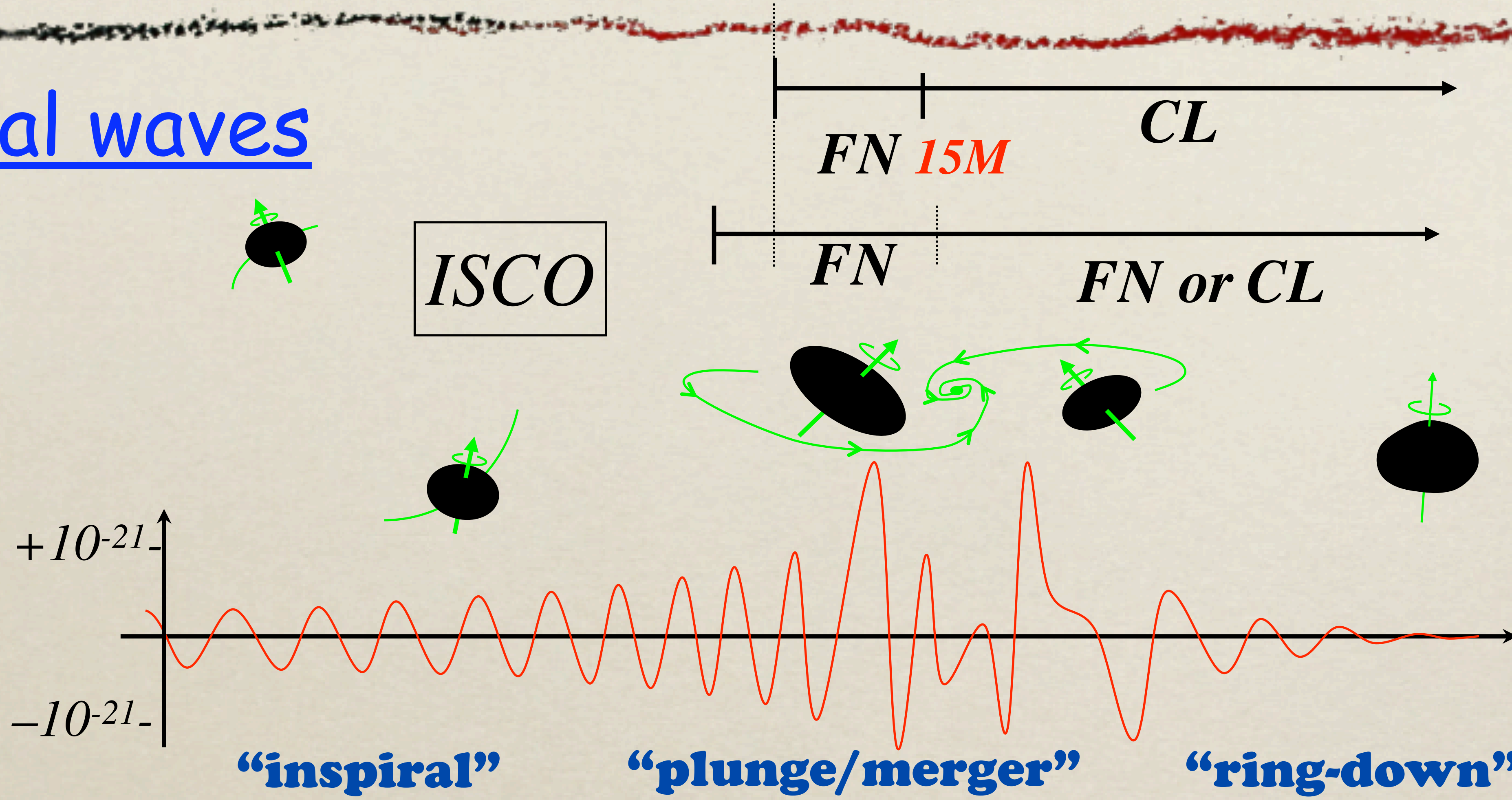
Introduction to Numerical Relativity

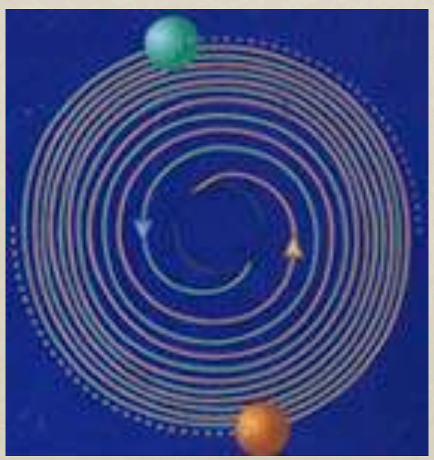
Roberto De Pietri
(Parma University)

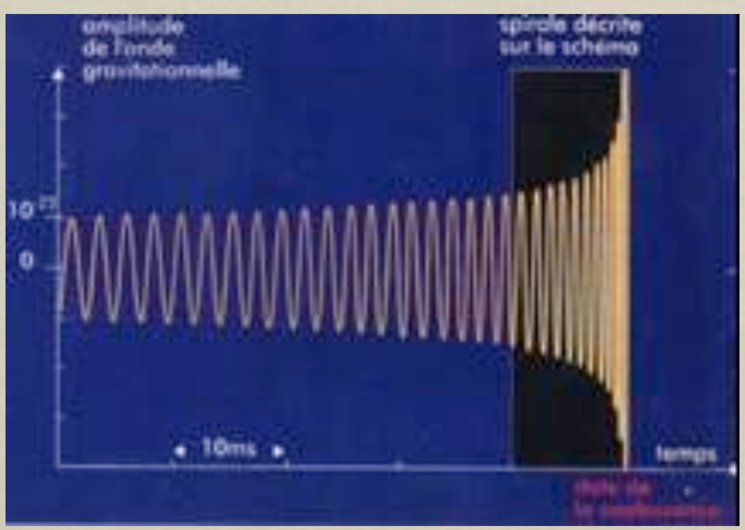
Lecture 2

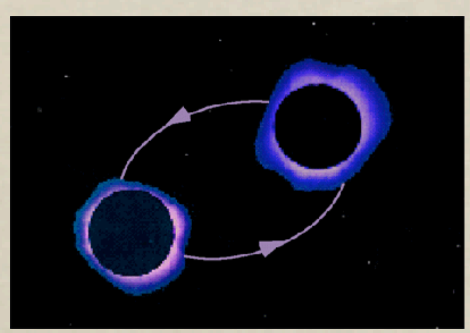
Two-bodies problem and GWs

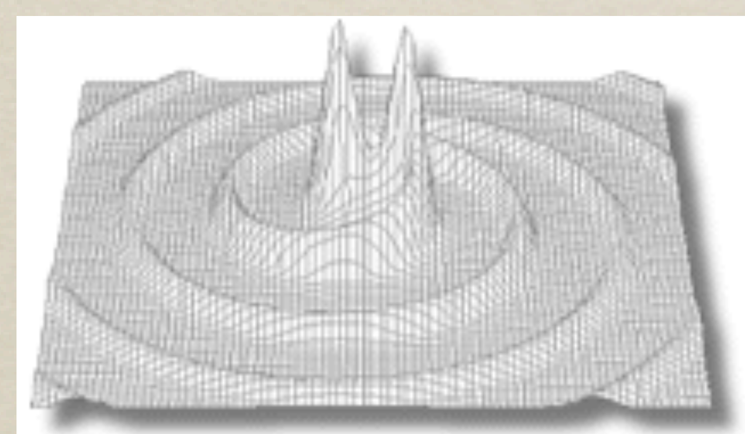
Gravitational waves







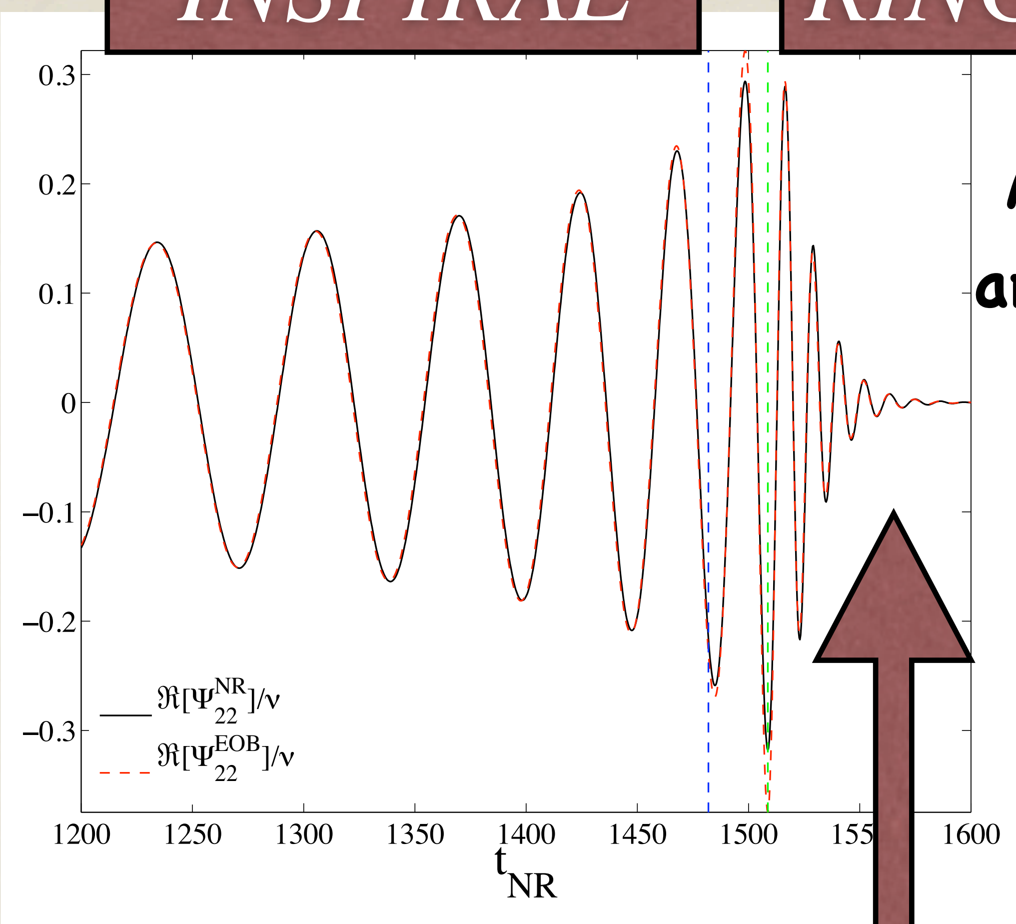




Black Hole Merger results

INSPIRAL

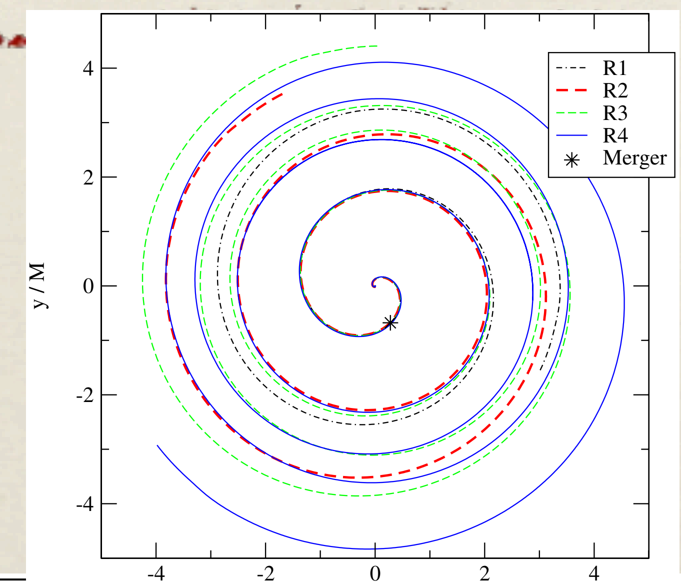
RING DOWN

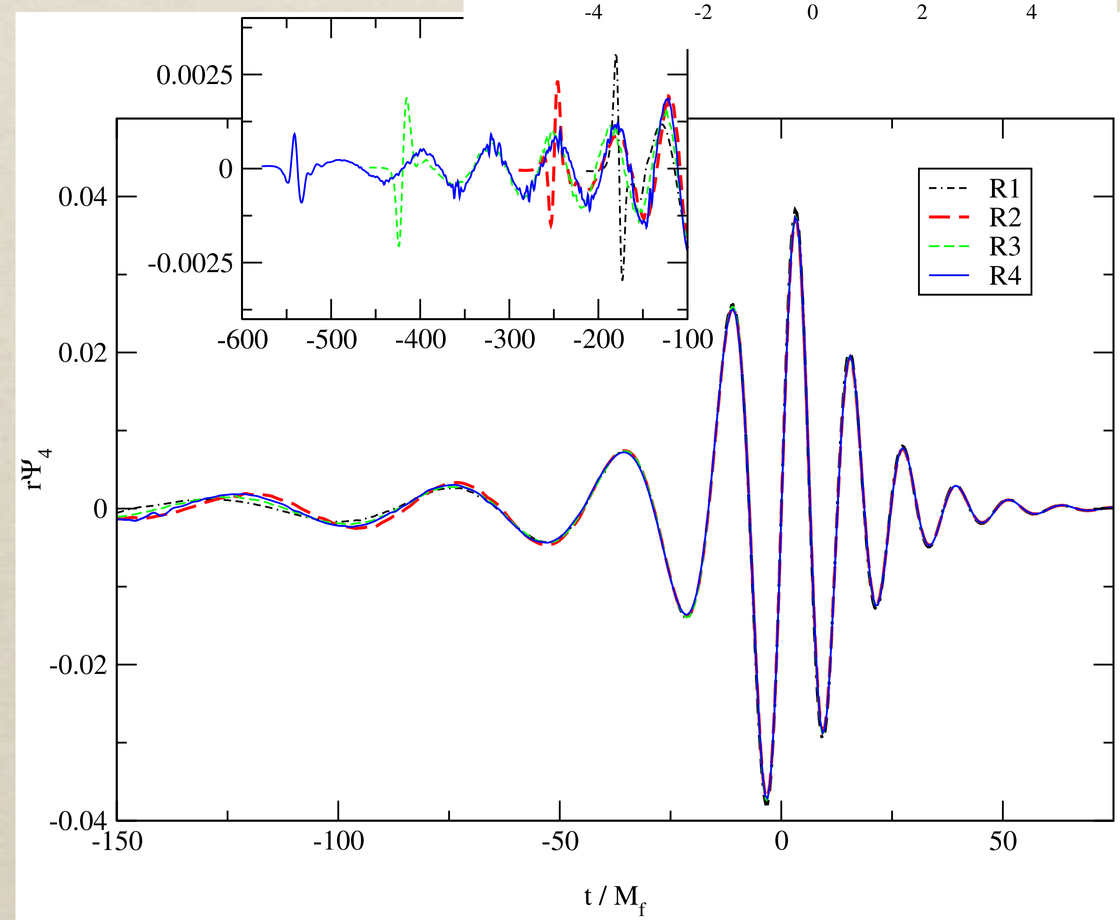


AEI simulation

arXiv:0712.3003

PHYSICAL REVIEW D **73,** 104002 (2006)



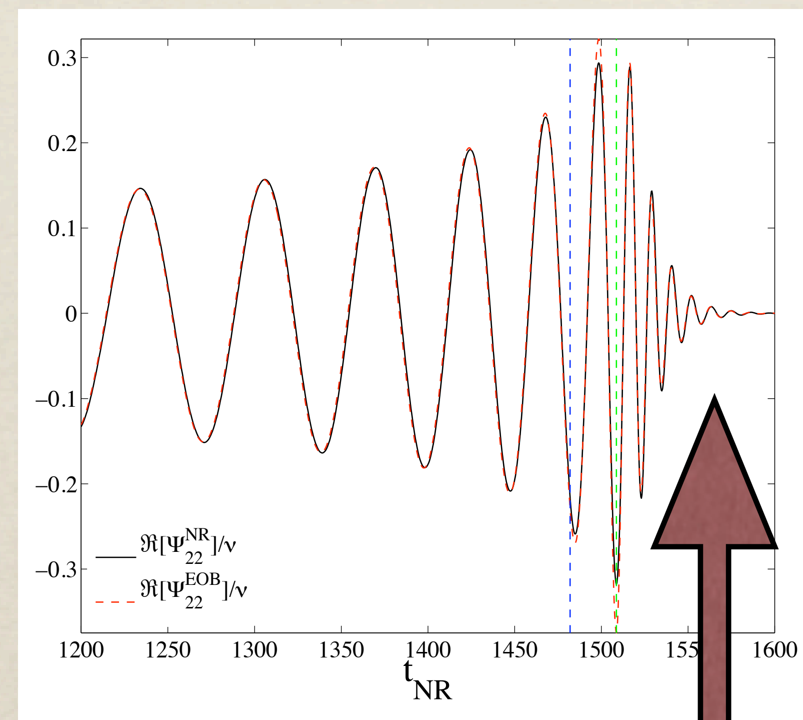


Exponentially dumped oscillation (QNM)

Inspiral part of the signal: .....

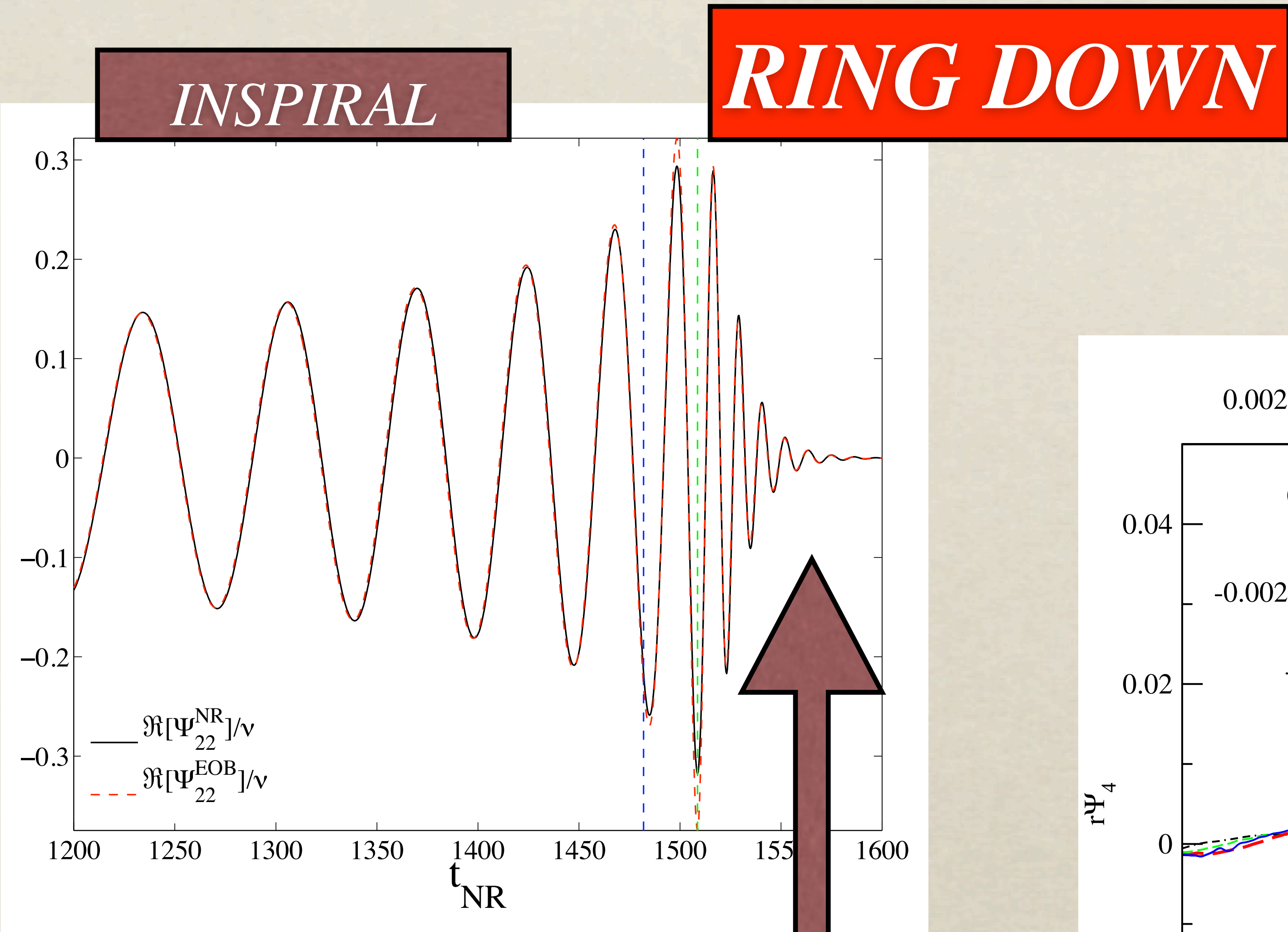
INSPIRAL | RING DOWN

- Post Newtonian approximation ....
- ... Damour EOB (Effective one body) waveforms for the two bodies problem.
- See Damour-Nagar about matching Numerical-Relativity waveform and EOB ones.

