PARMA 5

5

CHAOS AND SYMMETRY

IN

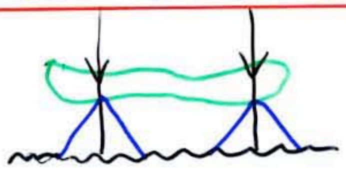
'STRING COSMOLOGY'

COSMOLOGICAL SINGULARITIES

- NOT AS A MODEL OF THE EARLY UNIVERSE

- BUT AS A TOOL FOR PROBING THE STRUCTURE OF M-THY, AND, IN PARTICULAR, FOR SEARCHING FOR HIDDEN SYMMETRIES

NEAR SPACELIKE SINGULARITY LIMIT



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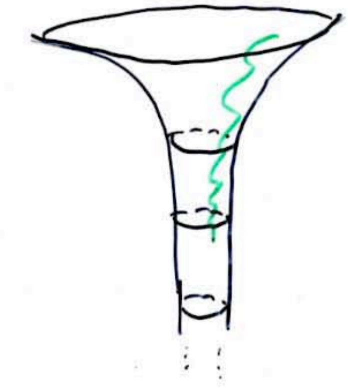
NEAR HORIZON LIMIT



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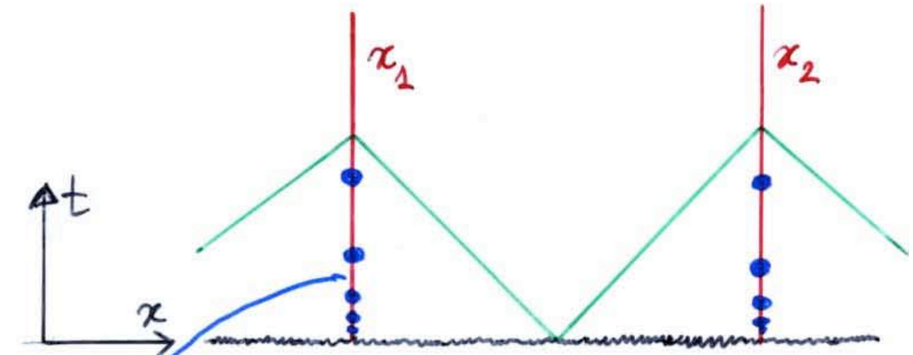
HOLOGRAPHIC CORRESPONDENCE

STRINGS ON $AdS_5 \times S_5 \longleftrightarrow CFT_4$

CHAOS IN (SUPER) GRAVITY

- Belinskii, Khalatnikov, Lifshitz (BKL) 1969

infinitely 'oscillatory' behaviour of generic inhomogeneous solution of $R_{\mu\nu} = 0$ near $t = 0$ in $D = 4$



BKL expansion: $\partial_x \ll \partial_t$

$$ds^2 \simeq -dt^2 + \sum_a A_a^2(t) \left(e^a_i(x)\, dx^i \right)^2$$

SUCCESSIVE KASNER EPOCHS: $A_a(t) \sim t^{p_a}$

- CHAOTIC BEHAVIOUR:
  - OF SUCCESSIVE p_a's : Lifshitz, Lifshitz, Khalatnikov '71
    Khalat, Lif., Khanin, Schur, Sinai '85
  - OF ASSOCIATED HAMILTONIAN BILLIARD: Misner, Chitre, Kirillov, ...

- Demaret, Henneaux, Spindel 1985

BKL chaos disappears in $D \geqslant 11$! monotonic Kasner-like

T.D., Henneaux, Rendall, Weaver '02

- T.D., Henneaux 2000

BOSONIC SECTORS OF $D = 11$ SUPERGRAVITY, AS WELL AS $D = 10$ STRING THEORIES (I, IIA, IIB, HO, HE) ARE ALL BKL CHAOTIC

HAMILTONIAN APPROACH TO BKL BEHAVIOUR

Misner, Chitre, Kirillov, Kirillov-Melnikov, TD Henneaux Nicolai

$$\text{ADM}: S_{ADM} = \int d^d x \left[\pi^{ij} \dot{g}_{ij} \left(+ \pi^{ijk} \dot{A}_{ijk} + \cdots \right) - \tilde{N} \mathcal{H} - \cdots \right]$$

CONJUGATE MOMENTUM

SPATIAL METRIC

HAMILTONIAN CONSTRAINT

LOCAL
+ IWASAWA
DECOMPOSITION

$$\boxed{g_{ij}(t,\vec{x}) = \sum_a e^{-2\beta^a(t,\vec{x})} \mathcal{N}^a_i(t,\vec{x}) \mathcal{N}^a_j(t,\vec{x})}$$

LOGARITHMS OF "SCALE FACTORS"

$$A_a(t,\vec{x}) \equiv e^{-\beta^a}$$

UPPER TRIANGULAR "OFF DIAGONAL" METRIC COMPONENTS

$$\mathcal{N}^a_i = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

NEW CANONICAL VARIABLES: $\left(\beta^a(t\vec{x}), \pi_a(t\vec{x}) \right) ; \left(\mathcal{N}^i_a(t\vec{x}), \mathcal{P}^a_i(t\vec{x}) \right) \ldots$

SIMPLIFICATION OF DYNAMICS AS $t \to 0$

ALL "OFF-DIAGONAL" VARIABLES FREEZE: $\mathcal{N}^i_a, \mathcal{P}^a_i, A_{abc}, \pi^{abc}, \ldots$

ONLY "DIAGONAL" ONES HAVE A COMPLICATED DYNAMICS : $(\beta^a, \pi_a) + (\varphi, \pi_\varphi)$ IF $\exists$

WHICH IS APPROXIMATELY GOVERNED BY ODE'S : "TODA-LIKE" SYSTEM

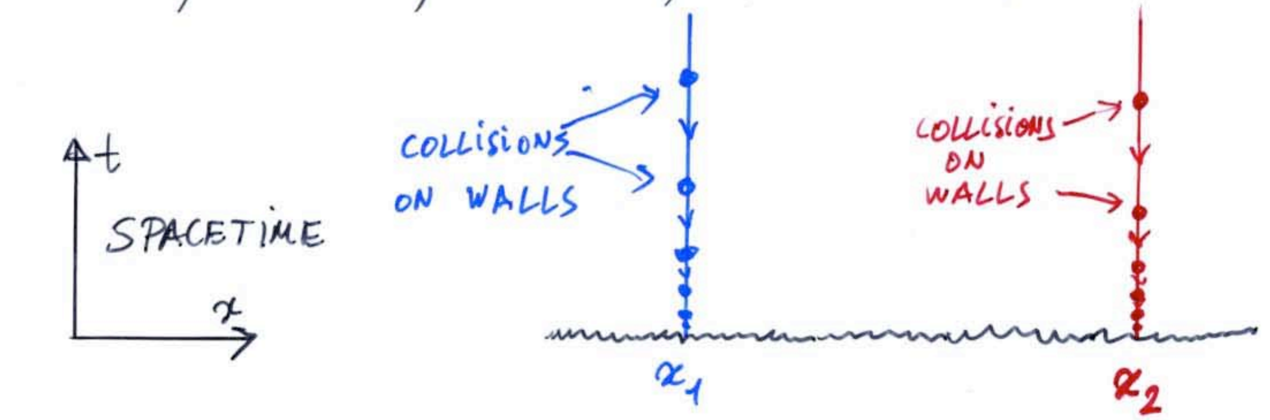
$$\boxed{\mathcal{H}_{RED} \simeq G_{ab} \pi^a \pi^b + \sum_A c_A e^{-2 w_A(\beta)}}$$

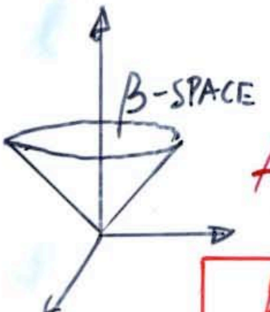
$c_A(\mathcal{N}, \mathcal{P}, A_{abc}, \pi^{abc} \ldots) \to$ LIMIT

