS

(Not yet complete)

Supergravity is Back!

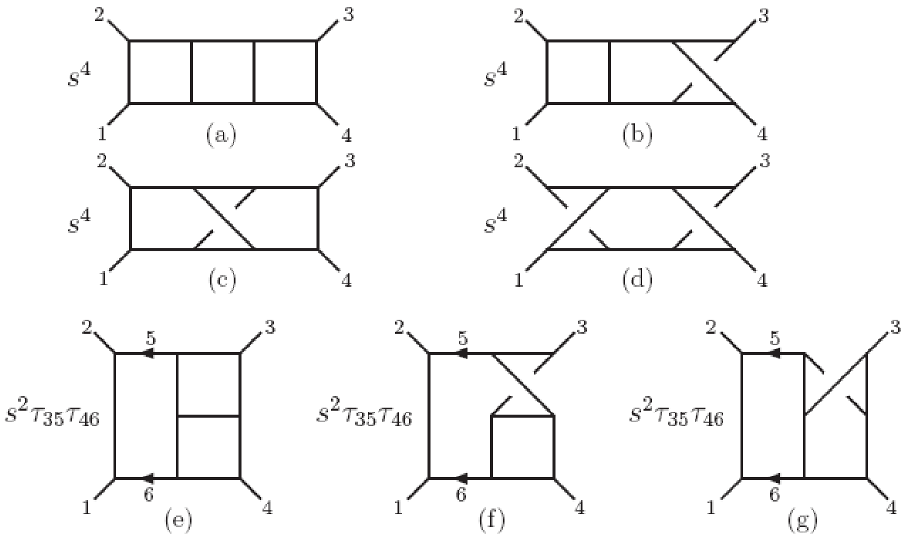
Some recent work on UV properties:

- **Powerful new tools: Unitarity method. Instead of debating we calculate!** ZB, Dixon, Dunbar, Kosower (BDDK); ZB, Dixon, Dunbar, Perelstein, Rozowsky; ZB, Carrasco, Johansson, Kosower
- **Double copy of gravity in terms of gauge theory.** Kawai, Lewellen, Tye; ZB, Carrasco, Johansson (BCJ)
- **String dualities restrict supergravity divergences.** Green, Vanhove, Russo
- **Field theory versions of string theory used to explore divergences (Berkovits pure spinors).** Bekovits, Green, Vanhove, Russo; Bjornsson and Green
- **Better understanding of symmetries.** Arkani-Hamed, Cachazo, Kaplan; Bossard, Howe, Stelle; Beisert, Elvang, Freedman, Kiermaier, Stieberger; Kallosh, Ramond; Bossard, Nicolia; Kallosh

Complete Three-Loop Result

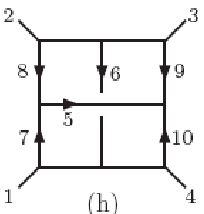
ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

Obtained via on-shell unitarity method:

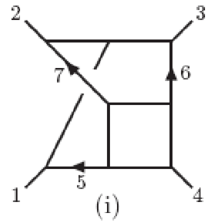


$$\tau_{ij} = 2k_i \cdot k_j$$

$$\begin{aligned}&(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2\\&+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\&+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\&+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})\end{aligned}$$



$$\begin{aligned}&(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})\\&-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})\\&+l_5^2s^2t+l_6^2st^2-\tfrac{1}{3}l_7^2stu\end{aligned}$$



Three loops is not only ultraviolet finite it is "superfinite"— finite for *D* < 6.

All UV cancellations exposed manifestly

No UV divergence in sight.

A More Recent Opinion

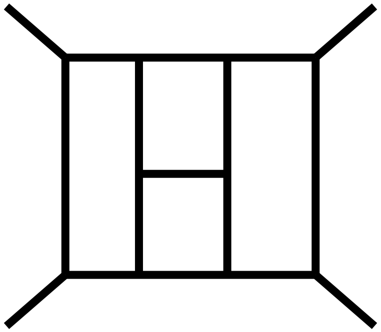
Back in 2009 Bossard, Howe and Stelle had a look at the question of finiteness in supergravity

At the time, best available understanding of symmetries:

In particular … suggest that maximal supergravity is likely to diverge at four loops in *D* = 5 and at five loops in *D* = 4 …

Bossard, Howe, Stelle (2009)

Bottles of wine were at stake!