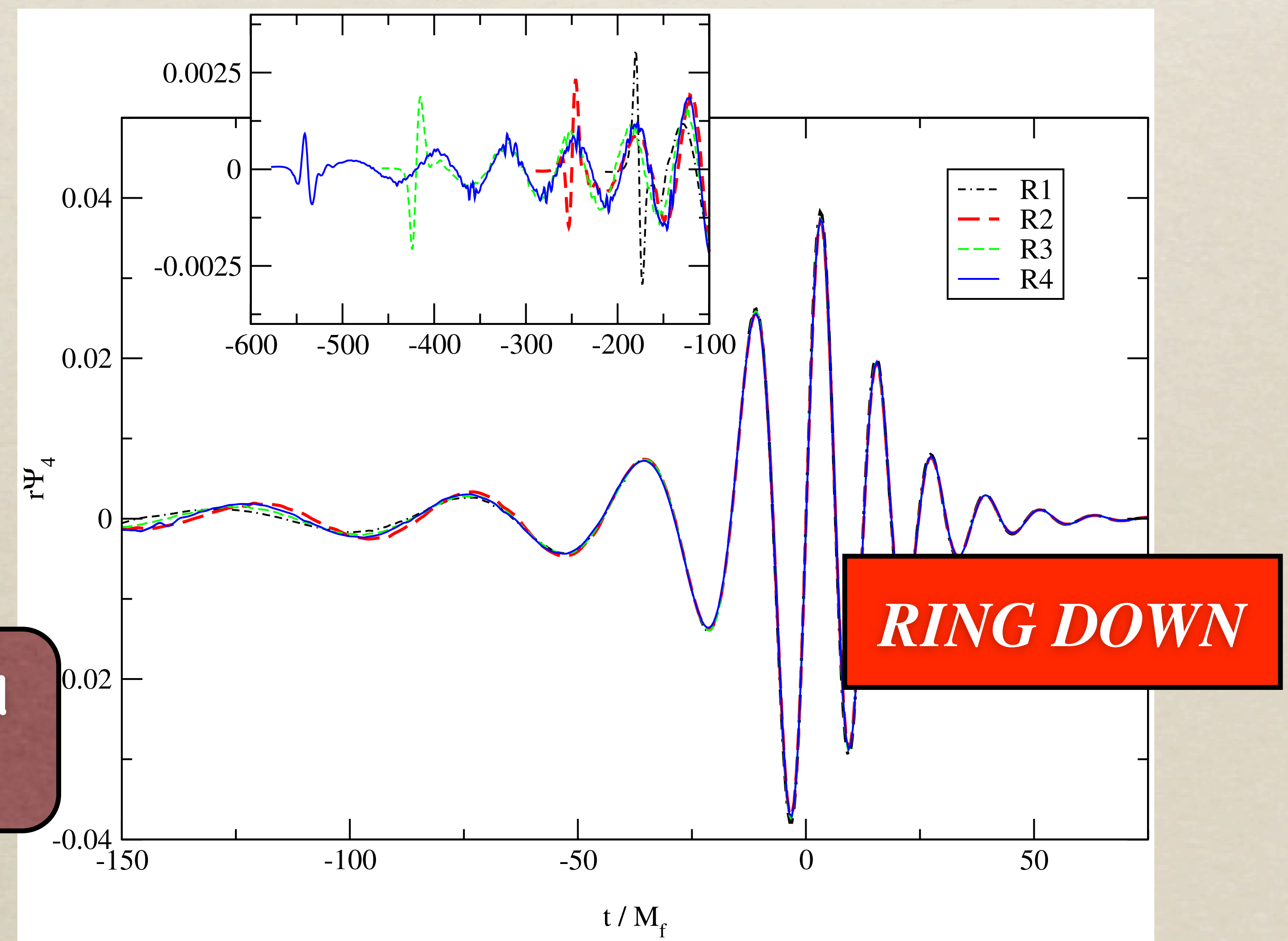


Exponentially dumped oscillation (QNM)

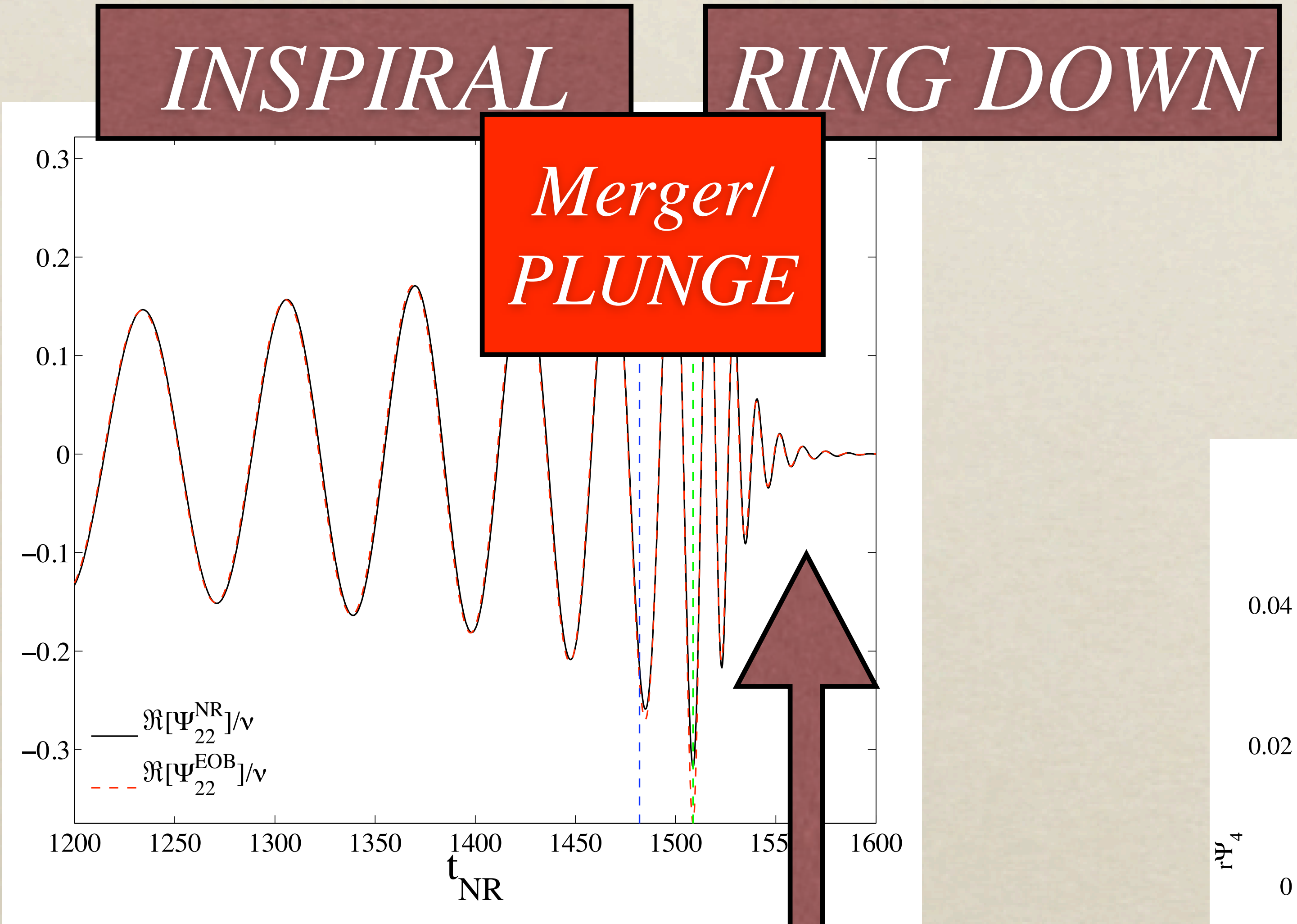
QNM part of the signal: 1D codes



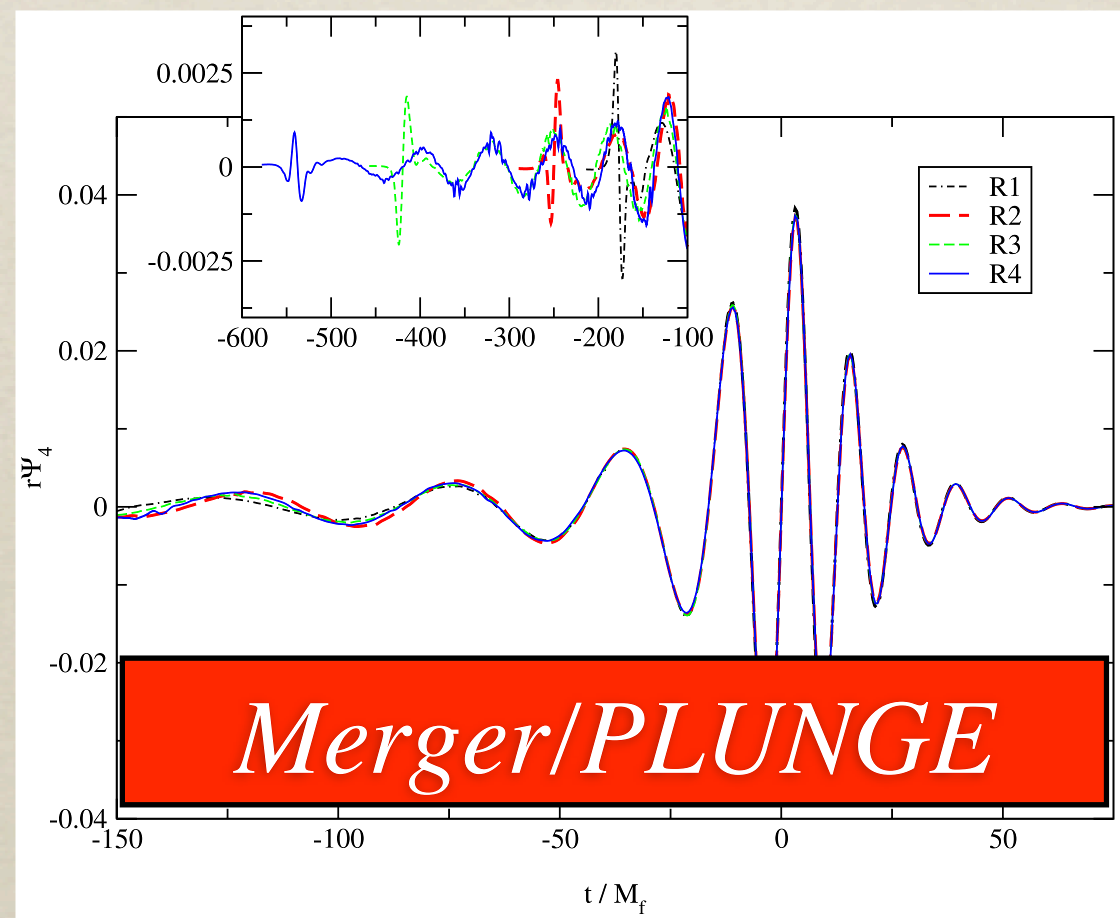
Exponentially dumped oscillation (QNM)



MERGER part of the signal: 3D codes



Exponentially dumped oscillation (QNM)



Numerical General Relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$ Einstein Equations

$$\nabla_\mu T^{\mu\nu} = 0$$ Conservation of energy momentum

$$\nabla_\mu(\rho\, u^\mu) = 0$$ Conservation of baryon density

$$p = p(\rho, \epsilon)$$ Equation of state

- Introduce a foliation of space-time
- write as a 3+1 evolution equation
- solve them on a computer !

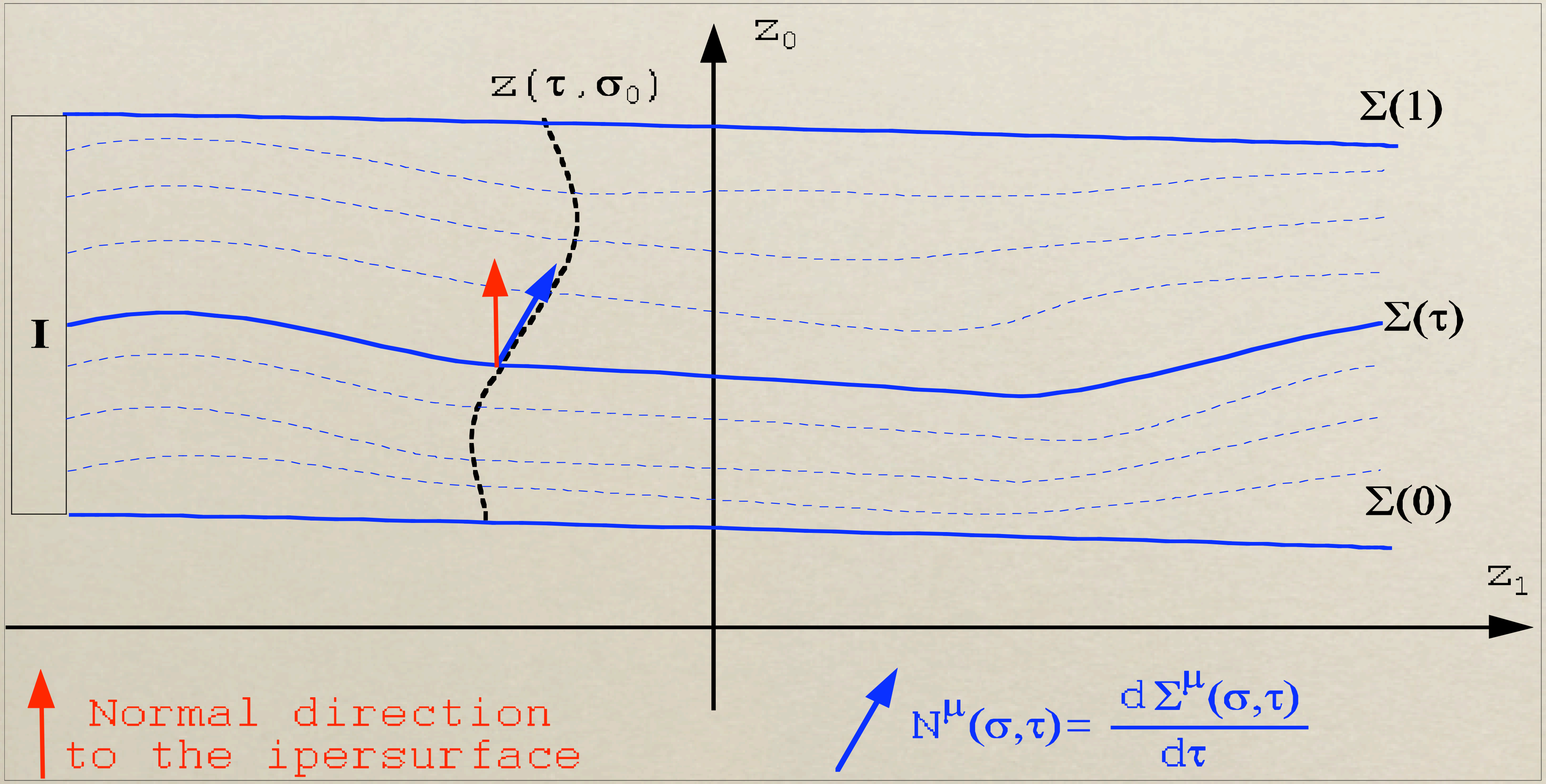
$$T^{\mu\nu} = (\rho(1+\epsilon) + p) u^\mu u^\nu + p g^{\mu\nu}$$

Why Numerical Relativity is hard!

- **No obviously "better" formulation of Einstein's equations**
  - ADM, conformal decomposition, first-order hyperbolic form,.... ???
- **Coordinates (spatial and time) do not have a special meaning**
  - this gauge freedom need to be carefully handled
  - gauge conditions must avoid singularities
  - gauge conditions must counteract "grid-stretching"
- **Einstein's Field equations are highly non-linear**
  - Essentially unknown in this regime
- **Physical singularity are difficult to deal with**

3+1 formulation

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$



α :: lapse

β^i :: shift vector

γ_{ij} :: 3-metric

$$N^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\alpha}\left(\frac{\partial}{\partial t} - \beta^j \frac{\partial}{\partial x^j}\right)$$

ADM evolution

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (2.1)$$

6 equations for the metric

$$\begin{aligned} \partial_t K_{ij} = & -\nabla_i \nabla_j \alpha + \alpha \Big[R_{ij} + K\, K_{ij} - 2K_{im} K^m_j \\ & -8\pi \left(S_{ij} - \frac{1}{2}\gamma_{ij} S \right) - 4\pi \rho_{\rm ADM} \gamma_{ij} \Big] \\ & +\beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m. \end{aligned} \quad (2.2)$$

+6 equations for the time-coordinate derivative of the metric (extrinsic curvature)

Hamiltonian + Momentum constraints

$${}^{(3)}R + K^2 - K_{ij}K^{ij} - 16\pi\rho_{\rm ADM} = 0$$

+1 constrain equation

$$\nabla_j K^{ij} - \gamma^{ij}\nabla_j K - 8\pi j^i = 0$$

+3 constrain equation

ADM evolution is not stable !

- **Use BSSN rewriting of the evolution equation**

$$\partial_t\varphi = -\tfrac{1}{6}\alpha K + \beta^i\partial_i\varphi + \tfrac{1}{6}\partial_i\beta^i$$

$$\partial_t K = -g^{ij}\nabla_i\nabla_j\alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \tfrac{1}{3}K) + \beta^i\partial_i K$$

$$\partial_t\tilde{g}_{ij} = -2\alpha K_{ij} + \tilde{g}_{jk}\partial_i\beta^k + \tilde{g}_{ik}\partial_j\beta^k - \tfrac{2}{3}\tilde{g}_{ij}\partial_k\beta^k$$

$$\partial_t\tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha(\Gamma^i_{jk}\tilde{A}^{jk} - \tfrac{2}{3}\tilde{g}^{ij}\partial_j K + 6\tilde{A}^{ij}\partial_j\varphi) +$$

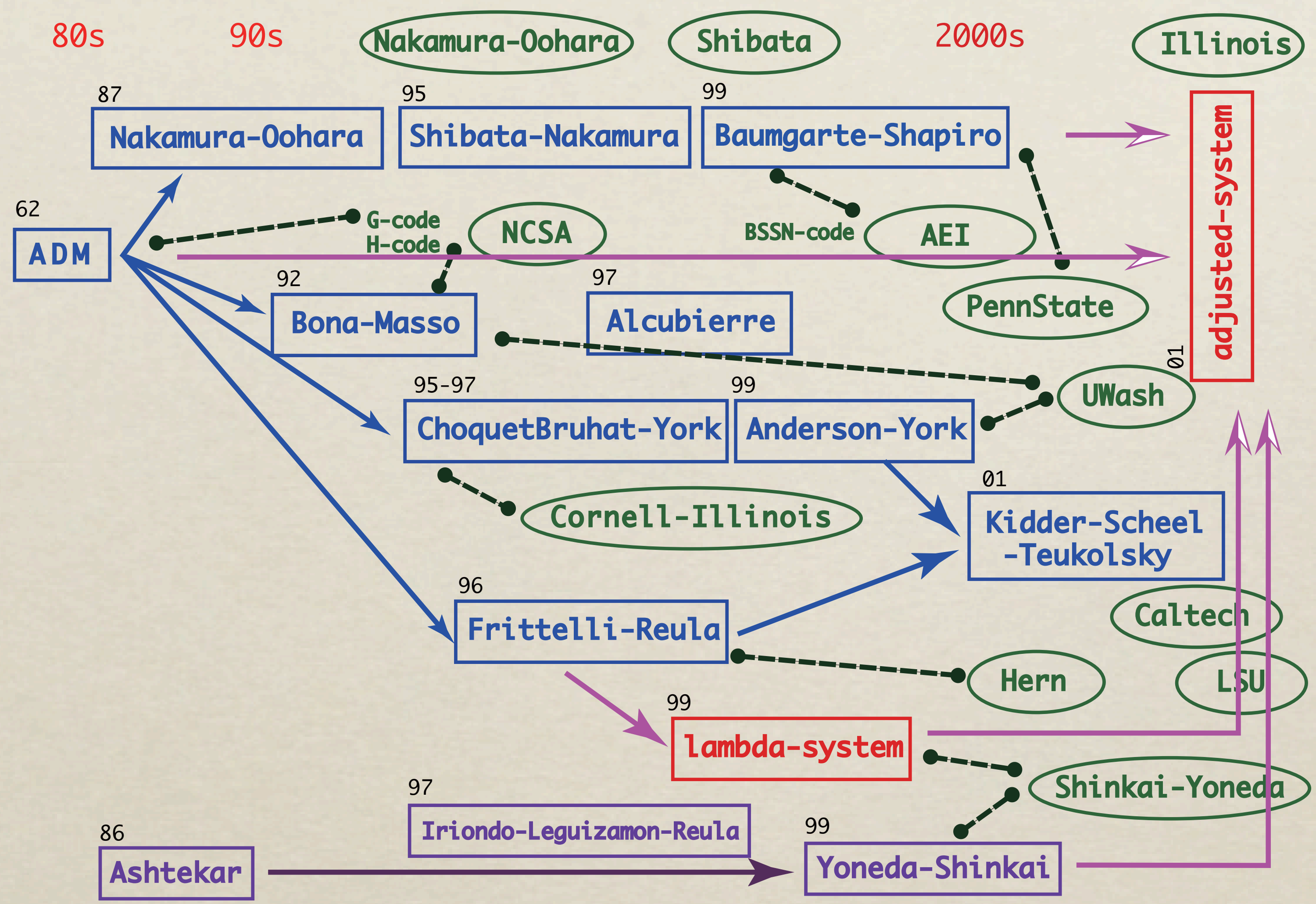
$$+\beta^k\partial_k\tilde{\Gamma}^i - \tilde{\Gamma}^k\partial_k\beta^i + \tfrac{2}{3}\tilde{\Gamma}^i\partial_k\beta^k + \tfrac{1}{3}\tilde{g}^{ij}\partial_j\partial_k\beta^k + \tilde{g}^{jk}\partial_j\partial_k\beta^i$$

$$\partial_t\tilde{A}_{ij} = e^{-4\varphi}(-(\nabla_i\nabla_j\alpha)^{TF} + \alpha R^{TF}_{ij}) + \alpha\left(\tilde{A}_{ij}K - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) - \partial_i\partial_j\alpha +$$

$$+\beta^k\partial_k\tilde{A}_{ij} + \left(\tilde{A}_{ik}\partial_j + \tilde{A}_{jk}\partial_i\right)\beta^\kappa - \tfrac{2}{3}\tilde{A}_{ij}\partial_k\beta^k$$

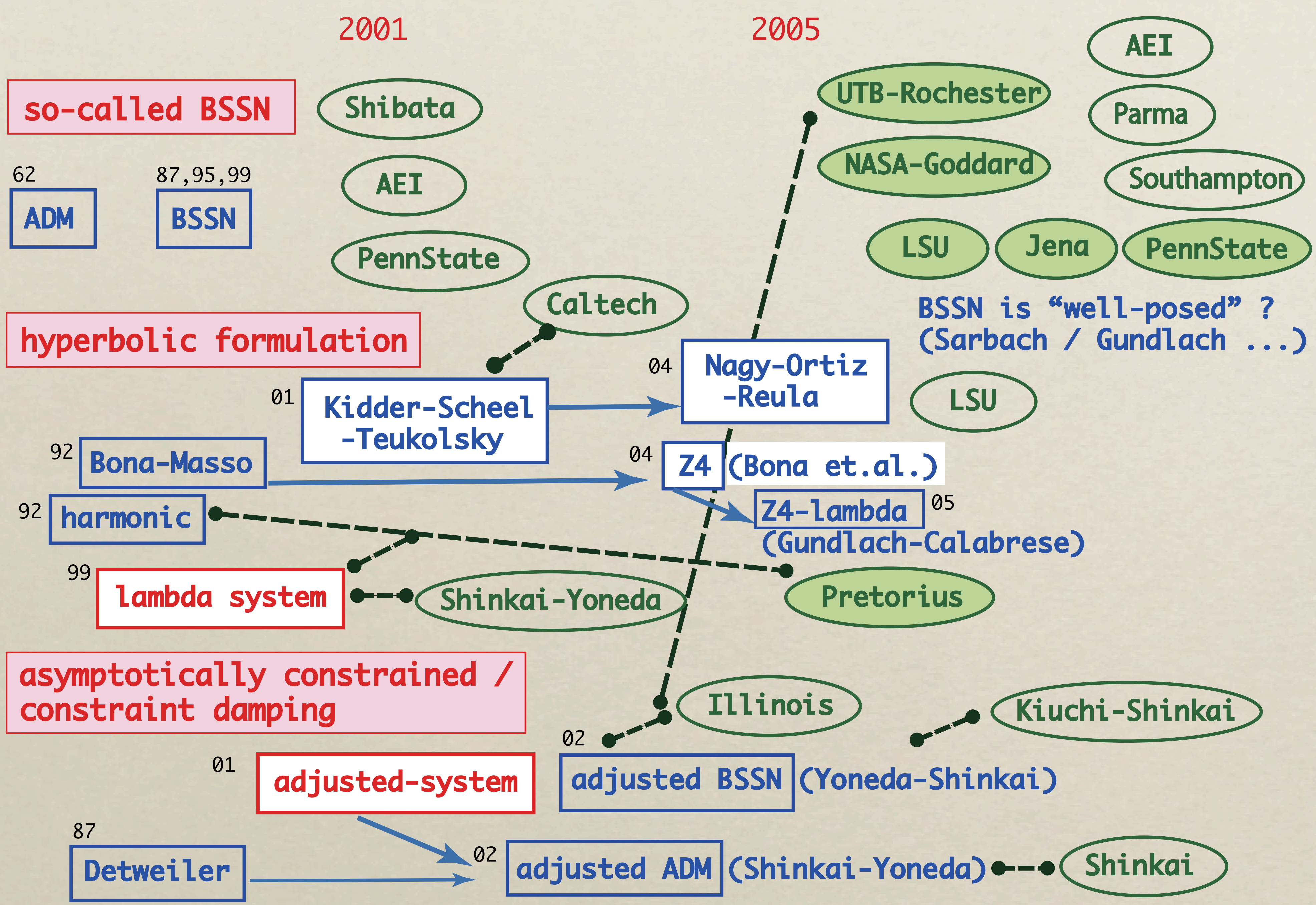
- **or Use Harmonic evolution equations**

Other schemes (beside BSSN)

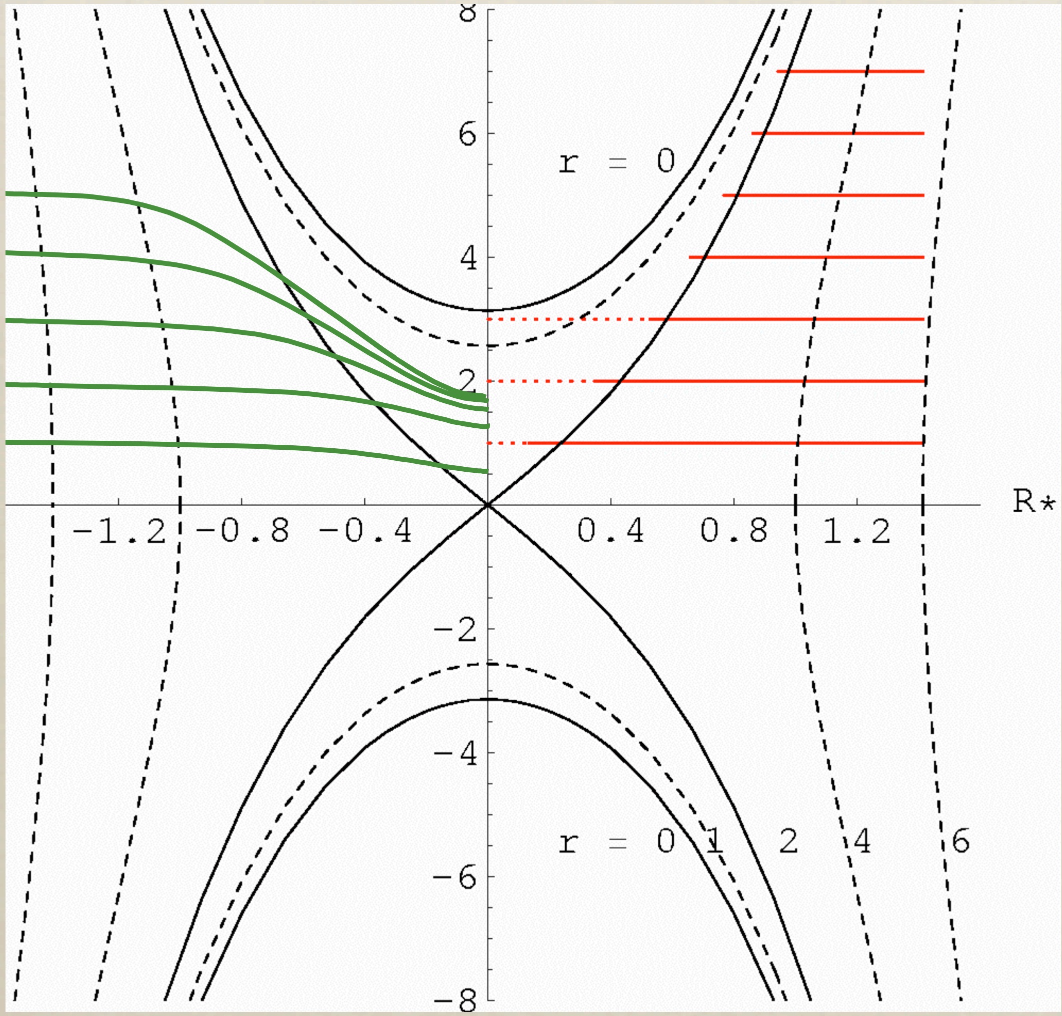


- See Hisa-aki Shinka, Formulations of the Einstein equations for numerical simulations, arXiv:0805.0068 for a review.