EXPONENTIAL WALLS

COSMOLOGICAL BILLIARDS

Chitre, Misner, Kirillov, Kirillov-Melnikov, TD Henneaux Nicolai

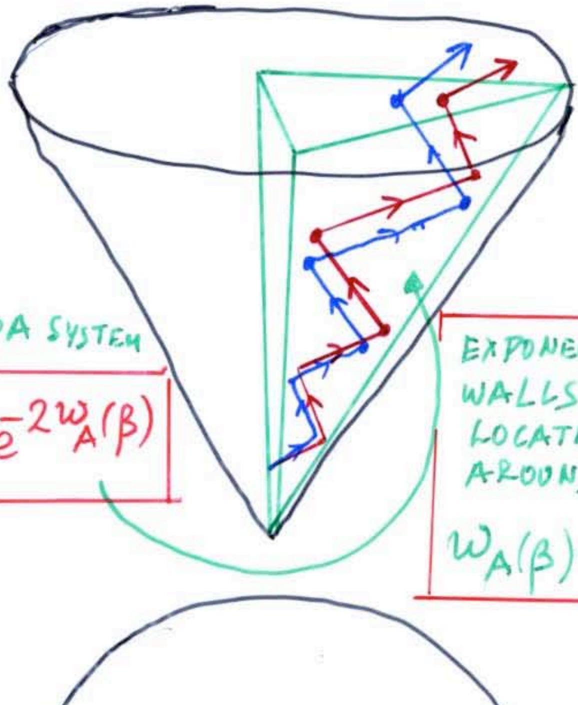




$$A_a \equiv e^{-\beta^a}$$

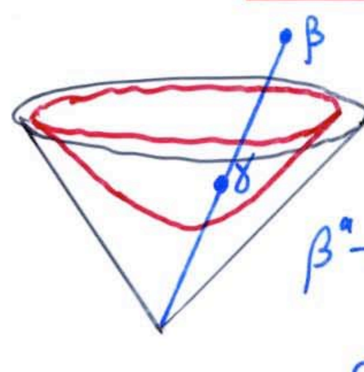
ASYMPTOTIC TODA SYSTEM

$$\boxed{\mathcal{H}_{RED} \simeq G_{ab}\pi^a\pi^b + \sum_A c_A e^{-2w_A(\beta)}}$$



EXPONENTIAL WALLS LOCATED AROUND

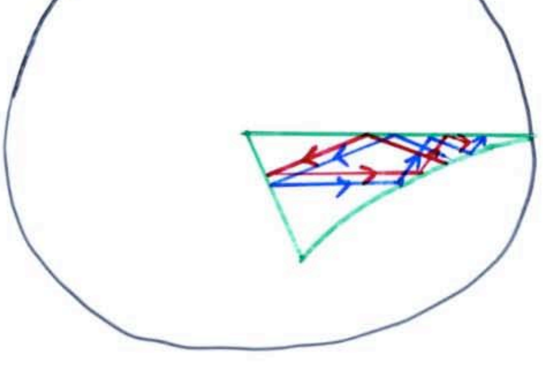
$$\boxed{w_A(\beta)=0}$$



HYPERBOLIC SPACE

$$\beta^a \to \gamma^a = \frac{\beta^a}{\sqrt{-G_{bc}\beta^b\beta^c}}$$

$$G_{bc}\gamma^b\gamma^c = -1$$



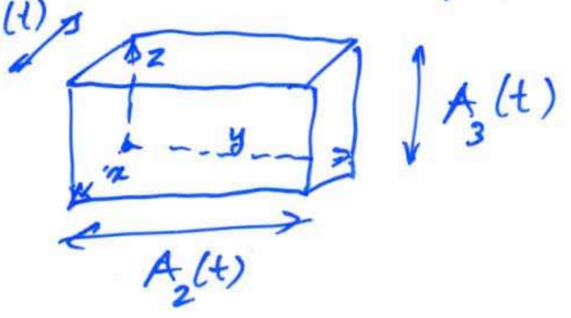
ESI

ANISOTROPIC
SPATIAL
GEOMETRY

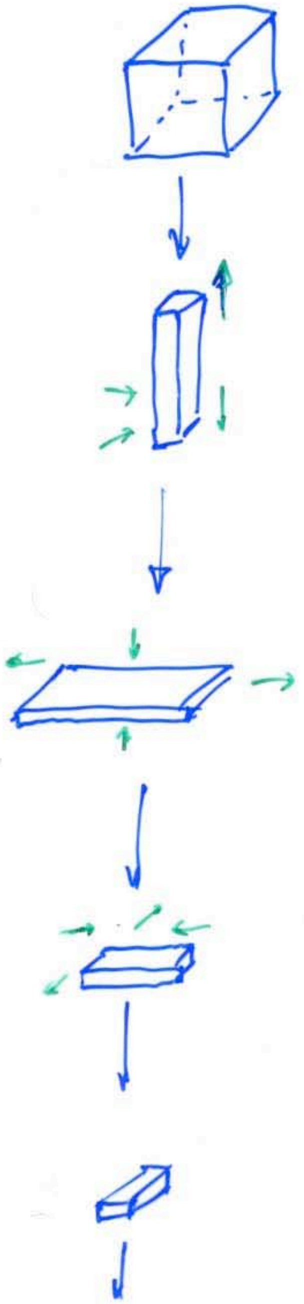
e.g. $ds^2 = A_1^2(t,x)\,\omega_1^2 + A_2^2(t)\,\omega_2^2 + A_3^2(t)\,\omega_3^2$

(LIMITING) IWASAWA FRAME

"BOX"



AS $t \to 0$



SYMMETRY IN GRAVITY

- Ehlers 1959 $GR_{D=4}$ WITH 1 KILLING VECTOR $k = \frac{\partial}{\partial x^3}$

$\Rightarrow$ CONTINOOUS SYMMETRY

$SL(2,\mathbb{R})_E$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

$$\boxed{Z' = \frac{aZ+b}{cZ+d}}$$

$ad-bc=1$

$a,b,c,d \in \mathbb{R}$

$$Z = B + i\Delta \quad \Delta \equiv g_{33} = k^2$$

NON-LOCAL DUAL TO $A_\mu = g_{3\mu}/g_{33}$

$$\varepsilon_{\mu\nu\lambda} \nabla^\lambda B = \Delta^2 (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

- Matzner, Misner 1967 GR_4 WITH 2 COMMUTING KILLING V. k_1, k_2

$\Rightarrow$ CONTINUOUS SYMMETRY

$SL(2,\mathbb{R})_{MM}$

LINKED TO FREEDOM IN $k'_a = \Lambda_a^b k_b$

LOCAL IN GRAVITY VARIABLES

- Geroch 1972 GR_4 WITH 2 COMMUTING K.V. $\oplus\ \epsilon_{\mu\nu\rho\sigma} \overset{1}{k}{}^{\mu} \overset{2}{k}{}^{\nu} \nabla^{\rho} \overset{a}{k}{}^{\sigma} = 0$

INTERPLAY OF $SL(2,\mathbb{R})_E \times SL(2,\mathbb{R})_{MM}$: INFINITE DIMENSIONAL LIE GROUP

- Julia 1981; Breitenlohner-Maison '84, '87; Belinskii-Zakharov...

GEROCH GROUP $= \widehat{SL(2,\mathbb{R})} = A_1^{(1)}$

AFFINE KAC-MOODY EXTENSION OF $SL(2,\mathbb{R})$

SYMMETRY IN SUPERGRAVITY

- Cremmer, Julia 1979 $\text{SUGRA}_{D=11}$ WITH

- 7 COMMUTING KILLING VECTORS : SYMMETRY
  $D=11 \rightarrow D=4$ E_7

- 8 COMM. KILLING V. E_8
  $D=11 \rightarrow D=3$ Cremmer-Julia
  Marcus-Schwarz '83

- 9 COMM. KILLING V. $E_9 = E_8^{(1)}$
  $D=11 \rightarrow D=2$ Nicolai '87

- ? 10 KV $\overset{?}{\Longrightarrow}$? E_{10}
  ? $D=11 \rightarrow D=1$
  Julia
  Nicolai
  Mizoguchi
  Ganor

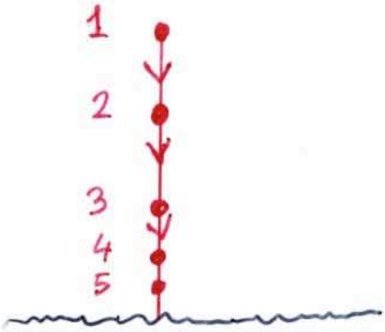
- ?? $D=11 \rightarrow D=0$?? $\overset{??}{\Longrightarrow}$ E_{11}
  Nicolai
  West
  Kleinschmidt.Schakenburg West
  Englert Houart