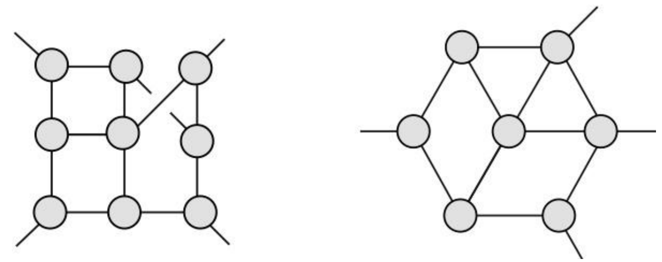
Four-Loop Construction

ZB, Carrasco, Dixon, Johansson, Roiban

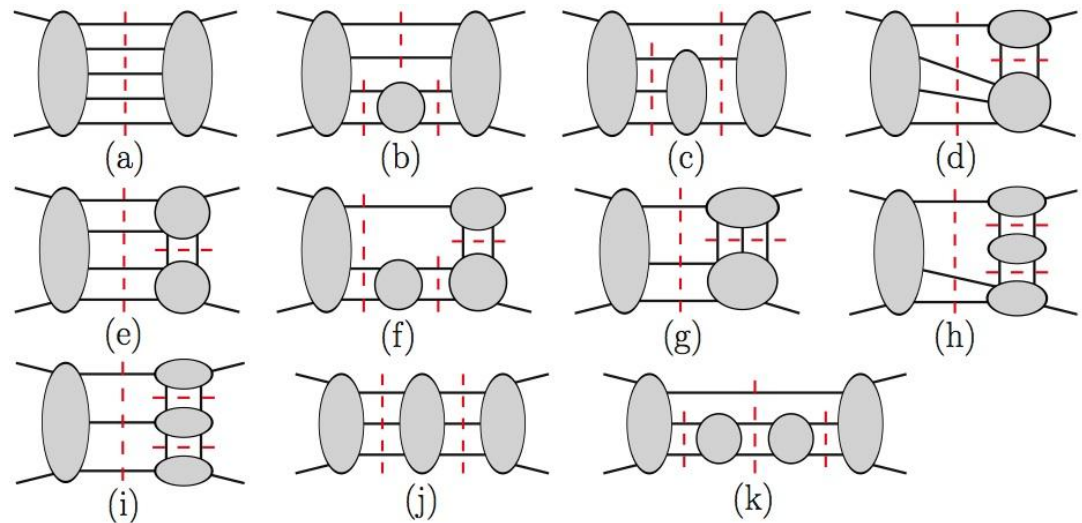
numerator

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

Determine numerators from 2906 maximal and near maximal cuts



Completeness of expression confirmed using 26 generalized cuts sufficient for obtaining the complete expression



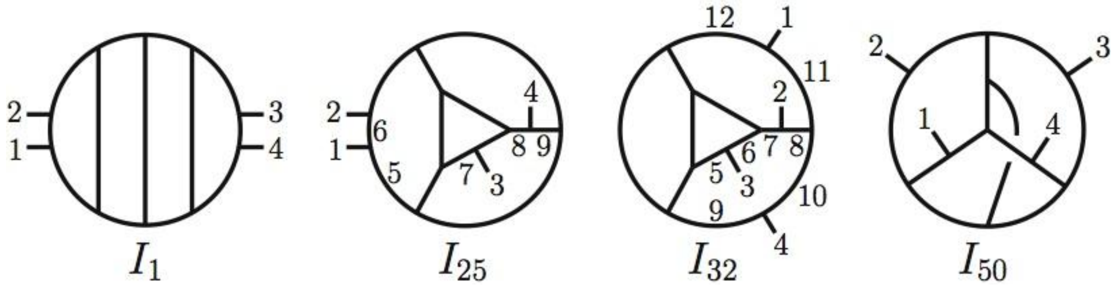
11 most complicated cuts shown

Today with BCJ it is trivial to construct the amplitude, but this is the way we prove it to be correct.

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

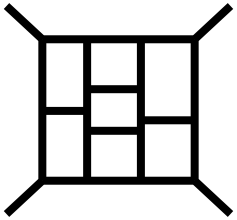
UV finite for *D* < 11/2
It's very finite!

Originally took more than a year.
Double copy discovered by doing this calculation!
Today with the double copy we can reproduce it in a few days!

Recent Status of Divergences

Consensus that in *N* = 8 supergravity trouble starts at 5 loops and by 7 loops we have valid UV counterterm in *D* = 4 under all known symmetries (suggesting divergences) .

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Bjornsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



For *N* = 8 sugra in *D* = 4:

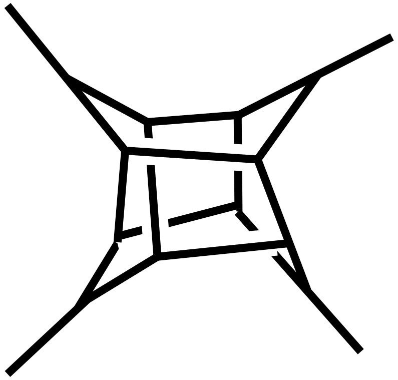
- **All counterterms ruled out until 7 loops!**
- **But D^8R^4 apparently available at 7 loops (1/8 BPS) under all known symmetries. (No known nonrenormalization theorem)**

Bossard, Howe, Stelle and Vanhove

Based on this a reasonable person would conclude that *N*=8 supergravity almost certainly diverges at 7 loops in *D*=4

N = 8 Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~100s terms each

Place your bets:

- **At 5 loops in D = 24/5 does N = 8 supergravity diverge?**
- **At 7 loops in D = 4 does N = 8 supergravity diverge?**

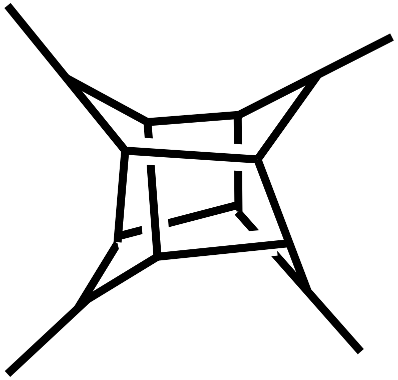
5 loops

Kelly Stelle:
British wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

$N = 8$ Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~100s terms each

Place your bets:

- **At 5 loops in $D = 24/5$ does $N = 8$ supergravity diverge?**
- **At 7 loops in $D = 4$ does $N = 8$ supergravity diverge?**

7 loops

David Gross:
California wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

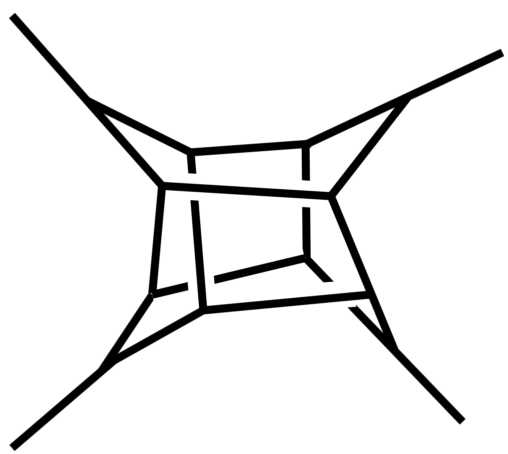
Calculation of $\mathcal{N} = 4$ sYM 5 Loop Amplitude

Key step for $\mathcal{N} = 8$ supergravity is construction of complete nonplanar 5 loop integrand of $\mathcal{N} = 4$ sYM theory. This is now finished but still need to find BCJ form).