Situation NOW: from 0805.0068



The problem of foliations.....



- Schwarzschild in Novikov Coordinates
- Geodesic slicing $(\alpha = 1,\ \beta^i = 0)$
- Singularity avoiding $(\beta^i = 0)$
- excision/puncture evolution

$$\text{1+log} \qquad \partial_t \alpha = -2\alpha(K - K_0)$$

$$\text{Gamma-driver} \qquad \partial_t^2 \beta^i = \frac{3}{4}\alpha \partial_t \tilde{\Gamma}^i - 2\partial_t \beta^i$$

Code Used

- **CACTUS/BSSN:** (www.cactuscode.org)
  Mainly developed at AEI (Golm, Germany) and LSU (USA)

- **WHISKY:** (http://www.aei-potsdam.mpg.de/~hawke/Whisky.html)

  Whisky is a code to evolve the equations of hydrodynamics on curved space. It is being written by and for members of the EU Network on Sources of Gravitational Radiation and is based on the Cactus Computational Toolkit.

- **Gauge choice** for the lapse and shift variables:

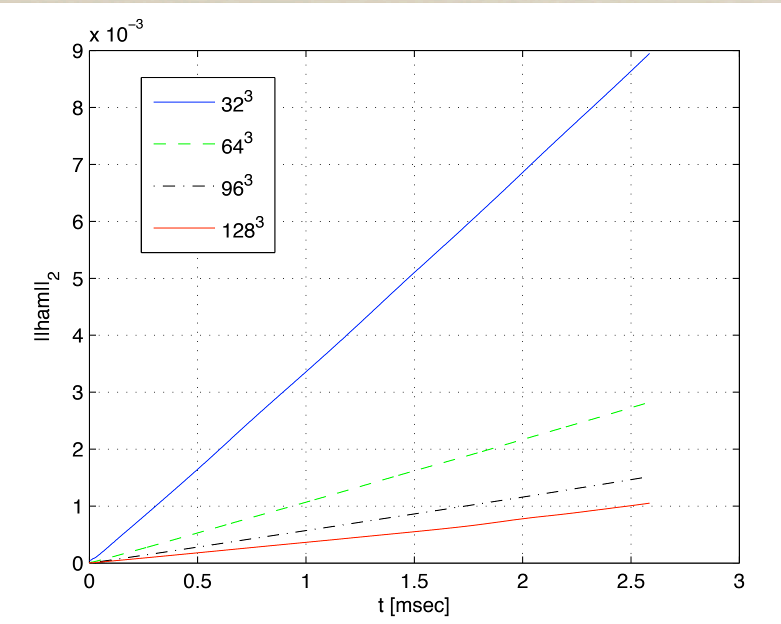
$$\text{1+log} \qquad \partial_t \alpha = -2\alpha(K - K_0)$$

$$\text{Gamma-driver} \qquad \partial_t^2 \beta^i = \frac{3}{4}\alpha \partial_t \tilde{\Gamma}^i - 2\partial_t \beta^i$$

Cactus ⇨ Infrastructure + GR

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \tag{2.1}$$

$$\begin{aligned}\partial_t K_{ij} = & -\nabla_i \nabla_j \alpha + \alpha \Big[R_{ij} + K\,K_{ij} - 2K_{im}K^m_j \\ & -8\pi \left(S_{ij} - \frac{1}{2}\gamma_{ij} S \right) - 4\pi\rho_{\rm ADM}\gamma_{ij} \Big] \\ & +\beta^m \nabla_m K_{ij} + K_{im}\nabla_j \beta^m + K_{mj}\nabla_i \beta^m .\end{aligned} \tag{2.2}$$



Hamiltonian + Momentum constraints

$$^{(3)}R + K^2 - K_{ij}K^{ij} - 16\pi\rho_{\rm ADM} = 0$$

$$\nabla_j K^{ij} - \gamma^{ij}\nabla_j K - 8\pi j^i = 0$$

WHISKY ⇨ Matter evolution

Write hydrodynamic equation in a flux conservative form [*] J. A. Font, Living Rev. Relativity 6, 4 (2003).

Use HRSC methods to solve the equations

$$\nabla_\mu T^{\mu\nu} = 0\,,$$
$$\nabla_\mu(\rho u^\mu) = 0\,.$$

$$\partial_t \mathbf{q} + \partial_i \mathbf{f}^{(i)}(\mathbf{q}) = \mathbf{s}(\mathbf{q})$$

$$\mathbf{q} \equiv (D, S^i, \tau)$$

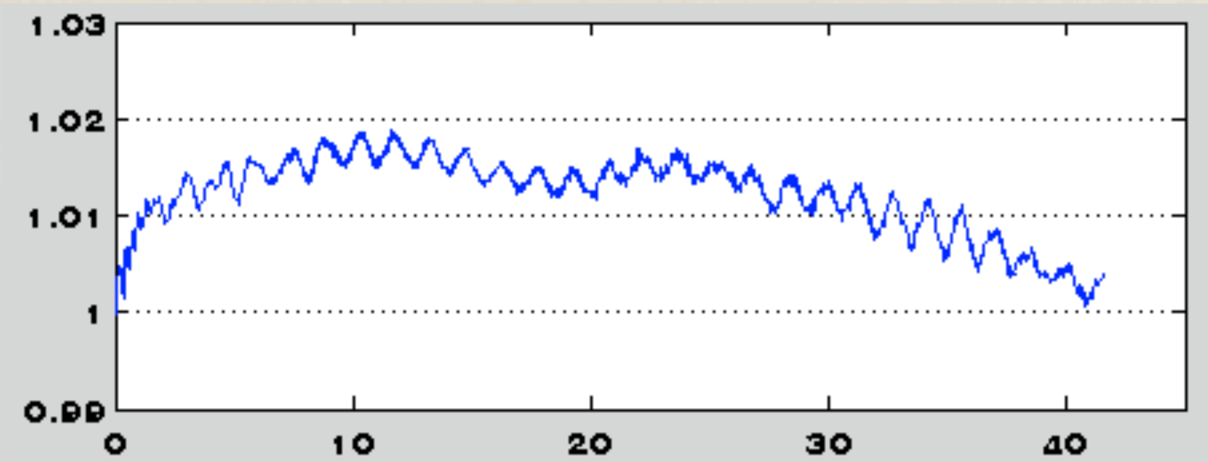
$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

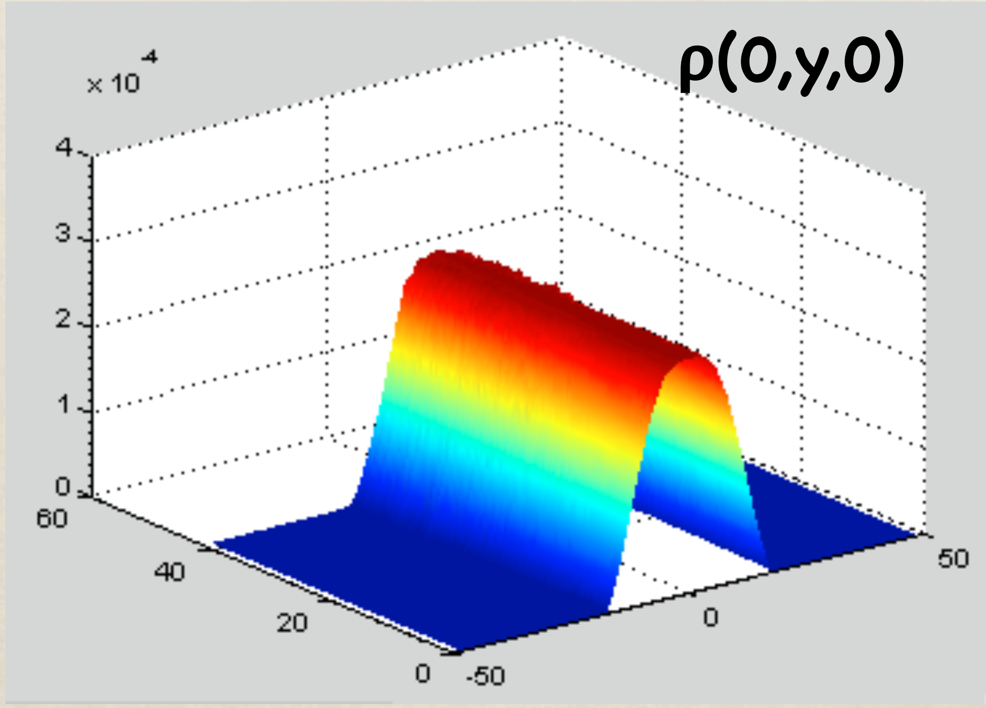
$$h = 1 + \varepsilon + \frac{p}{\rho}$$

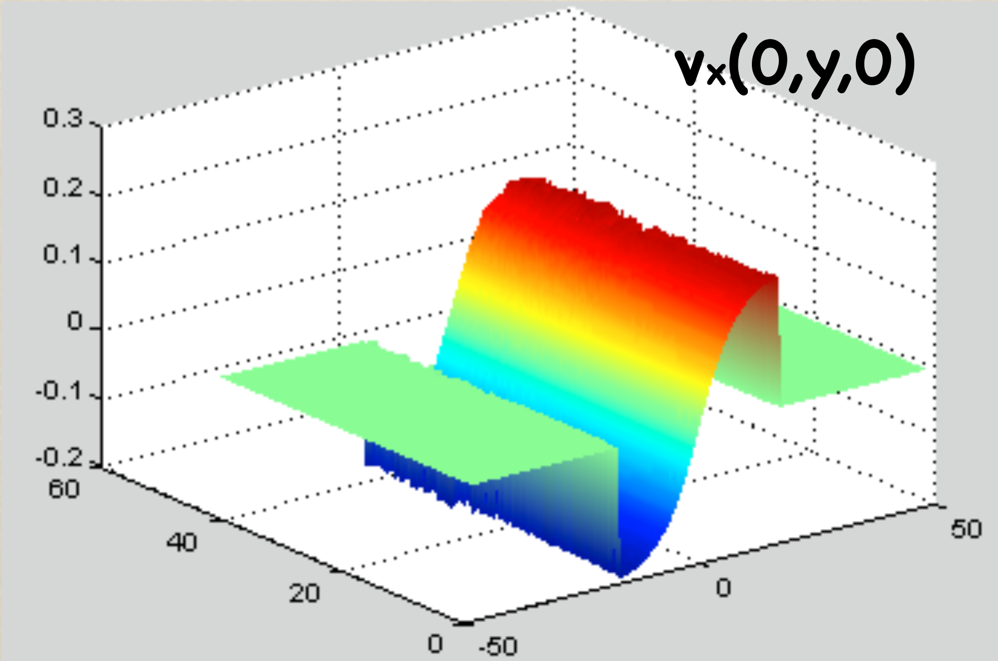
$$D \equiv \rho^* = \sqrt{\gamma} W \rho\,,$$
$$S^i \equiv \sqrt{\gamma}\rho h W^2 v^i\,,$$
$$\tau \equiv \sqrt{\gamma}\left(\rho h W^2 - p\right) - D$$

Stable evolutions of stable star!

$\rho_c(t)/\rho_c(0)$ as function of t for stable model A10







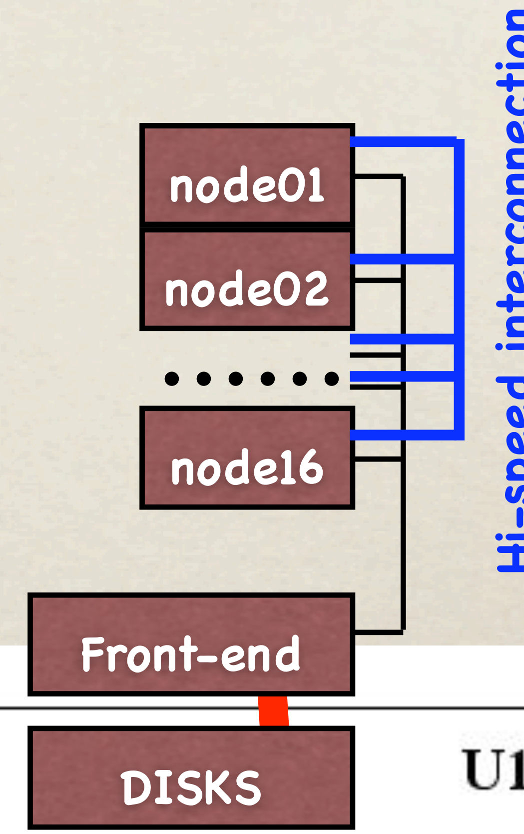
	β	Full GR	CFC [1]
A9	0.189	791 Hz	809 Hz
A10	0.223	674 Hz	685 Hz

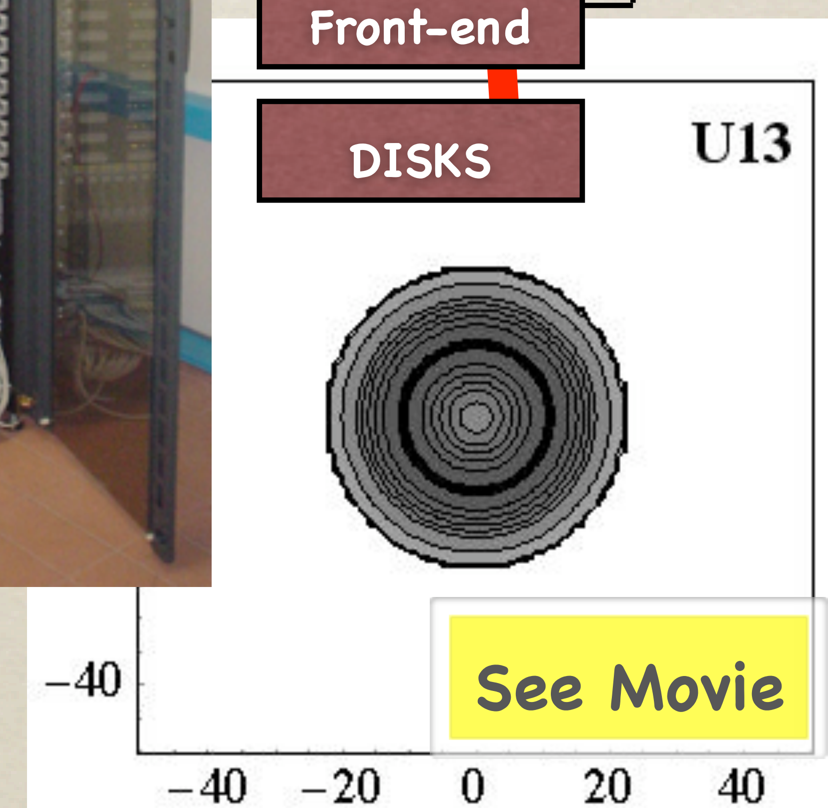
[1] Dimmelmeier, Stergioulas, Font: astro-ph/0511394

Computers for Numerical Relativity

- standard workstation nodes: e.g., biprocessor Opteron/Intel with 4-8 GBytes of RAM
- Fast interconnection, e.g., Infiniband
- A front-end workstation
- MPI communication Library
- Huge storage space to save results of the simulations

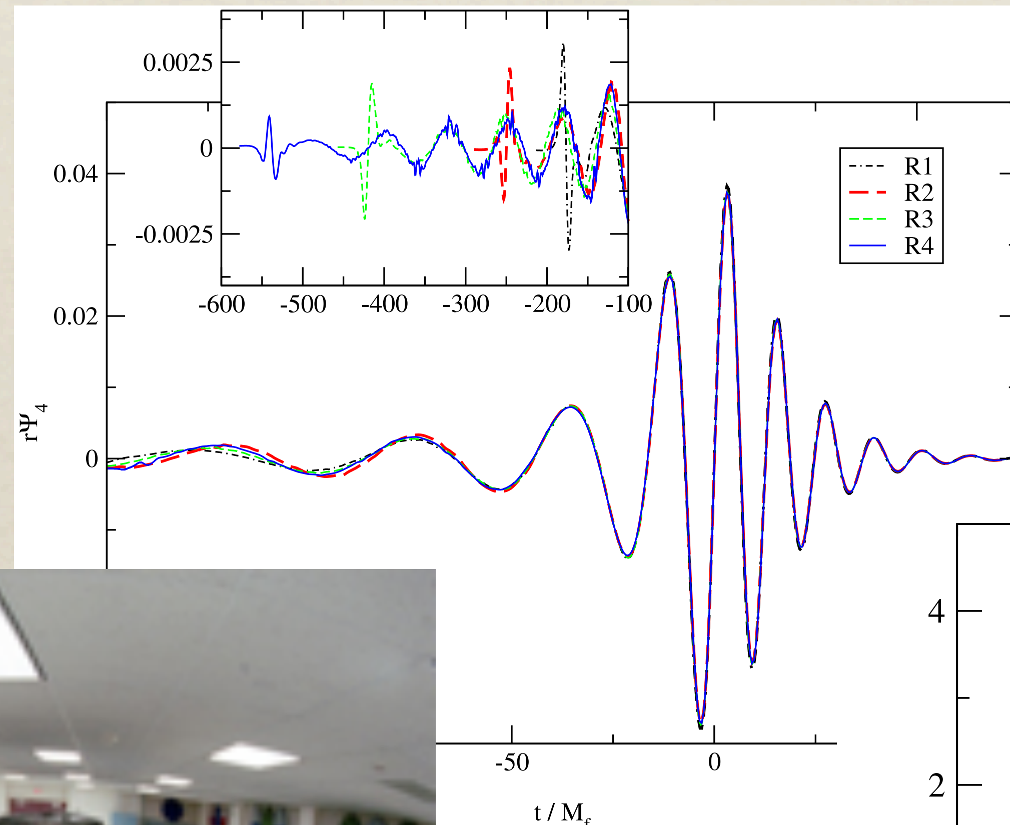
- WE NEED A LOT OT MEMORY !



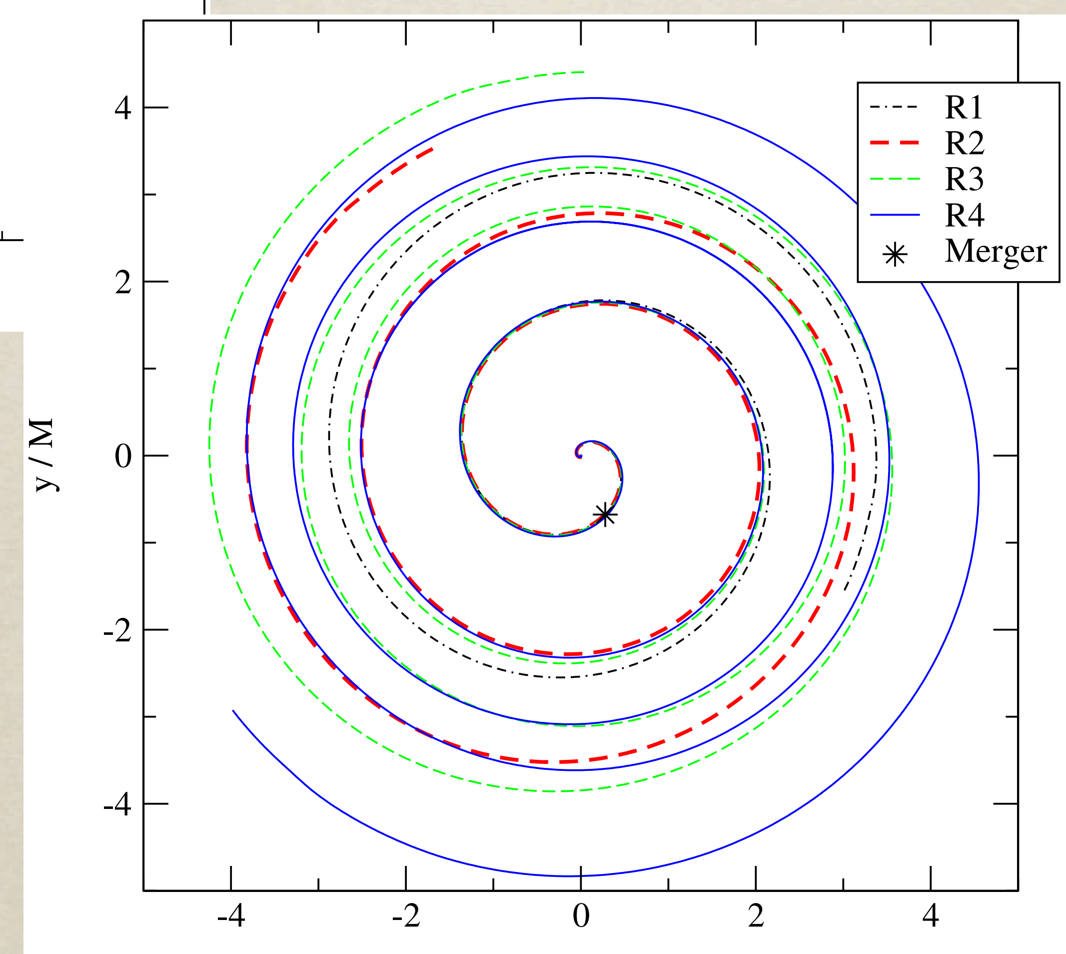


Computers for Numerical Relativity

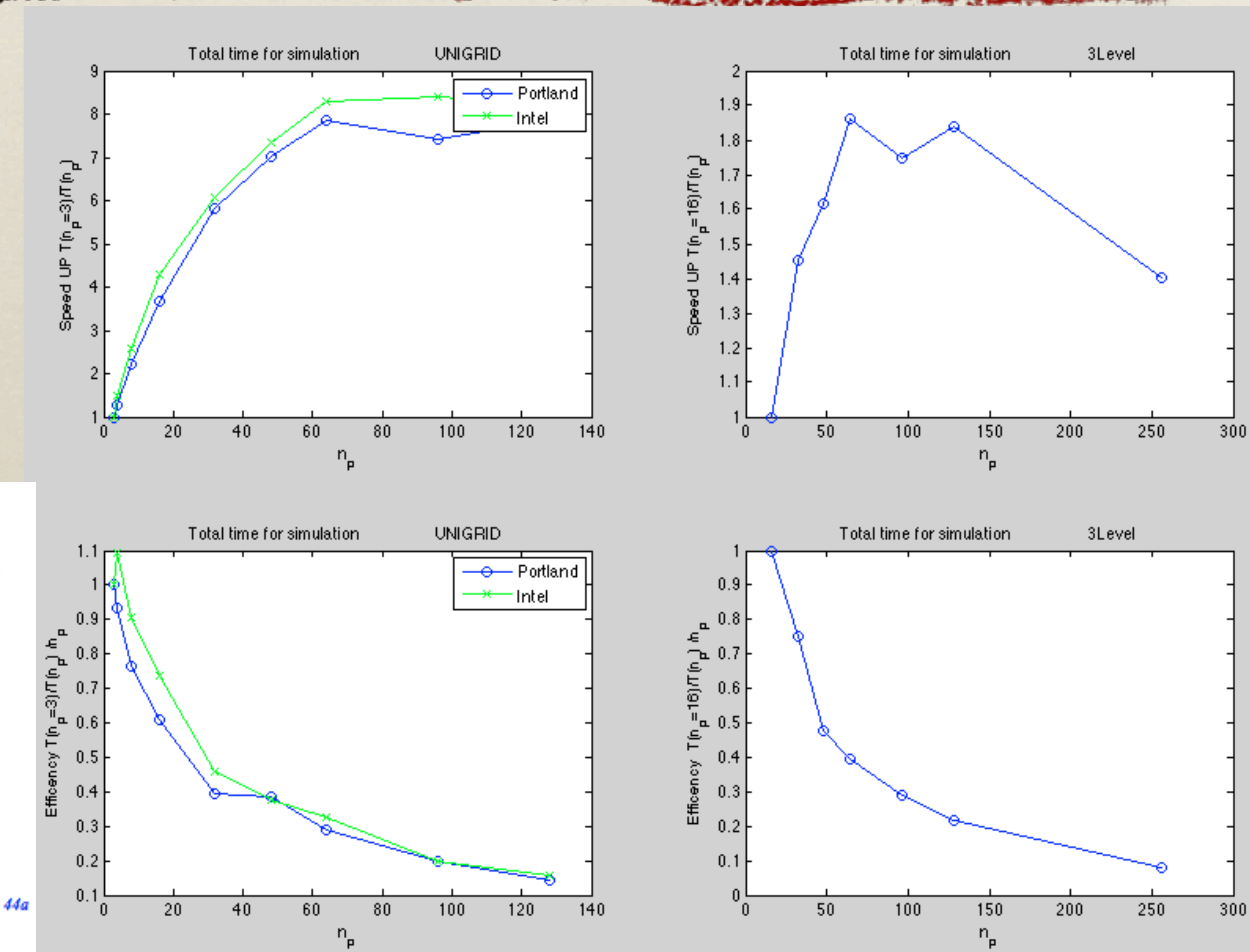
NASA Goddard (2005)



