Towards Holographic Duality for Condensed Matter

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Preface

My goal for these lectures is to convince you that string theory may be useful for condensed matter physics.

The systems about which we can hope to say something using string theory have in common strong coupling.

This makes our usual techniques basically useless.

goal for first lecture:

AdS/CFT solves certain strongly-coupled quantum field theories in terms of simple (gravity) variables.

Real systems with strong coupling

We've developed enough confidence in these techniques to try to apply them to questions about real strongly-coupled systems.

Like what?

- quark-gluon plasma at RHIC (Yaron Oz' lectures?)
- fermions at unitarity (*e.g.* cold atoms with Feshbach-tuned interactions)
- non-Fermi liquid metals (*e.g.* high T_c, heavy fermion phase transitions)

What about standard techniques?

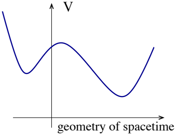
perturbation theory (requires one to perturb about the right description)

even cond-mat is 'particle physics': reliance on quasiparticles.

monte carlo simulation (obstructed by sign problem here)

A word about string theory

String theory is a (poorly-understood) quantum theory of gravity
which has a ‘landscape’ of **many** groundstates
some of which look like our universe
($3+1$ dimensions, particle physics...)
most of which don’t.



A difficulty for particle physics, a virtue for many-body physics:
by AdS/CFT, each groundstate (with $\Lambda < 0$) describes a universality class of critical behavior and its deformations
This abundance mirrors ‘landscape’ of many-body phenomena.

An opportunity to connect string theory and experiment.
We are learning about string theory and about the duality.

Outline

1. Holographic duality with a view toward condensed matter
   [review: JM, 0909.0518]

2. Gravity duals of non-relativistic QFTs
   [Son, 0803.3972
   Koushik Balasubramanian, JM, 0803.4053
   Allan Adams, KB, JM, 0807.1111
   KB, JM, 1007.2184]

3. Non-Fermi liquids from non-holography
   [D. Mross, JM, H. Liu, T. Senthil, 1003.0894]

4. Non-Fermi liquids from holography
   [Hong Liu, JM, David Vegh, 0903.2477
   Tom Faulkner, HL, JM, DV, 0907.2694
   TF, Gary Horowitz, JM, Matt Roberts, DV, 0911.3402
   TF, Nabil Iqbal, HL, JM, DV, 1003.1728 and to appear]

5. (If time allows:) Strongly correlated topological insulators
   [J. Maciejko, X. Qi, A. Karch, and S.-C. Zhang, 1004.3628
   B. Swingle, M. Barkeshli, JM, T. Senthil, 1005.1076]

Holographic duality with a view toward condensed matter

Bold assertions

[Horowitz-Polchinski, gr-qc/0602037]

a) Some ordinary quantum many-body systems are actually quantum theories of gravity in extra dimensions
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What can this mean??

Two hints:

1. The Renormalization Group (RG) is local in scale
2. Holographic Principle

Old-school universality

experimental universality (late 60s):
same critical exponents from very different systems.
Near a (continuous) phase transition (at $T = T_c$), scaling laws:
observables depend like power laws on the distance from the critical point.
e.g. ferromagnet near the Curie transition (let $t \equiv \frac{T_c - T}{T_c}$)

$$\text{specific heat: } c_v \sim t^{-\alpha}$$
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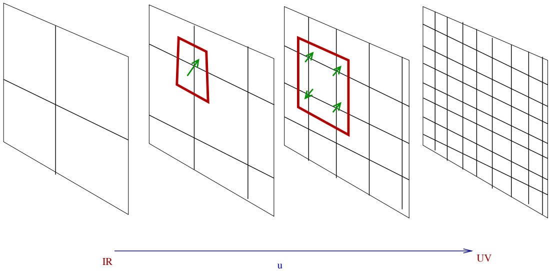
water near its liquid-gas critical point:

$$\text{specific heat: } c_v \sim t^{-\alpha}$$
$$\text{compressibility: } \chi \sim t^{-\gamma}$$

with the same α, γ!

Renormalization group idea

This phenomenon is explained by the Kadanoff-Wilson idea:



$$e.g.: \; H = \sum_{ij} J_{ij} S_i S_j$$

Idea: measure the system with coarser and coarser rulers.

Let 'block spin' = average value of spins in block.

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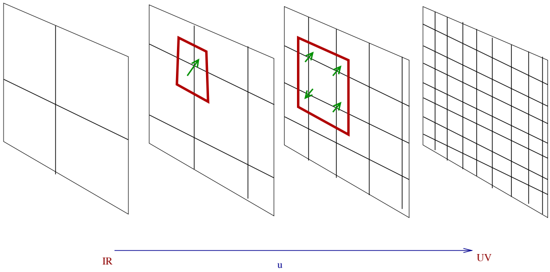

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Idea: measure the system with coarser and coarser rulers.

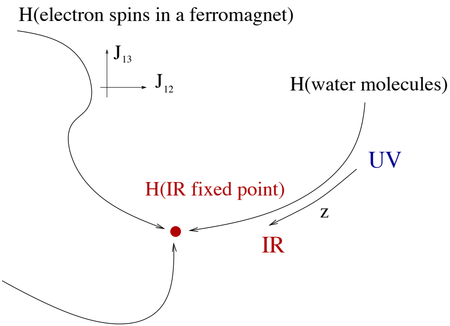
Let 'block spin' = average value of spins in block.

Define a Hamiltonian $H(u)$ for block spins so long-wavelength observables are the same.

$\longrightarrow$ a flow on the space of hamiltonians: $H(u)$

Fixed points of the RG are scale-invariant

This procedure (the sums) is hard to do in practice.



Many microscopic theories will flow to the same fixed-point $\longrightarrow$ same critical scaling exponents.

The fixed point theory is scale-invariant:
if you change your resolution you get the same picture back.

Hint 1: RG is local in scale

QFT $=$ family of trajectories on the space of hamiltonians: $H(u)$

at each scale u, expand in symmetry-preserving local operators $\{\mathcal{O}_A\}$

$$H(u) = \int d^{d-1}x \sum_A g_A(u)\mathcal{O}_A(u,x)$$

[*e.g.* suppose the dof is a scalar field. then $\{\mathcal{O}_A\} = \{(\partial\phi)^2, \phi^2, \phi^4, \ldots\}$]

since $H(u)$ is determined by a step-by-step procedure,

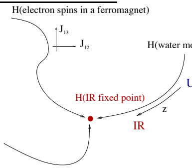
$$u\partial_u g = \beta_g(g(u)) \ .$$

for each coupling g

locality in scale: β_g depends only on $g(u)$.

Def: near a fixed point,

β_g is determined by the *scaling dimension* Δ of $\mathcal{O}$:



$$\mathcal{O}^a(x,u_1) \sim \left(\frac{u_1}{u_2}\right)^{\Delta} \mathcal{O}(x,u_2)$$

ops of large Δ $(> d$, "irrelevant")

become small in IR (as $u \to 0$).

Hint 2: Holographic principle

holographic principle: in a gravitating system, max entropy in region V $\propto$ area of ∂V in planck units. ['t Hooft, Susskind 1992]

recall: max entropy $S_{MAX} \sim \ln \dim \mathcal{H} \propto \# dof$.

in an ordinary system with local dofs $S_{MAX} \propto V$

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to see that gravitating systems are different, we combine two facts:

fact 1: BH has an entropy $\propto$ area of horizon in planck units.

$$S_{BH} = \frac{A}{4 G_N}$$

in $d+1$ spacetime dimensions, $G_N \sim \ell_p^{d-1}$ $\longrightarrow$ S_{BH} dimless.

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Whence fact 1?

Black holes have a temperature [Hawking] *e.g.* $T_H = \frac{1}{8\pi G_N M}$ for schwarzchild

Consistent thermodynamics requires us to assign them an entropy:

$dE_{\rm BH} = T_H dS_{\rm BH}$ for schwarzchild, $E_{\rm BH} = M$, $A = 4\pi(4M^2G^2)$ gives $(\star)$

'Generalized 2d Law': $S_{total} = S_{\rm ordinary\ stuff} + S_{\rm BH}$

Hint 2: Holographic principle, cont'd

fact 2: dense enough matter collapses into a BH

Hint 2: Holographic principle, cont'd

fact 2: dense enough matter collapses into a BH
$1 + 2 \longrightarrow$ in a gravitating system,
max entropy in a region of space $=$
entropy of the biggest black hole that fits.

$$S_{max} = S_{BH} = \frac{1}{4\pi G_N} \times \text{horizon area}$$

Idea [Bekenstein, 1976]: consider a volume V with area A in a flat region of space.

suppose the contrary: given a configuration with
$S > S_{\rm BH} = \frac{A}{4G_N}$ but $E < E_{\rm BH}$ (biggest BH fittable in V)
then: throw in junk (increases S and E) until you make a BH.
S decreased, violating 2d law.

punchline: gravity in $d+2$ dimensions has the same number of degrees of freedom as a QFT in *fewer* $(d+1)$ dimensions.

1+2

combining these hints, we conjecture:

$$\begin{array}{c}\text{gravity}\\ \text{in a space with an extra dim}\\ \text{whose coord is the energy scale}\end{array} \quad \stackrel{?}{=} \quad \text{QFT}$$

to make this more precise, we consider a simple case
(AdS/CFT) [Maldacena, 1997]
in more detail.

AdS/CFT

a relativistic field theory, scale invariant ($\beta_g = 0$ for all nonzero g)

$$x^\mu \to \lambda x^\mu \quad \mu = 0...d-1, \quad u \to \lambda^{-1} u$$

u is the energy scale, RG coordinate

with d-dim'l Poincaré symmetry: Minkowski $ds^2 = -dt^2 + d\vec{x}^2$

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Most gen'l $d+1$ dim'l metric w/ Poincaré plus scale inv.

$$AdS_{d+1}: \quad ds^2 = \frac{u^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + L^2 \frac{du^2}{u^2} \qquad L \equiv \text{'}AdS \text{ radius'}$$

If we rescale space and time and move in the radial dir,

the geometry looks the same (isometry).

copies of minkowski space of varying 'size'.

(Note: this metric also has conformal symmetry $SO(d,2)$

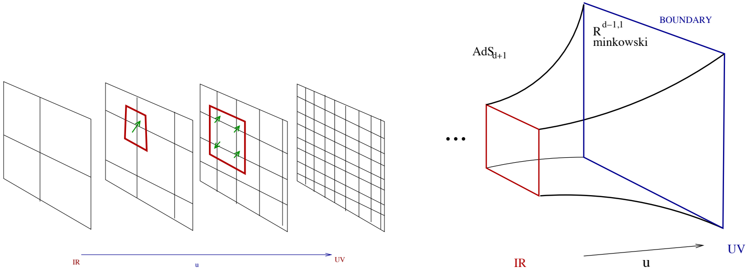
$\exists$ gravity dual $\Longrightarrow$ "Polchinski's Theorem" for any d.)

another useful coordinate:

$$z \equiv \frac{L^2}{u} \qquad ds^2 = L^2 \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}$$

$[u] =$ energy, $[z] =$ length ($c = \hbar = 1$ units).

Geometry of AdS continued



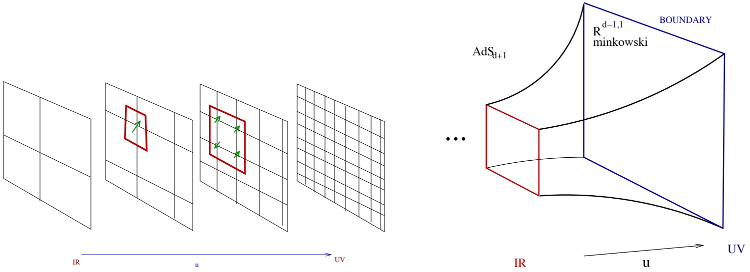
The extra ('radial') dimension is the resolution scale.

(The bulk picture is a hologram.)

preliminary conjecture:

$$\mathrm{CFT}_d \stackrel{?}{=} \text{gravity on } AdS_{d+1} \text{ space}$$

Geometry of *AdS* continued



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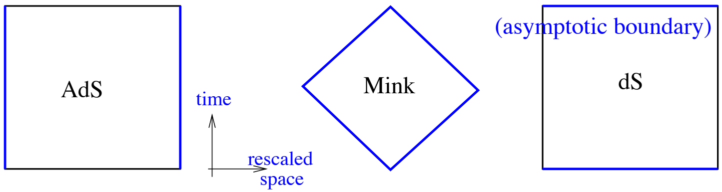
crucial refinement:

in a gravity theory the metric fluctuates.

$\longrightarrow$ what does 'gravity in AdS' mean ?!?

Geometry of *AdS* continued

AdS has a boundary (where $u \to \infty, z \to 0$, 'size' of Mink blows up).
massless particles reach it in finite time.
$\implies$ must specify boundary conditions there.
the fact that the geometry is *AdS* near there is one of these boundary conditions.
different from Minkowski space or (worse) de Sitter:



so: some $\text{CFT}_d \stackrel{?}{=}$ gravity on *asymptotically* AdS_{d+1} space
(we will discuss the meaning of this '=' much more)

Preview of dictionary

"bulk" $\leftrightsquigarrow$ "boundary"

fields in AdS_{d+1} $\leftrightsquigarrow$ operators in CFT

(Note: operators in CFT don't make particles.)

mass $\leftrightsquigarrow$ scaling dimension

$$m^2 L^2 = \Delta(\Delta - d)$$

a simple bulk theory with a small # of light fields	$\leftrightsquigarrow$	CFT with a small # of ops of small Δ (like rational CFT)

What to calculate

some observables of a QFT (Euclidean for now):
vacuum correlation functions of local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

standard trick: make a generating functional $Z[J]$ for these correlators by perturbing the action of the QFT:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_A J_A(x)\mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x)$$

$$Z[J] = \langle e^{-\int \mathcal{L}_J}\rangle_{CFT}$$

$J_A(x)$: arbitrary functions (sources)

$$\langle \prod_n \mathcal{O}_n(x_n)\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z\Big|_{J=0}$$

Hint: $\mathcal{L}_J$ is a *UV* perturbation – near the boundary, $z \to 0$

Holographic duality made quantitative

[Witten; Gubser-Klebanov-Polyakov (GKPW)]

$$Z_{QFT}[\mathrm{sources}] = Z_{\mathrm{quantum\ gravity}}\left[\text{boundary conditions at } u \longrightarrow \infty\right]$$

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&\approx e^{-S_{\text{bulk}}[\text{boundary conditions at } u\to\infty]}\Big|_{\text{saddle of } S_{\text{bulk}}} \\
J = \phi_0 & \qquad \text{"}\phi \overset{u\to\infty}{\longrightarrow} \phi_0\text{"}
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What's S_{bulk}? AdS solves the EOM for

$$S_{\text{bulk}} = \frac{1}{\# G_N} \int d^4x \sqrt{g}\,(\mathcal{R} - 2\Lambda + \ldots)$$

(... = fields which vanish in groundstate, more irrelevant couplings.)

expansion organized by decreasing relevance

$\Lambda = -\frac{d(d-1)}{2L^2}$ note tuning!

$\mathcal{R} \sim \partial^2 g \implies G_N \sim \ell_p^{d-1}$

gravity is classical if $L \gg \ell_p$.

This is what comes from string theory (when we can tell)

at low E and for $\frac{1}{L} \ll \frac{1}{\sqrt{\alpha'}} \equiv \frac{1}{\ell_s}$ ($\frac{1}{\sqrt{\alpha'}}$ = string tension)

(One basic role of string theory here: fill in the dots.)

Conservation of evil

large AdS radius $L \leftrightsquigarrow$ strong coupling of QFT

(avoids an immediate disproof – obviously a perturbative QFT isn't usefully an extra-dimensional theory of gravity.)

a special case of a

Useful principle (Conservation of evil):
different weakly-coupled descriptions
have non-overlapping regimes of validity.

strong/weak duality: hard to check, very powerful

Info goes both ways: once we believe the duality, this is our best definition of string theory.

Holographic counting of degrees of freedom

[Susskind-Witten]

$$S_{max} = \frac{\text{area of boundary}}{4G_N} \overset{?}{=} \# \text{ of dofs of QFT}$$

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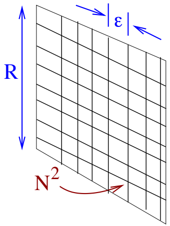
$$\text{yes :} \qquad \infty = \infty$$

need to regulate two divergences: dofs at every point in space (UV) (# dofs $\equiv N^2$) ,
spread over $\mathbb{R}^{d-1}$ (IR).
counting in QFT$_d$:

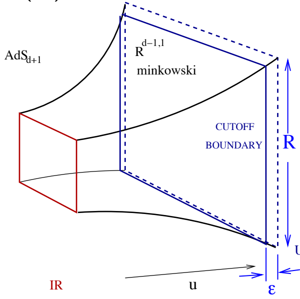
$$S_{max} \sim \left(\frac{R}{\epsilon}\right)^{d-1} N^2$$



counting in AdS$_{d+1}$: at fixed time: $ds^2_{AdS} = L^2 \frac{dz^2 + d\vec{x}^2}{z^2}$

$$A = \int_{bdy,\ z\ fixed} \sqrt{g}\, d^{d-1}x = \int_{\mathbb{R}^{d-1}} \sqrt{g}\, d^{d-1}x \left(\frac{L}{z}\right)^{d-1} |_{z\to 0}$$

$$A = \int_0^R d^{d-1}x \frac{L^{d-1}}{z^{d-1}}|_{z=\epsilon} = \left(\frac{RL}{\epsilon}\right)^{d-1}$$



The holographic principle

then says that the maximum entropy in the bulk is

$$\frac{A}{4G_N} \sim \frac{L^{d-1}}{4G_N}\left(\frac{R}{\epsilon}\right)^{d-1}.$$

$$\boxed{\frac{L^{d-1}}{G_N} = N^2}$$

lessons:

1. parametric dependence on R checks out.
2. gravity is classical if QFT has lots of dofs/pt: $N^2 \gg 1$

$$Z_{QFT}[\text{sources}] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } r\to\infty]}|_{\text{extremum of } I_{\text{bulk}}}$$

classical gravity (sharp saddle) $\leftrightsquigarrow$ many dofs per point, $N^2 \gg 1$

Confidence-building measures

Why do we believe this enough to try to use it to do physics?