416 such diagrams with ~1000s terms each

ZB, Carrasco, Johansson, Roiban (2012)

We are well on our way to calculate the UV properties of $\mathcal{N} = 8$ supergravity at five loops.



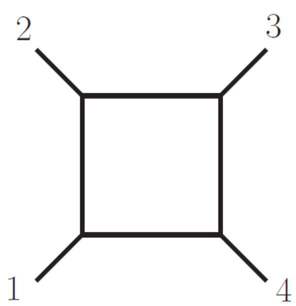
Half-Maximal Supergravity

Fine, but do you have any example where a divergence vanishes but for which there is no apparant symmetry explanation?

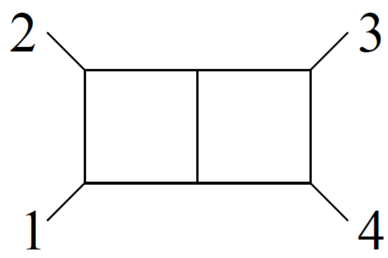
Yes, half maximal (16 supercharge) supergravity:

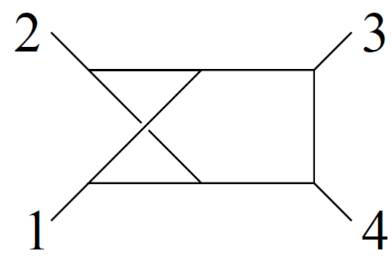
- **2 loops in $D = 5$**
- **3 loops in $D = 4$**

Half Maximal Supergravity in D > 4



ZB, Davies, Dennen, Huang





No surprises at one loop:

- **Finite for $D < 8$**
- **R^4 divergence in $D = 8$**
- **F^4 four-matter multiplet amplitude diverges in $D = 4$ -**

Very instructive to understand from double-copy vantage point

A two-loop surprise:

- **Finite in $D = 5$ with seemingly valid R^4 counterterm.**

A three loop surprise:

- **Finite for $D = 4$ with seemingly valid R^4 counterterm.**

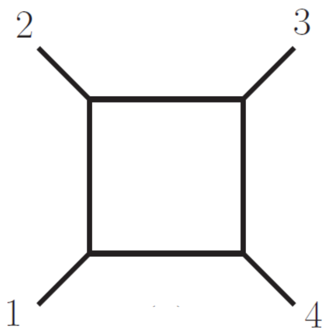
We now go through these examples

One-Loop Warmup in Half-Maximal sugra

ZB, Davies, Dennen, Huang

Generic color decomposition:

$$\mathcal{A}_Q^{(1)} = ig^4 \Big[c_{1234}^{(1)} A_Q^{(1)}(1,2,3,4) + c_{1342}^{(1)} A_Q^{(1)}(1,3,4,2) + c_{1423}^{(1)} A_Q^{(1)}(1,4,2,3) \Big]$$



Q = # supercharges **Q = 0 is pure non-susy YM**

To get Q +16 supergravity take 2nd copy *N* = 4 sYM

***N* = 4 sYM numerators independent of loop momenta**

$$n_{1234} = n_{1342} = n_{1423} = st A^{\text{tree}}_{Q=16}(1,2,3,4) \quad c_{1234}^{(1)} \to n_{1234}$$

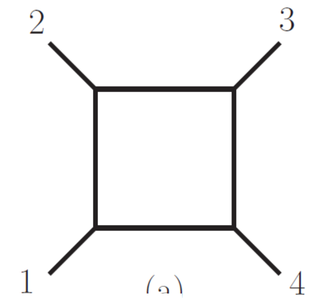
$$\mathcal{M}_{Q+16}^{(1)} = i\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}(1,2,3,4)\Big[A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big]$$

Note *exactly* the same combination as in *U*(1) decoupling identity.

One-loop divergences in pure YM

ZB, Davies, Dennen, Huang

Go to a basis of color factors:



$C_A = 2\, N_c$ for $SU(N_c)$

$b_1^{(0)}$ and $b_2^{(0)}$: tree color tensors

$b_1^{(1)}$: independent 1 loop color tensor

$$\mathcal{A}_Q^{(1)} = ig^4 \Big[b_1^{(1)} \Big(A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big) - \frac{1}{2} C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2} C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3) \Big]$$

Q supercharges (mainly interested in $Q = 0$)

***D* = 4 F^2 counterterm: 1-loop color tensor *not* allowed.**

***D* = 6 F^3 counterterm: 1-loop color tensor *not* allowed.**

$$F^3 = f^{abc} F^{a\mu}{}_{\nu} F^{b\nu}{}_{\sigma} F^{c\sigma}{}_{\mu}$$

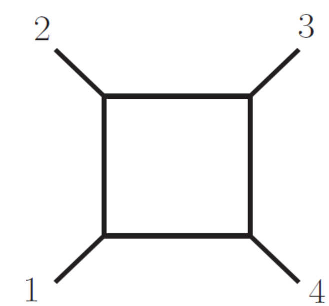
(a) (b)

$$A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big|_{\text{div.}} = 0$$

One-Loop Warmup in Half-Maximal Sugra

ZB, Davies, Dennen, Huang

$$\mathcal{M}^{(1)}_{Q+16} = i\left(\frac{\kappa}{2}\right)^4 st A^{\text{tree}}_{Q=16}(1,2,3,4)\Big[A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)\Big]$$



Cases where one-loop color tensor appear.
These give supergravity divergences.