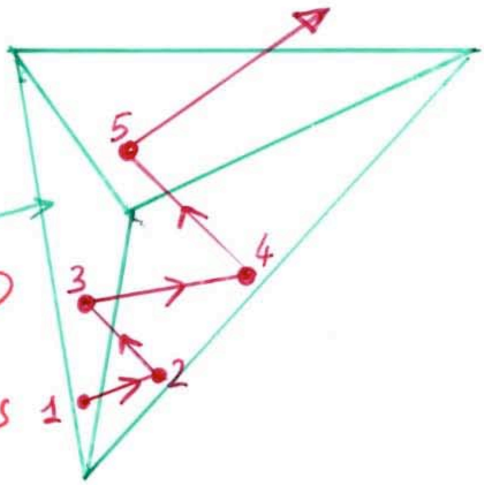
HIDDEN SYMMETRY IN BKL CHAOS

- T.D., Henneaux 2001

∃ HIDDEN SYMMETRY STRUCTURE IN THE COSMOLOGICAL CHAOS OF $SUGRA_{11}$, AS WELL AS THE VARIOUS $D=10$ STRING THEORIES

EFFECT OF 'BKL COLLISION' ⟷ REFLECTION IN HYPERPLANE ORTHOGONAL TO 'SIMPLE ROOT' OF A HYPERBOLIC KAC-MOODY (LIE) ALGEBRA





WALLS

EXPONENTIAL WALLS $e^{-2w_A(\beta)}$ $w_A(\beta)=0$

$\updownarrow$

SIMPLE ROOTS $\alpha_a(\beta)$

BKL-LIKE CHAOTIC OSCILLATIONS ⟷ CHAOTIC BILLIARD MOTION WITHIN WEYL CHAMBER OF KAC-MOODY ALG.

$D=11$ SUGRA ⟷ Weyl group of E_{10}

$D=n+1$ GR ⟷ Weyl group of AE_n

T.D. Henneaux, Julia, Nicolai '01

USUAL $D=4$ GR ⟷ KAC-MOODY $\widehat{\widehat{AE_3}} = \widehat{SL(2,\mathbb{R})} =$ GEROCH

HYPERBOLIC

FINITE-DIMENSIONAL (SIMPLE) LIE ALGEBRAS

SU(2) ~ SO(3)

$$[J_x, J_y] = i\, J_z$$
$$[J_y, J_z] = i\, J_x$$
$$[J_z, J_x] = i\, J_y$$

OR, BETTER,

$$\boxed{J_\pm \equiv J_x \pm i J_y}$$

'RAISING GENERATOR'

$$[J_z, J_+] = + J_+$$
$$[J_z, J_-] = - J_-$$

'LOWERING GENERATOR'

TO BE DIAGONALIZED: 'CARTAN GENERATOR'

SU(3)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4, \lambda_5, \lambda_6, \lambda_7, \quad \lambda_8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

- CAN DIAGONALIZE TWO GENERATORS $h_1, h_2 \sim \lambda_3, \lambda_8$
- AND THEN FIND ANALOGS OF 'RAISING' AND 'LOWERING' GENERATORS

$$\begin{cases} [h_1, E_1^+] = + E_1^+ \\ [h_2, E_1^+] = 0 . E_1^+ \end{cases} \quad \begin{cases} [h_1, E_2^+] = -\frac{1}{2} E_2^+ \\ [h_2, E_2^+] = \frac{\sqrt{3}}{2} E_2^+ \end{cases} \quad \begin{cases} [h_1, E_3^+] = +\frac{1}{2} E_3^+ \\ [h_2, E_3^+] = \frac{\sqrt{3}}{2} E_3^+ \end{cases}$$

GENERAL (SIMPLE) LIE ALGEBRA

RANK

∃ CARTAN SUBALGEBRA: LINEAR SPACE $\mathbb{R}^{r}$

$$[h', h''] = 0 \qquad \mathfrak{h} = \{ \beta^a h_a \, ; \ a = 1, 2, \dots, r \}$$

r independent Cartan gators

COORDINATES IN CARTAN SPACE

$$h = \sum_{a=1}^{r} \beta^a h_a$$

∃ TRIANGULAR DECOMPOSITION $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$

LOWERING GATORS F_α CARTAN RAISING GATORS $E\alpha$

degeneracy index

$$\boxed{[h , E^{(s)}_\alpha] = \alpha(h) \, E^{(s)}_\alpha}$$

Cartan Raising ROOT α

≡ EIGENVALUE OF ad_h

AS A LINEAR FORM OF $h \in \mathfrak{h}$

$$h = \beta^a h_a \Rightarrow \alpha(h) = \alpha_a \beta^a = \alpha(\beta)$$

$$[h , F^{(s)}_\alpha] = -\alpha(h) \, F^{(s)}_\alpha$$

$$[E^{(s)}_\alpha , E^{(t)}_\beta] = c^{(stu)}_{\alpha\beta} \, E^{(u)}_{\alpha+\beta}$$

'SIMPLE ROOTS' $\alpha_1, \dots, \alpha_r$

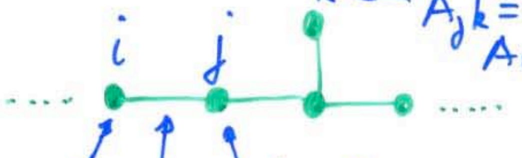
ANY ROOT $\alpha = \sum_{j=1}^{r} \underset{\text{INTEGER}}{n_j} \alpha_j$

IE ANY RAISING $E_\alpha = [E_{\alpha_1}[E_{\alpha_1}[\dots[E_{\alpha_2}[\dots[E_{\alpha_3}\dots]]]$

CARTAN MATRIX

$$\alpha_j(h_i) \equiv A_{ij} \in \mathbb{Z}$$

DYNKIN DIAGRAM



QUADRATIC FORMS

CARTAN $A_{ij} \to \langle h, h\rangle \to \langle x, x\rangle$

KAC-MOODY ALGEBRAS

GIVEN CARTAN MATRIX A_{ij}

$i, j = 1, 2, \dots, r$

$\updownarrow$

ENCODED IN

$$[h_i, h_j] = 0$$

$$[h_i, e_j] = A_{ij} e_j$$

$$[h_i, f_j] = -A_{ij} f_j$$

$$[e_i, f_j] = \delta_{ij} h_j$$

$$\mathrm{ad}_{e_i}^{1-A_{ij}} e_j = 0$$

$$\mathrm{ad}_{f_i}^{1-A_{ij}} f_j = 0$$

+ JACOBI IDENTITIES

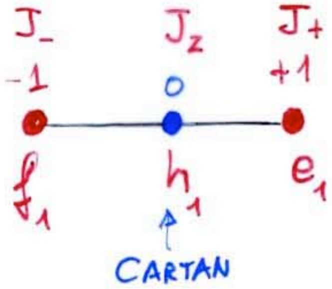
SU(2)

ROOT DIAGRAM

CARTAN MATRIX

DYNKIN DIAGRAM

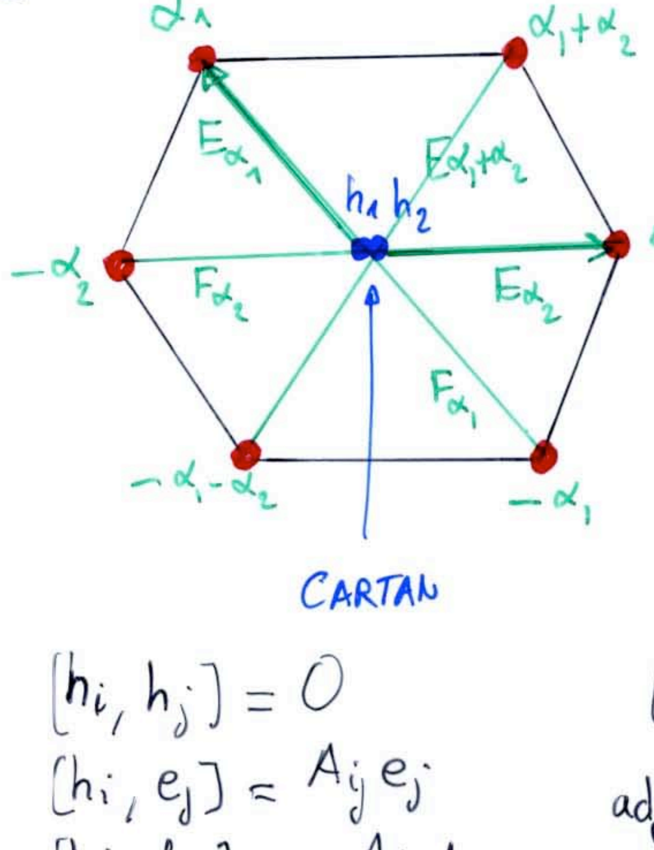
RANK=1 → 1 DIM

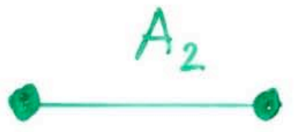


$$(2)$$

SU(3)