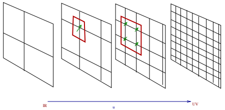
- 1. **Many** detailed checks in special examples
  examples: relativistic gauge theories (fields are $N \times N$ matrices), with extra symmetries (conformal invariance, supersymmetry)
  checks: 'BPS quantities,' integrable techniques, some numerics
- 2. sensible answers for physics questions
  rediscoveries of known physical phenomena: *e.g.* color confinement, chiral symmetry breaking, thermo, hydro, thermal screening, entanglement entropy, chiral anomalies, superconductivity, ...
  Gravity limit, when valid, says who are the correct variables.
  Answers questions about thermodynamics, transport, RG flow, ...
  in terms of geometric objects.
- 3. applications to quark-gluon plasma (QGP)
  benchmark for viscosity, hard probes of medium, approach to equilibrium

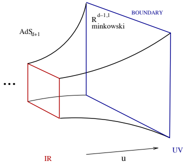
Simple pictures for hard problems, an example

Bulk geometry is a spectrograph separating the theory by energy scales.

$$ds^2 = w(z)^2 \left(-dt^2 + d\vec{x}^2\right) + \frac{dz^2}{z^2}$$

CFT: bulk geometry goes on forever, warp factor $w(z) = \frac{L}{z} \to 0$:





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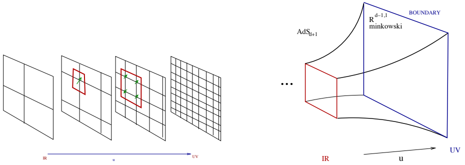
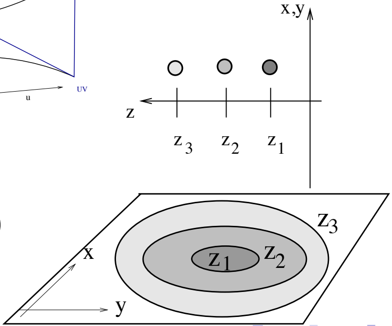




$t, \vec{x}$ are the field theory time and space coordinates.

$$(\text{size})_{FT} = \frac{1}{w(z)}(\text{proper size})$$

$$E_{FT} \sim i\partial_t = w(z) E_{\text{proper}}$$

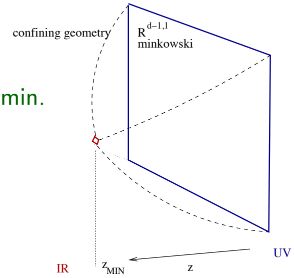
The role of the warp factor, cont'd

Model with a gap:

geometry ends smoothly, warp factor $w(z)$ has a min.

if IR region is missing,

no low-energy excitations, energy gap.



A word about large N^2

most prominent example: 't Hooft limit of $N \times N$ matrix fields X.
physical operators are $\mathcal{O}_k = \mathrm{tr}\, X^k$

this accomplishes several related things:

- $$\langle \mathcal{O}\mathcal{O} \rangle \sim \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle + o\left(N^{-2}\right)$$

is the statement that *something* (the excitations created by $\mathcal{O}$) behaves classically.

- provides notion of single-particle states in bulk.
- makes saddle well-peaked $Z \sim e^{-N^2 I}$

important comment:

this is just the best-understood class of examples.
in other examples, the # of dofs goes like N^b, $b \neq 2$.
I'll always write N^2 as a proxy for this large number.

More dictionary

really a ϕ_a for every $\mathcal{O}^a$ in CFT. how to match?

1. organize into reps of conformal group
2. single-trace operators correspond to 'elementary fields' in the bulk.

states from multitrace ops $(\mathrm{tr}\, X^k)^2|0\rangle$ — 2-particle states of ϕ.

3. simple egs fixed by symmetry:

• gauge fields in bulk A_μ – global currents J^μ in bdy

$S_{QFT} \ni \int A_\mu J^\mu$ (massless $A \leftrightsquigarrow$ conserved J)

• def of QFT stress tensor: response to change in metric on boundary $S_{QFT} \ni \int \delta g_{\mu\nu} T^{\mu\nu}$

$$\begin{array}{rcc} \text{energy momentum tensor: } T^{\mu\nu} & & \text{graviton: } g_{ab} \\ \text{global current: } J^\mu & & \text{Maxwell field: } A_a \\ \text{scalar operator: } \mathcal{O}_B & \leftrightsquigarrow & \text{scalar field: } \phi \\ \text{fermionic operator: } \mathcal{O}_F & & \text{fermionic field: } \psi\,. \end{array}$$

boundary conditions on bulk fields $\leftrightsquigarrow$ couplings in field theory

e.g.: boundary value of bulk metric $\lim_{r\to\infty} g_{\mu\nu}$

$=$ source for stress-energy tensor $T^{\mu\nu}$

different couplings in bulk action $\leftrightsquigarrow$ different field theories

Next: a few technical slides from which we can confirm our interpretation

$$u = \text{RG scale}$$

and see the machinery at work.

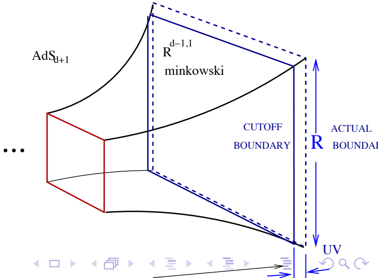
How to calculate

$$Z_{QFT}[\text{sources}] \approx e^{-N^2 I_{\text{bulk}}[\text{boundary conditions at } z \to 0]}\Big|_{\text{extremum of } I_{\text{bulk}}}$$

more explicitly:

$$\begin{aligned} Z_{QFT}[\text{sources}, \phi_0] &\equiv \langle e^{-\int \phi_0 \mathcal{O}} \rangle_{CFT} \\ &\approx e^{-N^2 I_{\text{bulk}}[\phi|\phi(z=\epsilon) \stackrel{?}{=} \phi_0]}\Big|_{\phi \text{ solves EOM of } I_{\text{bulk}}} \end{aligned}$$

As when counting dofs, we anticipate UV divergences
at the boundary $z \to 0$,
cut off the bulk at $z = \epsilon$
and set bc's there.



Example: scalar probe

Simple example: scalar field in the bulk. Natural (covariant) action:

$$\Delta S[\phi] = -\frac{\mathfrak{K}}{2}\int d^{d+1}x\sqrt{g}\left[g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2 + b\phi^3 + \dots\right]$$

$\mathfrak{K}$, a normalization constant: assume the theory of ϕ is weakly coupled, $\mathfrak{K} \propto N^2$.

$\left(\sqrt{g} = \sqrt{|\det g|} = \left(\frac{L}{z}\right)^{d+1}, \quad g^{AB} = \delta^{AB}z^2\right)$

We will study fluctuations around the solution $\phi = 0$, AdS.

(Recall: $\langle\mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^2 \ln Z|_{\underbrace{\phi_0 = 0}}$)

$\longrightarrow$ ignore interactions of ϕ for now.

Integrate by parts

$$S = -\frac{\mathfrak{K}}{2}\int_{\partial AdS} d^dx\,\sqrt{g}\,g^{zB}\phi\,\partial_B\phi - \frac{\mathfrak{K}}{2}\int\sqrt{g}\,\phi\left(-\Box + m^2\right)\phi + o(\phi^3)$$

From this expression we learn:

- the EOM for small fluctuations of ϕ is $(-\Box + m^2)\underline{\phi} = 0$
  (An underline will indicate fields which solve the equations of motion.)
- If $\underline{\phi}$ solves the equation of motion, the on-shell action
  $S[\underline{\phi}], \quad Z \equiv e^{-S[\underline{\phi}]}$
  is just given by the boundary term.

next: relate bulk masses and operator dimensions

$$\Delta(\Delta - d) = m^2 L^2$$

by studying the AdS wave equation near the boundary.

Wave equation in AdS

translational invariance in d dimensions, $x^\mu \to x^\mu + a^\mu$,

$$\text{Fourier}: \ \phi(z, x^\mu) = e^{ik_\mu x^\mu} f_k(z), \qquad k_\mu x^\mu \equiv -\omega t + \vec{k}\cdot\vec{x}$$

$$\begin{aligned} 0 &= (g^{\mu\nu} k_\mu k_\nu - \frac{1}{\sqrt{g}}\partial_z(\sqrt{g} g^{zz}\partial_z) + m^2) f_k(z) \\ &= \frac{1}{L^2}[z^2 k^2 - z^{d+1}\partial_z(z^{-d+1}\partial_z) + m^2 L^2] f_k(z), \end{aligned} \qquad (1)$$

we used $g^{AB} = (z/L)^2 \delta^{AB}$, $\quad \sqrt{g} = \sqrt{|\det\, g|} = \left(\frac{L}{z}\right)^{d+1}$.

Wave equation in *AdS*

translational invariance in d dimensions, $x^\mu \to x^\mu + a^\mu$,

$$\text{Fourier}: \ \phi(z, x^\mu) = e^{ik_\mu x^\mu} f_k(z), \qquad k_\mu x^\mu \equiv -\omega t + \vec{k}\cdot\vec{x}$$

$$\begin{aligned} 0 &= \left(g^{\mu\nu}k_\mu k_\nu - \frac{1}{\sqrt{g}}\partial_z(\sqrt{g}g^{zz}\partial_z) + m^2\right)f_k(z) \\ &= \frac{1}{L^2}[z^2k^2 - z^{d+1}\partial_z(z^{-d+1}\partial_z) + m^2L^2]f_k(z), \qquad (1) \end{aligned}$$

we used $g^{AB} = (z/L)^2\delta^{AB}$, $\quad \sqrt{g} = \sqrt{|\det g|} = \left(\frac{L}{z}\right)^{d+1}$.

Near boundary ($z \to 0$), power law solns, (spoiled by the z^2k^2 term).

Try $f_k = z^\Delta$ in (1):

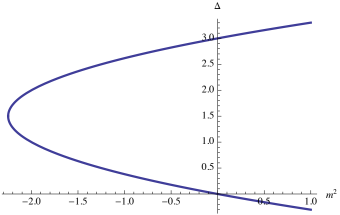
$$\begin{aligned} 0 &= k^2z^{2+\Delta} - z^{d+1}\partial_z(\Delta z^{-d+\Delta}) + m^2L^2z^\Delta \\ &= (k^2z^2 - \Delta(\Delta - d) + m^2L^2)z^\Delta, \end{aligned}$$

and for $z \to 0$ we get:

$$\Delta(\Delta - d) = m^2L^2 \qquad (2)$$

The two roots of (2) are $\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2L^2}$.

Comments



$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}.$$

- The solution proportional to z^{Δ_-} is bigger near $z \to 0$. $\to$ usually the source ('non-normalizable')
- $\Delta_+ > 0 \ \ \forall \ \ m$: z^{Δ_+} always decays near the boundary
- $\Delta_+ + \Delta_- = d$.

We want to impose boundary conditions that allow solutions. Leading $z \to 0$ behavior of generic solution: $\phi \sim z^{\Delta_-}$, we impose

$$\phi(x,z)|_{z=\epsilon} \stackrel{!}{=} \phi_0(x,\epsilon) = \epsilon^{\Delta_-} \phi_0^{Ren}(x),$$

where ϕ_0^{Ren} is a renormalized source field.

Wavefunction renormalization of $\mathcal{O}$ (Heuristic but useful)

Suppose: $(g_{\mu\nu} \overset{z\approx\epsilon}{=} \frac{dz^2}{z^2} + \gamma_{\mu\nu} dx^\mu dx^\nu$ defines the boundary metric γ .)

$$
\begin{aligned}
S_{bdy} &\ni \int_{z=\epsilon} d^d x \sqrt{\gamma}\, \phi_0(x,\epsilon)\mathcal{O}(x,\epsilon) \\
&= \int d^d x \left(\frac{L}{\epsilon}\right)^d (\epsilon^{\Delta_-}\phi_0^{Ren}(x))\mathcal{O}(x,\epsilon),
\end{aligned}
$$

where we have used $\sqrt{\gamma} = (L/\epsilon)^d$.

Demanding that this be finite as $\epsilon \longrightarrow 0$:

$$
\begin{aligned}
\mathcal{O}(x,\epsilon) &\sim \epsilon^{d-\Delta_-}\mathcal{O}^{Ren}(x) \\
&= \epsilon^{\Delta_+}\mathcal{O}^{Ren}(x),
\end{aligned}
$$

(we used $\Delta_+ + \Delta_- = d$)

The scaling dimension of $\mathcal{O}^{Ren}$ is $\Delta_+ \equiv \Delta$.

To confirm: $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$

Relevantness

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}$$

- If $m^2 > 0$: $\Delta \equiv \Delta_+ > d$, so $\mathcal{O}_\Delta$ is an irrelevant operator.

$$\Delta S = \int d^d x \; (\mathrm{mass})^{d-\Delta} \mathcal{O}_\Delta ,$$

the effects of such an operator go away in the IR, at energies $E < \mathrm{mass}$.

$\phi \sim z^{\Delta_-} \phi_0$ *is* this coupling.

It grows in the UV (small z). If ϕ_0 is a finite perturbation, it will back-react on the metric and destroy the asymptotic AdS-ness of the geometry: extra data about the UV will be required.

- $m^2 = 0 :\leftrightarrow \Delta = d$ means that $\mathcal{O}$ is marginal.
- If $m^2 < 0$: $\Delta < d$, so $\mathcal{O}$ is a relevant operator. Note that in AdS, $m^2 < 0$ is ok (*i.e.* not unstable) if m^2 is not too negative.

(Note: $\Delta(m)$ depends on the spin of the bulk field.)

So far: setting up machinery.

Next: make contact with physics (linear response, finite temperature).

Some big picture questions

1. What physics is contained in classical gravity duals?
   dissipation, entanglement, RG, what else?
2. What is the scope of this kind of relationship?
   Which systems have simple duals?
   (we'll address: deformations of CFT, non-relativistic CFT (NRCFT)
   open problem: lattice models?)
3. how close can we get to a lab system?

Recap

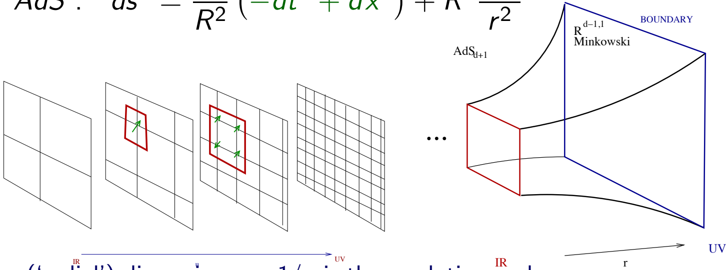
gravity in spacetimes$_{d+1}$
with timelike asymptotic boundaries $\quad \leftrightsquigarrow \quad QFT_d$

important special case:

gravity in AdS_{d+1} = d-dimensional conformal field theory (CFT)

isometries of AdS_{d+1} $\leftrightsquigarrow$ conformal symmetry

$$AdS: \quad ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + R^2\frac{dr^2}{r^2}$$



The extra ('radial') dimension $r = 1/z$ is the resolution scale.

fields in bulk $\leftrightsquigarrow$ (possibly-) running couplings

$$Z_{QFT}[\text{sources}, \phi_0] \approx e^{-S_{\text{bulk}}[\text{boundary conditions at } r\to\infty]}\big|_{\text{saddle of } S_{\text{bulk}}}$$

Vacuum of CFT, euclidean case

Return to the scalar wave equation in momentum space:

$$0 = [z^{d+1}\partial_z(z^{-d+1}\partial_z) - m^2L^2 - z^2k^2]f_k(z)$$

If $k^2 > 0$ (spacelike or Euclidean) the general solution is (a_K, a_I, integration consts):

$$f_k(z) = a_K z^{d/2} K_\nu(kz) + a_I z^{d/2} I_\nu(kz), \quad \nu = \Delta - \frac{d}{2} = \sqrt{(d/2)^2 + m^2L^2}.$$

In the interior of AdS ($z \to \infty$), the Bessel functions behave as

$$K_\nu(kz) \overset{z\to\infty}{\approx} e^{-kz} \qquad I_\nu(kz) \overset{z\to\infty}{\approx} e^{kz} \ .$$

regularity in the interior uniquely fixes $\underline{f}_k \propto K_\nu$.

Plugging this into the action S gives $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$

note: $\exists$ nonlinear uniqueness statement, 'Graham-Lee theorem'

Real-time

In Lorentzian signature with timelike k^2 $(\omega^2 > \vec{k}^2)$,
$\exists$ many solutions with the same UV behavior $(z \to 0)$, different IR behavior:

$$z^{d/2} K_{\pm\nu}(iqz) \overset{z\to\infty}{\approx} e^{\pm iqz} \qquad q \equiv \sqrt{\omega^2 - \vec{k}^2}$$

these modes oscillate near the Poincaré horizon.
this ambiguity reflects the multiplicity of real-time Green's f'ns.

Important example: retarded Green's function, describes causal response of the system to a perturbation.

Linear response: nothing fancy, just QM

The retarded Green's function for two observables $\mathcal{O}_A$ and $\mathcal{O}_B$ is

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = -i\int d^{d-1}x dt\ e^{i\omega t - ik\cdot x}\theta(t)\langle[\mathcal{O}_A(t,x), \mathcal{O}_B(0,0)]\rangle$$

$\theta(t) = 1$ for $t > 0$, else zero.

(We care about this because it determines what $\langle\mathcal{O}_A\rangle$ does if we kick the system via $\mathcal{O}_B$.)

the source is a time dependent perturbation to the Hamiltonian:

$$\delta H(t) = \int d^{d-1}x \phi_{B(0)}(t,x)\mathcal{O}_B(x)\,.$$

$$\begin{aligned} \langle\mathcal{O}_A\rangle(t,x) &\equiv \mathrm{Tr}\ \rho(t)\,\mathcal{O}_A(x) \\ &= \mathrm{Tr}\ \rho_0\, U^{-1}(t)\mathcal{O}_A(t,x)U(t) \end{aligned}$$

in interaction picture: $U(t) = Te^{-i\int^t \delta H(t')dt'}$ (*e.g.* $\rho_0 = e^{-\beta H_0}$)

Linear response, cont'd

linearize in small perturbation:

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(t,x) &= -i\mathrm{Tr}\ \rho_0\int^t dt'[\mathcal{O}_A(t,x),\delta H(t')]\\ &= -i\int^t d^{d-1}x' dt'\langle[\mathcal{O}_A(t,x),\mathcal{O}_B(t',x')]\rangle\phi_{B(0)}(t',x')\\ &= \int dx' G_R(x,x')\phi_B(x')\end{aligned}$$

fourier transform:

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)$$

Linear response, an example

perturbation: an external electric field, $E_x = i\omega A_x$
couples via $\delta H = A_x J^x$ where J is the electric current ($\mathcal{O}_B = J_x$)
response: the electric current ($\mathcal{O}_A = J_x$)

$$\delta\langle\mathcal{O}_A\rangle(\omega, k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k)\delta\phi_{B(0)}(\omega, k)$$

it's safe to assume $\langle J\rangle_{E=0} = 0$:

$$\langle\mathcal{O}_J\rangle(\omega, k) = G^R_{JJ}(\omega, k)A_x = G^R_{JJ}(\omega, k)\frac{E_x}{i\omega}$$

Ohm's law: $J = \sigma E$

$$\implies \textit{Kubo formula}: \qquad \boxed{\sigma(\omega, k) = \frac{G^R_{JJ}(\omega, k)}{i\omega}}$$

Holographic real-time prescription is easy

Claim [Son-Starinets 2002]: corresponds to the solution which at $z \to \infty$ describes stuff falling into the horizon

- Both the retarded response and stuff falling through the horizon describe things that *happen*, rather than *unhappen*.
- You can check that this prescription gives the correct analytic structure of $G_R(\omega)$ ([Son-Starinets] and all the hundreds of papers that have used this prescription).
- It has been derived from a holographic version of the Schwinger-Keldysh prescription [Herzog-Son, Maldacena, Skenderis-van Rees].