***D* = 8** $\frac{1}{\epsilon} c^{abcd} F^{a\mu\nu} F^{b}{}_{\mu\sigma} F^{c\sigma\rho} F^{d}_{\rho\mu}$ $\quad c^{abcd} \equiv \tilde{f}^{a\,e_1 e_2} \tilde{f}^{b\,e_2 e_3} \tilde{f}^{c\,e_3 e_4} \tilde{f}^{d\,e_4 e_1}$

one-loop color tensor allowed so no cancellations

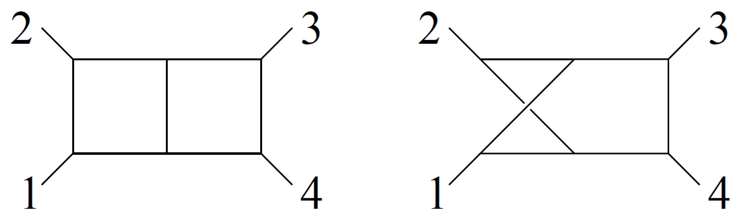
F^4 YM divergence ⟷ R^4 sugra divergence

***D* = 4 with matter:** $\frac{1}{\epsilon} c^{abcd} \phi^a \phi^b \phi^c \phi^d$

ϕ^4 YM divergence ⟷ F^4 matter sugra diverge

(shown long ago by Fischler)

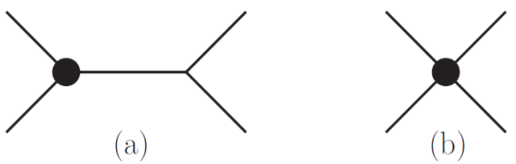
Two loop half maximal sugra in $D = 5$



ZB, Davies, Dennen, Huang

$$\mathcal{A}_Q^{(2)} = -g^6 \Big[c^{\rm P}_{1234} A_Q^{\rm P}(1,2,3,4) + c^{\rm P}_{3421} A_Q^{\rm P}(3,4,2,1) + c^{\rm NP}_{1234} A_Q^{\rm NP}(1,2,3,4) + c^{\rm NP}_{3421} A_Q^{\rm NP}(3,4,2,1) + \text{ cyclic} \Big]$$

$D = 5$ F^3 counterterm: 1,2-loop color tensors forbidden!
Demand this and plug into double copy:



1) **Go to color basis.**
2) **Demand no forbidden color tensors in pure YM divegence.**
3) **Plug into the BCJ double copy formula.**

$$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0$$

Half maximal supergravity four-point divergence vanishes because forbidden color tensor cancels in pure YM theory

Half Maximal Supergravity in $D = 5$

In a bit more detail:

ZB, Davies, Dennen, Huang

gauge theory

$$\mathcal{A}_Q^{(2)} = -g^6 \Big[c^{\rm P}_{1234} A^{\rm P}_Q(1,2,3,4) + c^{\rm P}_{3421} A^{\rm P}_Q(3,4,2,1) + c^{\rm NP}_{1234} A^{\rm NP}_Q(1,2,3,4) + c^{\rm NP}_{3421} A^{\rm NP}_Q(3,4,2,1) + \text{ cyclic} \Big]$$

gravity

$$\mathcal{M}^{(2)}_{Q+16} = -i \left(\frac{\kappa}{2}\right)^6 st A^{\rm tree}_{Q=16}(1,2,3,4) \Big[s\Big(A^{\rm (P)}_Q(1,2,3,4) + A^{\rm (NP)}_Q(1,2,3,4) + A^{\rm (P)}_Q(3,4,2,1) + A^{\rm (NP)}_Q(3,4,2,1)\Big) + \text{cyclic} \Big]$$

Equations that eliminate forbidden 2 loop color tensor:

$$\begin{aligned} 0 = {} & t(A^{\rm P}_Q(1,3,4,2) + A^{\rm P}_Q(1,4,2,3) + A^{\rm P}_Q(3,1,4,2) + A^{\rm P}_Q(3,2,1,4) \\ & + A^{\rm NP}_Q(1,3,4,2) + A^{\rm NP}_Q(1,4,2,3) + A^{\rm NP}_Q(3,1,4,2) + A^{\rm NP}_Q(3,2,1,4) \\ & + s(A^{\rm P}_Q(1,3,4,2) + A^{\rm P}_Q(3,1,4,2) + A^{\rm NP}_Q(1,3,4,2) + A^{\rm NP}_Q(3,1,4,2))\Big|_{D=5\,\text{div.}}\,, \\ 0 = {} & s(A^{\rm P}_Q(1,2,3,4) + A^{\rm P}_Q(1,3,4,2) + A^{\rm P}_Q(3,1,4,2) + A^{\rm P}_Q(3,4,2,1) \\ & + A^{\rm NP}_Q(1,2,3,4) + A^{\rm NP}_Q(1,3,4,2) + A^{\rm NP}_Q(3,1,4,2) + A^{\rm NP}_Q(3,4,2,1)) \\ & + t(A^{\rm P}_Q(1,3,4,2) + A^{\rm P}_Q(3,1,4,2) + A^{\rm NP}_Q(1,3,4,2) + A^{\rm NP}_Q(3,1,4,2))\Big|_{D=5\,\text{div.}} \end{aligned}$$

Plug into gravity double copy:

$$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0$$

A Conjecture

We can conjecture as well as others:

Conjecture: **(Q + 16) supercharge supergravity amplitude are finite when divergences in corresponding Q supercharge YM amplitudes carry only tree color tensors.**