PHYSICAL REVIEW D **73,** 104002 (2006)



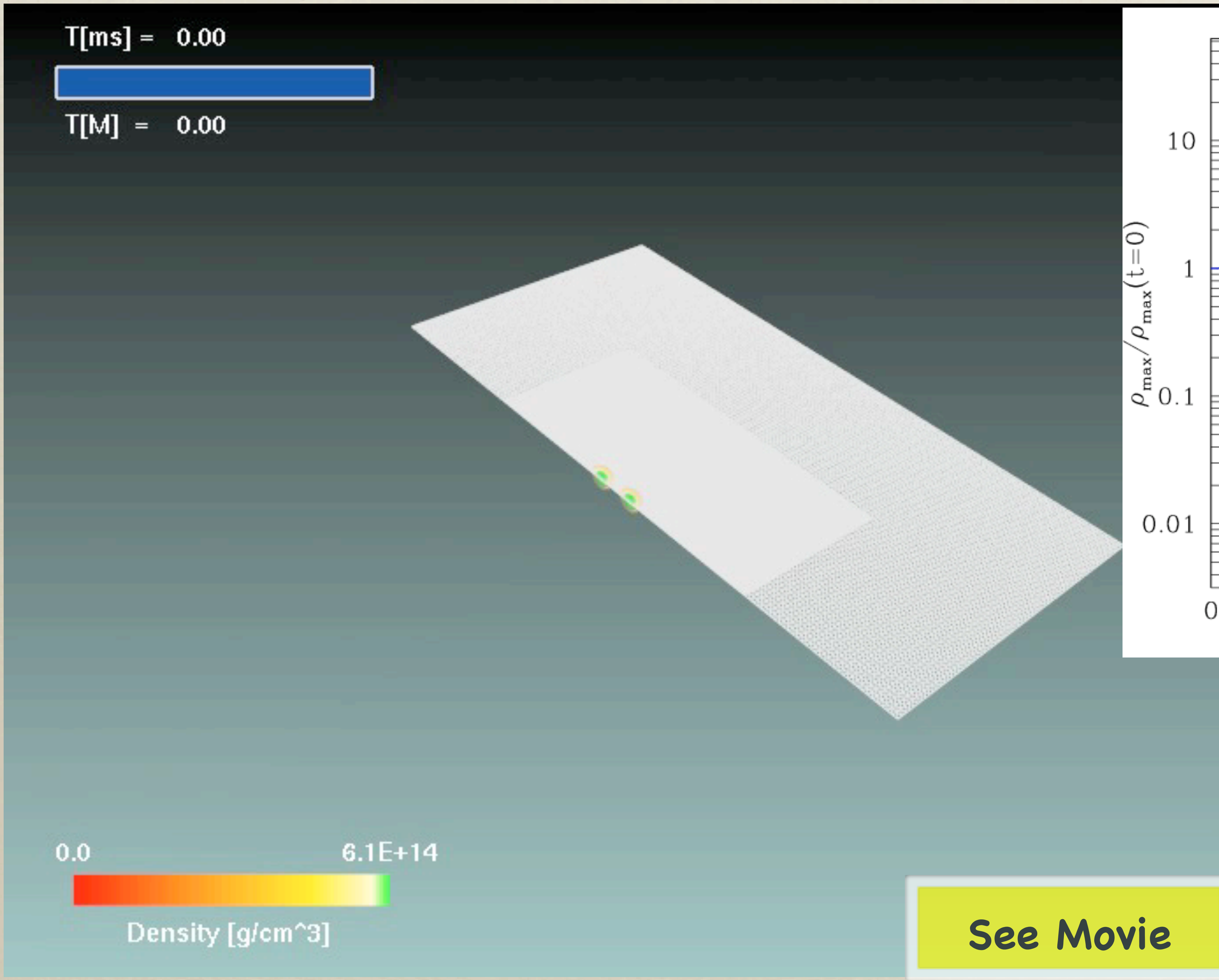
Code scaling on MPI clusters

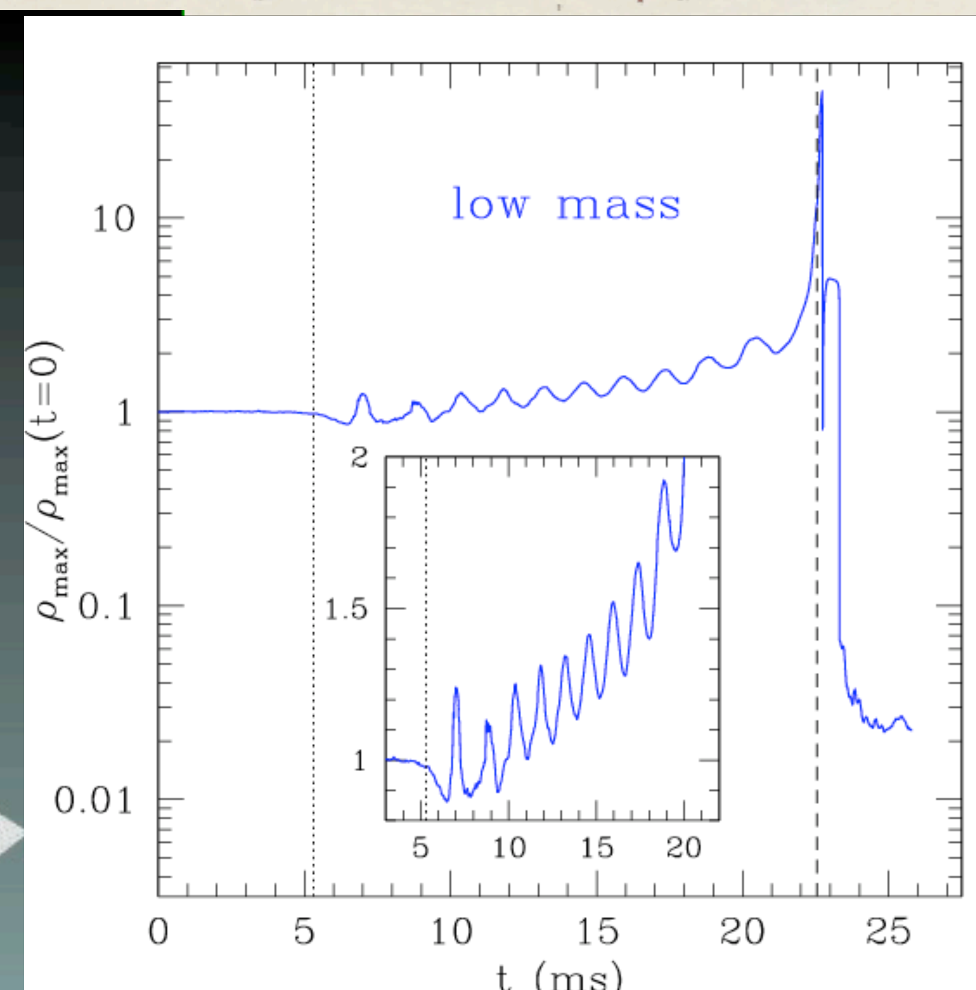


IBM BCX/5120, con 5120 processori
Lo scorso novembre, il sistema ha portato il Cineca alla 44a posizione nella prestigiosa lista TOP500 che raccoglie i server di calcolo più potenti al mondo.

Numerical relativity at work

Neutron star merger: low-mass merger to NS + disk





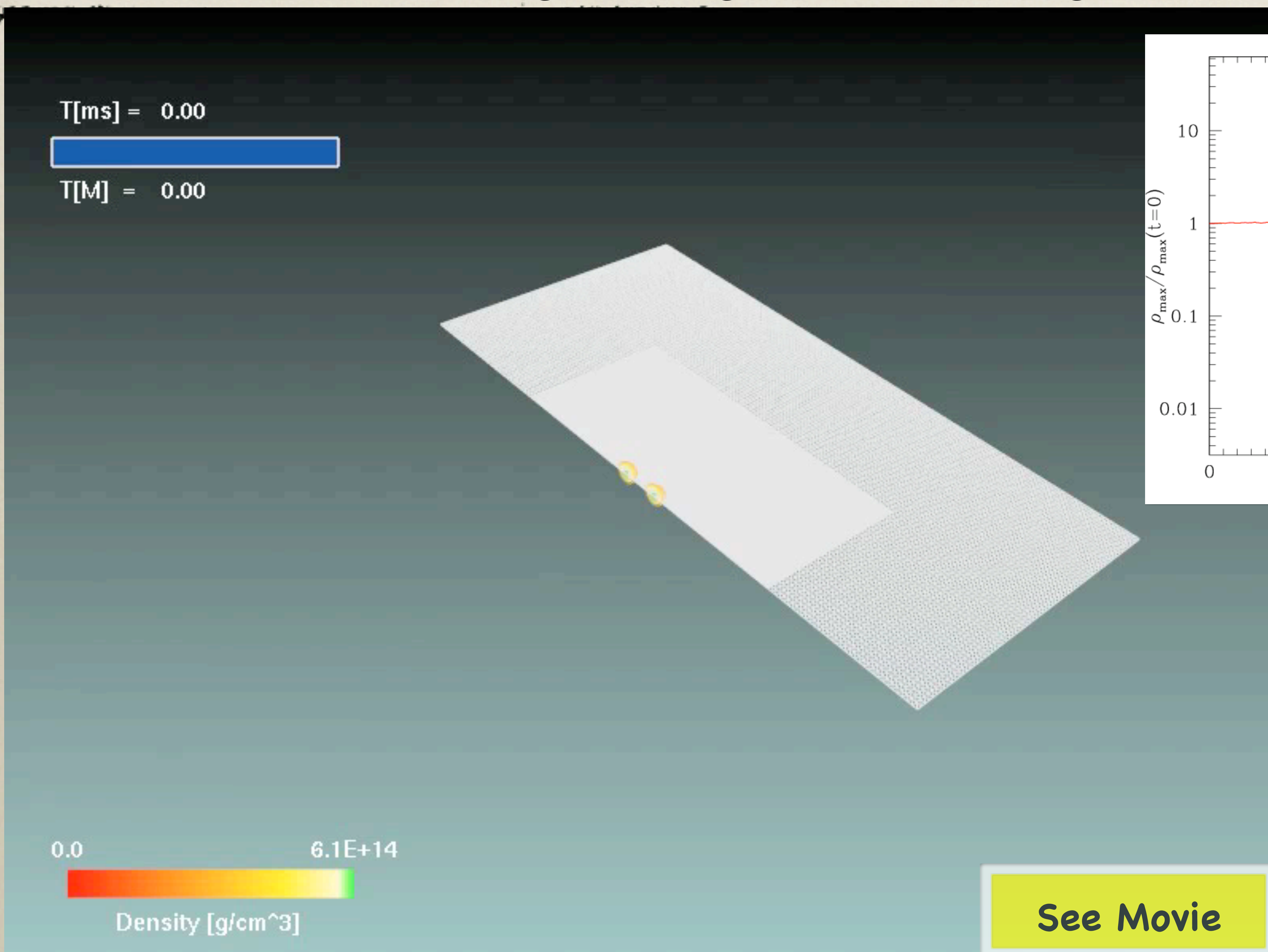
See Movie

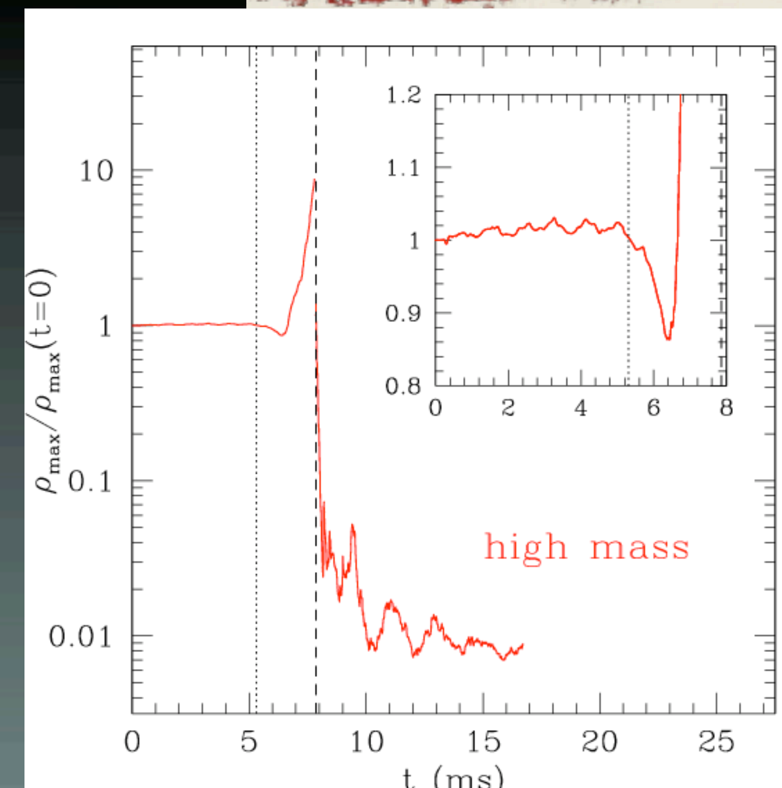
Credits: R. Kaehler & B. Giacommazzo & L. Rezzolla

http://arxiv.org/pdf/0804.0594

Numerical relativity at work

Neutron star merger: high-mass merger to BH + disk





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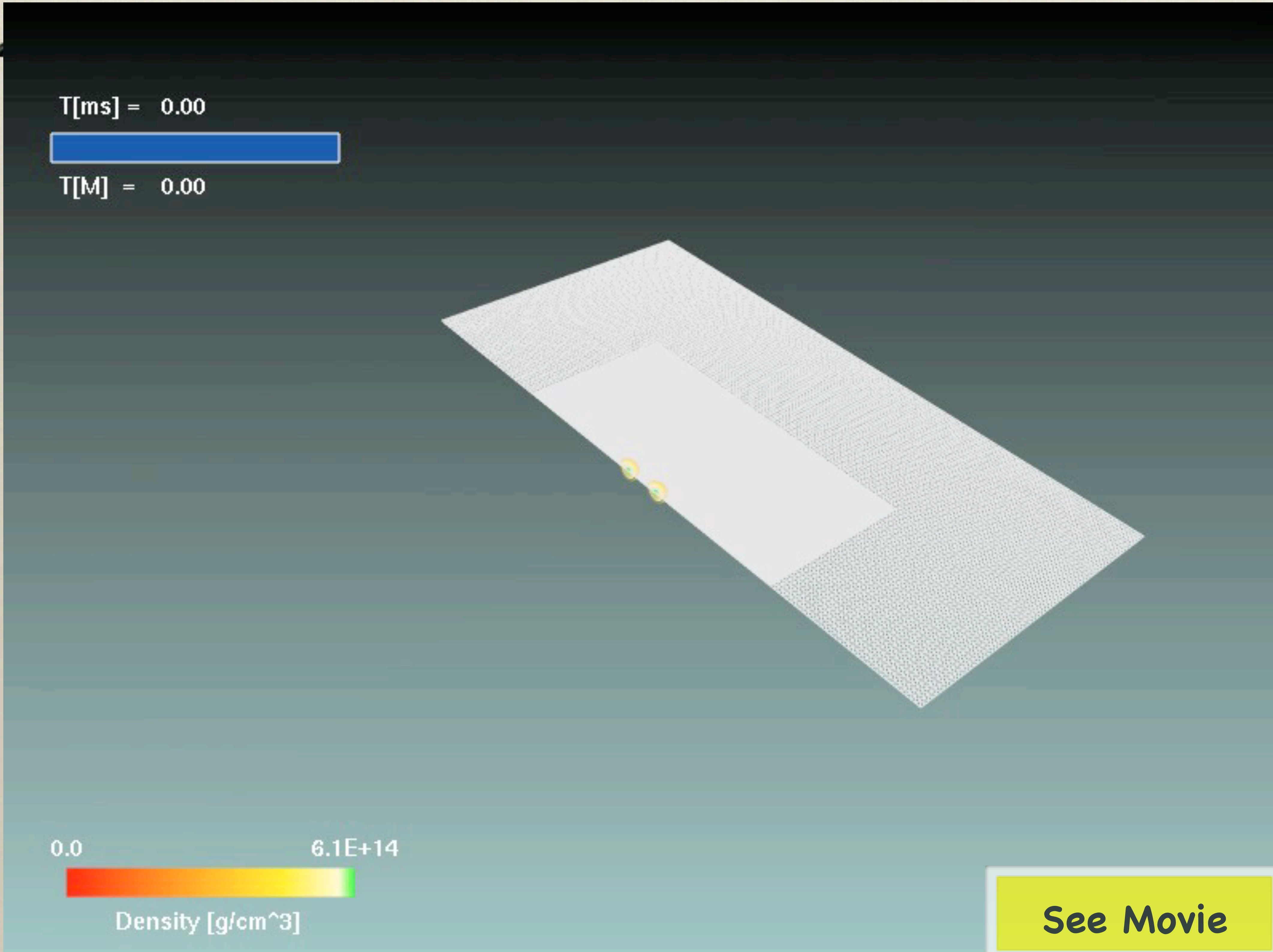


(Ideal Fluid EOS)

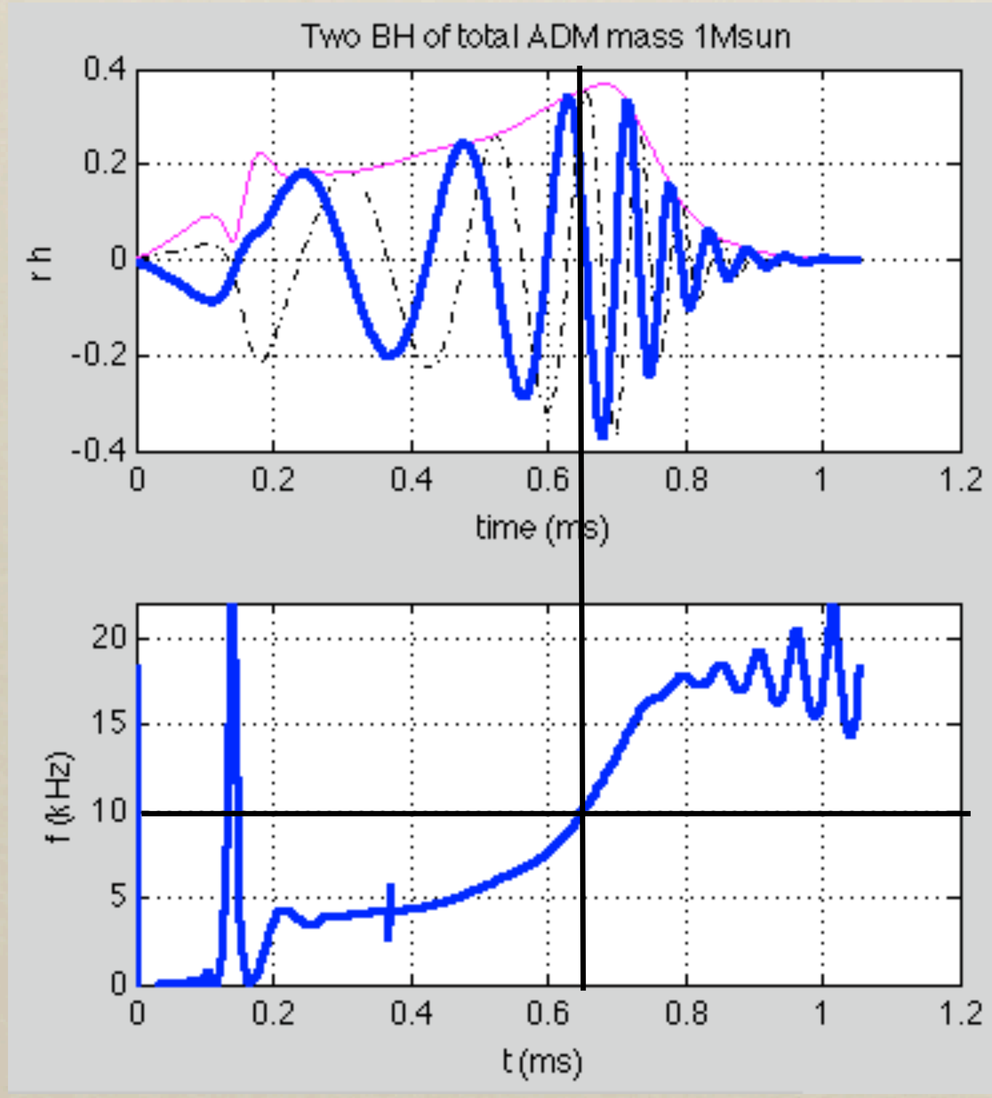
Credits: R. Kaehler & B. Giacommazzo & L. Rezzolla