RANK = 2 → 2 DIM





$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$i,j = 1 \dots r$

$$[h_i, h_j] = 0$$
$$[h_i, e_j] = A_{ij}\, e_j$$
$$[h_i, f_j] = -A_{ij}\, f_j$$

+ Jacobi identities

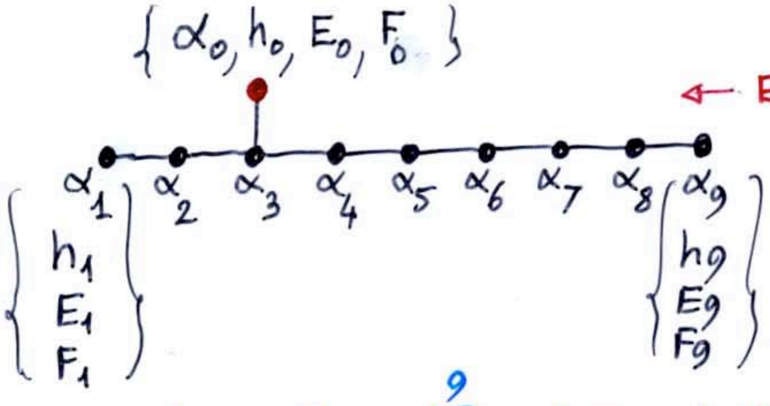
$$[e_i, f_j] = \delta_{ij}\, h_j$$

$$\left.\begin{aligned} \mathrm{ad}_{e_i}^{1-A_{ij}}\, e_j &= 0 \\ \mathrm{ad}_{f_i}^{1-A_{ij}}\, f_j &= 0 \end{aligned}\right\} \text{Serre relations}$$

E_{10}

rank 10 : dim $\mathfrak{h} = 10$ AND $\exists$ 10 basic raising g^{ators} E_{α_i}

10 SIMPLE ROOTS

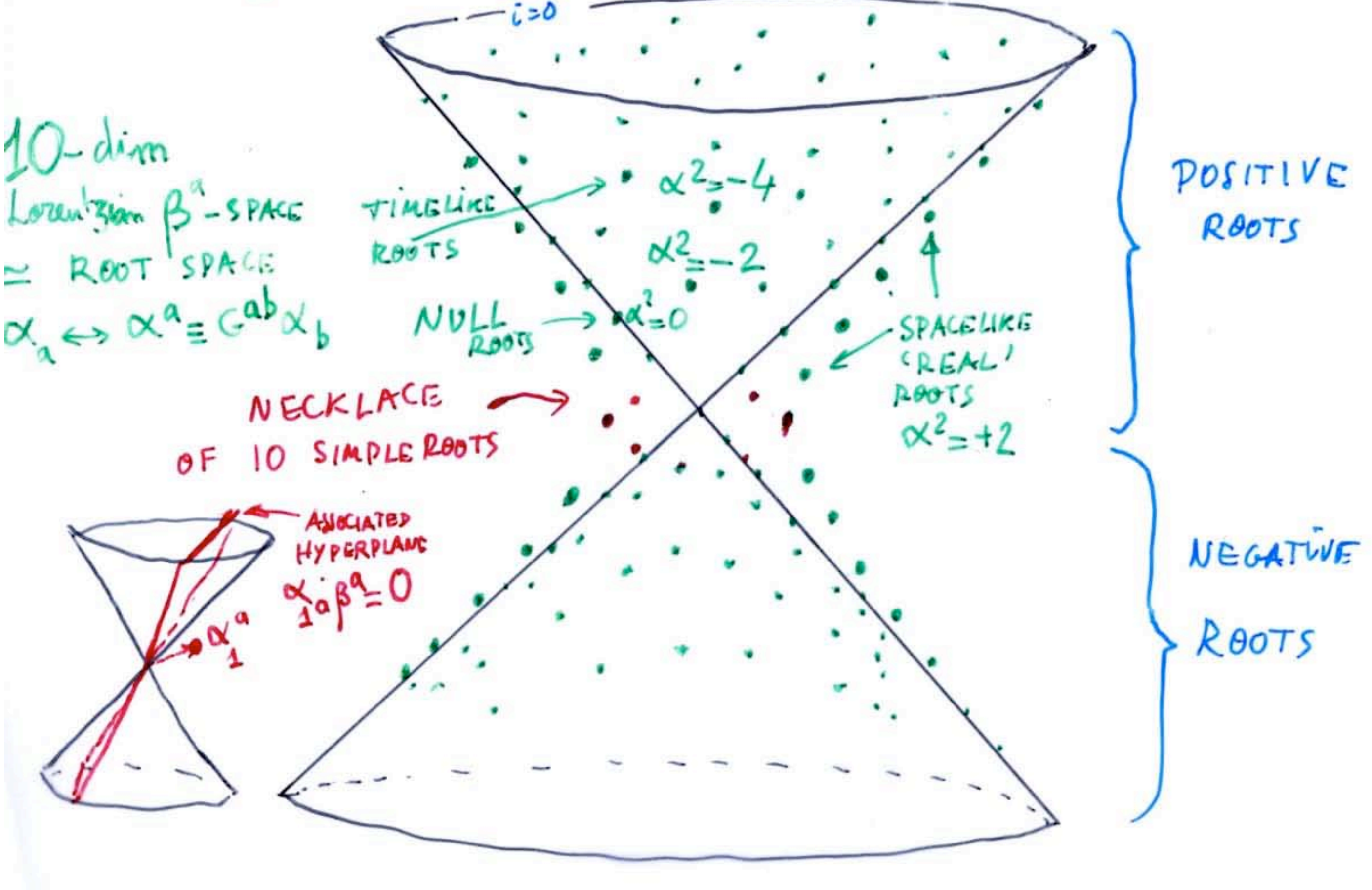


$\leftarrow E_0 = E^{123} \in E^{[abc]}$; $F_0 = F_{123} \in F_{[abc]}$

$\leftarrow$ WITH h_0 DEFINES GL_{10} SUBALGEBRA

$$\{\text{ALL ROOTS}\} = \underbrace{\left\{\alpha = \sum_{i=0}^{9} n^i \alpha_i \; ; n^i \in \mathbb{N}\right\}}_{\text{POSITIVE ROOTS}} \cup \underbrace{\left\{\alpha = -\sum_{i=0}^{9} n^i \alpha_i \; ; n^i \in \mathbb{N}\right\}}_{\text{NEGATIVE ROOTS}}$$

height $ht[\alpha] = \sum_{i=0}^{9} n_i$



KAC-MOODY ALGEBRAS

Lie algebra $\mathfrak{g} \equiv \mathfrak{g}(A)$

$\underset{\text{rank}}{\underset{\uparrow}{r}} \times r$ integer-valued Cartan matrix $\begin{cases} a_{ii} = +2 \\ -a_{ij} \in \mathbb{N}\ ;\ i \neq j \\ a_{ij} = 0 \Rightarrow a_{ji} = 0 \end{cases}$

eg

$$A_{E_{10}} = \begin{bmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 2 & -1 & & & & \\ & & & & -1 & 2 & -1 & & & \\ & & & & & -1 & 2 & -1 & & -1 \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & -1 & 2 & \\ & & & & & & -1 & & & 2 \end{bmatrix}$$

DYNKIN DIAGRAM :