Corollary: **$N \geq 4$ supergravity in $D = 4$ is ultraviolet finite.**

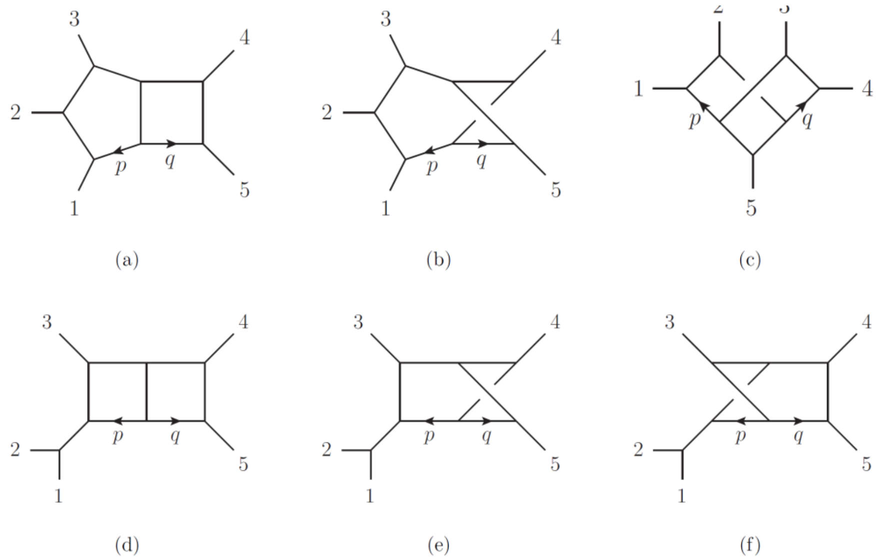
Conjecturing is easy. The nontrivial part is to prove (or disprove).

Remains a challenge to prove beyond above 1,2 loop examples because loop momenta appear in numerators of both copies.

But we still have the power to calculate!

Two loops and five points

Might the above have to do with the special property of no numerator loop momentum in *N* = 4 sYM? Two-loop five-point doesn't have this property.



Carrasco and Johansson Give us maximal sYM in BCJ format.

Half maximal sugra: Take other copy to be pure YM Feynman diagrams.

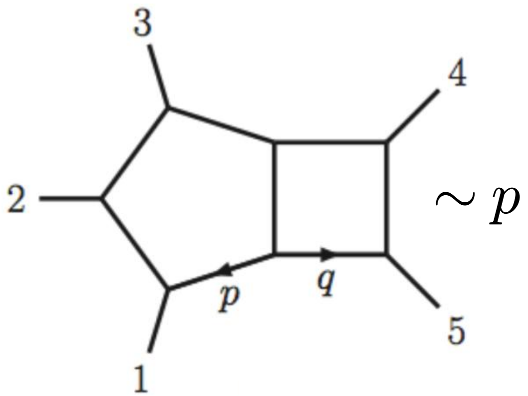
$\mathcal{I}^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills numerator
(a),(b)	$\frac{1}{4}\Big(\gamma_{12}(2s_{45}-s_{12}+\tau_{2p}-\tau_{1p})+\gamma_{23}(s_{45}+2s_{12}-\tau_{2p}+\tau_{3p})$ $+2\gamma_{45}(\tau_{5p}-\tau_{4p})+\gamma_{13}(s_{12}+s_{45}-\tau_{1p}+\tau_{3p})\Big)$
(c)	$\frac{1}{4}\Big(\gamma_{15}(\tau_{5p}-\tau_{1p})+\gamma_{25}(s_{12}-\tau_{2p}+\tau_{5p})+\gamma_{12}(s_{34}+\tau_{2p}-\tau_{1p}+2s_{15}+2\tau_{1q}-2\tau_{2q})$ $+\gamma_{45}(\tau_{4q}-\tau_{5q})-\gamma_{35}(s_{34}-\tau_{3q}+\tau_{5q})+\gamma_{34}(s_{12}+\tau_{3q}-\tau_{4q}+2s_{45}+2\tau_{4p}-2\tau_{3p})\Big)$
(d)-(f)	$\gamma_{12}s_{45}-\frac{1}{4}\Big(2\gamma_{12}+\gamma_{13}-\gamma_{23}\Big)s_{12}$

$$\tau_{ip}=2k_i\cdot p$$

Two loops five points double copy

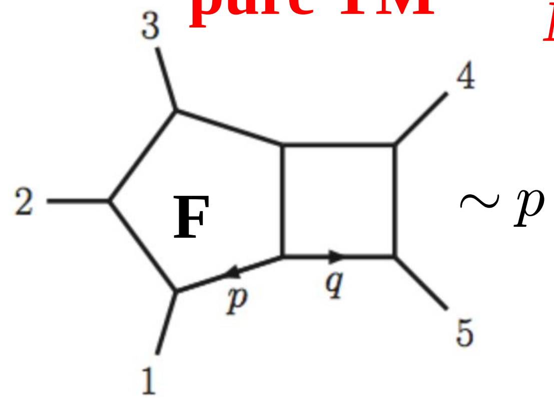
Half maximal supergravity: ($\mathcal{N}$ = 4 sYM) x ($\mathcal{N}$ = 0 YM)

$\mathcal{N}$ = 4 sYM



BCJ representation

pure YM



Feynman representation

$\mathcal{N}$ = 4 sugra quadratic divergent

$$\int_{D=5} (d^5 l)^2 \frac{k^6 l^8}{l^{16}}$$

- **To extract UV expand in small external momenta.**
- **Integrals have subdivergences which causes complications. But this was well understood 30 years ago by Vladimirov and by Marcus and Sagnotti.**

Five Points Two loops $D = 5$ half-max sugra

ZB, Davies, Dennen, Huang

Evaluation of the integrals gives the UV divergences:

Graph	$(\text{divergence})/(i\,\gamma_{12}\,\varepsilon_1\cdot\varepsilon_3\,\varepsilon_4\cdot\varepsilon_5\,k_1\cdot\varepsilon_2\,s_{12})$
(a)	$\frac{-64497+925D_s}{362880\sqrt{2}}\frac{1}{\epsilon}$
(b)	$\frac{820641-149788D_s}{1451520\sqrt{2}}\frac{1}{\epsilon}$
(c)	$\frac{-27555+8116D_s}{80640\sqrt{2}}\frac{1}{\epsilon}$
(d)	$\left(\frac{20605+912D_s}{53760\sqrt{2}}+\frac{-38+D_s}{240\sqrt{2}}\frac{s_{14}}{s_{13}}+\frac{655-161D_s}{1680\sqrt{2}}\frac{s_{23}}{s_{13}}+\frac{-5171-148D_s}{6720\sqrt{2}}\frac{s_{24}}{s_{13}}\right)\frac{1}{\epsilon}$
(e)	$\left(\frac{-71986+4511D_s}{241920\sqrt{2}}+\frac{935+6D_s}{6720\sqrt{2}}\frac{s_{14}}{s_{13}}+\frac{-907+342D_s}{6720\sqrt{2}}\frac{s_{23}}{s_{13}}+\frac{27859+844D_s}{60480\sqrt{2}}\frac{s_{24}}{s_{13}}\right)\frac{1}{\epsilon}$
(f)	$\left(\frac{-31847-8615D_s}{241920\sqrt{2}}+\frac{129-34D_s}{6720\sqrt{2}}\frac{s_{14}}{s_{13}}+\frac{-1713+302D_s}{6720\sqrt{2}}\frac{s_{23}}{s_{13}}+\frac{2335+61D_s}{7560\sqrt{2}}\frac{s_{24}}{s_{13}}\right)\frac{1}{\epsilon}$

$D = 5$

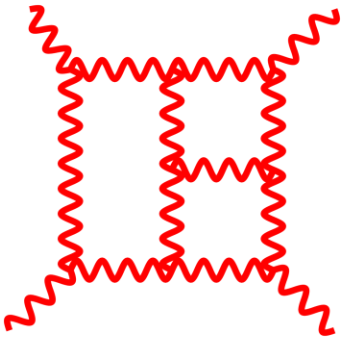
Sum over diagrams vanishes

D_s: state counting regularization parameter

- **While more complicated we see the same cancellations as we saw at four points (where no integration required).**
- **Potential R^4 and ϕR^4 counterterms in $D = 5$ half maximal supergravity have vanishing coefficients.**
- **R^4 full superspace integral. Seems to be no duality or susy explanation for vanishing -- see Bossard's talk.**

$\mathcal{N} = 4$ supergravity

ZB, Davies, Dennen, Huang



A no lose calculation:
Either we find first example of a divergence or once again we show an expected divergence is not present!

Motivated by Bossard, Howe, Stelle and Vanhove paper

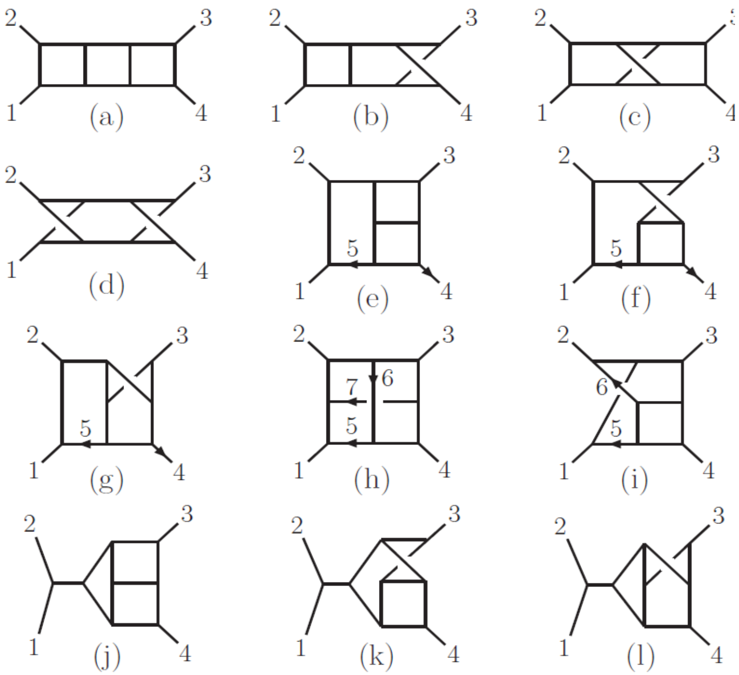
One year everyone believed that supergravity was finite. The next year the fashion changed and everyone said that supergravity was bound to have divergences even though none had actually been found. — *Stephen Hawking, 1994*

To this day no one has ever proven that *any* pure supergravity diverges in $D = 4$.

Three-loop construction

ZB, Davies, Dennen, Huang

$\mathcal{N} = 4$ sugra : ($\mathcal{N} = 4$ sYM) x ($\mathcal{N} = 0$ YM)



- For $\mathcal{N} = 4$ sYM copy use known BCJ representation.
- What representation should we use for pure YM side?

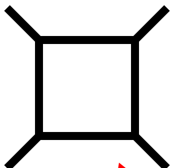
Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35} + \tau_{45} + t) - t\,(\tau_{25} + \tau_{45}) + u\,(\tau_{25} + \tau_{35}) - s^2 \right)/3$
(h)	$\left(s\,(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u) \right.$ $\left. + t\,(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2 \right)/3$
(i)	$\left(s\,(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t) \right.$ $\left. + t\,(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

$\mathcal{N} = 4$ sYM integrand

Multiloop N = 4 super-Yang-Mills

The duality satisfying forms of *N*=4 sYM amplitudes

One-loop: only box integral contributes.