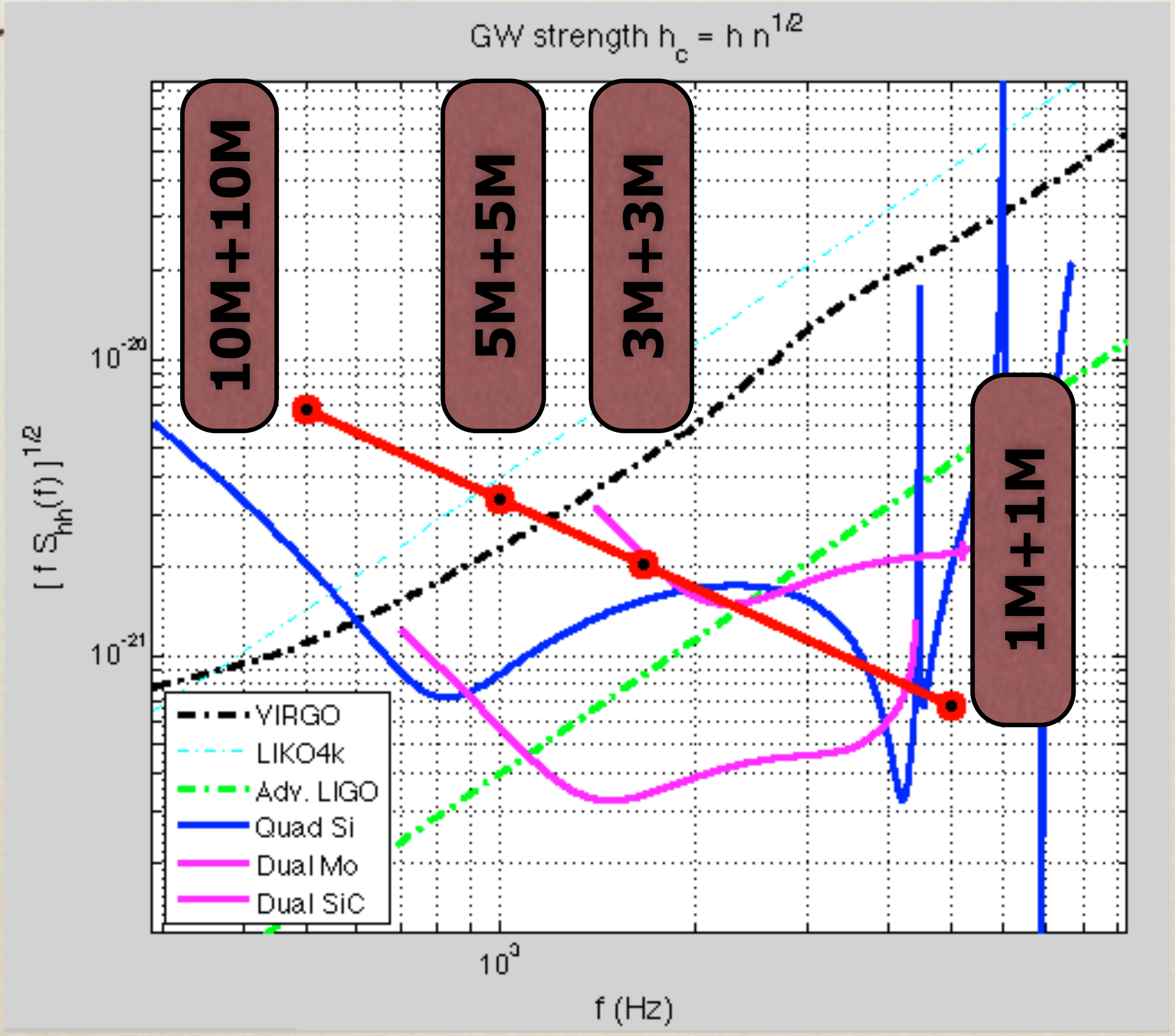
http://arxiv.org/pdf/0804.0594

Different EOS

SNR BH-BH @ 100Mpc

At 100 Mpc to be scaled by:
$$\sim 10^{-21} M/M_\odot$$

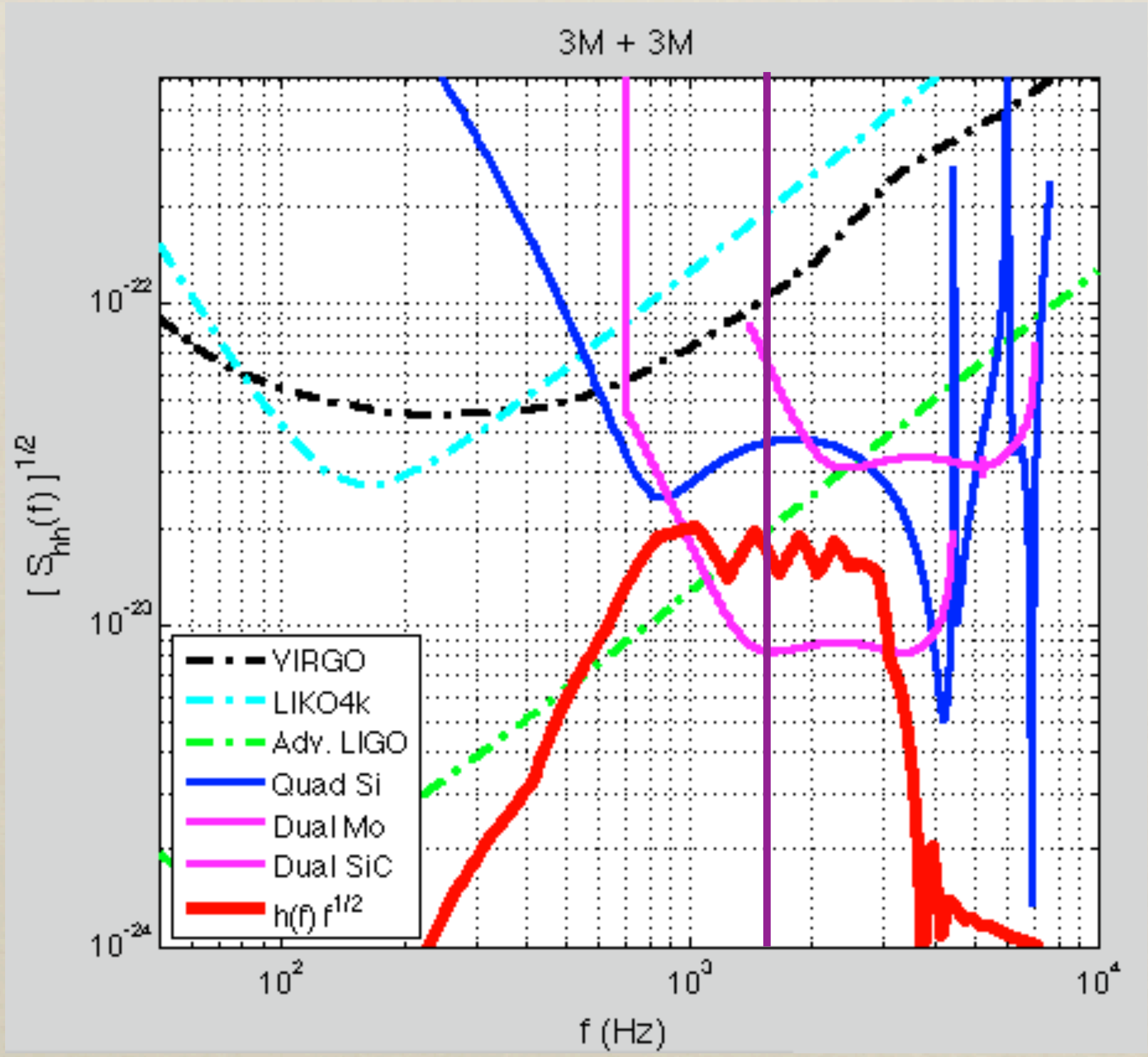




f to be scaled by: $0.5 M_\odot / M$

PHYSICAL REVIEW D 73, 061501(R) (2006) "Last orbit of binary black holes" M. Campanelli, C. O. Lousto, and Y. Zlochower

SNR 3M+3M @ 100Mpc



$$SNR^2 = 4 \int df \; \frac{|\tilde{h}(f)|^2}{S_{hh}(f)}$$

SNR Dual SiC=1.4

SNR QUAD Si=3.9

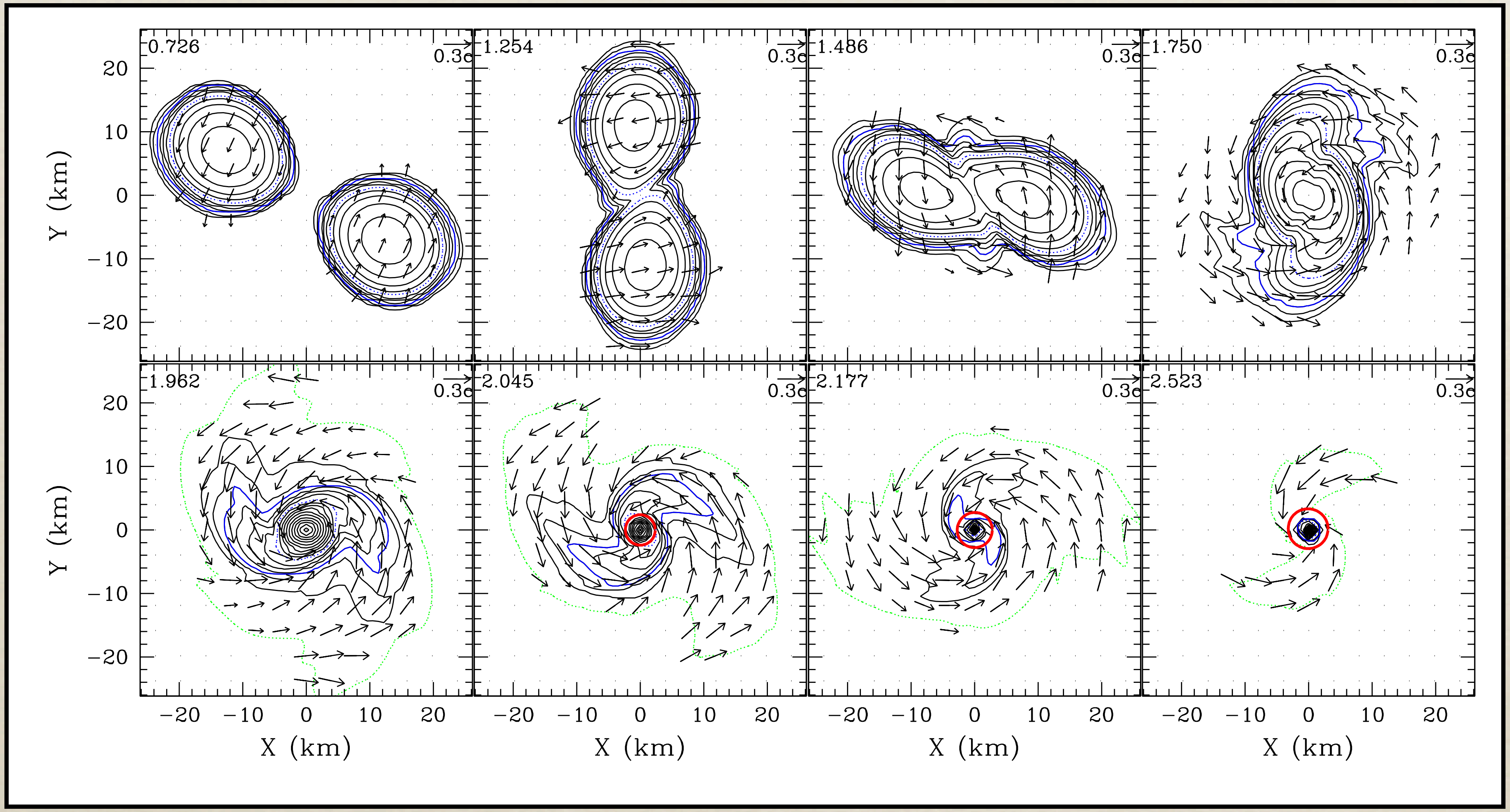
Two bodies merging of NS-NS

- Shibata et all., Phys.Rev. D71,084021 (2005)
- Shibata-Taniguchi, Phys.Rev. D73, 064027 (2006)

Model	$M_\infty (M_\odot)$
APR1313	1.30, 1.30
APR1214	1.20, 1.40
APR135135	1.35, 1.35
APR1414	1.40, 1.40
APR1515	1.50, 1.50
APR145155	1.45, 1.55
APR1416	1.40, 1.60
APR135165	1.35, 1.65
APR1317	1.30, 1.70
APR125175	1.25, 1.75
APR1218	1.20, 1.80
SLy1313	1.30, 1.30
SLy1414	1.40, 1.40
SLy135145	1.35, 1.45
SLy1315	1.30, 1.50
SLy125155	1.25, 1.55
SLy1216	1.20, 1.60

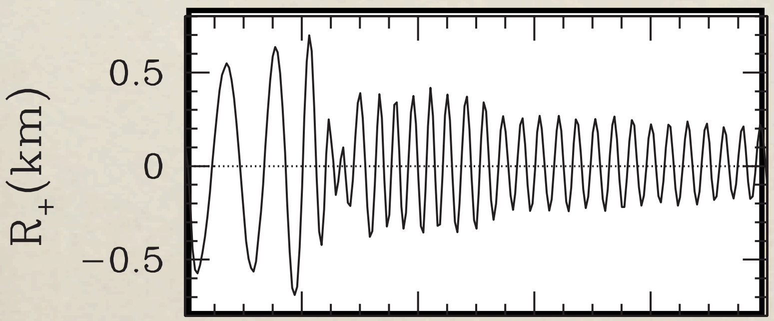
$$h_{\mathrm{gw}} \approx 10^{-22}\left(\frac{\sqrt{R_+^2 + R_\times^2}}{0.31\ \mathrm{km}}\right)\left(\frac{100\ \mathrm{Mpc}}{r}\right).$$

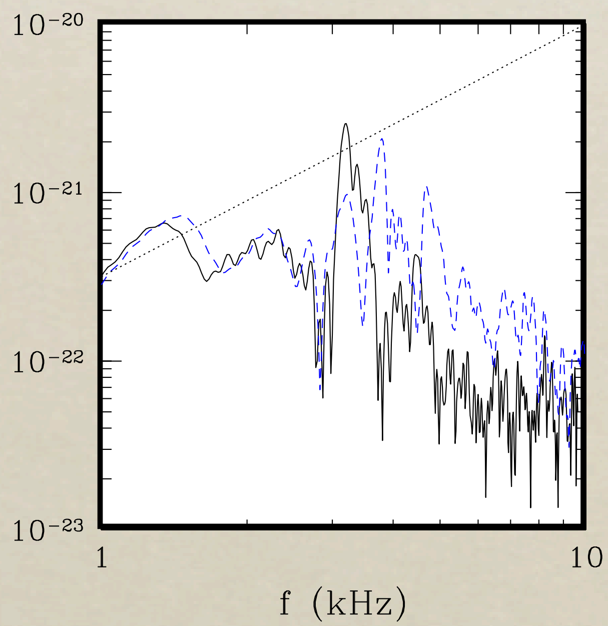
Simulation APR1515

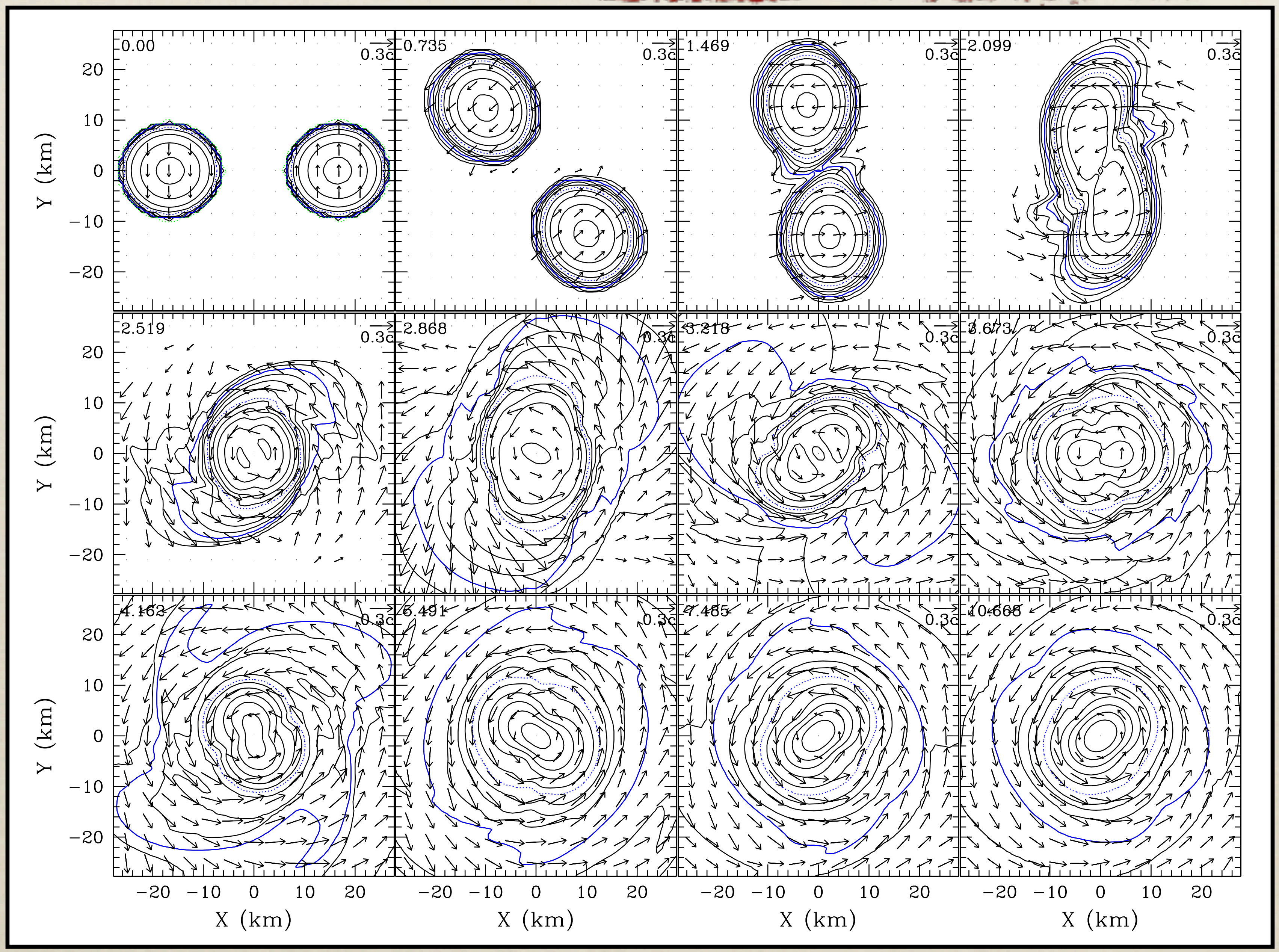


f_{merger} = 6.5 kHz

Simulation APR1313



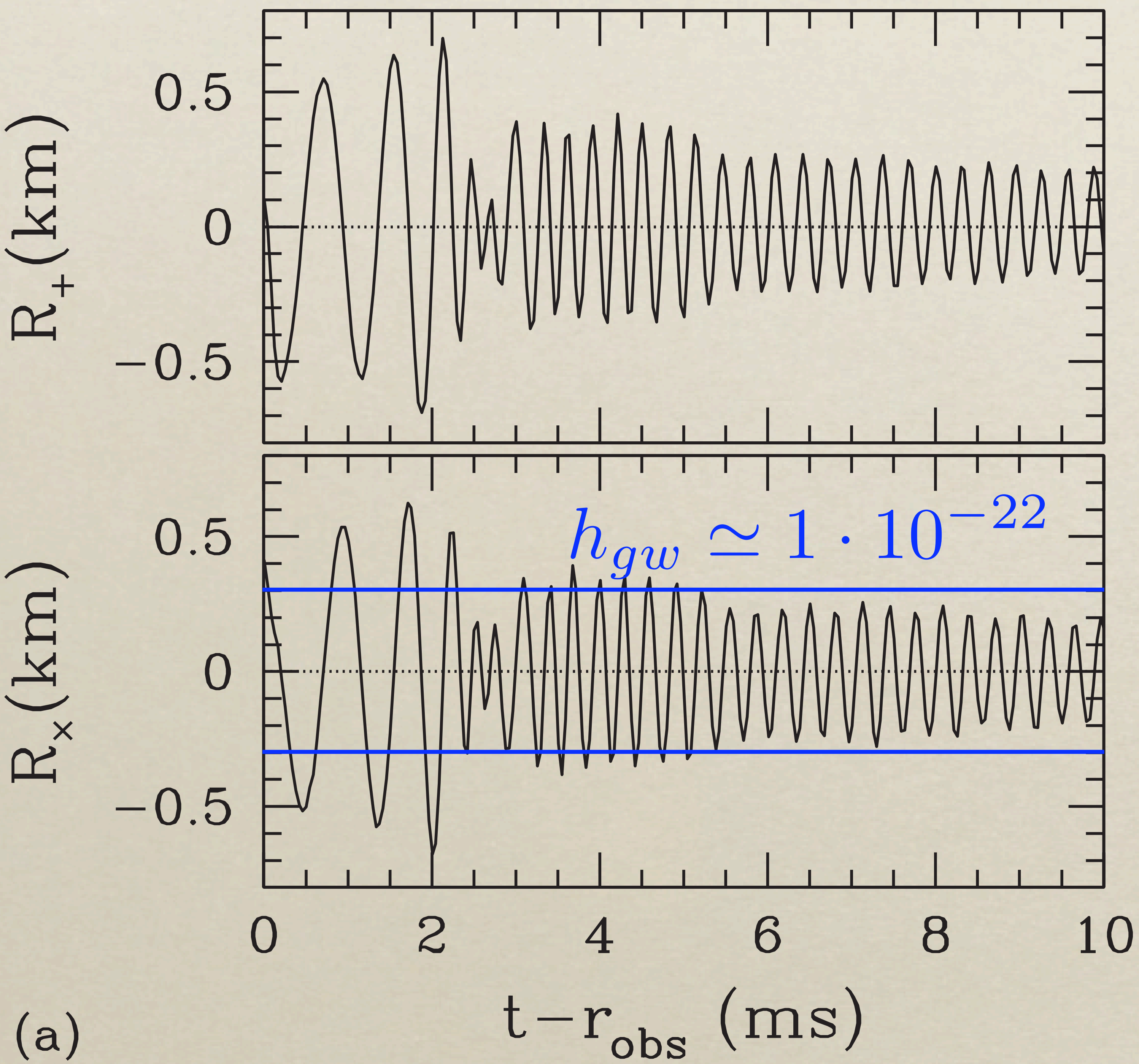


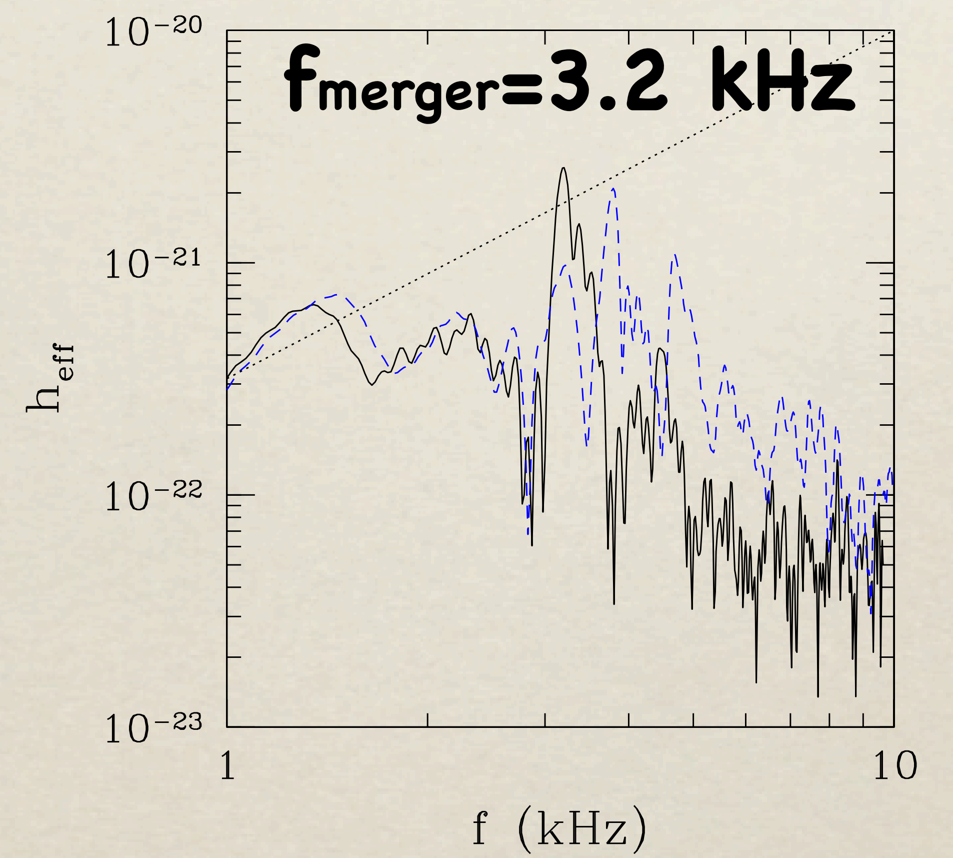


f_{merger}=3.2 kHz

$$h_{\rm eff} \equiv \sqrt{|\bar{R}_+|^2 + |\bar{R}_\times|^2} f$$
$$= 1.8 \times 10^{-21} \left(\frac{dE/df}{10^{51}\ {\rm erg/Hz}}\right)^{1/2} \left(\frac{100\ {\rm Mpc}}{r}\right),$$