The fact that stuff goes past the horizon and doesn't come out is what breaks time-reversal invariance in the holographic computation of G^R.

Here, the ingoing choice is $\phi(t,z) \sim e^{-i\omega t + iqz}$:

as t grows, the wavefront moves to larger z.

(the solution which computes causal response is $z^{d/2} K_{+\nu}(iqz)$.)

The same prescription, adapted to the black hole horizon, works in the finite temperature case.

What to do with the solution

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \approx \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right);$$

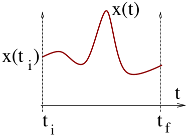
this formula defines the coefficient ϕ_1 of the subleading behavior of the solution.

All the information about G is in ϕ_0, ϕ_1.

recall: $Z[\phi_0] \equiv e^{-W[\phi_0]} \simeq e^{-S_{\text{bulk}}[\underline{\phi}]}\big|_{\phi \overset{z\to 0}{\to} z^{\Delta_-}\phi_0}$

confession: this is a euclidean eqn. next: a nice general trick. [Iqbal-Liu]

classical mechanics interlude: consider a particle in 1d with action $S[x] = \int_{t_i}^{t_f} dtL$. The variation of the action with respect to the initial value of the coordinate is the initial momentum:



$$\Pi(t_i) = \frac{\delta S}{\delta x(t_i)}, \quad \Pi(t) \equiv \frac{\partial L}{\partial \dot{x}} \quad . \tag{3}$$

Thinking of the radial direction of AdS as time, a mild generalization of (3): [Iqbal-Liu]

$$\langle \mathcal{O}(x) \rangle = \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = \lim_{z \to 0} \left(\frac{z}{L} \right)^{\Delta_-} \Pi(z, x)|_{\text{finite}},$$

where $\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)}$ is the bulk field-momentum with z treated as time.

two minor subtleties:

(1) the factor of z_-^Δ arises because of our renormalization of ϕ: $\phi \sim z^{\Delta_-}\phi_0$, so $\frac{\partial}{\partial \phi_0} = z^{-\Delta_-} \frac{\partial}{\partial \phi(z=\epsilon)}$.

(2) Π itself in general has a term proportional to the source ϕ_0

Linear response from holography

With these caveats, away from the support of the source:

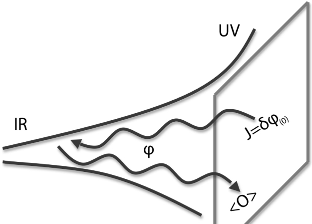
$$\langle \mathcal{O}(x) \rangle = \mathfrak{K} \frac{2\Delta - d}{L} \phi_1(x).$$

linearize in the size of the perturbing source:

$$\langle \mathcal{O}(x) \rangle = G_R \cdot \delta\phi_0$$

summary: The leading behavior of the solution encodes the source *i.e.* the perturbation of the *action* of the QFT.
The coefficient of the subleading falloff encodes the response [Balasubramanian et al, 1996].

$$\boxed{G \propto \frac{\delta\phi_1}{\delta\phi_0}}$$



[figure: Hartnoll, 0909.3553]

(Quasi)normal modes

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \overset{z\to 0}{\approx} \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right);$$

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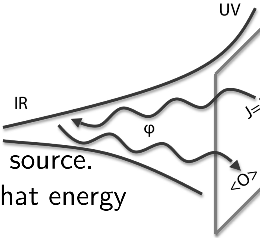


[figure: Hartnoll, 0909.3553]

G has poles when $\phi_1 \neq 0, \phi_0 = 0$: response without source.
this means that the system has an actual mode at that energy
(if $\omega \in \mathbb{C}$, 'quasinormal mode')

A useful visualization: 'Witten diagrams'

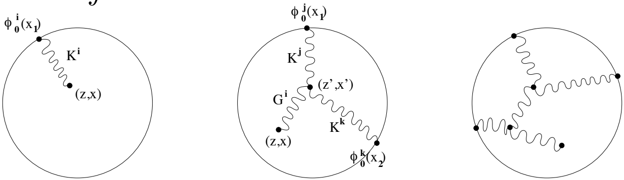
e.g. consider 3-point function, $\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^3 \ln Z|_{\phi_0=0}$.

cubic coupling matters:

$$(\Box - m_i^2)\phi_i(z,x) = b\phi_j\phi_k\epsilon^{ijk}$$

Solve perturbatively in ϕ_0: (K, G are Green's f'ns for $\Box - m_i^2$)

$$\begin{aligned}\underline{\phi}^i(z,x) &= \int d^dx_1 K^{\Delta_i}(z,x;x_1)\phi_0^i(x_1)\\ &+ \quad b\epsilon^{ijk}\int d^dx'dz'\sqrt{g}\,G^{\Delta_i}(z,x;z',x')\\ &\times \quad \int d^dx_1\int d^dx_2 K^{\Delta_j}(z',x';x_1)\phi_0^j(x_1)K^{\Delta_k}(z',x';x_2)\phi_0^k(x_2) + o(b^2\phi_0^3\end{aligned}$$



external legs $\leftrightarrow$ sources ϕ_0, vertices $\leftrightarrow$ bulk interactions

Finite temperature

AdS was scale invariant. sol'n dual to *vacuum* of CFT.
saddle point for CFT in an ensemble with a scale (some relevant perturbation) is a geometry which approaches AdS near the bdy:

$$ds^2 = \frac{L^2}{z^2}\left(-f\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right)$$

When the emblackening factor $f \overset{z\to 0}{\to} 1$ this is the Poincaré AdS metric.

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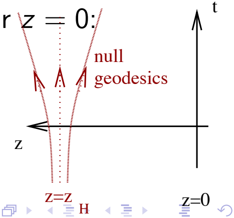
saddle point for CFT in an ensemble with a scale (some relevant perturbation) is a geometry which approaches AdS near the bdy:

$$ds^2 = \frac{L^2}{z^2}\left(-f\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right) \qquad f = 1 - \frac{z^d}{z_H^d}$$

When the emblackening factor $f \stackrel{z\to 0}{\to} 1$ this is the Poincaré AdS metric.
[exercise: check that this solves the same EOM as AdS.]
It has a horizon at $z = z_H$, where the emblackening factor $f \propto z - z_H$
Events at $z > z_H$ can't influence the boundary near $z = 0$:



Physics of horizons

Claim: geometries with horizons describe thermally mixed states.

Why: Near the horizon ($z \sim z_H$),

$$ds^2 \sim -\kappa^2 \rho^2 dt^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \quad \rho^2 \equiv \frac{2}{\kappa z_H^2}(z - z_H) + o(z - z_H)^2$$

$\kappa \equiv \frac{4}{|f'(z_H)|} = d/2z_H$ is called the 'surface gravity'

Ccontinue this geometry to euclidean time, $t \to i\tau$:

$$ds^2 \sim \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \tag{4}$$

which looks like $\mathbb{R}^{d-1} \times \mathbb{R}^2_{\rho,\kappa\tau}$ with polar coordinates $\rho, \kappa\tau$.

There is a deficit angle in this plane unless we identify

$$\kappa\tau \simeq \kappa\tau + 2\pi.$$

A deficit angle would mean nonzero Ricci scalar curvature, which would mean that the geometry is *not* a saddle point of our bulk path integral.

So: $T = \kappa/(2\pi) = 1/(\pi z_H)$.

(Note: this is the temperature of the Hawking radiation.)

Static BH describes thermal equilibrium

This identification on τ also applies at the boundary. If

$$ds^2_{bulk} \overset{z\to 0}{\approx} \frac{dz^2}{z^2} + \frac{L^2}{z^2} g^{(0)}_{\mu\nu} dx^\mu dx^\nu$$

then, up to a factor, the boundary metric is $g^{(0)}_{\mu\nu}$.
This includes making the euclidean time periodic.

$$A = \int_{z=z_H,\text{fixed } t} \sqrt{g}\, d^{d-1}x = \left(\frac{L}{z_H}\right)^{d-1} V$$

The Bekenstein-Hawking entropy is

$$S = \frac{A}{4G_N} = \frac{L^{d-1}}{4G_N}\frac{V}{z_H^{d-1}} = \frac{N^2}{2\pi}(\pi T)^{d-1} V = \frac{\pi^2}{2} N^2 V T^{d-1} \quad . \qquad (5)$$

The Bekenstein-Hawking entropy *density* is

$$s_{BH} = \frac{S_{BH}}{V} = \frac{a_{BH}}{4G_N}.$$

where $a_{BH} \equiv \frac{A}{V}$ is the 'area density' of the black hole.

QFT thermo from black holes cont'd

how to think about this:

$$Z_{CFT}(T) \approx e^{-S_{\rm bulk}^{\rm eucl}[\underline{g}]}$$

$\underline{g}$ is the saddle with the correct periodicity of eucl time at the bdy.

(warning: boundary terms in action are important)

$$Z_{CFT}(T) = e^{-\beta F}$$

$$-\frac{F}{V} = \frac{L^2}{16\pi G_N}\frac{1}{z_H^4} = \frac{\pi^2}{8}N^2 T^4.$$

QFT thermo from black holes cont'd

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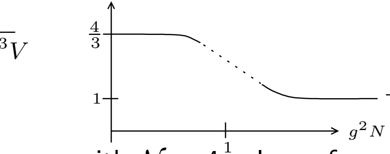

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$$-\frac{F}{V} = \frac{L^2}{16\pi G_N}\frac{1}{z_H^4} = \frac{\pi^2}{8} N^2 T^4 .$$

with $\mathcal{N}=4$ values of parameters, $F(\lambda=\infty) = \frac{3}{4}F(\lambda=0)$.

checks:

- $$\underbrace{S_{BH}}_{\text{horizon}} = \underbrace{-\frac{\partial F}{\partial T}}_{\text{integral over all spacetime}}$$

(relatedly: first law of thermo holds)

- $c_V > 0$ for AdS BH. (unlike schwarzchild in asymptotically flat space!)
- uniqueness of stationary BH ('no hair') $\leftrightsquigarrow$ few state variables in eq thermo

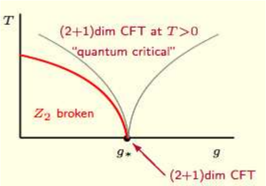
Sample application 1: quantum critical transport

Quantum critical points provide one situation where short-distance physics is unimportant.

And they present a challenge for ordinary QFT techniques:

The usual theory of transport (Boltzman eqn) depends on a description in terms of *particles*.

For $T > m(g) = \frac{1}{\xi(g)} \propto (g - g_c)^\nu$, CFT at finite T: excitations are not particles.



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Quantum critical transport from holography

Fluctuations of Maxwell field in AdS BH

$\longrightarrow$ Density-density response function (or longitudinal conductivity)

in *some* thermal CFT

$$0 = \frac{\delta S}{\delta A_\nu(\omega, q, r)} \propto \partial_\mu \left(\sqrt{g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \right) \quad A = e^{-i\omega t + iqx} \left(dt A_t(r) + dx A_x(\right.$$

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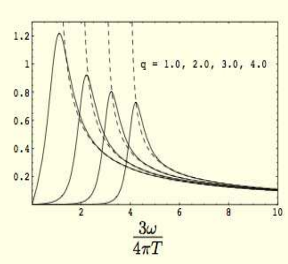

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new hydro result, predicted by AdS/CFT.

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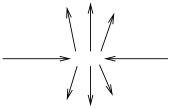

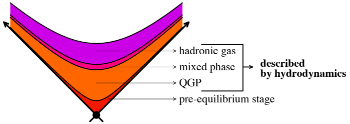

[Heller-Janik-Peschanski]

after time τ_{th}: locally thermal distribution and hydrodynamics.

At RHIC: τ_{th} much smaller than perturbation theory answer.
(τ_{th} affects measurement of viscosity:
good elliptic flow requires both low η and early applicability of hydro)

Thermal equilibrium of CFT stuff $\leftrightsquigarrow$ AdS black hole
$T \leftrightsquigarrow$ location of horizon $r = r_H$

Local thermal equilbrium (hydro) $\leftrightsquigarrow$ slowly-varying deformations of AdS BH: $r = r_H(\vec{x}, t)$. [Janik-Peschanski, Bhattacharyya et al]

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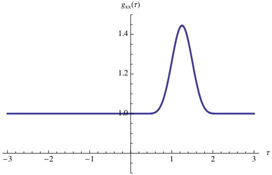
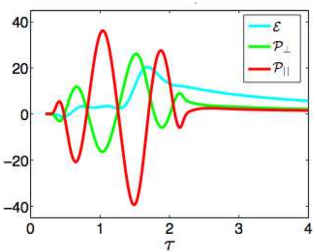
input: 

output: 

black hole forms from vacuum initial conditions.

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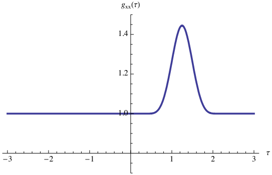
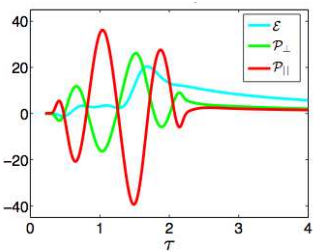
input: 

output: 

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far-reaching consequence: gravity as an entropic force. [E. Verlinde, 1001....]

Here we should pause.

Towards
Physical Applications of Holographic Duality
Parts 3 and 4

John McGreevy, MIT

What to do with the solution

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \approx \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right);$$

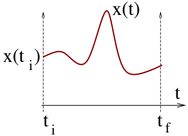
this formula defines the coefficient ϕ_1 of the subleading behavior of the solution.

All the information about G is in ϕ_0, ϕ_1.

recall: $Z[\phi_0] \equiv e^{-W[\phi_0]} \simeq e^{-S_{\rm bulk}[\underline{\phi}]}\big|_{\phi \overset{z\to 0}{\to} z^{\Delta_-}\phi_0}$

confession: this is a euclidean eqn. next: a nice general trick. [Iqbal-Liu]

classical mechanics interlude: consider a particle in 1d with action $S[x] = \int_{t_i}^{t_f} dtL$. The variation of the action with respect to the initial value of the coordinate is the initial momentum:



$$\Pi(t_i) = \frac{\delta S}{\delta x(t_i)}, \quad \Pi(t) \equiv \frac{\partial L}{\partial \dot{x}} \quad . \tag{1}$$

Thinking of the radial direction of AdS as time, a mild generalization of (1): [Iqbal-Liu]

$$\langle \mathcal{O}(x) \rangle = \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = \lim_{z \to 0} \left(\frac{z}{L}\right)^{\Delta_-} \Pi(z, x)|_{\text{finite}},$$

where $\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)}$ is the bulk field-momentum with z treated as time.

two minor subtleties:

(1) the factor of z_-^Δ arises because of our renormalization of ϕ: $\phi \sim z^{\Delta_-}\phi_0$, so $\frac{\partial}{\partial \phi_0} = z^{-\Delta_-} \frac{\partial}{\partial \phi(z=\epsilon)}$.

(2) Π itself in general has a term proportional to the source ϕ_0

Linear response from holography

With these caveats, away from the support of the source:

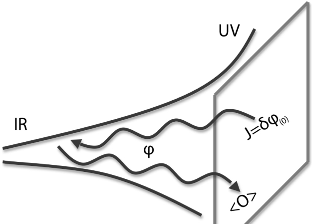
$$\langle \mathcal{O}(x) \rangle = \mathfrak{K} \frac{2\Delta - d}{L} \phi_1(x).$$

linearize in the size of the perturbing source:

$$\langle \mathcal{O}(x) \rangle = G_R \cdot \delta\phi_0$$

summary: The leading behavior of the solution encodes the source *i.e.* the perturbation of the *action* of the QFT.
The coefficient of the subleading falloff encodes the response [Balasubramanian et al, 1996].

$$\boxed{G \propto \frac{\delta\phi_1}{\delta\phi_0}}$$



[figure: Hartnoll, 0909.3553]

(Quasi)normal modes

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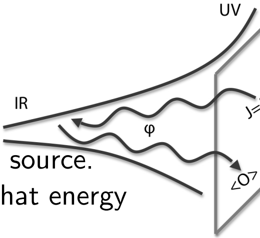


[figure: Hartnoll, 0909.3553]

G has poles when $\phi_1 \neq 0, \phi_0 = 0$: response without source.
this means that the system has an actual mode at that energy
(if $\omega \in \mathbb{C}$, 'quasinormal mode')

A useful visualization: 'Witten diagrams'

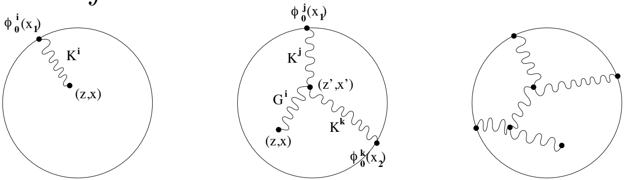
e.g. consider 3-point function, $\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle = \left(\frac{\delta}{\delta\phi_0}\right)^3 \ln Z|_{\phi_0=0}$.

cubic coupling matters:

$$(\Box - m_i^2)\phi_i(z,x) = b\phi_j\phi_k\epsilon^{ijk}$$

Solve perturbatively in ϕ_0: (K, G are Green's f'ns for $\Box - m_i^2$)

$$\begin{aligned}\underline{\phi}^i(z,x) &= \int d^dx_1 K^{\Delta_i}(z,x;x_1)\phi_0^i(x_1) \\ &+ \quad b\epsilon^{ijk}\int d^dx'dz'\sqrt{g}\,G^{\Delta_i}(z,x;z',x') \\ &\times \quad \int d^dx_1 \int d^dx_2 K^{\Delta_j}(z',x';x_1)\phi_0^j(x_1)K^{\Delta_k}(z',x';x_2)\phi_0^k(x_2) + o(b^2\phi_0^3\end{aligned}$$



external legs $\leftrightarrow$ sources ϕ_0, vertices $\leftrightarrow$ bulk interactions

Finite temperature

AdS was scale invariant. sol'n dual to *vacuum* of CFT.
saddle point for CFT in an ensemble with a scale (some relevant perturbation) is a geometry which approaches AdS near the bdy:

$$ds^2 = \frac{L^2}{z^2}\left(-f\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right)$$

When the emblackening factor $f \overset{z\to 0}{\to} 1$ this is the Poincaré AdS metric.