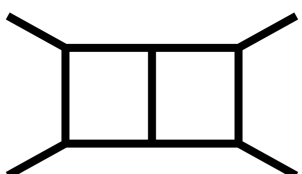
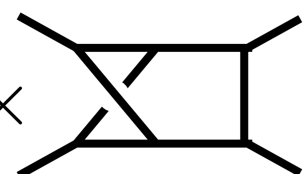
$$st A_4^{\text{tree}} \times$$  + perms

$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$

***N* = 4 sYM box numerator**

Scalar box integral. Same as in ϕ^3 theory

Two loops: only double box integrals contribute.

$$s^2 t A_4^{\text{tree}} \times$$  $$+ s^2 t A_4^{\text{tree}} \times$$  + perms

Scalar double-box integral. Same as in ϕ^3 theory

Amazing simplicity: "*N* = 4 sYM is hydrogen atom of gauge theory"

Multiloop $N = 4$ supergravity

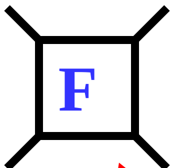
$N = 4$ sugra : ($N = 4$ sYM) x ($N = 0$ YM)

Does it work? Test at 1, 2 loops

All pure supergravities finite at 1,2 loops

One-loop: keep only box Feynman diagrams

$$st A_4^{\text{tree}} \times$$



$$+ \text{perms} = \frac{0}{\epsilon} = \text{finite}$$

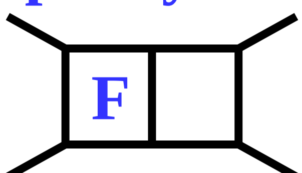
$N = 4$ sYM box numerator

$N = 0$ Feynman diagram, including ghosts

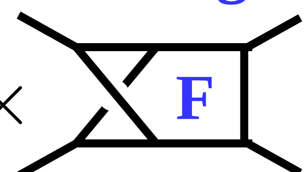
Becomes gauge invariant after permutation sum.

Two-loop: keep only double box Feynman diagrams

$$s^2 t A_4^{\text{tree}} \times$$



$$+ s^2 t A_4^{\text{tree}} \times$$



$$+ \text{perms} = \frac{0}{\epsilon}$$

Feynman diagram including ghosts

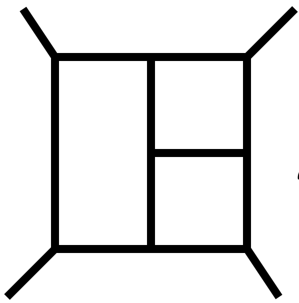
Get correct results. Who would have imagined multiloop gravity calculations this simple?

Three-Loop Construction

Now apply the construction to three loops.

N = 4 sugra : (*N* = 4 sYM) x (*N* = 0 YM)

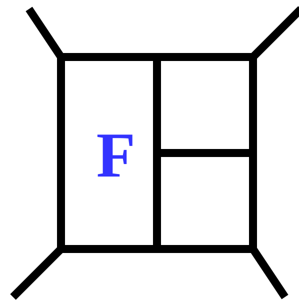
N = 4 sYM



$\sim l \cdot k \, s^2 t A_4^{\text{tree}}$

BCJ representation

pure YM



$\sim (\varepsilon_i \cdot l)^4 \; l^4$

Feynman representation

N = 4 sugra linear divergent

$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

simple to see finite for *N*=5,6 sugra

Will find that *N* = 4 supergravity better behaved than pure YM.

Numerator: $k^7 l^9 + k^8 l^8$ + finite

log divergent

Series expand in external momenta *k* and integrate

Dealing With Subdivergences

The integrals have subdivergences, greatly complicating evaluation

The problem was solve nearly 30 years ago. Marcus, Sagnotti (1984)

Recursively subtract all subdivergences.

regulator dependent

reparametrize subintegral

$$\mathcal{S}\left[\int \prod_{i=1}^{L} dp_i\, I\right] \equiv \text{Div}\left[\int \prod_{i=1}^{L} dp_i\, I\right] - \sum_{l=1}^{L-1} \sum_{\substack{l-\text{loop}\\ \text{subloops}}} \text{Div}\left[\int \prod_{j=l+1}^{L} dp'_j\, \mathcal{S}\left[\int \prod_{i=1}^{l} dp'_i\, I\right]\right]$$

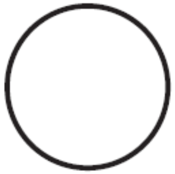
Regulator Independent

Nice consistency check: all log(*m*) terms must cancel

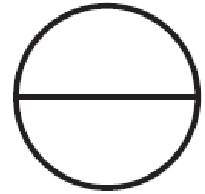
Extracting UV divergence in the presence of UV subdivergences and IR divergences is a well understood problem.