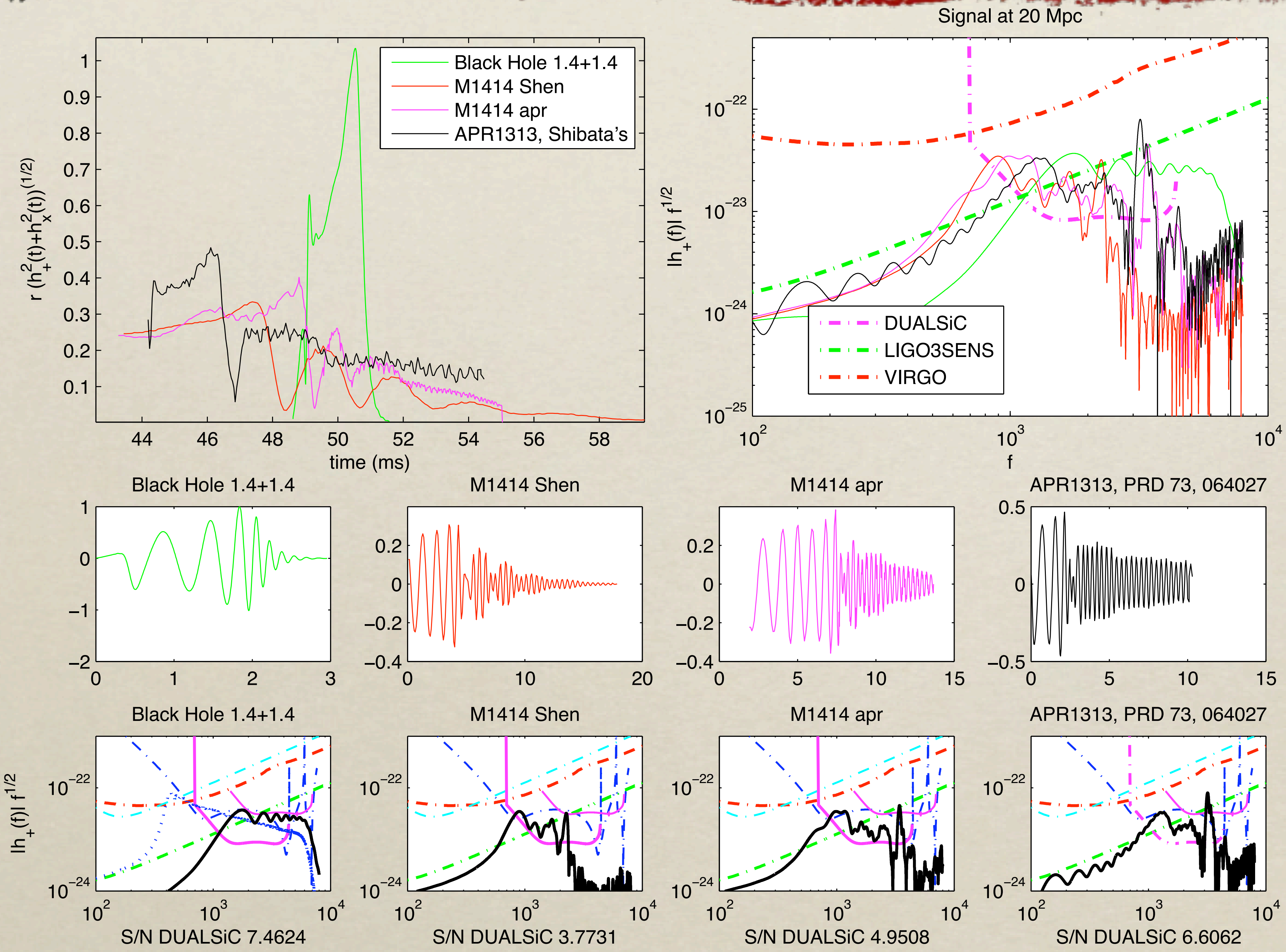
Simulation APR1313



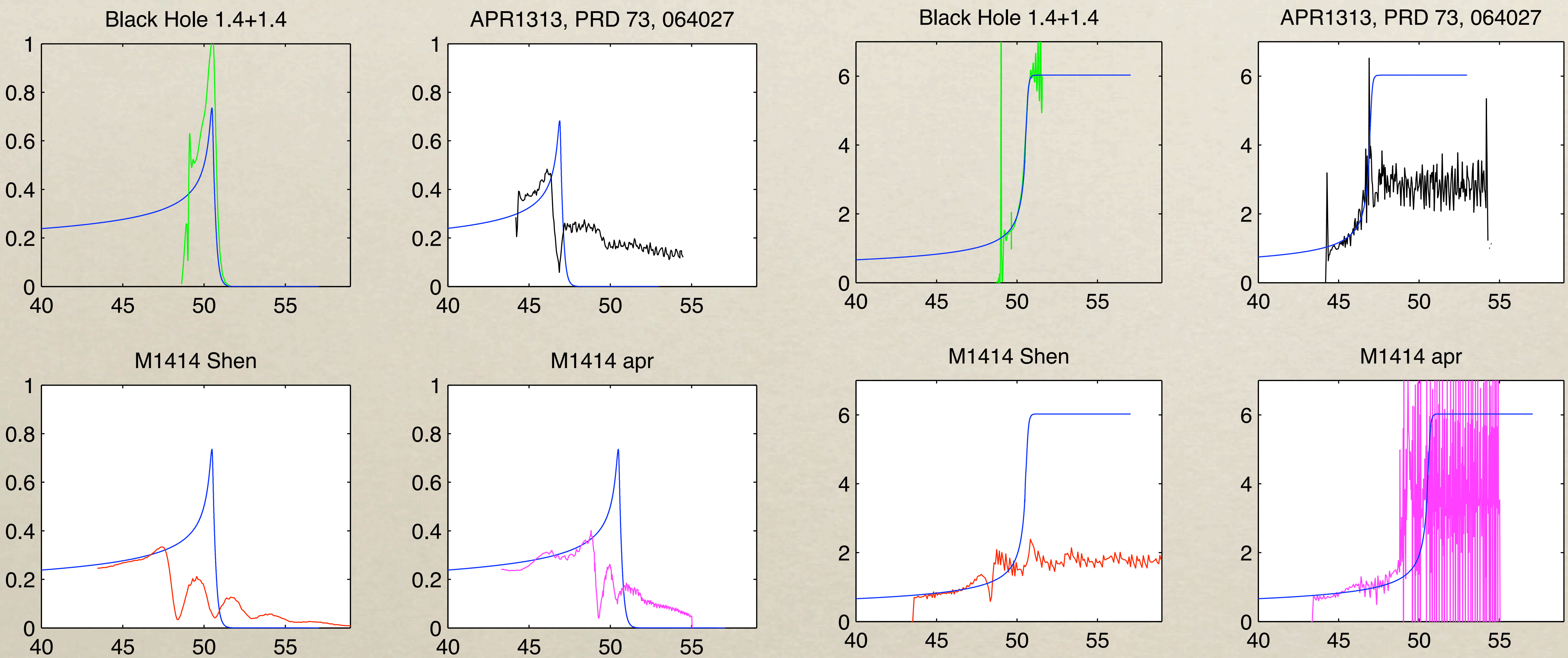


$$\begin{aligned} h_c &= h\sqrt{n} \\ &\simeq 2 \cdot 10^{-22}\sqrt{20 \times 3.2} \\ &= 1.6 \cdot 10^{-21} \end{aligned}$$

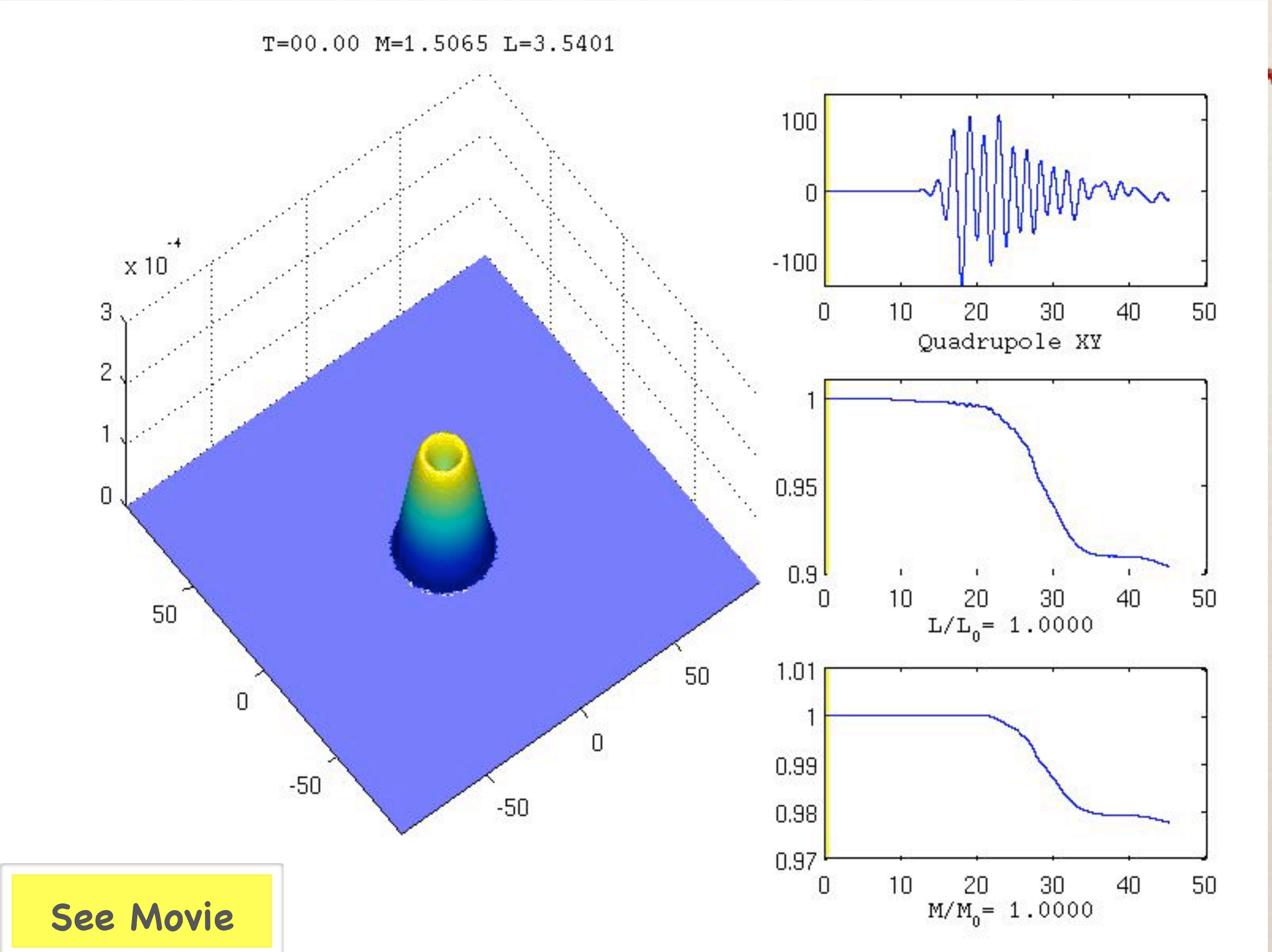
S/N for the merger phase



Blu-line is EOB



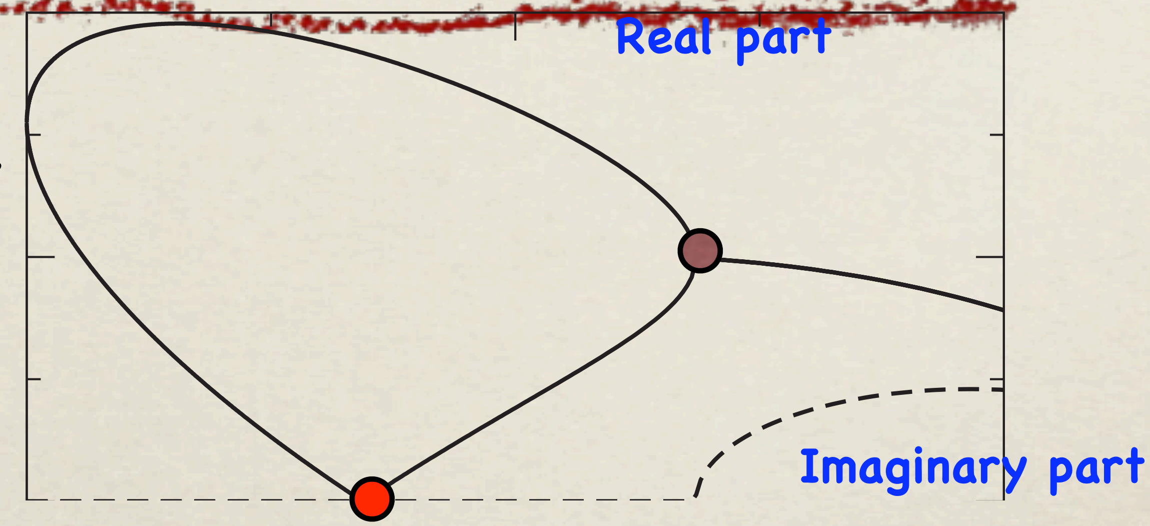
Numerical relativity at work



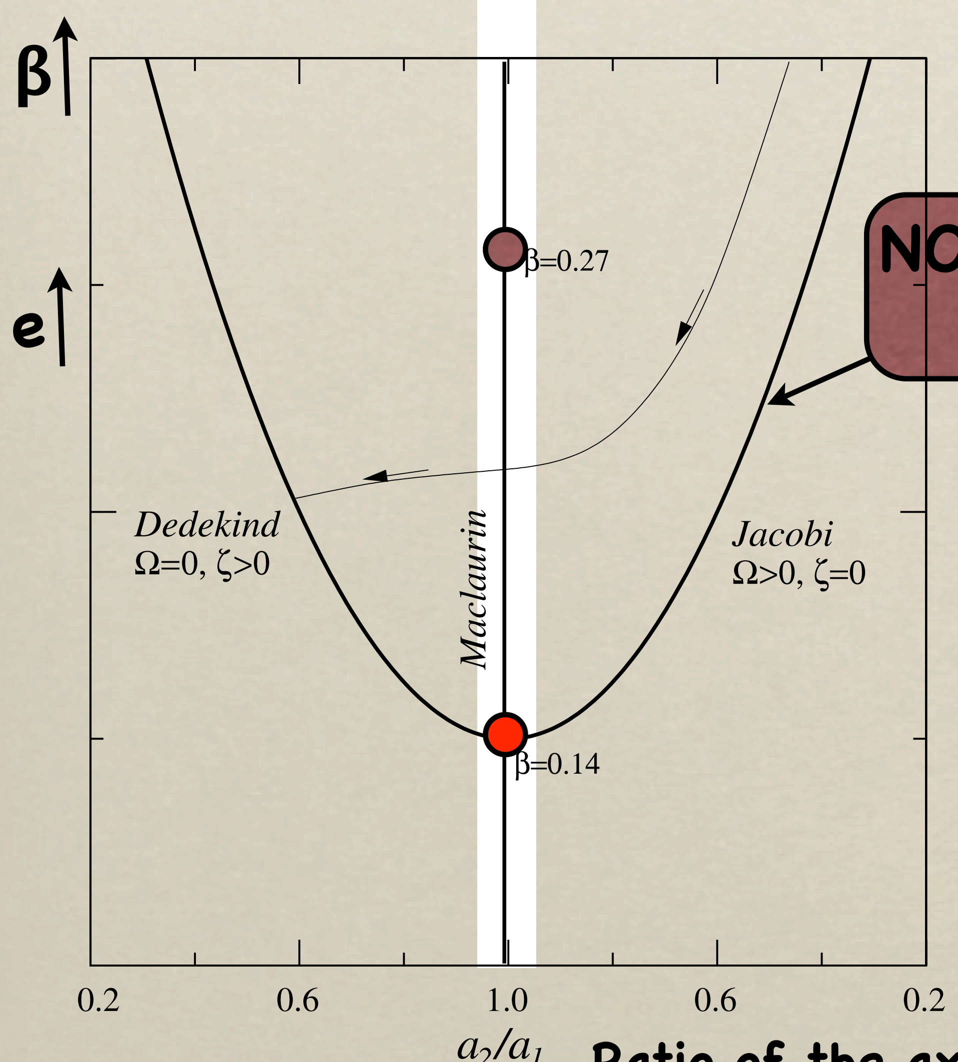
See Movie

Elipsoidal figures of equilibrium (Newtonian)

Eigenvalue of the m=2 mode



Axisymmetric configuration



Ratio of the axes on the xy plane

NON-Axisymmetric configuration

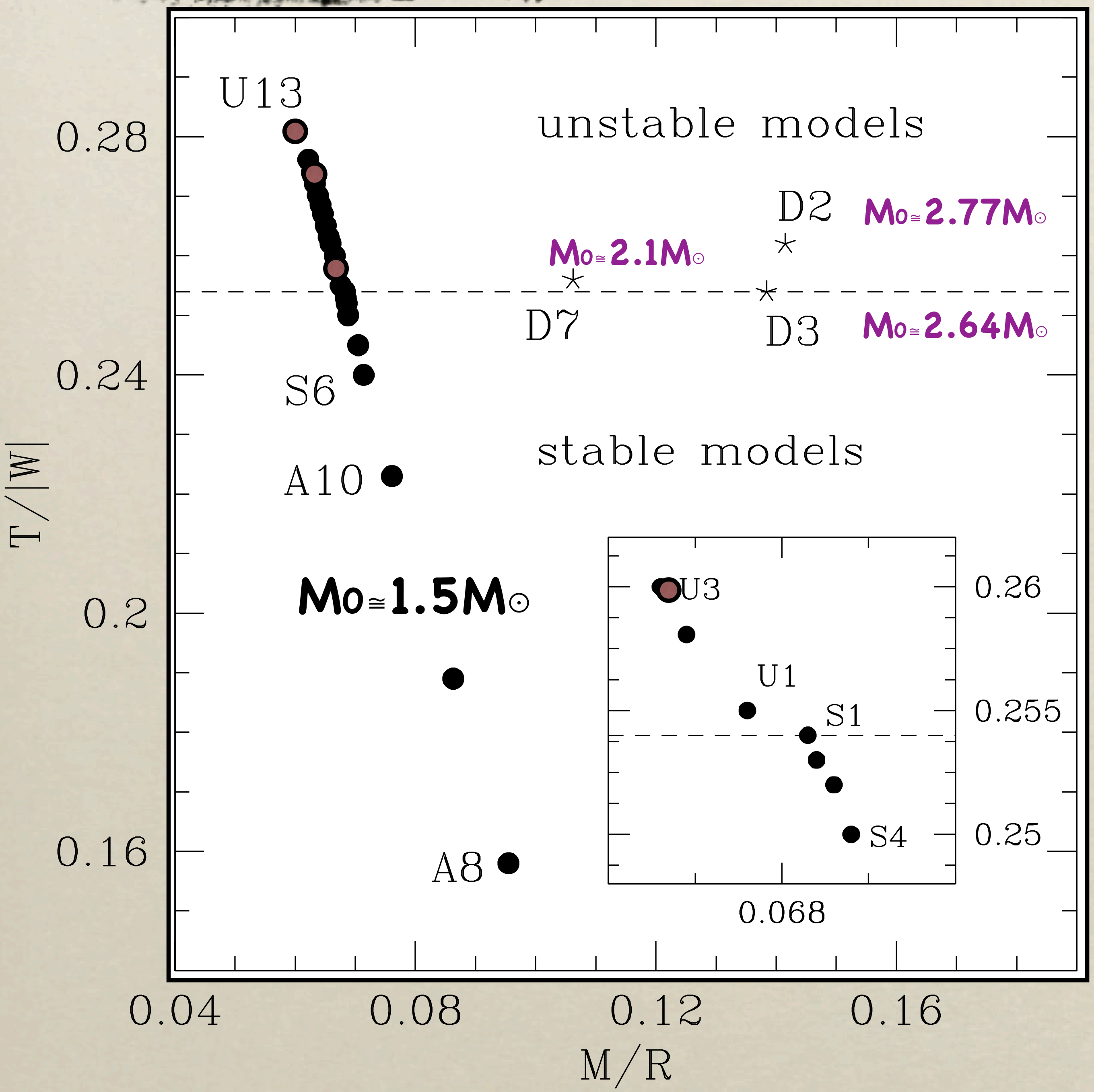
$$\sigma = \Omega(e) \pm \sqrt{4B_{11}(e) - \Omega^2(e)}$$

$$\beta(e) = \frac{\mathrm{T}}{|\mathrm{W}|} = -1 + \frac{3}{2e^2} - \frac{3\sqrt{1-e^2}}{2e\arcsin(e)}$$

$$\Omega^2 = \frac{-6\,\left(1-e^2\right)}{e^2} + \frac{2\,\left(3-2\,e^2\right)\,\sqrt{1-e^2}\,\arcsin(e)}{e^3}$$

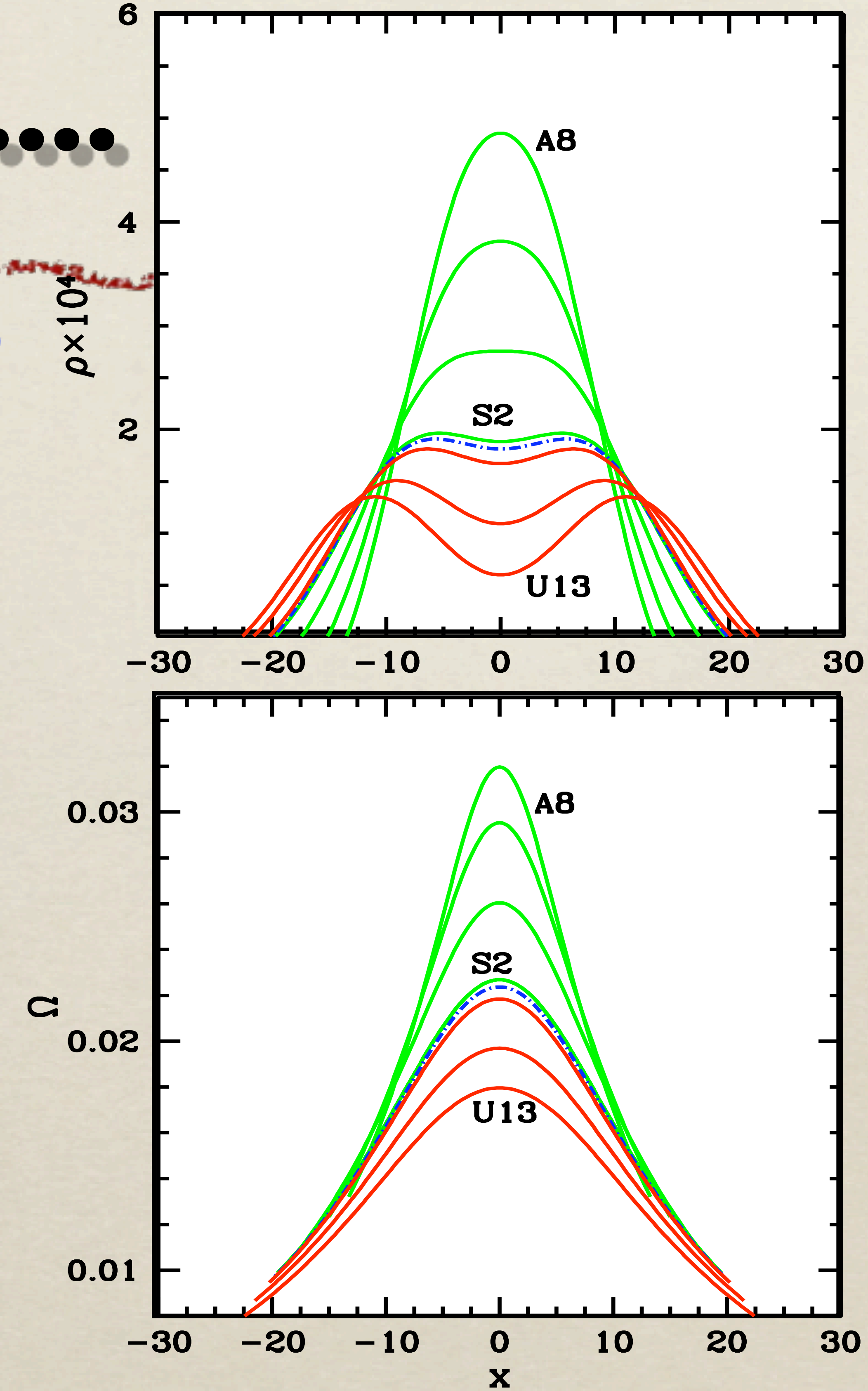
$$B_{11} = \frac{3\,e - 5\,e^3 + 2\,e^5 + \sqrt{1-e^2}\,\left(-3+4\,e^2\right)\,\arcsin(e)}{4\,e^5}$$

Simulated models .....