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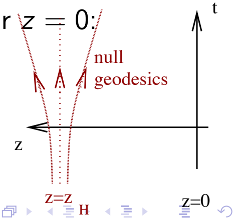
$$ds^2 = \frac{L^2}{z^2}\left(-f\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right) \qquad f = 1 - \frac{z^d}{z_H^d}$$

When the emblackening factor $f \overset{z \to 0}{\to} 1$ this is the Poincaré AdS metric.
[exercise: check that this solves the same EOM as AdS.]
It has a horizon at $z = z_H$, where the emblackening factor $f \propto z - z_H$
Events at $z > z_H$ can't influence the boundary near $z = 0$:



Physics of horizons

Claim: geometries with horizons describe thermally mixed states.

Why: Near the horizon $(z \sim z_H)$,

$$ds^2 \sim -\kappa^2 \rho^2 dt^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \quad \rho^2 \equiv \frac{2}{\kappa z_H^2}(z - z_H) + o(z - z_H)^2$$

$\kappa \equiv \frac{4}{|f'(z_H)|} = d/2z_H$ is called the 'surface gravity'

Ccontinue this geometry to euclidean time, $t \to i\tau$:

$$ds^2 \sim \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \qquad (2)$$

which looks like $\mathbb{R}^{d-1} \times \mathbb{R}^2_{\rho,\kappa\tau}$ with polar coordinates $\rho, \kappa\tau$.

There is a deficit angle in this plane unless we identify

$$\kappa\tau \simeq \kappa\tau + 2\pi.$$

A deficit angle would mean nonzero Ricci scalar curvature, which would mean that the geometry is *not* a saddle point of our bulk path integral.

So: $T = \kappa/(2\pi) = 1/(\pi z_H)$.

(Note: this is the temperature of the Hawking radiation.)

Static BH describes thermal equilibrium

This identification on τ also applies at the boundary. If

$$ds^2_{bulk} \overset{z \to 0}{\approx} \frac{dz^2}{z^2} + \frac{L^2}{z^2} g^{(0)}_{\mu\nu} dx^\mu dx^\nu$$

then, up to a factor, the boundary metric is $g^{(0)}_{\mu\nu}$.
This includes making the euclidean time periodic.

$$A = \int_{z=z_H,\text{fixed } t} \sqrt{g}\, d^{d-1}x = \left(\frac{L}{z_H}\right)^{d-1} V$$

The Bekenstein-Hawking entropy is

$$S = \frac{A}{4G_N} = \frac{L^{d-1}}{4G_N} \frac{V}{z_H^{d-1}} = \frac{N^2}{2\pi} (\pi T)^{d-1} V = \frac{\pi^2}{2} N^2 V T^{d-1} \quad . \tag{3}$$

The Bekenstein-Hawking entropy *density* is

$$s_{BH} = \frac{S_{BH}}{V} = \frac{a_{BH}}{4G_N}.$$

where $a_{BH} \equiv \frac{A}{V}$ is the 'area density' of the black hole.

QFT thermo from black holes cont'd

how to think about this:

$$Z_{CFT}(T) \approx e^{-S_{\rm bulk}^{\rm eucl}[\underline{g}]}$$

$\underline{g}$ is the saddle with the correct periodicity of eucl time at the bdy.

(warning: boundary terms in action are important)

$$Z_{CFT}(T) = e^{-\beta F}$$

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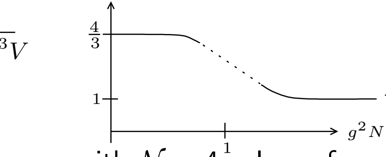

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with $\mathcal{N} = 4$ values of parameters, $F(\lambda = \infty) = \frac{3}{4}F(\lambda = 0)$.

checks:

- $$\underbrace{S_{BH}}_{\text{horizon}} = \underbrace{-\frac{\partial F}{\partial T}}_{\text{integral over all spacetime}}$$

(relatedly: first law of thermo holds)

- $c_V > 0$ for AdS BH. (unlike schwarzchild in asymptotically flat space!)
- uniqueness of stationary BH ('no hair') $\leftrightsquigarrow$ few state variables in eq thermo

Sample application 1: quantum critical transport

Quantum critical points provide one situation where short-distance physics is unimportant.

And they present a challenge for ordinary QFT techniques:

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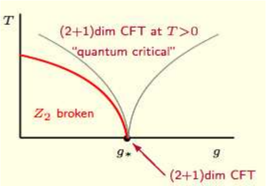

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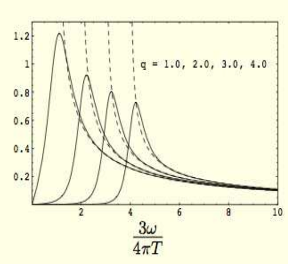

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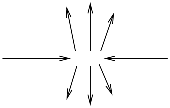

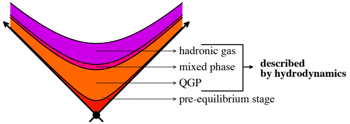

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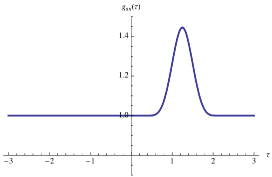
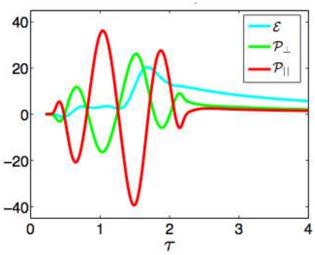
input: 

output: 

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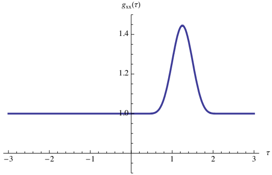
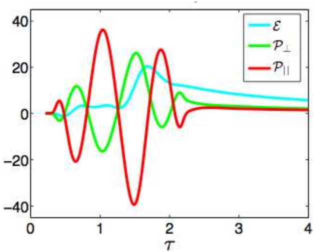
input: 

output: 

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$\mathcal{N} = 4$ SYM is a CFT, (a supersymmetric, relativistic gauge theory)

each of these red words is bad from our point of view.

The $\mathcal{N} = 4$ SYM action is schematically

$$\mathcal{L}_{\mathsf{SYM}} \sim \mathrm{tr}\left(F^2 + (D\Phi)^2 + i\bar{\Psi}\Gamma \cdot D\Psi + g^2[\Phi,\Phi]^2 + ig\bar{\Psi}[\Phi,\Psi]\right)$$

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this gauge theory comes with 2 parameters:

a coupling constant $\lambda = g^2 N$ (with $\beta_\lambda \equiv 0$)

an integer, the number of colors N.

An example of a theory with a known gravity dual

$\mathcal{N}=4$ SYM is a CFT, (a supersymmetric, relativistic gauge theory)

each of these red words is bad from our point of view.

The $\mathcal{N}=4$ SYM action is schematically

$$\mathcal{L}_{\mathsf{SYM}} \sim \mathrm{tr}\,\left(F^2 + (D\Phi)^2 + i\bar{\Psi}\Gamma\cdot D\Psi + g^2[\Phi,\Phi]^2 + ig\bar{\Psi}[\Phi,\Psi]\right)$$

this gauge theory comes with 2 parameters:

a coupling constant $\lambda = g^2 N$ (with $\beta_\lambda \equiv 0$)

an integer, the number of colors N.

$$\boxed{\mathcal{N}=4\ \mathrm{SYM}_{N,\lambda}} = \boxed{\text{IIB strings in } AdS_5\times S^5 \text{ of size } \lambda, \hbar = 1/N}$$

[Maldacena 1997]

• large N makes gravity classical (suppresses splitting and joining of strings)

• strong coupling (large λ) makes the geometry big.

'IIB strings in ...' specifies a list of bulk fields and interactions.

$\exists$ *infinitely many* other examples of dual pairs [e.g. Hanany, Vegh et al...]

Remarks on the role of supersymmetry (susy)

- susy constrains the form of interactions.
  fewer candidates for dual.
- in susy theories, $\exists$ more coupling-independent quantities, hence $\exists$ more checks.
- susy allows *lines* of fixed points (*e.g.* $\mathcal{N} = 4$ SYM)
  coupling $=$ dimensionless parameter
- for these applications, susy is broken by finite T, μ, anyway. it's not clear what influence it has on the resulting states.

Remarks on the role of string theory

1. What are consistent ways to UV complete our gravity model?
   - So far, no known constraints that aren't visible from EFT.
   - Suggests interesting resummations of higher-derivative terms, protected by stringy symmetries.
     e.g. the DBI action $L_{DBI} \sim \sqrt{1 - F^2}$ is 'natural' in string theory because its form is protected by the T-dual Lorentz invariance.
2. What is a microscopic description of the dual QFT?
   - Such a description is crucial for the detailed checks that make us believe the duality.
   - A weak coupling limit needn't exist (isolated fixed points are generic).
   - A Lagrangian description needn't exist
     (*e.g.* minimal models) gravity plus matter in AdS provides a much more direct construction of CFT.
   - Honesty: Any L_{micro} that we would get from string theory is so far from $L_{Hubbard}$ anyway that it isn't clear how it helps.

Lessons for how to use AdS/CFT to do physics

- critical exponents depend on 'landscape issues'
  (parameters in bulk action)
- thermodynamics is not so different between weak and strong coupling
  (in examples: $\mathcal{N} = 4$ SYM, lattice QCD)
- transport is very different
  transport by weakly-interacting quasiparticles is less effective

$$\left(\frac{\eta}{s}\right)_{\text{weak}} \sim \frac{1}{g^4 \ln g} \quad \gg \quad \left(\frac{\eta}{s}\right)_{\text{strong}} \sim \frac{1}{4\pi}.$$

Gravity duals of non-relativistic CFTs

(towards cold atoms at unitarity)

Motivation

Some relativistic CFTs have an effective description at strong coupling in terms of gravity (more generally strings) in AdS space.

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(solutions of strong-coupling problems, quantum gravity experiments)

Some laboratory systems have critical points described by relativistic CFTs.

– QCD a little above T_c acts like a CFT

– some quantum-critical condensed matter systems have emergent lightcones

Alternative approach (later): ask questions which don't care about the short-distance symmetries.

More precisely

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  and which can be created in a laboratory more convenient than RHIC.

- Piles of atoms have a rest frame.

(even if present, lightcone need not be shared by different degrees of freedom.)

So, in searching for experiments with which string theory has some interface, it's worth noting that:

non-relativistic CFTs exist.

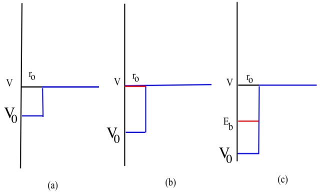
Cold atoms at unitarity

Most of the work on AdS/CFT involves relativistic CFTs.

Strongly-coupled Galilean-invariant CFTs exist, even experimentally.

[Zwierlein et al, Hulet et al, Thomas et al]

Consider nonrelativistic fermionic particles ('atoms') interacting via a short-range attractive two-body potential $V(r)$, *e.g.*:



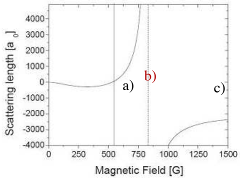
Case (b): σ saturates bound on scattering cross section from unitarity

Range of interactions $\to 0$, scattering length $\to \infty \implies$ no scale.

Lithium atoms

have a boundstate with a different magnetic moment.

Zeeman effect $\implies$ scattering length can be controlled using an external magnetic field:



Strongly-coupled NRCFT

The fixed-point theory ("fermions at unitarity") is a strongly-coupled nonrelativistic CFT ('Schrödinger symmetry') [Nishida-Son].

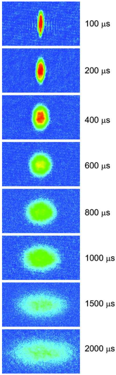
universality: it also describes neutron-neutron scattering [Mehen-Stewart-Wise]

Two-body physics is completely solved.

Many body physics is mysterious.

Experiments: very low viscosity, $\frac{\eta}{s} \sim \frac{5}{4\pi}$ [Thomas, Schafer]

$\longrightarrow$ strongly coupled.



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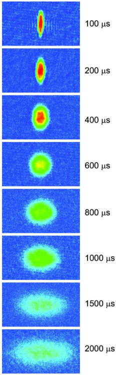

AdS/CFT?

Clearly we can't approximate it as a *relativistic* CFT.

Different hydro: conserved particle number.

A holographic description?

Method of the missing box

AdS : relativistic CFT

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$\boxed{?}$: galilean-invariant CFT

A holographic description?

Method of the missing box

AdS : relativistic CFT

$\boxed{?}$: galilean-invariant CFT

Note restriction to Gal.-invariance $\partial_t - \vec{\nabla}^2$
distinct from: Lifshitz-like fixed points $\partial_t^2 - (\vec{\nabla}^2)^2$
are not relativistic, but have antiparticles.
gravity duals of those: S. Kachru, X. Liu, M. Mulligan, 0808.1725

before guessing what's in the box, more about this symmetry and its realizations

Galilean scale invariance

$i, j = 1 \ldots d$ spatial dims (sorry for the notation change)

Symmetries of free schödinger equation $i\partial_t \psi = \partial_x^2 \psi$

Galilean symmetry:

translations P_i, rotations M_{ij}, time translations H, Galilean boosts K_i, number or mass operator N:

$[K_i, P_j] = \delta_{ij} iN$ (in 'non-relativistic natural units': $\hbar = M = 1$)

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dilatations D: $[D, P] = -iP$ (D measures length dimensions)

$[D, H] = -izH$ ($z \equiv$ dynamical exponent: $x \to \lambda x, \quad t \to \lambda^z t$)

closure of algebra $\longrightarrow$ $[D, K] = i(z-1)K, \quad [D, N] = i(z-2)N.$

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Schrödinger symmetry:

In the special case $z = 2$, there is an additional conformal generator, $C = ITI$

$$[M_{ij}, C] = 0, \quad [K_i, C] = 0, \quad [D, C] = -2iC, \quad [H, C] = -iD.$$

comments

- there's only *one* special conformal symmetry, not $d+1$ like in relativistic case.
- we're using 'non-relativistic natural units' where $\hbar = M = 1$, so $\hat{N}$ measures particle number or mass.
- this 'schrödinger' algebra $\subset SO(d+1,2)$
  (the relativistic conformal group)
- [Nishida-Son] irreps of Schrod ($z=2$) labelled by $\Delta_0, N_0 \equiv \ell$.
- [Tachikawa] unitarity bound: $\Delta \geq \frac{d}{2}$ (independent of spin.)

QFT realization

free fermions (or free bosons): $S_0 = \int dt d^d x \left(\psi^\dagger i \partial_t \psi + \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi \right)$

$$n(\vec{x}) \equiv \psi^\dagger \psi, \;\; \vec{j}(\vec{x}) \equiv -\frac{i}{2} \left(\psi^\dagger \vec{\nabla} \psi - \vec{\nabla} \psi^\dagger \psi \right)$$

$$N = \int d^d x \; n(\vec{x}), \;\; P_i = \int j_i(\vec{x}), \;\; M_{ij} = \int (x_i j_j(\vec{x}) - x_j j_i(\vec{x}))$$

$$K_i = \int x_i n(\vec{x}), \;\; D = \int x_i j_i(\vec{x}), \;\; C = \int \frac{x^2 n(\vec{x})}{2}$$

satisfy all the commutation relations not involving the Hamiltonian.
With $H_0 = \int d\vec{x} \frac{1}{2} \nabla_i \psi^\dagger \nabla_i \psi$, ψ saturates unitarity bound.

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satisfy all the commutation relations not involving the Hamiltonian.
With $H_0 = \int d\vec{x} \frac{1}{2} \nabla_i \psi^\dagger \nabla_i \psi$, ψ saturates unitarity bound.
towards interacting NRCFT:

$$\Delta S = \frac{1}{2} \int dt \int d\vec{x} d\vec{y} \psi^\dagger(\vec{x}) \psi^\dagger(\vec{y}) \quad \underbrace{V\left(|\vec{x}-\vec{y}|\right)}_{\equiv V(r)} \quad \psi(\vec{y})\psi(\vec{x})$$

geometric realization

A metric whose isometry group is the schrödinger group:

$$L^{-2} ds^2_{\mathrm{Schr}^z_d} = \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2} - 2\beta^2 \frac{dt^2}{r^{2z}}$$

$\boxed{?}$ = 'schrödinger space' [Son; Balasubramanian, JM]

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$\boxed{?}$ = 'schrödinger space' [Son; Balasubramanian, JM]
Compare to AdS in light-cone coordinates:

$$ds^2_{AdS_{d+3}} = \frac{-d\tau^2 + dy^2 + d\vec{x}^2 + dr^2}{r^2}$$

$$= \frac{2d\xi dt + \vec{dx}^2 + dr^2}{r^2}$$

without the β^2 term, ∂_t is lightlike.

comments

1. only $z = 2$ has conformal symmetry.
2. if $\xi \in \mathbb{R}$, we can scale away $2\beta^2$ by (remnant of boost) $\begin{cases} t \mapsto \frac{t}{\sqrt{2}\beta} \\ \xi \mapsto \sqrt{2}\beta\xi \end{cases}$

   but discrete spectrum requires compact $\xi \simeq \xi + L_\xi$

   $\frac{\beta}{L_\xi}$ is an invariant parameter $\sim M$.
3. dual to *vacuum* of a gal. inv't field theory (no antiparticles!).

   the ξ-circle is *null*. (light winding modes?)

   (this is the phase of the wavefunction of a state with no particles!)

   at finite temperature or density, not so.
4. all curvature scalars are constant.
5. this spacetime is conformal to a pp-wave.

   conformal boundary is one-dimensional.

   [Berenstein-Nastase, Hubeny-Rangamani]

   Nevertheless, we will compute correlators of a CFT with d spatial dims.

What holds it up?