Chevalley-Serre generators : $\underbrace{\{h_i, e_i, f_i\}_{i=1\dots r}}$ $\sim (J_z, J_+, J_-)$ for $SU(2) \sim A_1$

$[h_i, h_j] = 0 \leftarrow$ Cartan subalgebra

$[e_i, f_j] = \delta_{ij}\, h_j$

\+ Jacobi $[a,[b,c]] + \cdots = 0$

$[h_i, e_j] = A_{ij}\, e_j$

\+ Serre relations

$\mathrm{ad}_{e_i}^{1-A_{ij}}\, e_j = 0 = \mathrm{ad}_{f_i}^{1-A_{ij}}\, f_j$

$[h_i, f_j] = -A_{ij}\, f_j$

FIRST FACT

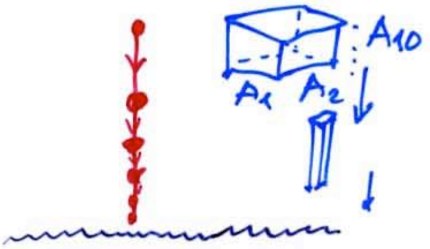
$D=11$ $\qquad$ $F_4 = dA_3$

BOSONIC
$SUGRA_{11}$

$$S=\int d^D x\,\sqrt{-G}\left[R(G)-\frac{1}{2\cdot 4!}F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho}\right]+\frac{1}{(12)^4}\int A_3\wedge F_4\wedge F_4$$

$\Downarrow$

CHAOTIC BEHAVIOUR
NEAR A COSMOLOGICAL SING.



$\Downarrow$

BILLIARD DESCRIPTION
OF THIS CHAOS

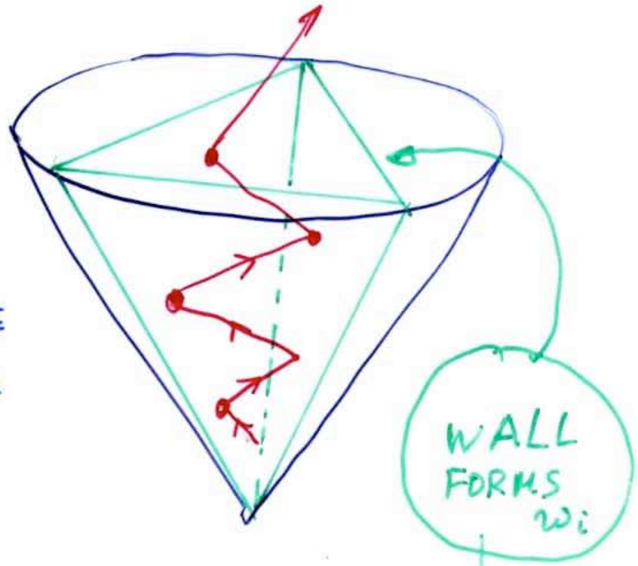
$\Downarrow$

TODA-LIKE SYSTEM

IN
β - SPACE

$A_a \equiv e^{-\beta^a}$

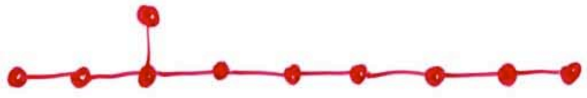
$$\mathcal{H}=G_{ab}\pi^a\pi^b+\sum_A c_A\, e^{-2w_A(\beta)}$$



GEOMETRY OF 'COSMOLOGICAL BILLIARD'

RESCALED GRAM MATRIX $\quad 2\,\dfrac{w_i\cdot w_j}{w_i\cdot w_i}$

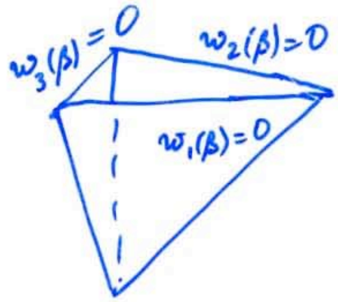
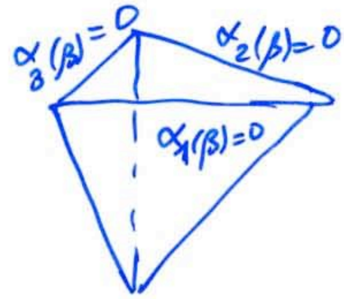
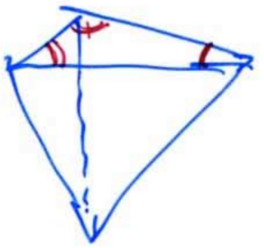
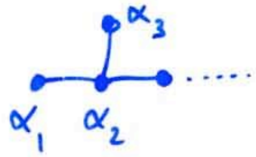
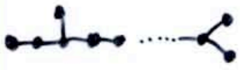
$\Longleftrightarrow$



$\equiv$ E_{10} DYNKIN DIAGRAM

COSMOLOGICAL SINGULARITIES AND KAC-MOODY ALGEBRAS

DICTIONARY

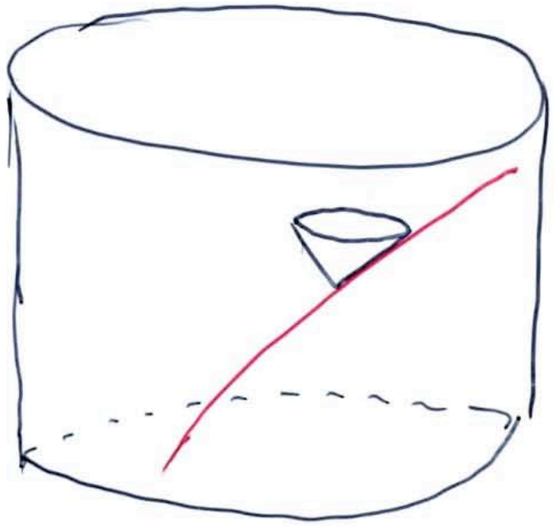
GRAVITY		KAC-MOODY	
LOGARITHMIC SCALE FACTORS	$\beta^a = -\ln A_a$	CARTAN ELEMENT	$h = \sum_{a=1}^{r} \beta^a h_a$
DOMINANT COSMOLOGICAL WALLS	$e^{-2w_i(\beta)}$	SIMPLE ROOTS	$\alpha_i(\beta)$
COSMOLOGICAL BILLIARD		WEYL CHAMBER	
GEOMETRY OF COSMO. BILLIARD		DYNKIN DIAGRAM	
RESCALED GRAM MATRIX	$2\frac{w_i \cdot w_j}{w_i \cdot w_i}$	CARTAN MATRIX	$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i} = \begin{pmatrix} 2 & -1 & 0 & \\ -1 & 2 & \cdots & -1 \\ \vdots & \cdot & 2 & \\ -2 & & & \ddots \end{pmatrix}$
PURE GRAVITY IN $D = d+1$			$AE_d = A^{\wedge\wedge}_{d-2}$
SUGRA_{11} ; IIA_{10} , IIB_{10}			E_{10}
I_{10} ; HET_{10}			BE_{10}
BOSONIC STRING_D			DE_D

HIDDEN SYMMETRY GROUP ?