Integral Basis

Using FIRE we obtain a basis of integrals:

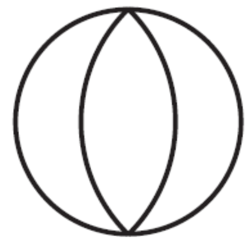


$$(m^2)^{1-\epsilon}\left(\frac{1}{\epsilon}+1+\left(1+\frac{1}{2}\zeta_2\right)\epsilon\right)$$



$$(m^2)^{1-2\epsilon}\left(\frac{3}{2\epsilon^2}+\frac{9}{2\epsilon}+\frac{21}{2}+\frac{3}{2}\zeta_2-2c\right)$$

$$(m^2)^{1-3\epsilon}\left(\frac{1}{\epsilon^3}+\frac{17}{3\epsilon^2}+\left(\frac{67}{3}+\frac{3}{2}\zeta_2-4c\right)\frac{1}{\epsilon}\right)$$



$$(m^2)^{2-3\epsilon}\left(\frac{2}{\epsilon^3}+\frac{23}{3\epsilon^2}+\left(\frac{35}{2}+3\zeta_2\right)\frac{1}{\epsilon}\right)$$

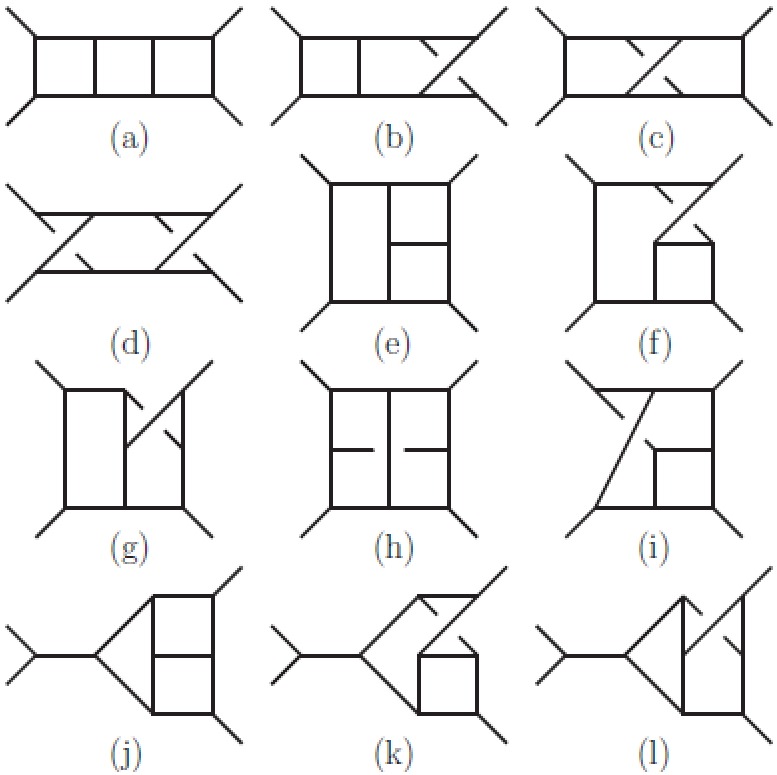
$$(m^2)^{-3\epsilon}\left(\frac{2\zeta_3}{\epsilon}\right)$$

$$c=\sqrt{3}\,\mathrm{Im}\left(\mathrm{Li}_2(e^{i\pi/3})\right)$$

Use Mellin-Barnes resummation of residues method of Davydychev and Kalmykov on all but last integral. Last one doable by staring at paper from Grozin or Smirnov's book (easy because no subdivergences). In paper from Czakon

The $N = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	(divergence)$/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

Individual diagrams gauge dependent

Sum over diagrams gauge invariant

All divergences cancel completely in sum over diagrams!

Surprise: it's actually UV finite

Once again we prove that there are more cancellations than expected

Summary

- A new duality conjectured between color and kinematics.
- When duality between color and kinematics manifest, gravity integrands follow *immediately* from gauge-theory ones.
- Double copy gives us a powerful way to explore the UV properties of gravity theories.
- Surprises, contrary to symmetry considerations:
  — $Q = 16$ supergravity in D=5 has no 2-loop 4-point divergences.
  — $N = 4$ sugra in D=4 has no 3-loop 4-point divergences.
- Power counting using known symmetries and their known consequences can be misleading. Concrete example.
- Concrete examples directly linking cancellations of divergences in forbidden color factor of pure YM to those of half max supergravity

The double copy + unitarity formalism gives us good reasons to believe that $N \geq 4$ pure supergravity theories are UV finite. More importantly it give us the tools to decisively test this.

List of Papers

Research Articles:

- ZB, L. Dixon , R. Roiban, hep-th/0611086
- ZB, J.J. Carrasco, L. Dixon, H. Johansson, D. Kosower, R. Roiban, hep-th/0702112.
- ZB, J.J.M. Carrasco , L.J. Dixon, Henrik Johansson, R. Roiban, arXiv:0808.4112.
- ZB, J.J.M. Carrasco, H. Ita, H. Johansson, R. Roiban, arXiv:0903.5348.
- ZB, J.J.M. Carrasco, L.J. Dixon, H. Johansson, R. Roiban, arXiv:0905.2326.
- ZB, J. J. M. Carrasco, H.Johansson, arXiv:1004.0476
- ZB, T. Dennen, Y.t. Huang and M. Kiermaier, arXiv:1004.0693.
- ZB, J.J.M. Carrasco , L.J. Dixon, Henrik Johansson, R. Roiban, arXiv:1201.5366
- ZB, S. Davies, T. Dennen, and Y.-t. Huang, : arXiv:1202.3423

Review Articles:

- Z. Bern, gr-qc/0206071
- Z. Bern, J. J. M. Carrasco and H. Johansson, 0902.3765 [hep-th]
- H. Nicolai, Physics, 2, 70, (2009).
- R. P. Woodard, arXiv:0907.4238 [gr-qc].
- L. Dixon, arXiv:1005.2703 [hep-th].

Further Reading

Hermann Nicolai, *Physics Viewpoint*, "Vanquishing Infinity"
http://physics.aps.org/articles/v2/70

Anthony Zee, *Quantum Field Theory in a Nutshell*,
2nd Edition is first textbook to contain modern formulation of scattering and commentary on new developments. 4 new chapters.

Z. Bern, L. J. Dixon, D. A. Kosower
May 2012 cover story of *Scientific American*

Some amusement

YouTube: Search "Big Bang DMV", first hit, 20 sec into the clip