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} - \delta^t_\mu \delta^t_\nu g_{tt} \mathcal{E}$$

$\Lambda = -\frac{(d+1)(d+2)}{2L^2}$: CC $\qquad$ $\mathcal{E}$: a constant energy density ('dust')

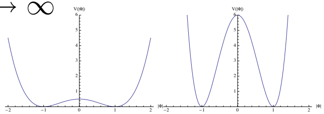
A realization of the dust: metric is sourced by *e.g.* the ground state of an Abelian Higgs model in its broken phase.

$$S = \int d^{d+3}x \sqrt{g} \left(-\frac{1}{4} F^2 + \frac{1}{2} |D\Phi|^2 - V\left(|\Phi|^2\right) \right)$$

with $D_a \Phi \equiv (\partial_a + ieA_a)\Phi$, with a Mexican-hat potential

$$V\left(|\Phi|^2\right) = g \left(|\Phi|^2 - \frac{z(z+d)}{e^2} \right)^2 + \Lambda$$

extreme type II limit : $g \to \infty \Longrightarrow m_h^2 \to \infty$



$$L_{bulk} = -\frac{1}{4} F^2 - \frac{m^2}{2} A^2 - \Lambda, \qquad m^2 = z(z+d)$$

Holographic dictionary

Basic entry: bulk fields $\leftrightarrow$ operators in dual QFT

Irreps of schrod labelled by $\Delta, \hat{N} = \ell$, so we work at fixed ξ-momentum, ℓ: $\phi(r, t, \vec{x}, \xi) = f_{\omega,k,\ell}(r) e^{i(\ell\xi - \omega t + \vec{k}\cdot\vec{x})} \leftrightarrow \mathcal{O}_{\ell,\Delta}(\omega, \vec{k})$ scalar operator.

Consider a probe scalar field:

$S[\phi] = -\int d^{d+1}x\sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right).$

or: δg_y^x also satisfies this equation

Scalar wave equation in this background:

$$\left(-r^{d+3}\partial_r\left(\frac{1}{r^{d+1}}\partial_r\right) + r^2(2l\omega + \vec{k}^2)\ + r^{4-2z}l^2 + m^2\right) f_{\omega,\vec{k},l}(r) = 0.$$

For $z \leq 2$, the behavior of the solution near the boundary $(r \sim 0)$ is:

$$f \propto r^{\Delta}, \qquad \Delta_{\pm} = 1 + \frac{d}{2} \pm \sqrt{\left(1 + \frac{d}{2}\right)^2 + m^2 + \delta_{z,2}l^2}.$$

For $z > 2$, not power law. (??!)

some basic checks (focus on $z = 2$)

1) $\Delta_+ + \Delta_- = d + 2$ matches dimensional analysis on

$$S_{bdy} \ni \int dt d^d x \; \phi_0 \mathcal{O}$$

(ϕ_0 is the source for $\mathcal{O}$)

$$[x] = -1, [t] = -2, [\phi_0] = \Delta_-, [\mathcal{O}] = \Delta_+.$$

2) unitarity bound $\Delta \geq \frac{d}{2}$ matches requirement on m to prevent bulk tachyon instability (analog of BF-bound).

3) the correlators are of the expected form

$$\rightarrow \langle \mathcal{O}_1(x,t) \mathcal{O}_2(0,0) \rangle \propto \delta_{\Delta_1, \Delta_2} \theta(t) \frac{1}{|\epsilon^2 t|^\Delta} e^{-iMx^2/2|t|}$$

consistent with (determined by [Nishida-Son]) NR conformal Ward ids.

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But: the vacuum of a galilean-invariant field theory is extremely boring: no antiparticles! no stuff!

How to add stuff?

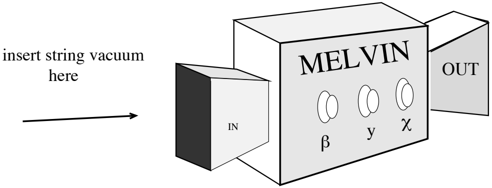
A holographic description of more than zero atoms?

A black hole (BH) in schrödinger spacetime.

[A. Adams, K. Balasubramanian, JM; Maldacena et al; Rangamani et al]

Here, string theory was extremely useful:

A solution-generating machine named Melvin [Ganor et al]



IN: $AdS_5 \times S^5$ OUT: schrodinger $\times S^5$

→ A hint about *which* NRCFTs we are describing:

we can also feed $\mathcal{N} = 4$ SYM to Melvin.

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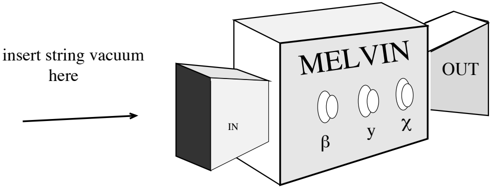

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$\rightarrow$ A hint about *which* NRCFTs we are describing:

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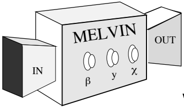
IN: AdS_5 BH $\times S^5$ OUT: schrodinger BH $\times$ squashed S^5

“Null Melvin Twist”

is a machine 

which generates new type II SUGRA solutions from old Ganor et al, Gimon et al. (with different asymptotics)

Previous work: dials set to ‘highly non-commutative’.

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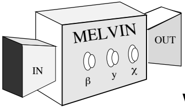
is a machine 

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Previous work: dials set to 'highly non-commutative'.

Choose two killing vectors $(\partial_y, \partial_\chi)$ and:

1. Boost along y with boost parameter γ
2. T-dualize along y.
3. Twist: replace $\chi \to \chi + \alpha y$, α constant
4. T-dualize back along y
5. Boost back by $-\gamma$ along y
6. Scaling limit: $\gamma \to \infty$, $\alpha \to 0$ keeping $\beta = \frac{1}{2}\alpha e^{\gamma}$ fixed.

Schrödinger spacetime in string theory

Input solution of type IIB supergravity: $AdS_5 \times S_5$

$$ds^2 = \frac{-d\tau^2 + dy^2 + d\vec{x}^2 + dr^2}{r^2} + ds^2_{S_5} \qquad \vec{x} \equiv (x^1, x^2).$$

$ds^2_{S_5} = ds^2_{\mathbb{P}^2} + \eta^2$. $\eta \equiv d\chi + \mathcal{A} =$ vertical one-form of Hopf fibration

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Feeding this to the melvinizer gives:

$$ds^2 = \frac{1}{r^2}\left(-\left(1 + \frac{\beta^2}{r^2}\right) d\tau^2 + \left(1 - \frac{\beta^2}{r^2}\right) dy^2 + 2\frac{\beta^2}{r^2} d\tau dy + d\vec{x}^2 + dr^2\right) + ds^2_{S_5}$$

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Defining $\xi \equiv \frac{1}{2\beta}(y - \tau)$, $t \equiv \beta(\tau + y)$, and reducing on the 5-sphere:

$$\longrightarrow ds^2 = \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2} - \frac{dt^2}{r^4} \qquad (\mathrm{Schr}^{z=2}_{d=2})$$

The ten-dimensional metric is sourced by

$$B = \beta r^{-2} \eta \wedge (d\tau + dy), \;\; F_5 = (1 + \star)\mathrm{Vol}(S^5) \;\; \xrightarrow{5d} A = r^{-2} dt, \;\; m^2 = 8, \;\; \Lambda.$$

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$$ds^2 = \frac{1}{r^2}\left(-\left(1+\frac{\beta^2}{r^2}\right)d\tau^2 + \left(1-\frac{\beta^2}{r^2}\right)dy^2 + 2\frac{\beta^2}{r^2}d\tau dy + d\vec{x}^2 + dr^2\right) + ds^2_{S_5}$$

Defining $\xi \equiv \frac{1}{2\beta}(y-\tau)$, $t \equiv \beta(\tau+y)$, and reducing on the 5-sphere:

$$\longrightarrow ds^2 = \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2} - \frac{dt^2}{r^4} \qquad (\mathrm{Schr}^{z=2}_{d=2})$$

The ten-dimensional metric is sourced by

$$B = \beta r^{-2}\eta \wedge (d\tau + dy), \;\; F_5 = (1+\star)\mathrm{Vol}(S^5) \;\; \xrightarrow{5d} A = r^{-2}dt, \;\; m^2 = 8, \;\; \Lambda.$$

– No higgs field, alas.

– This can be done for $S^5 \to$ any Sasaki-Einstein 5-manifold. – It works similarly for the AdS BH, but then the sphere gets squashed.

Thermodynamics

BH is saddle point of $Z = \mathrm{tr}\ e^{-\frac{1}{T}(H-\mu N)} = \mathrm{tr}\ e^{-\frac{1}{T}\left(i\partial_\tau - \mu i \partial_\xi\right)}$
Temperature & Chemical Potential: euclidean regularity requires
$it \simeq it + \frac{n}{T}, \xi \simeq \xi + L_\xi \mu n \quad \Longrightarrow \quad T = \frac{\kappa}{2\pi} = \frac{1}{\pi\beta r_H}, \mu = -\frac{1}{2\beta^2}$

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Entropy: $S_{BH} = \dfrac{1}{4G_N} \dfrac{L_y}{r_H^3} = V L_\xi \dfrac{\pi^2 N^2 T^3}{16\mu^2}$

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Mystery: we are forced to add *extrinsic* boundary terms for the massive gauge field: $S_{\mathrm{bdy}} \ni \int n^\mu A_\nu F^{\mu\nu}$

The required coefficient is exactly the one that changes the boundary conditions on A_μ from Dirichlet to Neumann.

Boundary stress tensor

$$S_{\rm bdy} = \int \sqrt{\gamma}\left(\Theta + c_0 + c_1\Phi + c_2\Phi^2 + n^\mu A^\mu F_{\mu\nu}\left(c_5 + c_6\Phi\right)\right)$$

Vary metric at boundary:

$$T^\mu_\nu = -\frac{2}{\sqrt{\gamma}}\frac{\delta S_{\rm onshell}}{\delta\gamma^\nu_\mu} = \Theta^\mu_\nu - \delta^\mu_\nu\Theta - {\rm c.t.}|_{bdy} \qquad \Theta = \text{extrinsic curvature}$$

Fix counterterm coeffs w/

–Ward identity: $2E = dP$ $=$ residual bulk gauge symmetries

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$$\longrightarrow \quad \mathcal{E} = \frac{E}{V} = -\int\sqrt{\gamma}\,T^t_t = \frac{1}{16\pi G r_H^4} = \frac{\pi^2 N^2}{64} L_\xi \frac{T^4}{\mu^2}$$

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Who is T^ξ_t? Just as T^χ_μ is the R-charge current,

Density: $$\rho = \int\sqrt{\gamma}\,T^\xi_t = \frac{\beta^2}{16\pi G r_H^4} = \frac{\pi^2 N^2 T^4}{32\mu^3} L_\xi$$

Note: T^ξ_ξ, $T^t_\xi = \infty$ with naive falloffs on $\delta_{\mu\nu}$. We don't care about these anyway.

Results so far

This black hole gives the thermo and hydro of some NRCFT ('dipole theory' [Ganor et al]).

$$\text{Einstein gravity} \quad \overset{[Iqbal-Liu]}{\Longrightarrow} \qquad \frac{\eta}{s} = \frac{1}{4\pi}.$$

Satisfies laws of thermodynamics, correct scaling laws, correct kubo relations.

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But it's a very different class from unitary fermions:

$$F \sim \frac{T^4}{\mu^2}, \quad \mu < 0$$

(note: scaling symmetry $\implies F \sim T^{\frac{D+2}{2}} g(T/\mu)$)
Q: why is $g(x) = x^2$?
A [MMTv5]: a) if solution arises from DLCQ, an extra (boost) symmetry: $t \to \alpha t, \xi \to \alpha^{-1}\xi \quad \implies T \to \frac{T}{\alpha}, \mu \to \frac{\mu}{\alpha^2}, F \to F \implies F(T,\mu) = g\left(\frac{\mu}{T^2}\right)$
b) melvin twist doesn't change planar amplitudes
(bulk explanation: symmetry of tree-level string theory
boundary explanation: 'non-commutative phases' cancel)

New gravity realizations of Schrod

This is *not* a necessary consequence of $\exists$ gravity dual.

[K Balasubramanian, JM, to appear]

Unnecessary assumption: All of Schrod must be realized geometrically.

We now know how to remove this assumption, can find more realistic models.

Gravity solutions with a ξ dimension are like DLCQ of rel CFT: periodically identify $x^+ \equiv x + t$. Clear from *e.g.* [Barbon-Fuertes]

(Schrod_D is the subgroup of $SO(d+1,2)$ which is preserved.)

We thought the ξ direction was required since $[K, P] = iN$

LHS must be realized geometrically.

This action has solutions with $Schrod_d$ asymptotics:

$$S_{d+2} = \int d^d x dt dr \sqrt{g} \left[R - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4} (dB)^2 - \frac{3}{2} (\partial\sigma)^2 \right]$$

$$ds^2 = e^{\sigma} \left(-Q \frac{dt^2}{r^6} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} \right), \quad B = Q \frac{f dt}{r^4}, \quad e^{2\sigma} = \frac{r^2}{Q}$$

Realization of symmetries

Symmetry generators of the lower dimensional theory realize Schröd:

- Particle Number:

$$B \to B + d\lambda, \quad \Phi \to e^{i\ell\lambda}\Phi$$

  (we take Φ to vanish in the solution shown above.) ℓ is the mass of the associated particle.

- Translations and rotations are realized as-usual by isometries.
- Galilean Boosts act by:

$$K^i = -t\partial_i + \text{Gauge shift}$$

  where the gauge transformation parameter is $\lambda = \frac{1}{2}v^2 t + \vec{v}\cdot\vec{x}$.

$$t \to t, \quad \vec{x} \to \vec{x} - \vec{v}t, \quad \varphi \to \varphi + \ell\left(\frac{1}{2}v^2 t + \vec{v}\cdot\vec{x}\right),$$

  where $\Phi \equiv e^{i\varphi}|\Phi|$ Role of ξ played by φ.

- Scale symmetry acts by

$$D = -2t\partial_t - x^i\partial_i - r\partial_r + \text{shift in } \sigma;$$

wave equation:

$$\left(-\omega^2 r^6 + m^2 + r^2(2\ell\omega + k^2)\right)\Phi - r^{d+3}\partial_r\left(r^{-d-1}\partial_r\Phi\right) = 0 \qquad (\star)$$

$(\star)$ is the eom from:

$$S_{\mathrm{probe}}[\Phi] = \int \sqrt{g}\left[|(\partial - i\ell B)\Phi|^2 - \left(\ell^2 e^{-3\sigma} + m^2 e^{-\sigma}\right)|\Phi|^2\right] \qquad .$$

coupling to scalar σ req'd to realize Schröd.

First term in $(\star)$ is unimportant for the boundary behavior ($r \to 0$), but does spoil the Schrödinger invariance of the equation.

Note: we can't find a solution which preserves all of schrod

(recall our surprise at finding a vacuum solution earlier)

But, black hole solution:

$$ds^2 = e^{\sigma}\left(-Qf\frac{dt^2}{r^6} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2 f}\right), \quad B = Q\frac{fdt}{r^4}, \quad e^{2\sigma} = \frac{r^2}{Q}$$

where $f = 1 - r^4/r_H^4$

Same thermo as before (obtained by dim'l reduction and scaling).

Another system which realizes schrod

$$S_4^E = \int d^4x(-g_4)^{1/2}\left[R_4 - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4}(dB)^2 - \frac{3}{2}(\partial\sigma)^2 - \frac{1}{2}(\partial\Psi)^2\right]$$

A solution with asymptotic *Sch* symmetry:

$$ds_E^2\left(\widehat{Sch}\right) = e^{\sigma}\left(-QK_x^2\frac{dt^2}{r^6} + K_x\frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2}\right)$$

$$B = Q\frac{(1 - r^4/r_0^4)dt}{r^4}, \quad e^{2\sigma} = \frac{r^2}{Q}, \quad e^{2\Psi/\sqrt{5}} = \frac{1 - r^4/r_0^4}{1 + r^4/r_0^4}$$

$$K_x^2 = 1 - r^8/r_0^8.$$

Geometry ends at $r = r_0$ with a curvature singularity.

related solutions: [Gubser-Rocha, Goldstein-Kachru-Prakash-Trivedi]

This curvature singularity at $r = r_0$ can be resolved by oxidation!

Lift to ten and eleven dimensions

The action S_4^E is a consistent truncation of

$$S_5^E = \int d^5x(-g_5)^{1/2}\left[R_5 - 2\Lambda - \frac{1}{2}(\partial\Psi)^2\right]$$

which is a consistent truncation of type IIB supergravity [MMT].
Lift to 10d:

$$\begin{aligned} ds_{10}^2 &= ds_E^2\left(\widehat{Sch}\right) + e^{2\sigma}(d\xi + B)^2 + ds^2\left(S^5\right), \\ F_5 &= Q\left(\Omega_5 + \star\Omega_5\right), \text{ and} \\ e^{2\Phi} &= e^{2\Psi} \end{aligned} \tag{4}$$

is still singular at $r = r_0$.
T-dualize on the Hopf dir [Duff-Pope] and lift to 11d SUGRA:

$$ds_{11}^2 = e^{-\Psi/2}\left[ds_E^2\left(\widehat{Sch}\right) + e^{2\sigma}(d\xi + B)^2 + ds^2\left(\mathbb{CP}^2\right) + d\chi_1^2\right] + e^{4\Psi/3}d\chi_2^2$$

and $G_4 = \ldots$

Consequences of lift

The important point: the 11d geometry ends smoothly at $r = r_0$.

(like the geometries describing confining gauge theories.) [KS, MN]

This determines the boundary conditions on fields

(like origin of polar coords.)

The solution has non-zero energy, pressure, density and free energy, but has zero entropy (no horizon).

Real boundary conditions.

for example on B_μ, which computes current correlators.

This is a "Mott" "insulator":

$\rho \neq 0$ but there is a gap to charge excitations

("Mott": there are strong interactions,

and it's not a band insulator or an Anderson insulator)

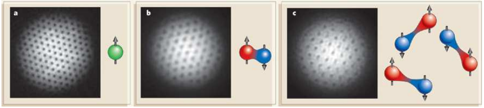
But: translation invariance $\Longrightarrow \sigma(\Omega) \propto \delta(\Omega)$.

As specified, it's a perfect conductor.

Conjecture: if we pinned down the center-of-mass mode, it would be an insulator.

Some future questions

- How close can we get to unitary fermions with a gravity dual?
- Can we realize the superfluid phase?



Should break ξ-isometry, cut off IR geometry.

Towards
Physical Applications of Holographic Duality
Parts 5 and 6

John McGreevy, MIT

Fermi Liquids and Non-Fermi Liquids

Fermi Liquids

Basic question: what is the ground state of a nonzero density of interacting fermions? ($\exists$ sign problem)

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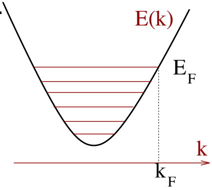
Lore: if it's a metal, it's a Fermi liquid [Landau, 50s].