K=100

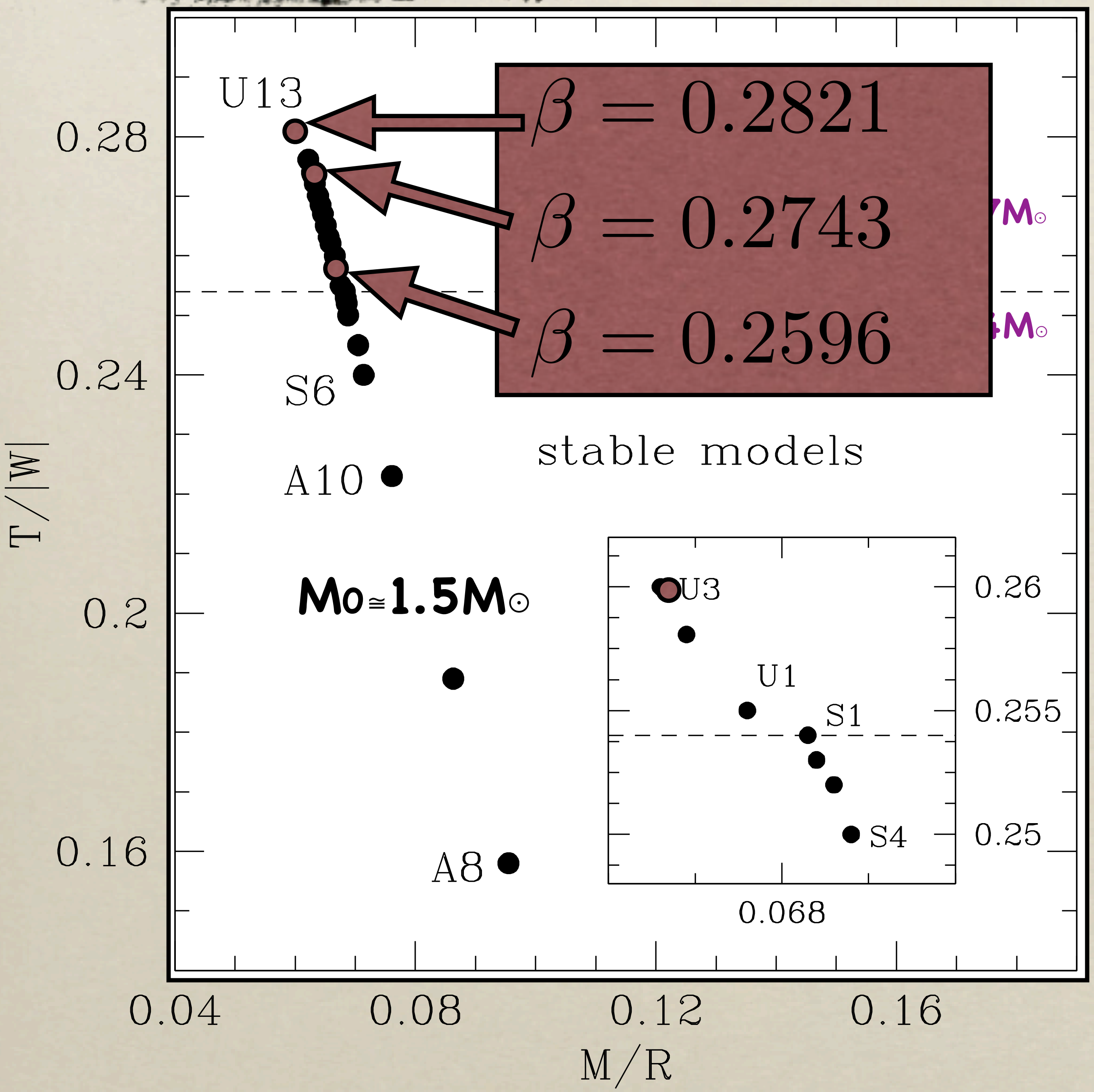
Γ=2



[1] Shibata, Baumgarte, Shapiro, ApJ. 542,(2000)453.

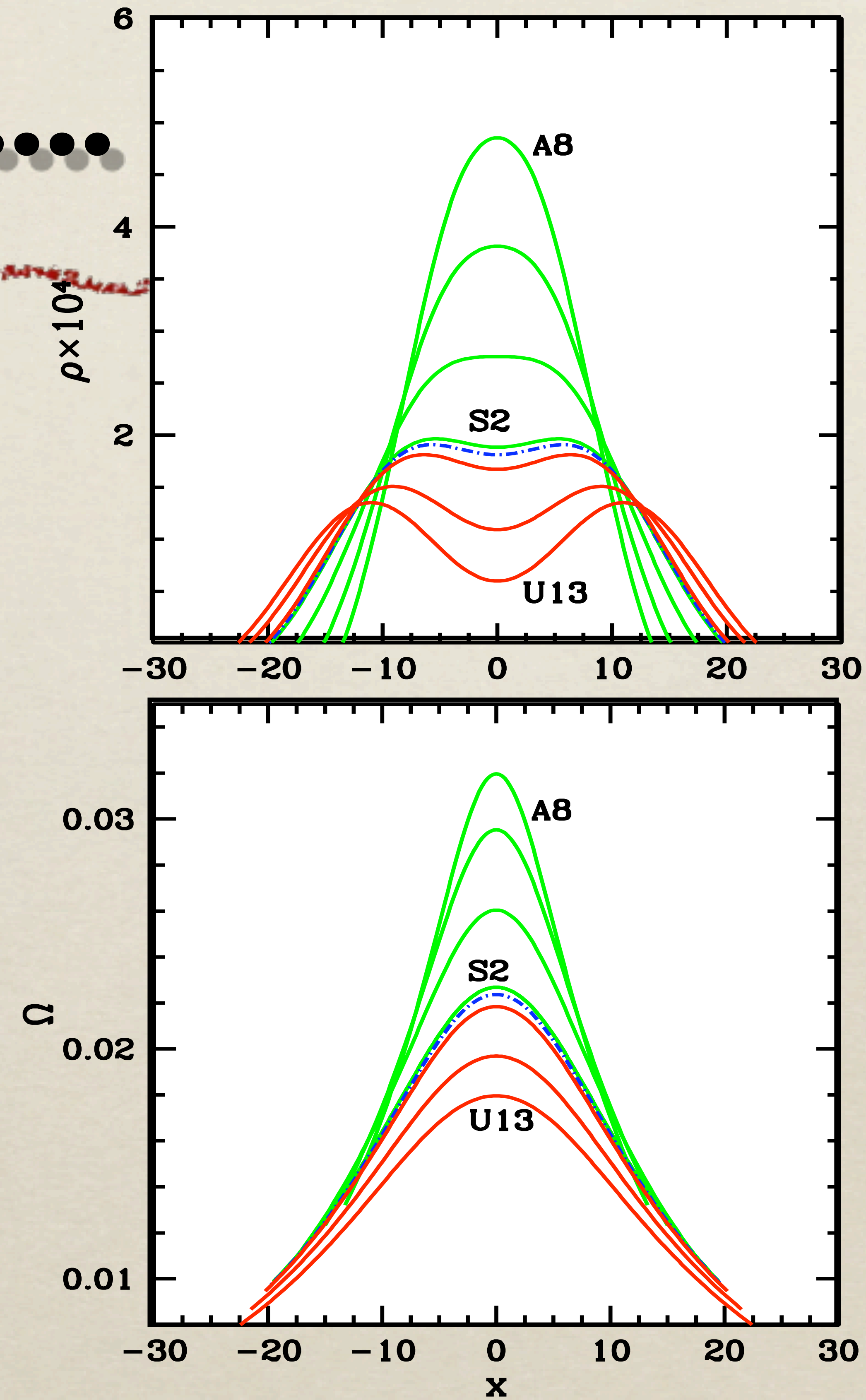
[2] Stergioulas, Apostolatos, Font: Mon. Not. R. Astron. Soc. 352(2004) 1089--1101

Simulated models .....



K=100

Γ=2

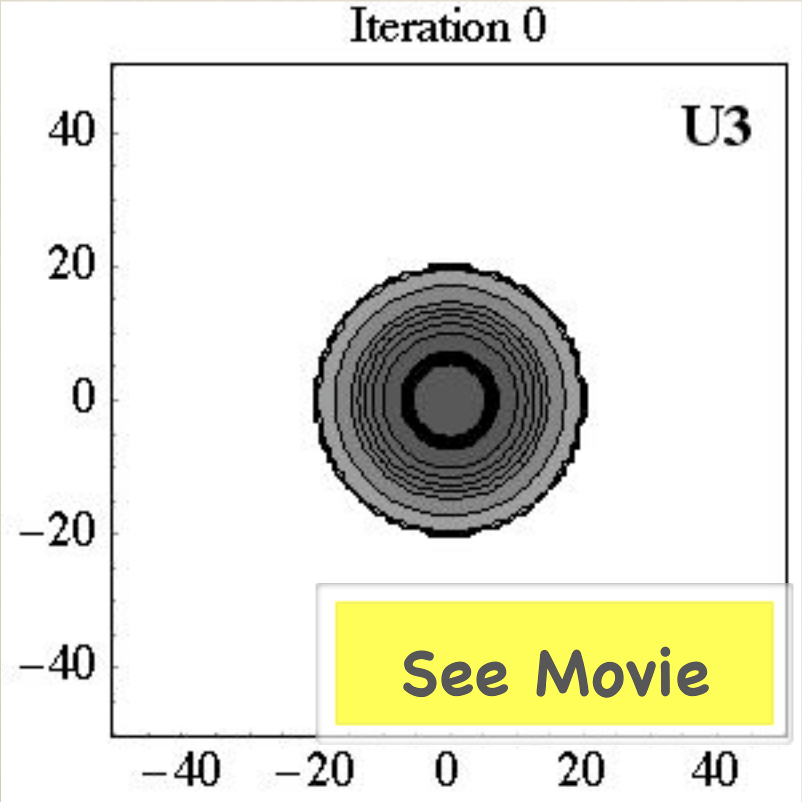


[1] Shibata, Baumgarte, Shapiro, ApJ. 542,(2000)453.

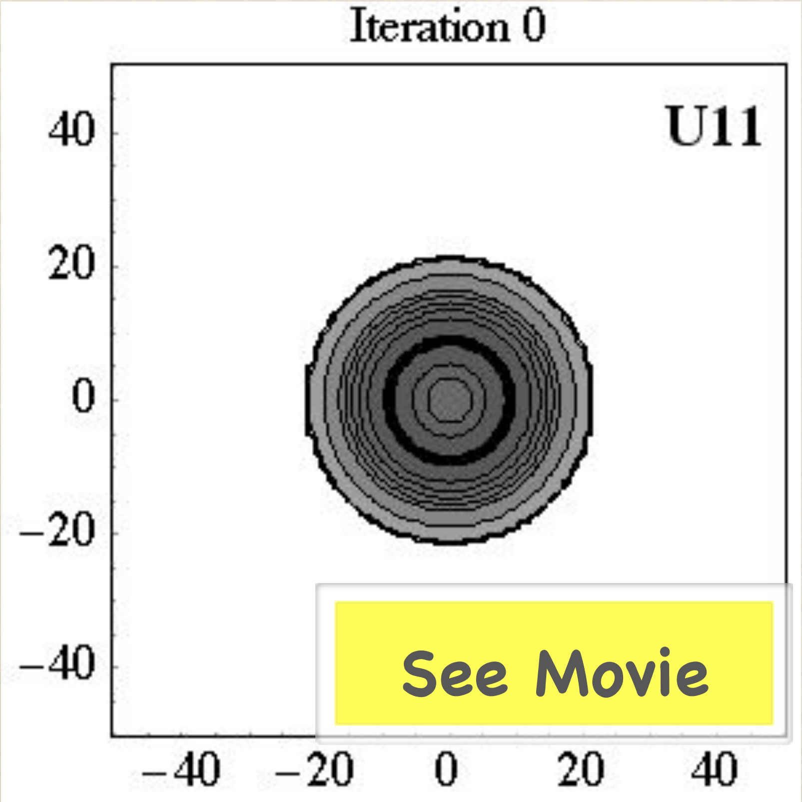
[2] Stergioulas, Apostolatos, Font: Mon. Not. R. Astron. Soc. 352(2004) 1089--1101

Movies

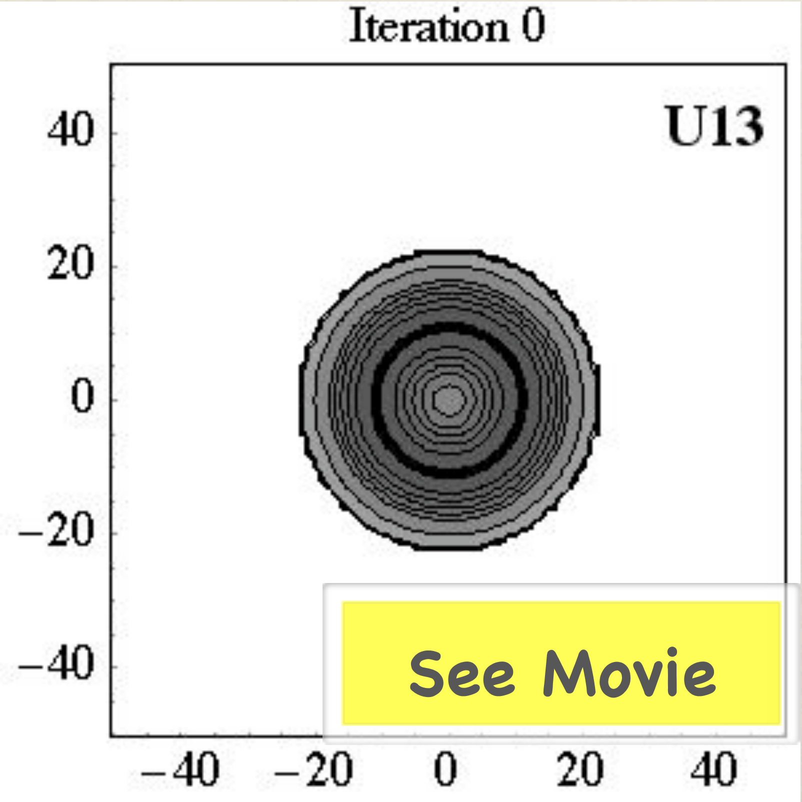
$\beta = 0.2596$



$\beta = 0.2743$



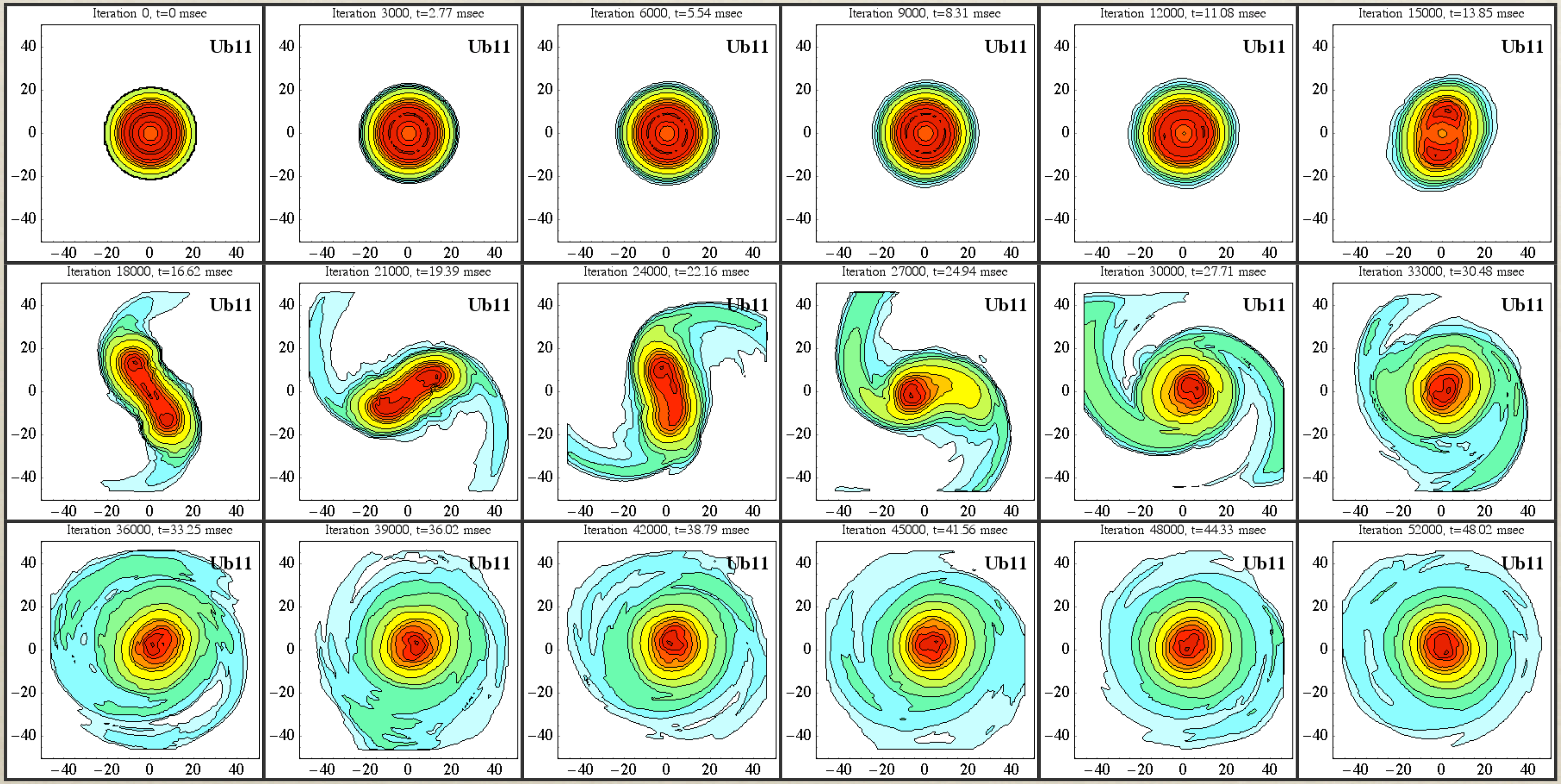
$\beta = 0.2812$



$\beta = 0.2743$

Simulation Ub11

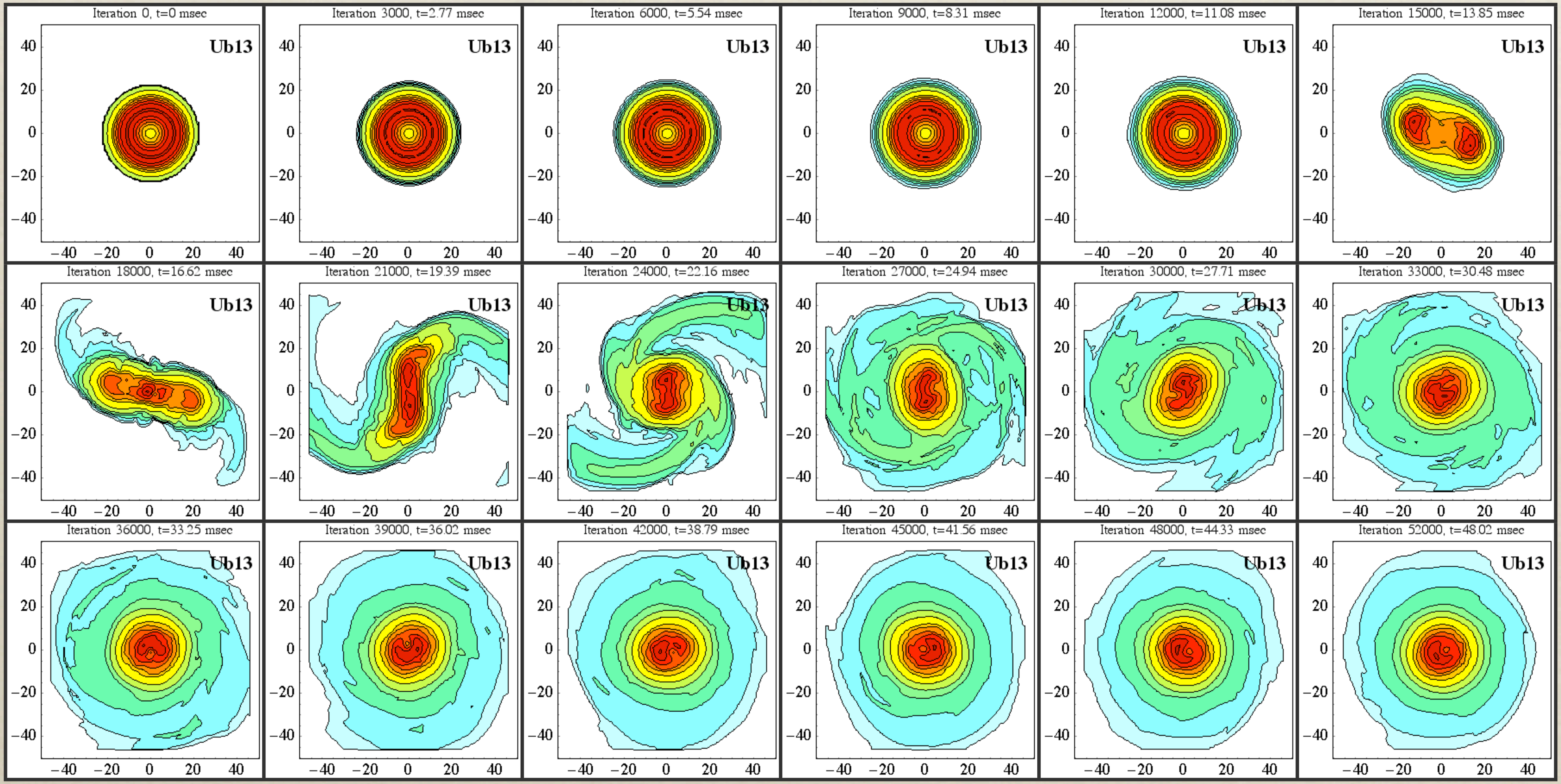
(Movies: http://www.fis.unipr.it/numrel/)



$\beta = 0.2821$

Simulation Ub13

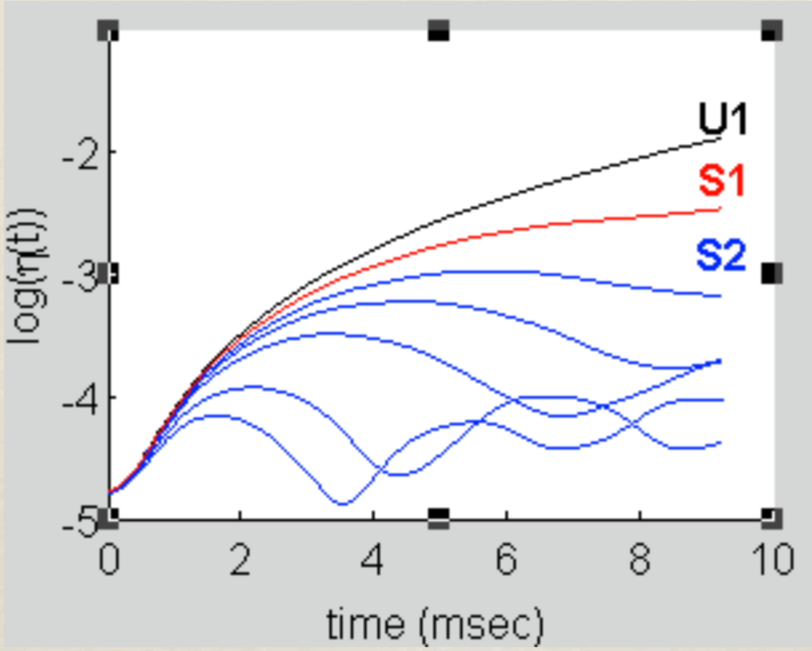
(Movies: http://www.fis.unipr.it/numrel/)