SOME EVIDENCE FOR A HOLOGRAPHIC CORRESPONDENCE

$$\mathrm{SUGRA}_{11} \longleftrightarrow \text{MASSLESS SPINNING PARTICLE ON COSET } E_{10}/K(E_{10})$$

$$\begin{matrix} G_{\mu\nu}(t,\vec{x}) \\ A_{\mu\nu\lambda}(t,\vec{x}) \\ \psi_{\rho}(t,\vec{x}) \end{matrix} \longleftrightarrow$$



$$S_{11}=\int d^{11}x\Big\{\frac{E}{4}R(G) - \frac{E}{48}(dA_3)^2+\frac{2}{(12)^4}F_4\wedge F_4\wedge A_3 - \frac{i}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{i}{96}\left(\bar{\psi}_\mu\Gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_\nu+12\bar{\psi}^\alpha\Gamma^{\beta\gamma}\psi^\delta\right)F_{\alpha\beta\gamma\delta}+\dots\Big\}$$

+ LOOP CORRECTIONS

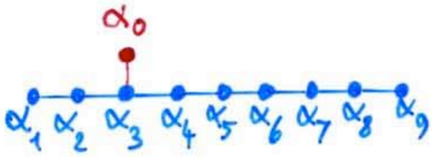
$$\longleftrightarrow$$

$$S_1^{\mathrm{COSET}}=\int dt\Big\{\frac{1}{4n}\langle P(t)|P(t)\rangle - i\left(\Psi(t)|\mathcal{D}^{vs}\Psi(t)\right)_{vs} + \left(\chi(t)|P(t)\odot\Psi(t)\right)_{s}\Big\}$$

T.D., Henneaux, Nicolai '02; T.D, Kleinschmidt, Nicolai '06; de Buyl, Henneaux, Paulot '06

Euros4

DECOMPOSING E_{10} W.R.T. $GL(10)$ SUBALGEBRA



← $SL(10) = A_9$ SUBALGEBRA

"LEVEL" ℓ : $\alpha = \ell\,\alpha_0 + \sum_{j=1}^{9} m^j \alpha_j$

$\ell = 0$ GL(10) GENERATORS $K^a{}_b$ $[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b$

$\ell = \pm 1$ $E^{[a_1 a_2 a_3]}$, $F_{[a_1 a_2 a_3]}$

3 INDICES

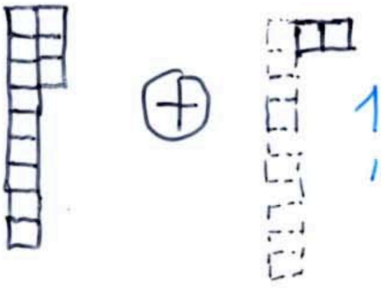
$\ell = \pm 2$ $E^{[a_1 \cdots a_6]}$, $F_{[a_1 \cdots a_6]}$

6 INDICES

$\ell = \pm 3$ $E^{[a_0 | a_1 \cdots a_8]}$, $F_{[a_0 | a_1 \cdots a_8]}$

9 INDICES

$\ell = \pm 4$

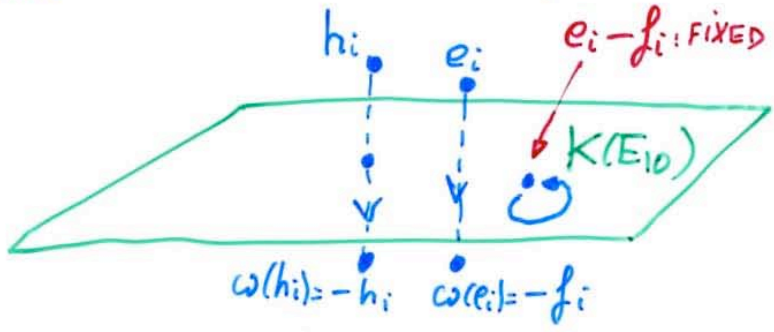


12 INDICES

•
•
•

$K(E_{10})$: MAXIMAL COMPACT SUBGROUP OF THE CANONICAL REAL FORM OF E_{10}

FIXED SET OF
CHEVALLEY INVOLUTION ω



$GL(10)$ DECOMPOSITION OF $K(E_{10})$

$J^{ab} = K^a{}_b - K^b{}_a$: $SO(10)$ $[J^{ab}, J^{cd}] = 4\,\delta^{[b}_{[c}\, J^{a]}{}_{d]}$

$J^{a_1a_2a_3} = E^{a_1a_2a_3} - F_{a_1a_2a_3}$ $\quad [J^{a_1a_2a_3}, J^{b_1b_2b_3}] = J^{a_1a_2a_3b_1b_2b_3} - 18\,\delta^{a_1b_1}\delta^{a_2b_2}J^{a_3b_3}$

$J^{a_1a_2\cdots a_6} = E^{a_1\cdots a_6} - F_{a_1\cdots a_6}$ $\quad [J^3, J^6] = J^9 + J^3$

$J^{a_0|a_1\cdots a_8} = E^{a_0|a_1\cdots a_8} - F_{a_0|a_1\cdots a_8}$ $\quad [J^3, J^9] = J^{12} + J^6$

⋮

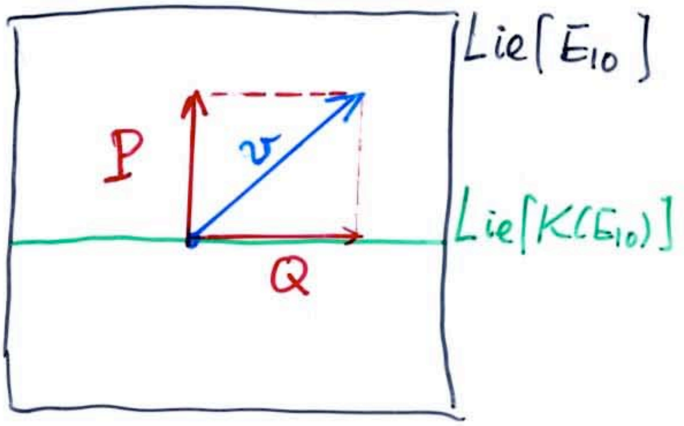
$D=1$ $E_{10}/K(E_{10})$ COSET MODEL

GROUP ELEMENT $g(t) \in E_{10}$

$\text{Lie}[E_{10}]$ 'VELOCITY':

$$\boxed{v \equiv \frac{dg}{dt}\, g^{-1}}$$

DECOMPOSING v



$$\underset{\text{Lie}(E_{10})}{\underset{\cap}{v}} \equiv \underset{\text{Lie}(K(E_{10}))^{\perp}}{\underset{\cap}{P}} + \underset{\text{Lie}(K(E_{10}))}{\underset{\cap}{Q}}$$

'VERTICAL' 'HORIZONTAL'

COSET ACTION

BOSONIC PART

'VERTICAL' PART OF VELOCITY $v = \dot{g}\, g^{-1}$

$$\boxed{S_{1\,\mathrm{BOS}}^{\mathrm{COSET}} = \int dt \, \frac{1}{4n} \langle P(t) \mid P(t) \rangle}$$

UNIQUE INVARIANT QUADRATIC FORM OF $Lie(E_{10})$

SIGNATURE: $\underset{\text{CARTAN}}{-+++++++++}\;;\;\underset{K(E_{10})^{\perp}}{+++++\cdots\cdots}\;;\;\underset{K(E_{10})}{\cancel{-\,-\,-\,-\,-\cdots}}$

SYMMETRY: $g(t) \longrightarrow \underset{\text{LOCAL } K(E_{10})}{k(t)}\; g(t)\; \underset{\text{GLOBAL } E_{10}}{g_0}$

$$P(t) \longrightarrow k(t)\, P(t)\, k(t)^{-1}$$

$$Q(t) \longrightarrow k(t)\, Q(t)\, k^{-1}(t) + \partial_t k\, k^{-1}$$

HORIZONTAL VELOCITY: ↑ $K(E_{10})$ CONNECTION

FERMIONIC PART

$$\boxed{S_{1\,\mathrm{FERM}}^{\mathrm{COSET}} = -\frac{i}{2}\int dt \left(\Psi(t) \mid \overset{vs}{D}\, \Psi(t)\right)_{vs} + \int dt \left(\chi(t) \mid P(t) \circ \Psi(t)\right)_{s}}$$

'VECTOR-SPINOR REPRES. OF $K(E_{10})$': $\Psi = (\psi_a, \overset{?}{\ldots})$

EXPLICIT PARAMETRIZATION OF $E_{10}/K(E_{10})$

$$g(t) = e^{h^a{}_b(t) K^b{}_a} \; e^{\frac{1}{3!} A_{a_1a_2a_3}(t) E^{a_1a_2a_3} + \frac{1}{6!} A_{a_1\cdots a_6} E^{a_1\cdots a_6} + \frac{1}{9!} A_{a_0|a_1\cdots a_8} E^{a_0|a_1\cdots a_8} + \cdots}$$