Recall:

if we had *free* fermions, we would fill single-particle energy levels $E(k)$ until we ran out of fermions: $\longrightarrow$



Low-energy excitations:

remove or add electrons near the fermi surface E_F, k_F.

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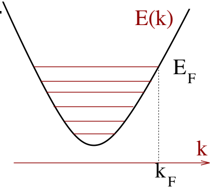

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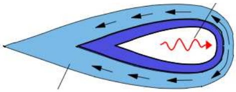
Idea [Landau]: The low-energy excitations of the interacting theory are still weakly-interacting fermionic, charged 'quasiparticles'

Elementary excitations are free fermions with some dressing:

$\xrightarrow{\textit{in medium}}$



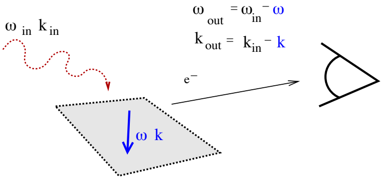
The standard description of metals

The metallic states that we understand well are described by Landau's Fermi liquid theory.

Landau quasiparticles $\rightarrow$ poles in single-fermion Green function G_R

at $k_\perp \equiv |\vec{k}| - k_F = 0$, $\omega = \omega_\star(k_\perp) \sim 0$: $\quad G_R \sim \dfrac{Z}{\omega - v_F k_\perp + i\Gamma}$

Measurable by ARPES (angle-resolved photoemission):



$$\text{Intensity} \propto \text{spectral density}: \quad A(\omega, k) \equiv \mathrm{Im}\, G_R(\omega, k) \overset{k_\perp \to 0}{\rightarrow} Z\delta(\omega - v_F k_\perp)$$

Landau quasiparticles are long-lived: width is $\Gamma \sim \omega_\star^2$.

residue Z (overlap with external e^-) is finite on Fermi surface.

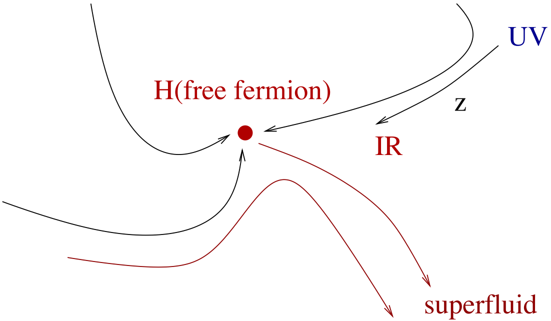
Reliable calculation of thermodynamics and transport relies on this.

Ubiquity of Landau fermi liquid

Physical origin of lore:
1. Landau FL successfully describes ^{3}He, all metals studied before $\sim 1980s$, ...

2. RG: Landau FL is stable under almost all perturbations.

[Shankar, Polchinski, Benfatto-Gallivotti 92]



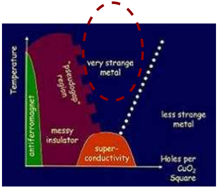
Effective Field Theory and the Fermi Surface

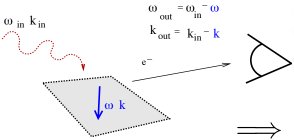
Polchinski, hep-th/9210046

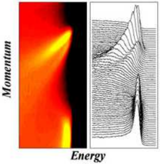
also
Benfatto-Gallivotti;
Shankar, RMP 66 (1994) 129

Non-Fermi liquids exist, but are mysterious

e.g.: 'normal' phase of optimally-doped cuprates: ('strange metal')







among other anomalies: ARPES shows gapless modes at finite k (FS!)

with width $\Gamma(\omega_\star) \sim \omega_\star$, vanishing residue $Z \overset{k_\perp \to 0}{\longrightarrow} 0$.

Working defintion of NFL:

Still a sharp Fermi surface (nonanalyticity of $A(\omega \sim 0, k \sim k_F)$)

but no long-lived quasiparticles.

[Anderson, Senthil] 'critical fermi surface'

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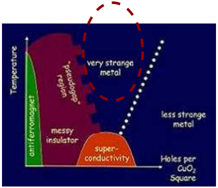

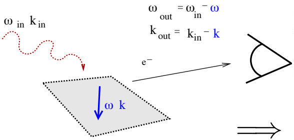

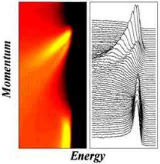

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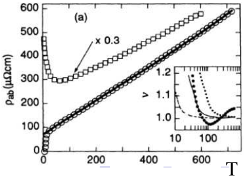
Most prominent

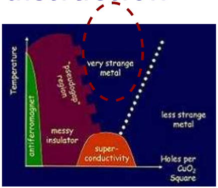
mystery of the strange metal phase:

e-e scattering: $\rho \sim T^2$, e-phonon: $\rho \sim T^5$,

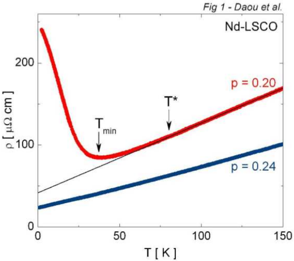
no known robust effective theory: $\rho \sim T$.



Superconductivity is a distraction



Look 'behind' superconducting dome by turning on magnetic field:



Strange metal persists to $T \sim 0$!

So we want to look for a theory of this intermediate-scale physics (like Fermi liquid theory).

Another source of NFL: how do fermi liquids die?

Some systems have both a Fermi liquid phase,
and a phase without a Fermi surface (Mott insulator).

e.g. spin-$\frac{1}{2}$ Hubbard model near half-filling:

$$H = \sum_{\langle ij \rangle} t \; c_i^\dagger c_j + U \sum_i n_i^\uparrow n_i^\downarrow$$

t: kinetic term $\quad$ U: on-site repulsion

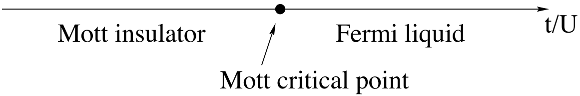


fig from [Senthil, 0803.4009]

$t/U \to \infty$: free electrons, FL.
$t/U \to 0$: each electron picks a site and sits there (Mott insulator).

Critical fermi surfaces

but:

Theorem [Luttinger]: *The volume inside the fermi surface is proportional to the number of electrons, which is conserved.*

It can't just shrink if the number of particles is fixed.

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At a continuous transition: "critical fermi surface" [Brinkman-Rice, Senthil]: $Z \to 0$.

$Z =$ jump in momentum space occupation number at the fermi momentum $n(k) = \int \frac{d\omega}{\pi} f(\omega) \mathrm{Im}\, G(\omega, k)$

$f(\omega) \equiv \frac{1}{e^{\beta\omega}+1}$, ω measured from μ.

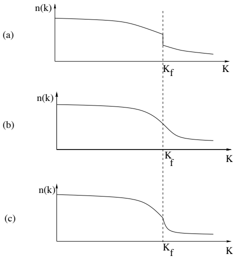
a) FL

b) mott insulator

c) critical fermi surface

$\partial_k^{(\ell)} n(k) = \infty$ for some ℓ

Z is like an order parameter for the FL phase.



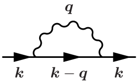
NFL from non-Holography

- Luttinger liquid (1+1-d) $G(k,\omega) \sim (k-\omega)^{2g}$ $\checkmark$
- numerics on Hubbard model
- loophole in RG argument:

couple a Landau FL perturbatively to a bosonic mode

(magnetic photon, slave-boson gauge field, statistical gauge field, ferromagnetism, SDW, Pomeranchuk order parameter...)

[Holstein et al, Baym et al, ... Halperin-Lee-Read, Polchinski, Altshuler-Ioffe-Millis, Nayak-Wilczek, Schafer-Schwenzer, Chubukov et al, Fradkin-Kivelson-Oganesyan, Metzner et al, S-S Lee, Metlitski-Sachdev, Mross et al]



$\rightarrow$ nonanalytic

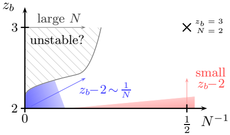
behavior in $G^R(\omega) \equiv \frac{1}{v_F k_\perp + \Sigma(\omega,k)}$ at FS:

$$\Sigma(\omega) \sim \begin{cases} \omega^{2/3} & d = 2+1 \\ \omega \log \omega & d = 3+1 \end{cases} \implies Z \overset{k_\perp \to 0}{\to} 0, \quad \frac{\Gamma(k_\perp)}{\omega_\star(k_\perp)} \overset{k_\perp \to 0}{\to} \mathrm{const}$$

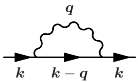
Fermi liquid killed by gapless boson

1. In these perturbative calculations, non-analytic terms $\propto$ control parameter $\Longrightarrow$
   perturbative answer is parametrically reliable $\leftrightarrow$
   effect is visible only at parametrically low temperatures.

2. Recently, the validity of the $1/N$ expansion has been questioned.
   [Sung-Sik Lee 0905.4532, Metlitski-Sachdev 1001.1153]
   A controlled perturbation expansion does exist in a slightly different theory.
   [David. Mross, JM, Hong Liu, Senthil, 1003.0894]



3. **Not strange enough:**
   These NFLs are not strange metals in terms of transport.
   FL killed by gapless bosons: small-angle scattering dominates $\Longrightarrow$
   (forward scattering does not degrade current)
   'transport lifetime' $\neq$ 'single-particle lifetime'
   i.e. in models with $\Gamma(\omega_\star) \sim \omega_\star$, $\rho \sim T^{\alpha > 1}$.



Holographic non-Fermi liquids:

Strange metal from black holes

based on:

Hong Liu, JM, David Vegh, 0903.2477

Tom Faulkner, HL, JM, DV, 0907.2694

TF, Gary Horowitz, JM, Matthew Roberts, DV, 0911.3402

TF, Nabil Iqbal, HL, JM, DV, 1003.1728 and to appear

see also: Sung-Sik Lee, 0809.3402

Cubrovic, Zaanen, Schalm, 0904.1933

Can string theory be useful here?

It would be valuable to have a non-perturbative description of such states in more than one dimension.

Gravity dual?

We're not going to look for a gravity dual of the whole material.

or of the Hubbard model.

Rather: lessons for principles of "non-Fermi liquid".

Basic question for the holographic descripion:

How to make a finite density of fermions?

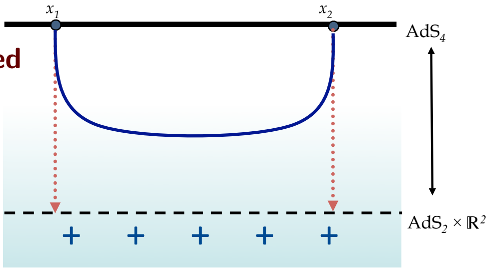
Outline

1. Strategy for holographic description
2. Fermion green functions, numerically
3. Analytic understanding of Fermi surface behavior
4. Charge transport
5. Stability of the groundstate
6. A framework for strange metal

Strategy to find a holographic Fermi surface

Consider any relativistic CFT with a gravity dual
a conserved $U(1)$ symmetry proxy for fermion number $\longrightarrow A_\mu$
and a charged fermion proxy for bare electrons $\longrightarrow \psi$.

Any $d > 1+1$, focus on $d = 2+1$.



CFT at finite density: **charged** black hole (BH) in AdS.
To find FS: [Sung-Sik Lee 0809.3402]
look for sharp features in fermion Green functions at finite momentum and small frequency.

To compute G_R: solve Dirac equation in charged BH geometry.

What we are doing, more precisely

Consider any relativistic CFT_d with

- an Einstein gravity dual $\mathcal{L}_{d+1} = \mathcal{R} + \frac{d(d-1)}{R^2}$

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Consider any relativistic CFT_d with

- an Einstein gravity dual $\mathcal{L}_{d+1} = \mathcal{R} + \frac{d(d-1)}{R^2} - \frac{2\kappa^2}{g_F^2} F^2 + \ldots$
- a conserved $U(1)$ current (proxy for fermion number)

$\rightarrow$ gauge field $F = dA$ in the bulk.

An ensemble with finite chemical potential for that current is described by the AdS Reissner-Nordstrom black hole:

$$ds^2 = \frac{r^2}{L^2}\left(-fdt^2 + d\vec{x}^2\right) + L^2 \frac{dr^2}{r^2 f}, \qquad A = \mu \left(1 - \left(\frac{r_0}{r}\right)^{d-2}\right) dt$$

$$f(r) = 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d}, \quad f(r_0) = 0, \quad \mu = \frac{g_F Q}{c_d L^2 r_0^{d-1}},$$

What we are doing, more precisely

Consider any relativistic CFT_d with

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- a charged fermion operator $\mathcal{O}_{\mathrm{F}}$ (proxy for bare electrons)

$\rightarrow$ spinor field ψ in the bulk $\mathcal{L}_{d+1} \ni \bar{\psi}\left(D_M \Gamma^M - m\right)\psi + \text{interactions}$

with $D_\mu \psi = \left(\partial_\mu + \frac{1}{4}\omega_\mu \cdot \Gamma - iqA_\mu\right)\psi$ $\left(\Delta = \frac{d}{2} \pm mL, \quad q = q\right)$

'Bulk universality': for two-point functions, the interaction terms don't matter!

Results only depend on q, Δ.

Comments about the strategy

- There are many string theory vacua with these ingredients. In specific examples of dual pairs (*e.g.* M2-branes $\Leftrightarrow$ M th on $AdS_4 \times S^7$), interactions and $\{q, m\}$ are specified.

  which sets $\{q, m\}$ are possible and what correlations there are is not clear.
- This is a large complicated system ($\rho \sim N^2$), of which we are probing a tiny part ($\rho_\Psi \sim N^0$).
- In general, both bosons and fermions of the dual field theory are charged under the $U(1)$ current: this is a Bose-Fermi mixture.

Computing G_R

Translation invariance in $\vec{x}, t \implies$ ODE in r.
Rotation invariance: $k_i = \delta_i^1 k$
Near the boundary, solutions behave as $(\Gamma^{\underline{r}} = -\sigma^3 \otimes 1)$

$$\psi \overset{r\to\infty}{\approx} a_\alpha r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_\alpha r^{-m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Matrix of Green's functions, has two independent eigenvalues:

$$G_\alpha(\omega, \vec{k}) = \frac{b_\alpha}{a_\alpha}, \quad \alpha = 1, 2$$

To compute G_R: solve Dirac equation in BH geometry,
impose infalling boundary conditions at horizon [Son-Starinets, Iqbal-Liu].

Dirac equation

$$\Gamma^a e_a{}^M \left(\partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab} - iqA_M \right) \psi - m\psi = 0$$

$$\Phi_\alpha \equiv (-g g^{rr})^{-1/4} \Pi_\alpha^{\hat{k}} \psi, \quad \psi = e^{-i\omega t + i k_i x^i} \psi_{\omega,k},$$

$$\boxed{\left(\partial_r + M\sigma^3\right) \Phi_\alpha = \left((-1)^\alpha K \sigma^1 + W i \sigma^2\right) \Phi_\alpha, \quad \alpha = 1, 2}$$

with

$$M \equiv m\sqrt{g_{rr}} = \frac{m}{r\sqrt{f}}, \quad K \equiv k\sqrt{\frac{g_{rr}}{g_{ii}}} = \frac{k}{r^2\sqrt{f}}, \quad W \equiv u\sqrt{\frac{g_{rr}}{g_{ii}}} = \frac{u}{r^2\sqrt{f}}.$$

$$u \equiv \sqrt{\frac{-g^{tt}}{g^{ii}}} \left(\omega + \mu_q \left(1 - \left(\frac{r_0}{r} \right)^{d-2} \right) \right)$$

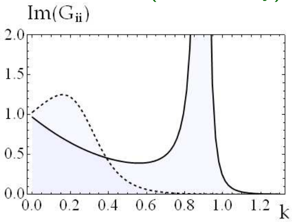
Eqn depends on q and μ only through $\mu_q \equiv \mu q$

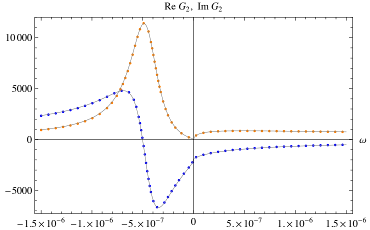
$\rightarrow$ ω is measured from the effective chemical potential, μ_q.

Results are in units of μ.

Fermi surface!

At $T = 0$, we find (numerically):





'MDC': $G(\omega = -0.001, k)$ 'EDC': $G(\omega, k = 0.9)$

For $q = 1, m = 0:\quad k_F \approx 0.918528499$

Fermi surface!

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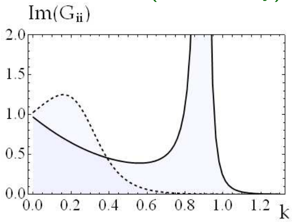

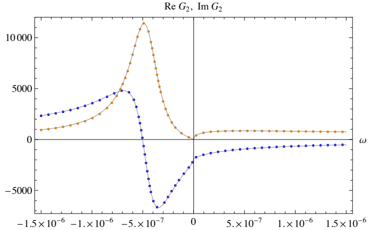

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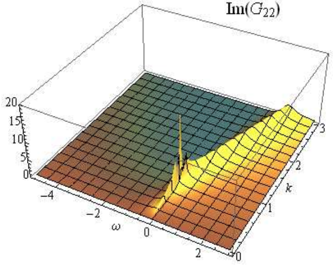


But it's not a Fermi liquid:

The peak moves
with dispersion relation $\omega \sim k_\perp^z$ with

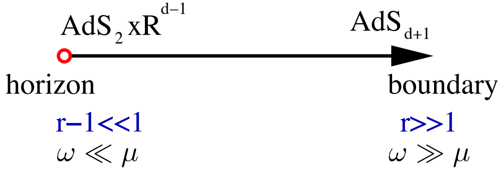
$$z = 2.09 \text{ for } q = 1, \Delta = 3/2.$$
$$z = 5.32 \text{ for } q = 0.6, \Delta = 3/2$$

and the residue vanishes.

Emergent quantum criticality

Whence these exponents?

Near-horizon geometry of black hole is $AdS_2 \times \mathbb{R}^{d-1}$.
The conformal invariance of this metric is **emergent**.
(We broke the microscopic conformal invariance with finite density.)



AdS/CFT says that the low-energy physics is governed by the dual **IR CFT**.

The bulk geometry is a picture of the RG flow from the CFT$_d$ to this NRCFT.

Analytic understanding of Fermi surface behavior: idea

$T > 0$: $G_R(\omega)$ analytic near $\omega = 0 \rightarrow$ can compute in series expansion. [Policastro-Son-Starinets]

$T = 0$: Expanding the wave equation in ω is delicate.