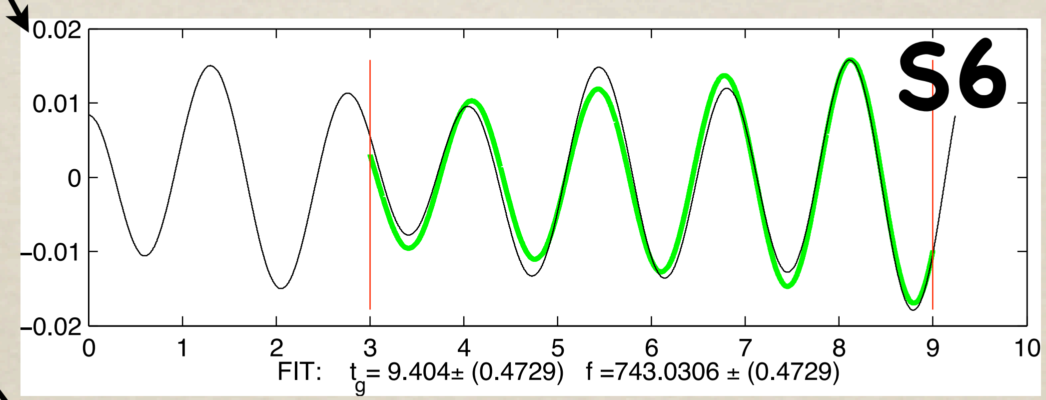


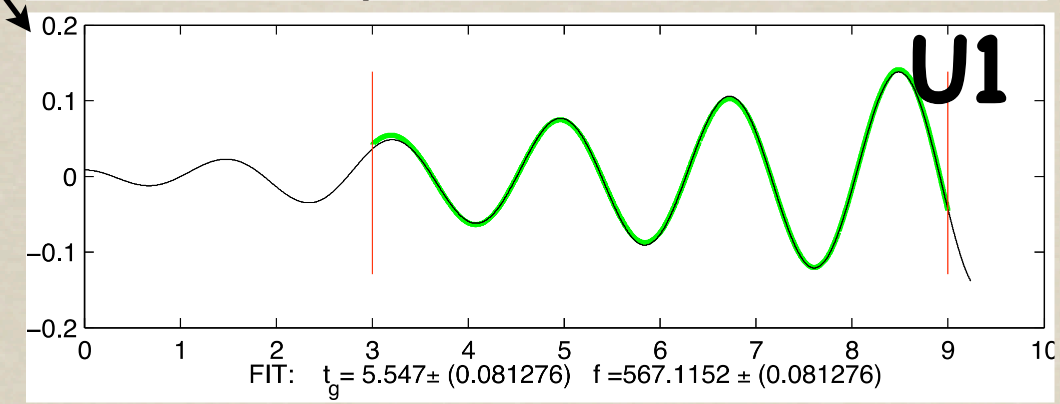
First Method

Model	β	notes	t_i ms	t_f ms	η (max)	τ_B (ms)	f_B Hz
S6	0.240	$\delta = .04$	3	9	0.02	——	740
S5	0.245	$\delta = .04$	3	9	0.02	——	705
S4	0.250	$\delta = .04$	3	9	0.03	——	656
S3	0.252	$\delta = .04$	3	9	0.04	——	611
S2	0.253	$\delta = .04$	3	9	0.05	——	588
S1	0.254	$\delta = .04$	3	9	0.09*	9.71	578
U1	0.255	$\delta = .04$	3	9	0.15*	5.26	567
S1	0.254		45	63	0.02	——	599
U1	0.255		45	63	0.13*	22.1	588

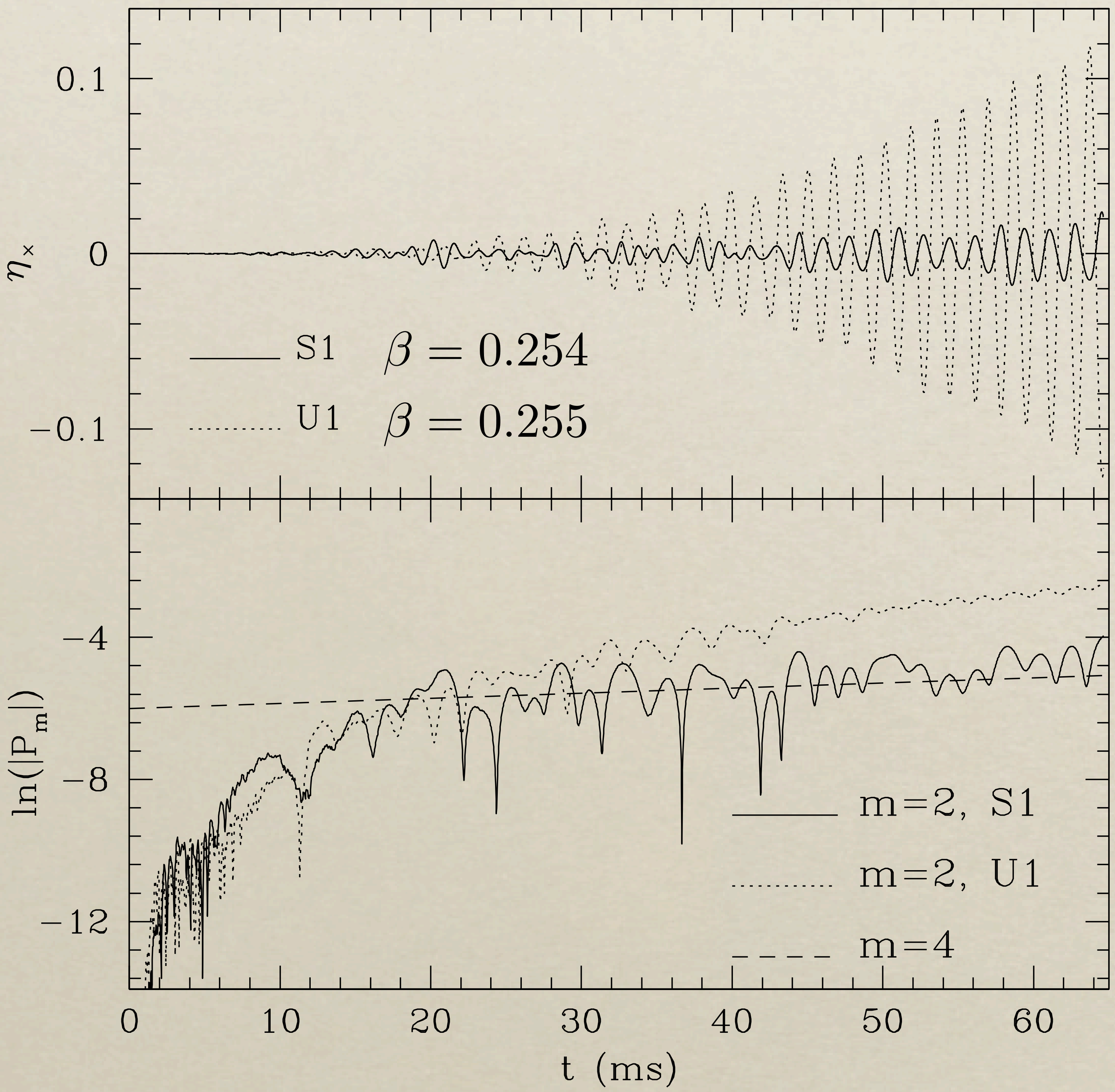
$$\delta\rho_2(x,y,z) = \delta_2 \left(\frac{x^2 - y^2}{r_e^2} \right) \rho \,,$$







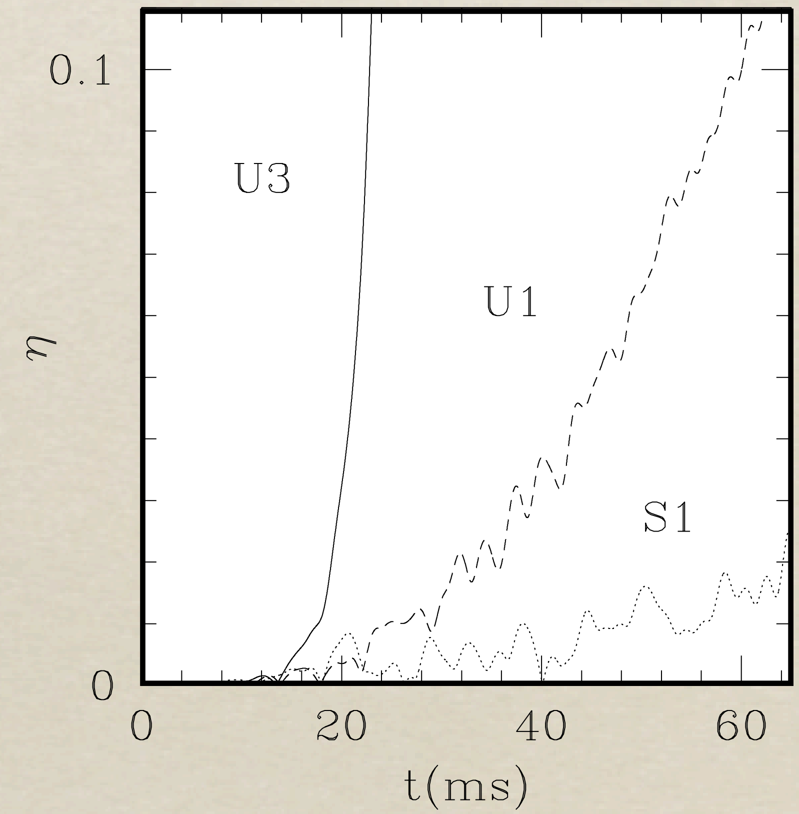
Un-perturbate dynamics at the threshold



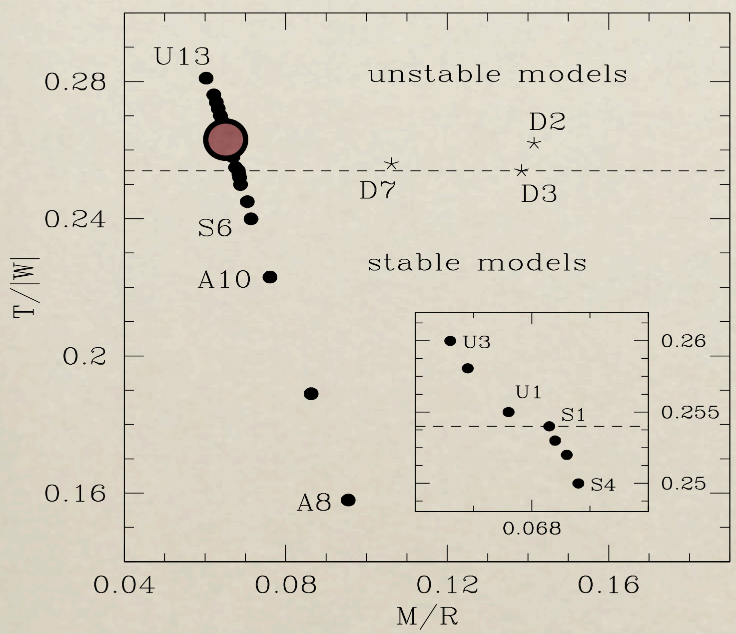
Unperturbed dynamics

Different value of β

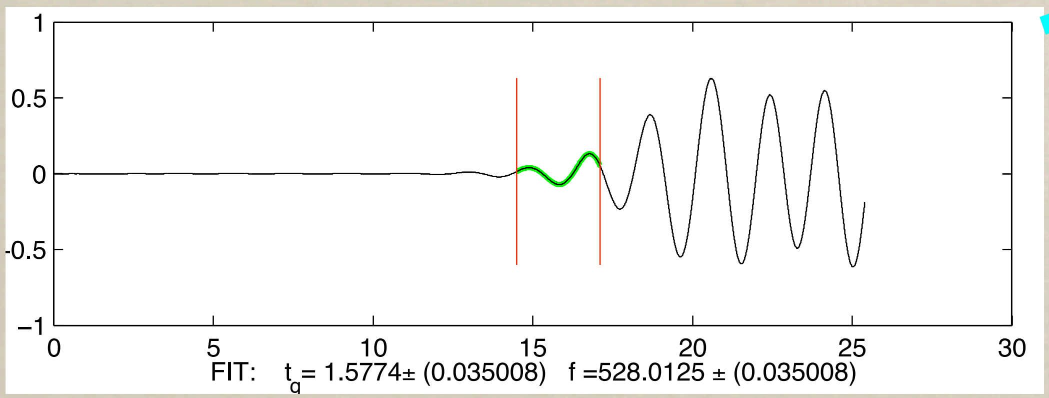
Importance of non linear coupling at the threshold

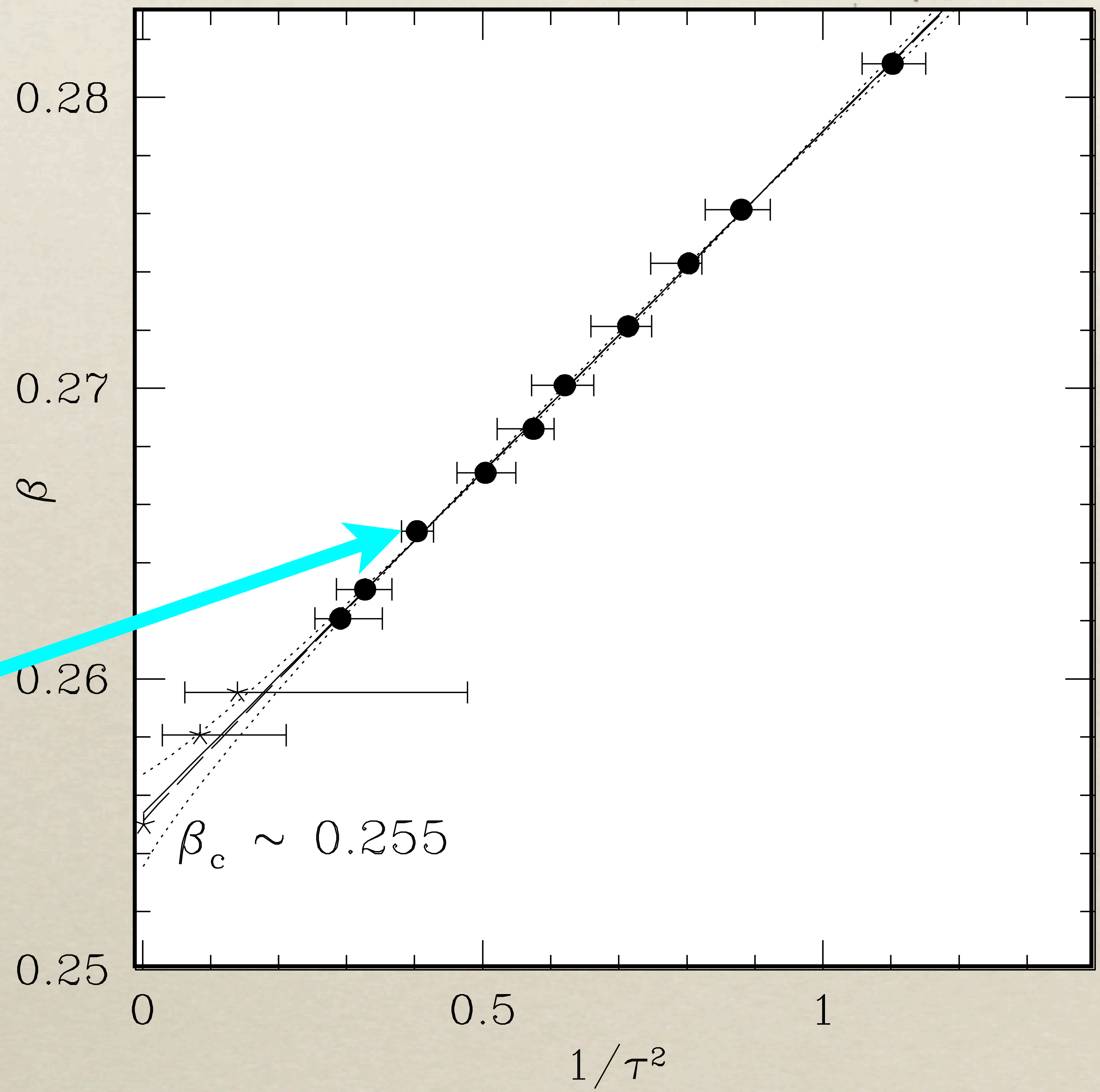


Second Method



U6 dx=0.625





$$\eta(t) = \eta_0\, e^{t/t_g} \sin(2\pi f\, t + \phi)$$

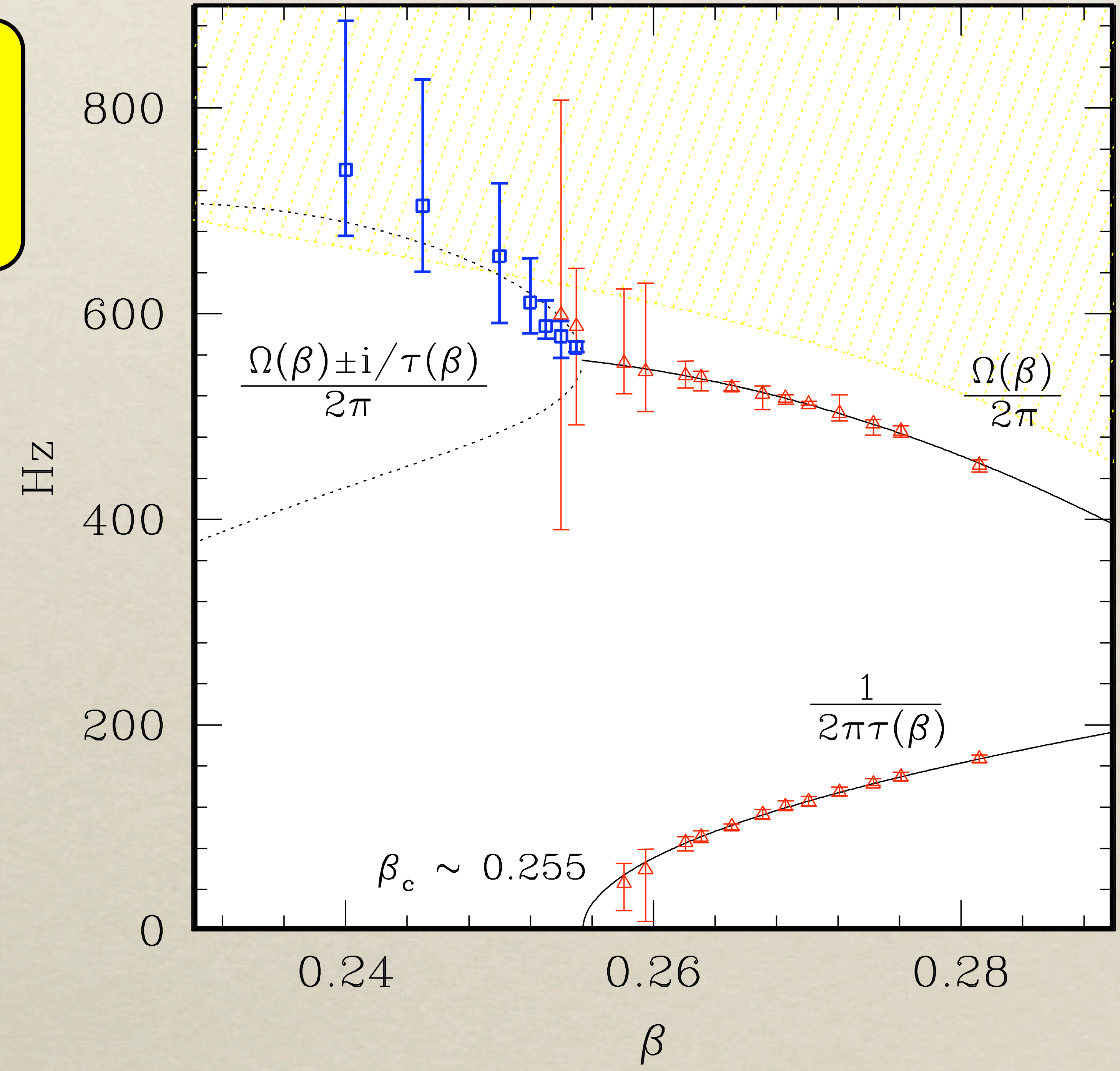
$$t_g = 1.5774(msec) \qquad f = 528Hz$$

Instability Diagram in full GR

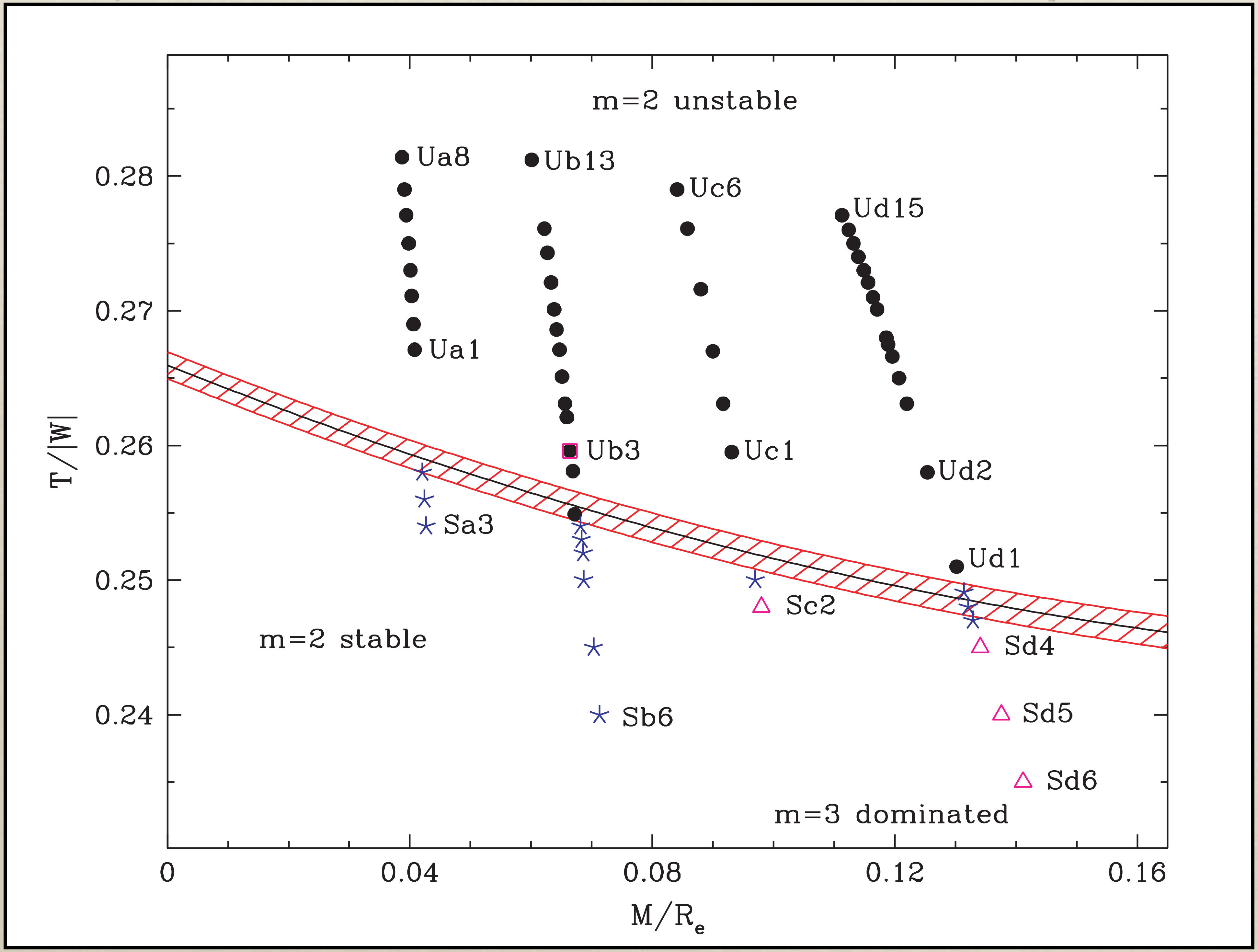
$$\Omega = f_c + f'_c(\beta - \beta_c) + f''_c(\beta - \beta_c)^2$$

$$\frac{1}{\tau} = \sqrt{k(\beta - \beta_c)}$$

Model	β	t_i ms	t_f ms	η (max)	τ_B (ms)	f_B Hz
U2	0.2581	16.9	22.4	0.3734	3.438	552
U3	0.2595	19.9	24.2	0.4241	2.678	544
U4	0.2621	15.3	18.3	0.5496	1.854	540
U5	0.2631	16.2	19.0	0.5788	1.748	538
U6	0.2651	14.5	17.1	0.6305	1.574	528
U7	0.2671	14.2	16.4	0.6694	1.408	522
U8	0.2686	12.2	14.3	0.7027	1.319	518
U9	0.2701	13.2	15.2	0.7223	1.269	512
U10	0.2721	13.7	15.6	0.7482	1.184	503
U11	0.2743	12.9	14.7	0.7749	1.116	493
U12	0.2761	12.0	13.7	0.7999	1.066	486
U13	0.2812	11.2	12.7	0.8551	0.952	453



Extending parameter space



Conclusions

- **Numerical relativity is ready to simulate real physics**
- **A lot of work to do:**
  - NS-NS merger with realistic EOS
  - MAGNETO-HYDRODYNAMICS
  - INSTABILITIES of isolated stars
  - Accretion driven collapse (of a NS to a BH)
  - ....