$GL(10) : K^a{}_b$

$h^a{}_b$

$g^{ab}(t) = (e^h)^a{}_c (e^h)^b{}_c$

$A_{a_1a_2a_3}$

$A_{a_1\cdots a_6}$

$A_{a_0|a_1\cdots a_8}$

$+ \cdots$

indices raised by g^{ab}

$$S_1^{E_{10}/K(E_{10})} = \int \frac{dt}{n(t)} \left[\frac{1}{4} \left(g^{ac} g^{bd} - g^{ab} g^{cd} \right) \dot{g}_{ab} \dot{g}_{cd} + \frac{1}{2} \frac{1}{3!} \dot{A}_{a_1a_2a_3} \dot{A}^{a_1a_2a_3} \right.$$
$$\left. + \frac{1}{2} \frac{1}{6!} DA_{a_1\cdots a_6} DA^{a_1\cdots a_6} + \frac{1}{2} \frac{1}{9!} DA_{a_0|a_1\cdots a_8} DA^{a_0|a_1\cdots a_8} + \cdots \right.$$

$$DA_{a_1\cdots a_6} = \dot{A}_{a_1\cdots a_6} + 10 \, A_{[a_1\cdots a_3} \dot{A}_{a_4\cdots a_6]}$$

$$DA_{a_0|a_1\cdots a_8} = \dot{A}_{a_0|a_1\cdots a_8} + 42 \, A_{\langle a_1\cdots a_3} \dot{A}_{a_4\cdots a_9\rangle} - 42 \, \dot{A}_{\langle a_1\cdots a_3} A_{a_4\cdots a_9\rangle}$$
$$+ 280 \, A_{\langle a_1\cdots a_3} A_{a_4\cdots a_6} \dot{A}_{a_7\cdots a_9\rangle}$$

$\langle \ldots \rangle$ = projection on Young

CORRESPONDENCE $E_{10}/K(E_{10})$ COSET ⟺ $SUGRA_{11}$

$$\mathcal{F}_4 = d\mathcal{A}_3$$

$$\mathcal{L}_1^{E_{10}} \sim (g^{-1}\dot{g})^2 + (\dot{A}_3)^2 + (\dot{A}_6 + A_3\dot{A}_3)^2 + (\dot{A}_9 + A_6\dot{A}_3 + A_3A_3\dot{A}_3)^2 + \dots$$

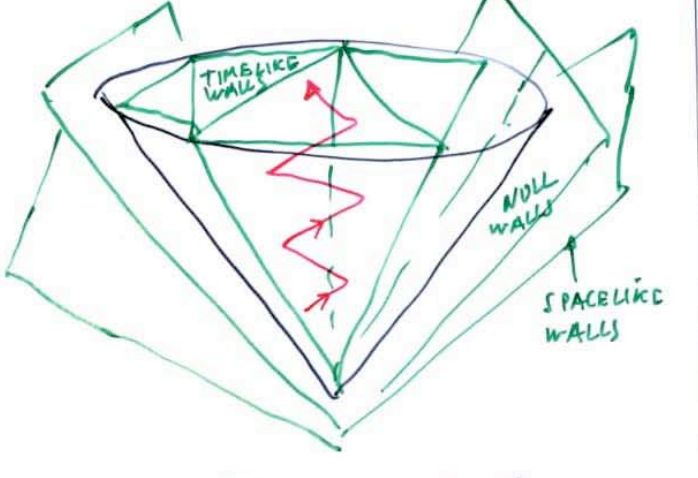
$$\Downarrow$$

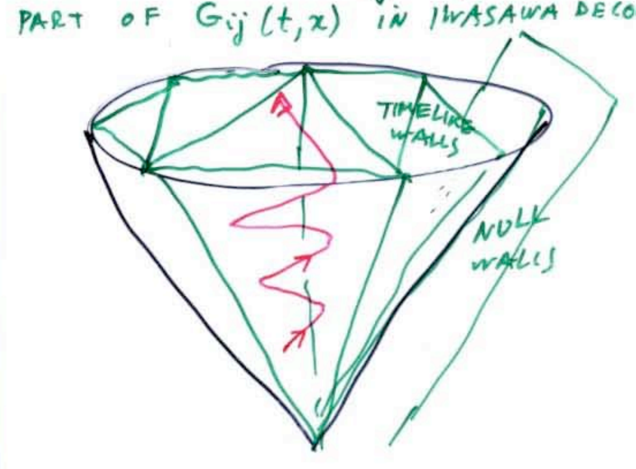
BILLIARD WITH INFINITE NUMBER OF EXPONENTIAL WALLS FOR CARTAN ELEMENT $\beta^{i}(t)$

$$\mathcal{L}_{11}^{SUGRA} = \int d^{11}x \left[\sqrt{-G}\left[R(G) - \frac{(d\mathcal{A}_3)^2}{48}\right] + \frac{1}{(12)^4}\mathcal{F}_4 \wedge \mathcal{F}_4 \wedge \mathcal{A}_3\right]$$

$$\Downarrow$$

BILLIARD WITH LARGE BUT FINITE # OF EXPONENTIAL WALLS FOR $\beta^a(t,x)$, DIAGONAL PART OF $G_{ij}(t,x)$ IN IWASAWA DECOMP.





$$\mathcal{H}_1 = \frac{1}{2}G^{\mu\nu}\pi_\mu\pi_\nu + \sum C_\alpha(q,p)\, e^{-2\alpha(\beta)}$$

$$\alpha(\beta) = \sum_i m^i \alpha_i(\beta) \qquad m^i \in \mathbb{N}$$

SIMPLE ROOTS

$$\mathcal{H}_{11} = \frac{1}{2}G^{\mu\nu}\pi_\mu\pi_\nu + \sum_A C_A(Q,P,\partial_x\beta,\partial_xQ,..) \times e^{-2w_A(\beta)}$$

$$w_A(\beta) = \sum_i m^i w_i(\beta)$$

DOMINANT WALLS

DICTIONARY

$$g^{ab}(t) = (e^h)^a{}_c (e^h)^b{}_c = G^{ab}(t,\vec{x}_0)$$

WRT A SPECIAL FRAME

$$\dot{A}_{a_1a_2a_3}(t) = \mathcal{F}_{0a_1a_2a_3}(t,\vec{x}_0)$$

$$\theta^a(x) = e^a_i(x)\,dx^i$$

$$DA^{a_1\cdots a_6}(t) = g^{a_1a_1'}\cdots g^{a_6a_6'}[\dot{A}_{a_1'\cdots a_6'} + 10A_{[3}\dot{A}_{3]}] = -\frac{1}{4!}\varepsilon^{a_1\cdots a_6 b_1\cdots b_4}\mathcal{F}_{b_1\cdots b_4}(t,\vec{x}_0)$$

$$DA^{b|a_1\cdots a_8}(t) = g^{\cdots}[\dot{A}_9 + 42A_3\dot{A}_6 + 280A_3A_3\dot{A}_3] = +\frac{3}{2}\varepsilon^{a_1\cdots a_8 b_1b_2}C^b{}_{b_1b_2}(\vec{x}_0)$$

$$d\theta^a = \frac{1}{2}C^a{}_{bc}\,\theta^b \wedge \theta^c$$

THE CORRESPONDENCE WORKS FOR ALL TERMS OF HEIGHT ≤ 29

$$\sum_i m^i \leq 29 \qquad \sum_i m^i \leq 29$$

HIGHER-ORDER M-THEORY CORRECTIONS AND E_{10}

$$S_{\text{M-THEORY}} = \int \frac{d^{11}x}{\ell_P^9} \left[R - F^2 + A \wedge F \wedge F \right]$$

$$+ \int \frac{d^{11}x}{\ell_P^3} \left[t_8 t_8 R^4 + \frac{2-1}{4} \varepsilon_8 \varepsilon_8 R^4 - 4 \varepsilon_{11} A_3 / t_2 R^4 \ldots \right] + \ldots$$