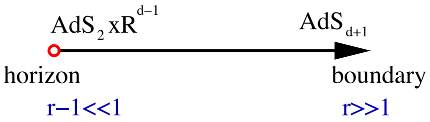
The ω-term dominates near the horizon.

Method of matched asymptotic expansions:

Find solution (in ω-expansion) in two regions of BH geometry (IR and UV), match their behavior in the region of overlap.

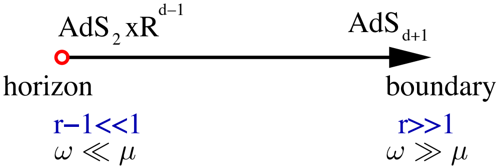
Familiar from the brane absorption calculations which led to AdS/CFT.

[Klebanov, Gubser, Maldacena, Strominger...]



Here: this ‘matching’ can be interpreted in the QFT as RG matching between UV and IR CFTs.

Matching regions



$$\textbf{Inner:} \quad \zeta \equiv \lambda \frac{L_2^2}{r-1} \quad \text{for} \quad \epsilon < \zeta < \infty$$

$$\textbf{Outer:} \quad \frac{\lambda L_2^2}{\epsilon} < r - 1$$

(L_2 is the 'AdS radius' of the IR AdS_2.)
and consider the limit

$$\lambda \to 0, \qquad \zeta = \text{finite}, \qquad \epsilon \to 0, \qquad \frac{\lambda R_2^2}{\epsilon} \to 0\ .$$

The boundary of $AdS_2 \times \mathbb{R}^2$ ($\zeta \to 0$) attaches to the near-horizon of the outer region.

Inner region (IR data)

$$ds^2 = \frac{R_2^2}{\zeta^2}\left(-d\tau^2 + d\zeta^2\right) + d\vec{x}^2, \quad A = d\tau \frac{e_d}{\zeta}$$

Wave equations for charged fields in AdS_2 are solvable.
Near the boundary:

$$\psi \overset{\zeta\to 0}{\approx} \mathcal{G}\zeta^{\nu} v_+ + \zeta^{-\nu} v_-$$

$$\nu_k \equiv R_2\sqrt{m^2 + k^2 - q^2/2}, \quad \delta_k = \frac{1}{2} + \nu_k$$

For a spinor in AdS_2, k is a parity-violating mass term $\tilde{m}\bar{\psi}\Gamma\psi$: $\tilde{m} \equiv k\frac{L_2}{r_0}$
$\Psi(\omega, k)$ matches onto some IR CFT operator $\mathcal{O}_k$ of dimension $\delta_k = \frac{1}{2} + \nu_k$, whose (retarded) two-point function is the

$$\text{IR CFT Green function}: \quad \mathcal{G}_k(\omega) = c(k)\omega^{2\nu_k}$$

$c(k) \in \mathbb{C}$, known.

Low-frequency expansion in outer region (UV data)

Basis of solutions at $\omega = 0$:

$$\psi_\alpha^{(0)\pm} \overset{r\to 1}{\approx} v_\pm (r-1)^{\mp\nu}$$

These two solutions match to the leading and subleading solutions in the near-horizon region.

$$\implies \quad \psi_\alpha = \psi_\alpha^+ + \mathcal{G}(\omega)\psi_\alpha^- .$$

$$\psi_\alpha^\pm = \psi_\alpha^{(0)\pm} + \omega\psi_\alpha^{(1)\pm} + \omega^2\psi_\alpha^{(2)\pm} + \dots, \quad \psi_\alpha^{(n)\pm} \overset{r\to\infty}{\approx} \begin{pmatrix} b_\alpha^{(n)\pm} r^{-m} \\ a_\alpha^{(n)\pm} r^m \end{pmatrix}.$$

$$\boxed{G_R(\omega,k) = K\frac{b_+^{(0)} + \omega b_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega)\left(b_-^{(0)} + \omega b_-^{(1)} + O(\omega^2)\right)}{a_+^{(0)} + \omega a_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega)\left(a_-^{(0)} + \omega a_-^{(1)} + O(\omega^2)\right)}}$$

Analytic understanding of Fermi surface behavior: results

$$\boxed{G_R(\omega,k) = \frac{b_+^{(0)} + \omega b_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega)\left(b_-^{(0)} + \omega b_-^{(1)} + O(\omega^2)\right)}{a_+^{(0)} + \omega a_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega)\left(a_-^{(0)} + \omega a_-^{(1)} + O(\omega^2)\right)}}$$

The location of the Fermi surface $(a_+^{(0)}(k = k_F) = 0)$ is determined by short-distance physics (analogous to band structure –
$a, b \in \mathbb{R}$ from normalizable sol'n of $\omega = 0$ Dirac equation in full BH)
but the low-frequency scaling behavior near the FS is universal
(determined by near-horizon region, IR CFT $\mathcal{G}$!).
$\mathcal{G} = c(k)\omega^{2\nu}$ is the retarded G_R of the op to which $\mathcal{O}_{\rm F}$ matches.
its scaling dimension is $\nu + \frac{1}{2}$, with (for $d = 2 + 1$)

$$\nu \equiv L_2\sqrt{m^2 + k_F^2 - q^2/2}$$

L_2 is the 'AdS radius' of the IR AdS_2.

Inner region (IR data) in more detail

$$\mathcal{G}_R(\omega) = \overbrace{e^{-i\pi\nu}\frac{\Gamma(-2\nu)\,\Gamma\left(1+\nu-iqe_d\right)}{\Gamma(2\nu)\,\Gamma\left(1-\nu-iqe_d\right)}\cdot\frac{\left(m+i\tilde{m}\right)L_2-iqe_d-\nu}{\left(m+i\tilde{m}\right)L_2-iqe_d+\nu}}^{c(k)}\left(2\omega\right)^{2\nu}$$

The AdS_2 Green's functions look like DLCQ of 1+1d CFT.

Leftmoving bit depends on q, rightmoving bit depends on ω.

qv [Azeyanagi et al, Guica et al, de Boer et al]

Inner region (IR data) in more detail

$$\mathcal{G}_R(\omega) = \overbrace{e^{-i\pi\nu}\frac{\Gamma(-2\nu)\,\Gamma\left(1+\nu-iqe_d\right)}{\Gamma(2\nu)\,\Gamma\left(1-\nu-iqe_d\right)} \cdot \frac{(m+i\tilde{m})\,L_2 - iqe_d - \nu}{(m+i\tilde{m})\,L_2 - iqe_d + \nu}}^{c(k)}\,(2\omega)^{2\nu}$$

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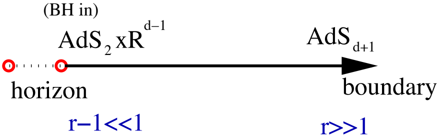


$T \neq 0$: near-horizon geometry is a BH in AdS_2

$\omega^{2\nu}$ is the $T \to 0$ limit of

$$T^{2\nu} g(\omega/T) = (2\pi T)^{2\nu}\frac{\Gamma\left(\frac{1}{2}+\nu-\frac{i\omega}{2\pi T}+iqe_d\right)}{\Gamma\left(\frac{1}{2}-\nu-\frac{i\omega}{2\pi T}+iqe_d\right)}$$

DLCQ of 1+1d CFT at $T > 0$.

Consequences for Fermi surface

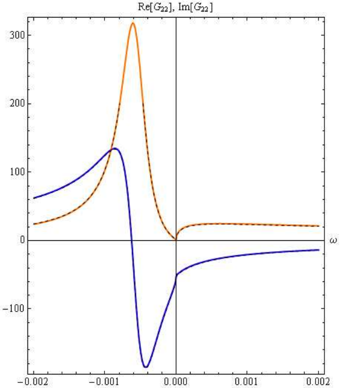
$$G_R(\omega, k) = \frac{h_1}{k_\perp - \frac{1}{v_F}\omega - h_2 c(k)\omega^{2\nu_{k_F}}}$$

$h_{1,2}, v_F$ real, UV data.

The AdS_2 Green's function

is the self-energy $\Sigma = \mathcal{G} = c(k)\omega^{2\nu}$!

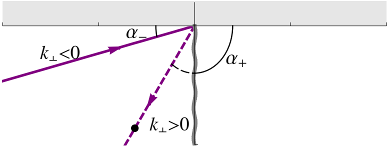
Correctly fits numerics near FS:



$\nu < \frac{1}{2}$: non-Fermi liquid

$$G_R(\omega, k) = \frac{h_1}{k_\perp - \frac{1}{v_F}\omega - h_2 \omega^{2\nu_{k_F}}}$$

$$\text{if } \nu_{k_F} < \frac{1}{2}, \quad \omega_\star(k) \sim k_\perp^z, \qquad z = \frac{1}{2\nu_{k_F}} > 1$$



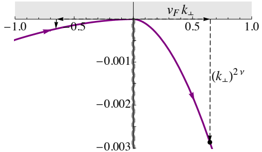
$$\frac{\Gamma(k)}{\omega_\star(k)} \xrightarrow{k_\perp \to 0} \text{const}, \qquad Z \propto k_\perp^{\frac{1-2\nu_{k_F}}{2\nu_{k_F}}} \xrightarrow{k_\perp \to 0} 0.$$

Not a stable quasiparticle.

$\nu > \frac{1}{2}$: Fermi liquid

$$G_R(\omega, k) = \frac{h_1}{k_\perp + \frac{1}{v_F}\omega + c\omega^{2\nu_{k_F}}} \qquad \omega_\star(k) \sim v_F k_\perp$$

c is complex.

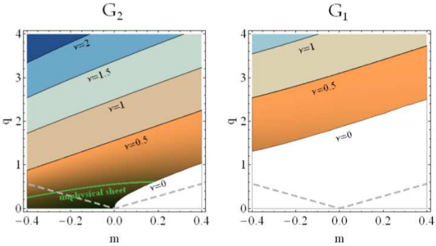


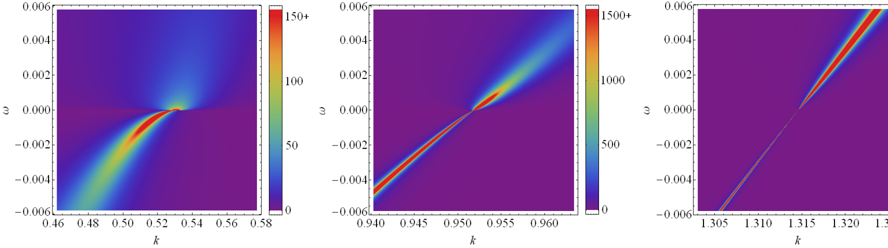
$$\frac{\Gamma(k)}{\omega_\star(k)} \propto k_\perp^{2\nu_{k_F}-1} \overset{k_\perp\to 0}{\longrightarrow} 0 \qquad Z \overset{k_\perp\to 0}{\longrightarrow} h_1 v_F.$$

A stable quasiparticle, but never Landau Fermi liquid.

summary

Depending on the dimension of the operator $(\nu + \frac{1}{2})$ in the IR CFT, we find Fermi liquid behavior (but not Landau) or non-Fermi liquid behavior:





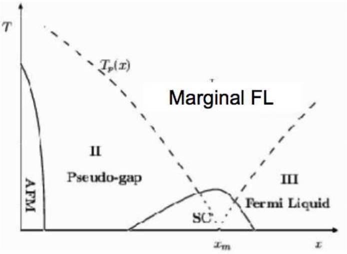
$\nu < \frac{1}{2}$ $\nu = \frac{1}{2}$ $\nu > \frac{1}{2}$

$\nu = \frac{1}{2}$: Marginal Fermi liquid

$$G_R \approx \frac{h_1}{k_\perp + \tilde{c}_1 \omega \ln \omega + c_1 \omega}, \quad \tilde{c}_1 \in \mathbb{R}, \quad c_1 \in \mathbb{C}$$

$$\frac{\Gamma(k)}{\omega_\star(k)} \overset{k_\perp \to 0}{\to} \text{const}, \qquad Z \sim \frac{1}{|\ln \omega_\star|} \overset{k_\perp \to 0}{\to} 0.$$

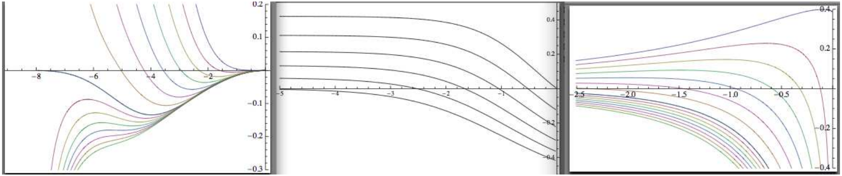
A well-named phenomenological model of high-T_c cuprates near optimal doping



[Varma et al, 1989].

UV data: where are the Fermi sufaces?

Above we supposed $a(k_F)_+^{(0)} = 0$. This happens at k_F: k s.t. $\exists$ normalizable, incoming solution at $\omega = 0$:
This black hole can acquire 'inhomogenous fermionic hair'



Schrodinger potential $V(\tau)/k^2$ at $\omega = 0$ for $m < 0, m = 0, m > 0$.

τ is the tortoise coordinate   Right ($\tau = 0$) is boundary; left is horizon.

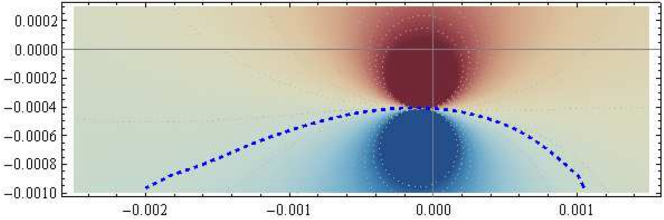
$k > qe_d$: Potential is always positive

$k < k_{osc} \equiv \sqrt{(qe_d)^2 - m^2}$: near the horizon $V(x) = \frac{\alpha}{\tau^2}$, with $\alpha < -\frac{1}{4}$ ("oscillatory region")

$k \in (qe_d, k_{osc})$: the potential develops a potential well, indicating possible existence of a zero energy bound state.

Note: *can* exist on asymp. flat BH [Hartman-Song-Strominger 0912]

Finite temperature movies



At finite T, the pole doesn't quite hit the real axis: thermal broadening.

$\min_k(\mathrm{Im}\,\omega_c) \simeq T$ (up to 1% accuracy).

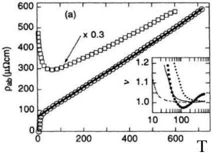
Branch cut from $\omega^{2\nu}$ approximated by a sequence of poles on neg Im axis. Like a dipole array approximates a capacitor.

Charge transport by holographic Fermi surfaces

[Tom Faulkner, Nabil Iqbal, Hong Liu, JM, David Vegh, 1003.1728 and to appear]

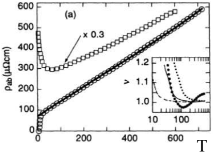
Charge transport

Most prominent mystery $\longrightarrow$
of strange metal phase: $\sigma_{\mathrm{DC}} \sim T^{-1}$
$(j = \sigma E)$



e-e scattering: $\sigma \sim T^{-2}$, e-phonon scattering: $\sigma \sim T^{-5}$, **nothing**: $\sigma \sim T^{-1}$

Charge transport



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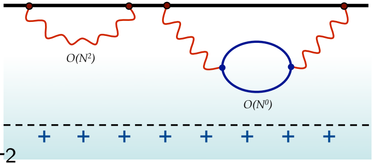
We can compute the contribution
to the conductivity from
the Fermi surface. [Faulkner, Iqbal, Liu, JM, Vegh]

Note: this is not the dominant contribution. $\rightarrow$



$$\sigma_{\rm DC} = \lim_{\omega \to 0} {\rm Im} \frac{1}{\omega} \langle j^x j^x \rangle (\omega, \vec{0}) = N^2 \frac{T^2}{\mu^2} + N^0 \left(\sigma^{\rm FS}_{\rm DC} + \ldots \right)$$

Charge transport by holographic non-Fermi liquids

slight complication: gauge field a_x mixes with metric perturbations.

There's a big charge density. Pulling on it with $\vec{E}$ leads to momentum flow.

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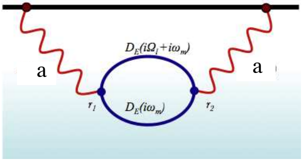
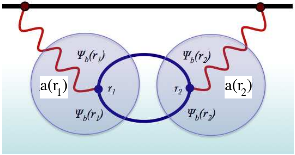


key step: $\mathrm{Im}\, D_{\alpha\beta}(\Omega, k; r_1, r_2) = \frac{\psi^b_\alpha(\Omega,k,r_1)\bar{\psi}^b_\beta(\Omega,k,r_2)}{W_{ab}} A(\Omega, k)$

bulk spectral density $\mathrm{Im}\, D$

1. ... is determined by bdy fermion spectral density, $A(\omega, k) = \mathrm{Im}\, G_R(\omega, k)$
2. ... factorizes on normalizable bulk sol'ns ψ^b



Figs by Nabil Iqbal

Charge transport by holographic non-Fermi liquids

like Fermi liquid calculation

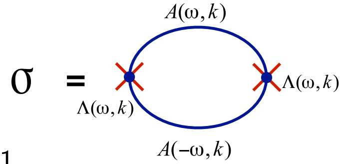
but with extra integrals over r, and no vertex corrections.

$$\sigma_{\mathrm{DC}}^{\mathrm{FS}} = C \int_0^\infty dkk \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{df}{d\omega} \Lambda^2(k,\omega) A^2(\omega,k)$$

$f(\omega) = \frac{1}{e^{\frac{\omega}{T}}+1}$: the Fermi distribution function

Λ: an effective vertex, data analogous to $v_F, h_{1,2}$.

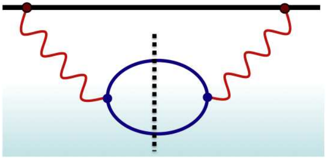
$\Lambda \sim q \int_{r_0}^{\infty} dr \sqrt{g} g^{xx} a_x(r,0) \frac{\bar{\psi}^b(r) \Gamma^x \psi^b(r)}{W_{ab}} \sim \text{const.}$

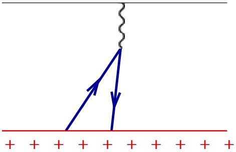


$$\int dk A(k,\omega)^2 \sim \frac{1}{T^{2\nu} g(\omega/T)}$$

scale out T-dependence $\implies \sigma^{DC} \sim T^{-2\nu}$.

Dissipation mechanism

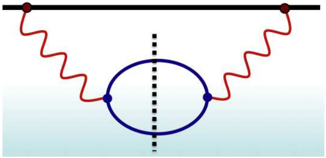


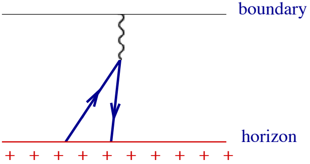


$\sigma_{\rm DC} \propto {\rm Im} \langle jj \rangle$ comes from fermions falling into the horizon.
dissipation of current is controlled by the decay of the fermions into the AdS_2 DoFs.
$\implies$ single-particle lifetime controls transport.

marginal Fermi liquid: $\nu = \frac{1}{2} \implies \boxed{\rho_{FS} = \left(\sigma^{DC}\right)^{-1} \sim T}\ .$

Dissipation mechanism





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marginal Fermi liquid: $\nu = \frac{1}{2} \Longrightarrow \boxed{\rho_{FS} = \left(\sigma^{DC}\right)^{-1} \sim T}$.

The optical conductivity $\sigma(\Omega)$ can distinguish the existence of quasiparticles ($\nu > \frac{1}{2}$) through the presence of a transport peak.

Stability of the Reissner-Nordstrom groundstate

Charged AdS black holes and frustration

Entropy density of black hole:

$$s(T=0) = \frac{1}{V_{d-1}} \frac{A}{4G_N} = 2\pi e_d \rho. \qquad \left(e_d \equiv \frac{g_F}{\sqrt{2d(d-1)}}\right)$$

This is a large low-energy density of states!

not supersymmetric ... lifted at finite N

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pessimism: $S(T=0) \neq 0$ violates third law of thermodynamics, unphysical, weird string-theorist nonsense.

optimism:

we're describing the state where the SC instability is removed by hand (**here:** don't include charged scalars, **expt:** large $\vec{B}$).

[Hartnoll-Polchinski-Silverstein-Tong, 0912.]: bulk density of fermions modifies extreme near-horizon region (out to $\delta r \sim e^{-N^2}$), removes residual entropy. (Removes non-analyticity in $\Sigma(\omega)$ for $\omega < e^{-N^2}\mu$)

Stability of the groundstate

Charged bosons: In many explicit dual pairs, $\exists$ charged scalars.

• At small T, they can condense spontaneously breaking the $U(1)$ symmetry, changing the background [Gubser, Hartnoll-Herzog-Horowitz].

spinor: $G_R(\omega)$ has poles only in LHP of ω [Faulkner-Liu-JM-Vegh, 0907]

scalar: $\exists$ poles in UHP $\quad \langle \mathcal{O}(t) \rangle \sim e^{i\omega_\star t} \propto e^{+\mathrm{Im}\,\omega_\star t}$

$\Longrightarrow$ growing modes of charged operator: holographic superconductor

[Gubser, Hartnoll-Herzog-Horowitz...]

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$\Longrightarrow$ growing modes of charged operator: holographic superconductor

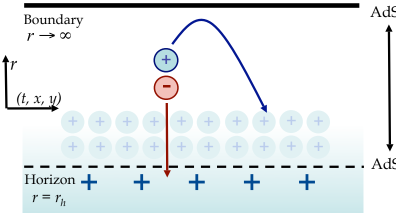
[Gubser, Hartnoll-Herzog-Horowitz...]

why: black hole *spontaneously* emits charged particles [Starobinsky, Unruh, Hawking].

AdS is like a box: they can't escape.

Fermi: negative energy states get filled.

Bose: the created particles then cause *stimulated emission* (superradiance).



A holographic superconductor is a "black hole laser".

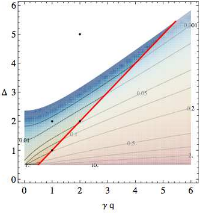
Stability of the groundstate, cont'd

- If their mass/charge is big enough, they don't condense.

[Denef-Hartnoll]

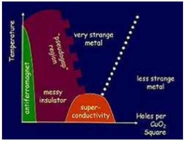
This is weird: a weakly-coupled charged boson at $\mu \neq 0$ will condense.

Finding such string vacua is like moduli stabilization.



- Many systems to which we'd like to apply this also have a superconducting region.

- So far we are describing the state behind and above the dome.



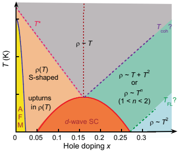
Aside: Other light bulk modes (*e.g.* neutral scalars) can also have an important effect on the groundstate

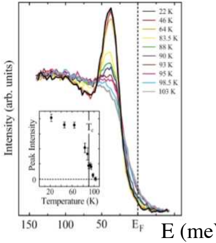
[Fareghbal-Gowdigere-Mosaffa-Sheikh-Jabbari Mulligan, Polchinski, Goldstein-Kachru-Prakash-Trivedi, Gubser-Rocha].

Photoemission ‘exp’ts’ on holographic superconductors

So far: a model of
some features of the normal state.

In SC state: a sharp peak forms in $A(k,\omega)$.





Photoemission ‘exp’ts’ on holographic superconductors

So far: a model of
some features of the normal state.
In SC state: a sharp peak forms in $A(k, \omega)$.

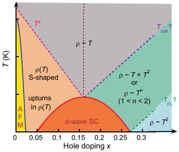
With a suitable coupling between ψ and φ,
the superconducting condensate
opens a gap in the fermion spectrum.

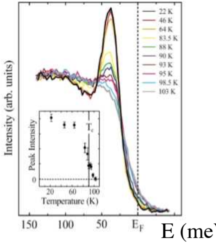
[Faulkner, Horowitz, JM, Roberts, Vegh]

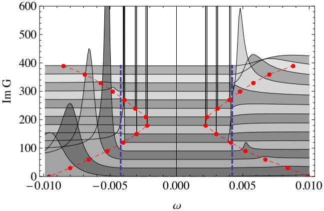
if $q_\varphi = 2q_\psi$ we can have

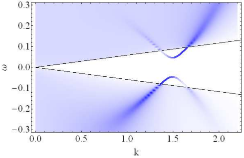
$$L_{\text{bulk}} \ni \eta_5 \varphi \bar{\psi} \mathcal{C} \Gamma^5 \bar{\psi}^T + \text{h.c}$$

The (gapped) quasiparticles
are exactly stable in a certain
kinematical regime
(outside the lightcone of the IR CFT) –
the condensate lifts the IR CFT modes
into which they decay.

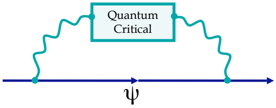








Framework for strange metal

a cartoon of the mechanism: 

a similar picture has been advocated by [Varma et al]

Comparison

- a Fermi surface coupled to a critical boson field

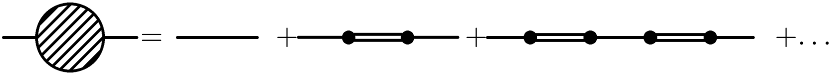
$$L = \bar{\psi}\left(\omega - v_F k\right)\psi + \bar{\psi}\psi a + L(a)$$

small-angle scattering dominates.

- a Fermi surface mixing with a bath of critical fermionic fluctuations with large dynamical exponent [TF-HL-JM-DV 0907.2694, Faulkner-Polchinski 1001...]

$$L = \bar{\psi}\left(\omega - v_F k\right)\psi + \bar{\psi}\chi + \psi\bar{\chi} + \bar{\chi}\mathcal{G}^{-1}\chi$$

χ: IR CFT operator



$$\langle\bar{\psi}\psi\rangle = \frac{1}{\omega - v_F k - \mathcal{G}} \qquad \mathcal{G} = \langle\bar{\chi}\chi\rangle = c(k)\omega^{2\nu}$$

$\nu \leq \frac{1}{2}$: the $\bar{\psi}\chi$ coupling is a relevant perturbation.

Concluding remarks

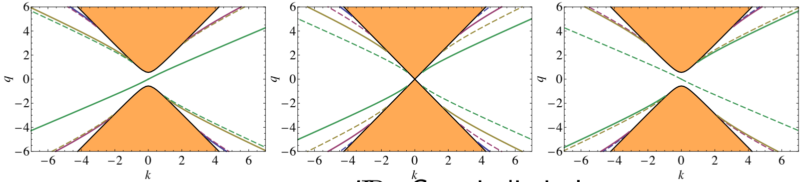
1. The green's function near the FS is of the form ('local quantum criticality', analytic in k.) found previously in perturbative calculations, but the nonanalyticity can be order one.
   This is an *input* of many studies (dynamical mean field theory)
2. [Sachdev, 1007...]: Slave-particle solution of large-d, large-spin limit of *random* antiferromagnet produces a very similar state [Sachdev-Ye].
3. [Denef-Hartnoll-Sachdev] The leading N^{-1} contribution to the free energy exhibits quantum oscillations in a magnetic field.
4. Main challenge: step away from large N. So far:
   - Fermi surface is a small part of a big system.
   - Fermi surface does not back-react on IR CFT.
   - IR CFT has $z = \infty$.

The end.

Thanks for listening.

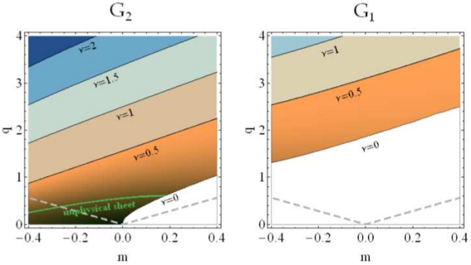
Where are the Fermi sufaces?

$m = -0.4,\ 0,\ 0.4$:



orange: 'oscillatory region': $\nu \in i\mathbb{R}$, G periodic in $\log \omega$

$$\delta_k = \frac{1}{2} + \nu_k, \qquad \nu_k = \frac{1}{\sqrt{6}}\sqrt{m^2 + k^2 - q^2/2}$$



Fermion poles always in LHP!

$$\arg c_k = \arg\left(e^{2\pi i\nu} \pm e^{-2\pi q e_d}\right) \qquad \mathcal{G} = c_k \omega^{2\nu}$$

$\pm$ for boson/fermion.

$$\omega_c^{2\nu} = \mathrm{real}\cdot\left(e^{-2\pi i\nu} - e^{-2\pi q e_d}\right).$$

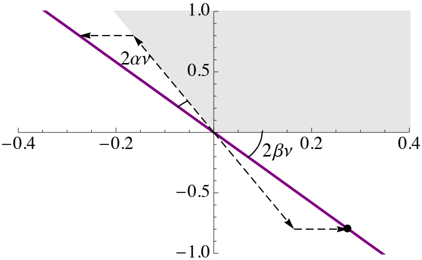


Figure: A geometric argument that poles of the fermion Green function always appear in the lower-half ω-plane: Depicted here is the $\omega^{2\nu}$ covering space on which the Green function is single-valued. The shaded region is the image of the upper-half ω-plane of the physical sheet.

fermi velocity

Think of $\omega = 0$ Dirac eqn as Schrödinger problem.

Like Feynman-Hellmann theorem: $\partial_k \langle H \rangle = \langle \partial_k H \rangle$

we can derive a formula for v_F in terms of expectation values in the bound-state wavefunction $\Phi^+_{(0)}$.

Let:

$$\langle \mathcal{O} \rangle \equiv \int_{r_\star}^{\infty} dr \sqrt{g_{rr}} \mathcal{O} \ ,$$

$$J^\mu \equiv \bar{\Phi}^+_{(0)} \ \partial_{k_\mu} \not{D}_{0,k_F} \Phi^+_{(0)} = \bar{\Phi}^+_{(0)} \Gamma^\mu \Phi^+_{(0)}$$

is the bulk particle-number current.

$$v_F = \frac{\langle J^1 \rangle}{\langle J^0 \rangle} = \frac{\int dr \sqrt{g_{rr} g^{ii}} \left(|y|^2 - |z|^2 \right)}{\int dr \sqrt{g_{rr}(-g^{tt})} \left(|y|^2 + |z|^2 \right)}.$$

$$\Phi = \begin{pmatrix} y \\ z \end{pmatrix}$$

Note: $\frac{g^{ii}}{-g^{tt}} = f(r) \leq 1$ implies that $v_F \leq c$.

fermi velocity

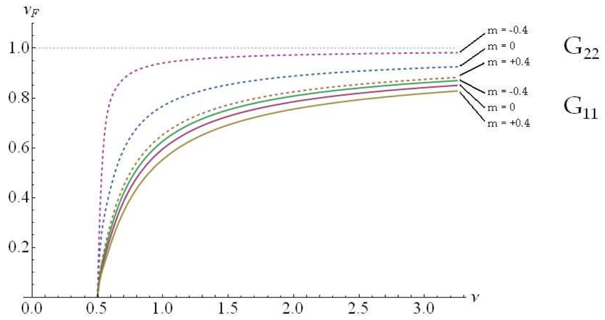


Figure: The Fermi velocity of the primary Fermi surface of various components as a function of $2\nu > 1$ for various values of m.

An explanation for the particle-hole symmetry

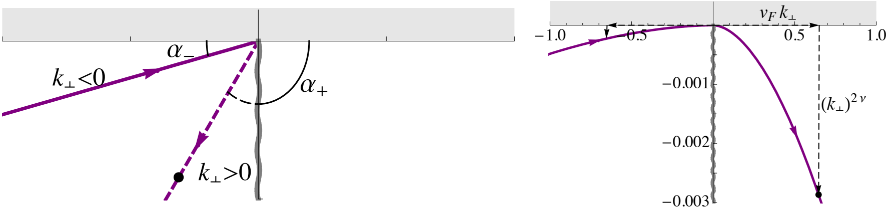


Figure: Left: Motion of poles in the $\nu < \frac{1}{2}$ regime. As k varies towards k_F, the pole moves in a straight line (hence $\Gamma \sim \omega_c$), and hits the branch point at the origin at $k = k_F$. After that, depending on $\gamma(k_F)$, it may move to another Riemann sheet of the ω-plane, as depicted here. In that case, no resonance will be visible in the spectral weight for $k > k_F$. Right: Motion of poles in the $\nu > \frac{1}{2}$ regime, which is more like a Fermi liquid in that the dispersion is linear in $k_\perp$; the lifetime is still never of the Landau form.

Note: the location of the branch cut is determined by physics:

at $T > 0$, it is resolved to a line of poles.

Oscillatory region

Above we assumed $\nu = R_2\sqrt{m^2 + k^2 - (qe_d)^2} \in \mathbb{R}$

$$\nu = i\lambda \quad \Leftrightarrow \text{Oscillatory region.}$$

This is when particle production occurs in AdS_2. [Pioline-Troost]
Effective mass below BF bound *in* AdS_2. [Hartnoll-Herzog-Horowitz]
$\mathrm{Re}\,\omega^{i2\lambda} = \sin 2\lambda \log \omega \implies$ periodic in $\log \omega$ with period $\frac{\pi}{|\nu|}$.
comments about boson case:
Net flux into the outer region $> 0 =$ superradiance of AdS RN black hole (rotating brane solution in 10d)
Classical equations know quantum statistics!
like: statistics functions in greybody factors
Required for consistency of AdS/CFT!
boson: particles emitted from near-horizon region, bounce off AdS_{d+1} boundary and return, causing further stimulated emission.
spinor: there is particle production in AdS_2 region, but net flux into the outer region is negative ('no superradiance for spinors').

oscillatory region and log-periodicity

When $\nu(k)$ is imaginary, $\mathcal{G} \sim \omega^{\nu}$ is periodic in $\log \omega$.

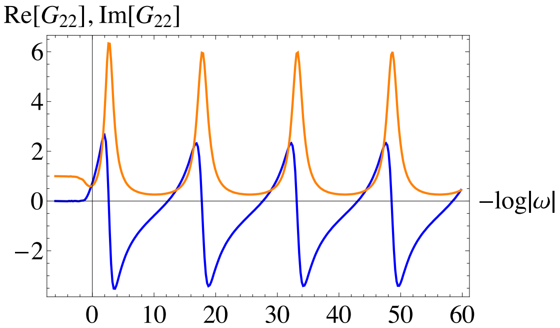


Figure: Both $\mathrm{Re}\, G_{22}(\omega, k = 0.5)$ (blue curve) and $\mathrm{Im}\, G_{22}(\omega, k = 0.5)$ (orange) are periodic in $\log \omega$ as $\omega \to 0$.

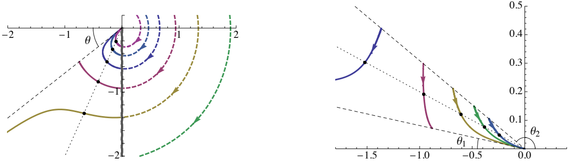
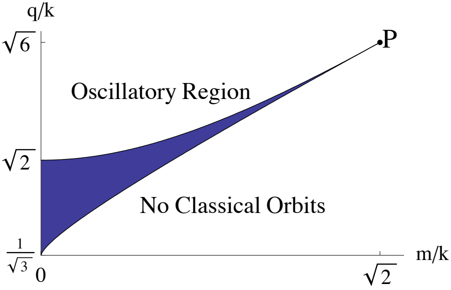


Figure: The motion of poles of the Green functions of spinors (left) and scalars (right) in the complex frequency plane. Both plots are for parameter values in the oscillatory region ($q = 1, m = 0$). In order to give a better global picture, the coordinate used on the complex frequency plane is $s = |\omega|^{\frac{1}{20}} \exp(i \arg(\omega))$. The dotted line intersects the locations of the poles at $k = k_0 = \ldots$, and its angle with respect to the real axis is determined by $\mathcal{G}(k, \omega)$. The dashed lines in the left figure indicate the motion of poles on another sheet of the complex frequency plane at smaller values of $k < k_0$. As k approaches the boundary of the oscillatory region, most of the poles join the branch cut. It seems that one pole that becomes the Fermi surface actually manages to stay in place. These plots are only to be trusted near $\omega = 0$.



Information from WKB. At large q, m, the primary Fermi momentum is given by the WKB quantization formula: $k_F \int_{s_-}^{s^+} ds \sqrt{V(s;\alpha,\beta)} = \pi$, where $\alpha \equiv \frac{q}{k}, \beta \equiv \frac{m}{k}$, s is the tortoise coordinate, and $s_\pm$ are turning points surrounding the classically-allowed region. For $k < q/\sqrt{3}$, the potential is everywhere positive, and hence there is no zero-energy boundstate. This line intersects the boundary of the oscillatory region at $k^2 + m^2 = q^2/2$ at the point $P = (\alpha, \beta) = (\sqrt{6}, \sqrt{2})$. Hence, only in the shaded (blue) region is there a Fermi surface. The exponent $\nu(k_F)$ is then given by $\nu(k_F) = \frac{\pi\sqrt{1+\beta^2-\alpha^2/2}}{\int ds \sqrt{V(s;\alpha,\beta)}}$. This becomes ill-defined at the point P, and interpolates between $\nu = 0$ at the boundary of the oscillatory region, and $\nu = \infty$ at $k = q/\sqrt{3}$.