$$R(DF)^2$$

$$+ \ldots + F8 \ldots$$

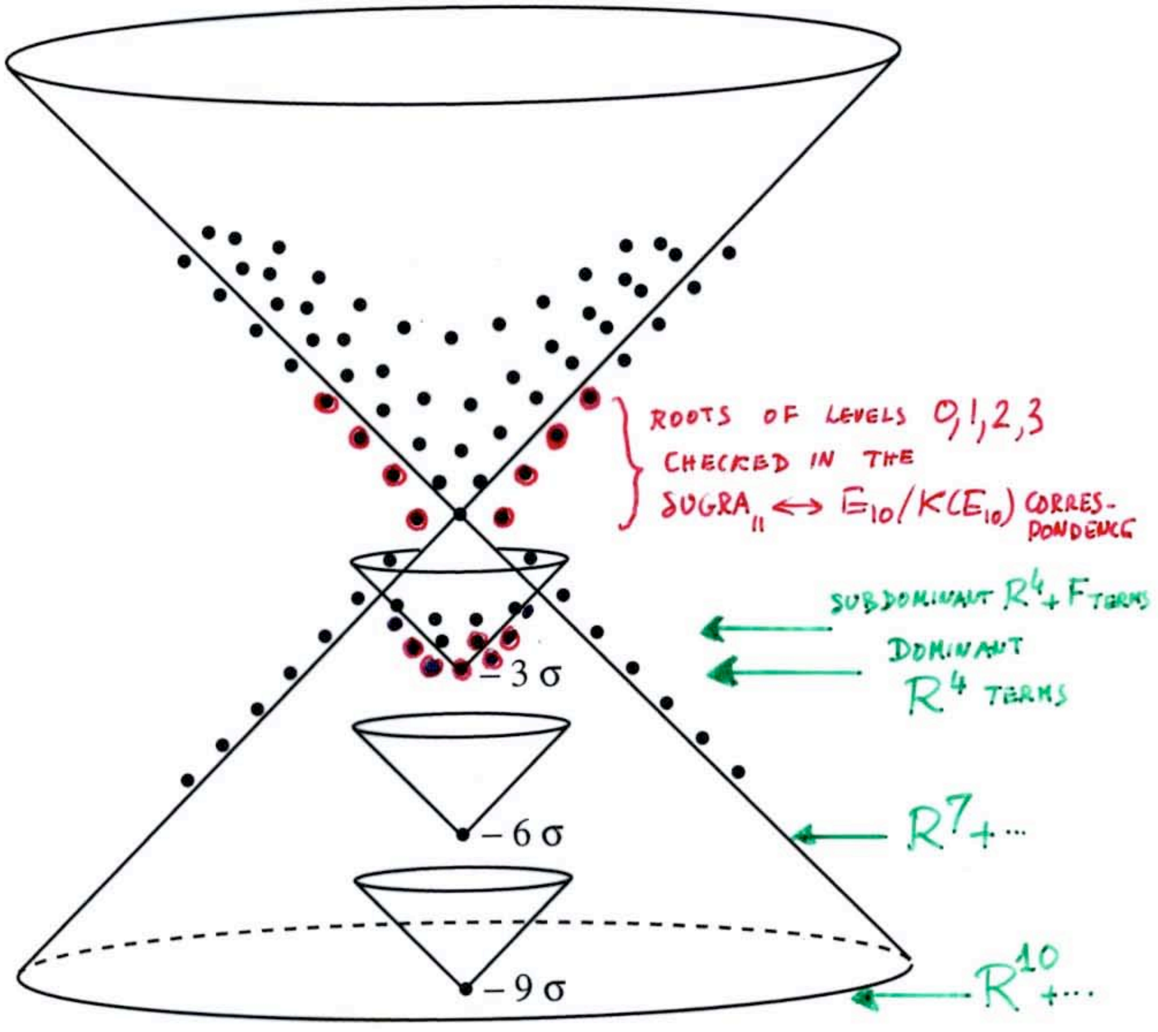
Green Schwarz '82, Sakai Tanii '87, Deser Seminara '99, Green Vanhove '97 ... Tseytlin
Green Gutperle Vanhove ... Duff, Liu Minasian

? EFFECT OF $R^4 + \ldots$ ON COSMOLOGICAL BILLIARD ?
(IN INTERMEDIATE ASYMPTOTICS)

$\Rightarrow$ ADD NEW WALLS, WHICH ARE FOUND TO STILL CORRESPOND TO SOME ROOTS OF E_{10}

(T.D., Nicolai, 2005)

ROOTS OF E_{10}



ROOTS OF LEVELS 0,1,2,3 CHECKED IN THE $SUGRA_{11} \leftrightarrow E_{10}/K(E_{10})$ CORRESPONDENCE

SUBDOMINANT $R^4 + F$ TERMS

DOMINANT R^4 TERMS

$R^7 + \cdots$

$R^{10} + \cdots$

CORRESPONDENCE AND FERMIONS

T.D., Kleinschmidt, Nicolai '06 ; de Buyl, Henneaux, Paulot '06

$$S_{1\,\mathrm{FERM}}^{\mathrm{COSET}} = -\frac{i}{2}\int dt\,\left(\Psi(t)\,|\,\overset{\mathrm{VS}}{\mathcal{D}}\Psi(t)\right)_{\mathrm{VS}} + \int dt\,\left(\chi(t)\,|\,P(t)\odot\Psi(t)\right)_{S}$$

$$S_{11\,\mathrm{FERM}}^{\mathrm{SUGRA}} = \int d^{11}x \left\{ -\frac{i}{2}\bar{\psi}_{\mu}(t,x)\,\Gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho} - \frac{i}{96}\left(\bar{\psi}_{\mu}\Gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_{\nu} + 12\,\bar{\psi}^{\alpha}\Gamma^{\alpha\beta}\psi^{\gamma}\right)\mathcal{F}_{\alpha\beta\gamma\delta} + \dots \right\}$$

EXTENDED DICTIONARY

COSET $\qquad\qquad$ SUGRA

$$e^{i}{}_{a} \equiv (\exp h)^{i}{}_{a} \quad \longleftrightarrow \quad \theta^{i}_{m}\, e_{(10)}{}^{m}{}_{a}$$

VECTOR-SPINOR REP. OF KE_{10} $\quad \Psi(t) = (\psi_a, \dots)\quad \psi_a \longleftrightarrow \psi^{(11)}_{\alpha} = E_{(11)}{}^{\mu}{}_{\alpha}\,\psi^{(11)}_{\mu}(t,\vec{x}_0)$

IN GAUGE

$$\psi^{(11)}_{0} = \Gamma_0\Gamma^{a}\psi^{(11)}_{a}$$

$\exists$ KE_{10} INVARIANT VECTOR-SPINOR QUADR. FORM

$$(\Psi\,|\,\Phi)_{\mathrm{VS}} = \psi_a^{T}\,\Gamma^{ab}\,\phi_b$$

VECTOR-SPINOR KE_{10}-INVARIANT CONNECTION $\overset{\mathrm{VS}}{\mathcal{D}}\Psi(t) = (\partial_t - \overset{\mathrm{VS}}{Q})\Psi(t)$ $\longleftrightarrow$ SUGRA FERMIONIC EOM

$$\hat{\mathcal{E}}_A := \Gamma^{B}\left[(D_A + \mathcal{F}_A)\psi_B^{(11)} - (D_B + \mathcal{F}_B)\psi_A^{(11)}\right] = 0$$

WITH $\mathcal{F}_A = +\frac{1}{144}\left[\Gamma_A^{BCDE} - 8\,\delta_A^{B}\,\Gamma^{CDE}\right] \times F^{(11)}_{BCDE}$

CONSTRAINTS

(Damour, Kleinschmidt, Nicolai '07)

THE CORRESPONDENCE

$$SUGRA_{11} \longleftrightarrow \text{MASSLESS PARTICLE ON } E_{10}/K(E_{10})$$

CONCERNS THE (10+1)-SPLIT, GAUGE-FIXED EVOLUTION EQS OF SUGRA

IN GAUGE: $\tilde{N} \equiv \frac{N}{\sqrt{g}} = 1$, $N_i \equiv g_{0i} = 0$, $A_{0ij} = 0$

NEED TO SUPPLEMENT EVOLUTION EQS BY CONSTRAINTS

$$\boxed{S_{SUGRA} = \int d^d x \left[\pi^{ij} \dot{g}_{ij} + \pi^{ijk} \mathring{A}_{ijk} - \tilde{N} \mathcal{H} - N^i \mathcal{H}_i - A_{0ij} \mathcal{G}^{ij} \right]}$$

HAMILTONIAN CONSTRAINT

MOMENTUM (SPATIAL DIFFEOS)

GAUSS

+ BIANCHI CONSTRAINTS FOR $R_{\mu\nu\rho\sigma}$ und $F_{\mu\nu\rho\sigma}$

$$\boxed{\text{SUGRA CONSTRAINTS} \overset{?}{\longleftrightarrow} \text{COSET CONSTRAINTS}}$$

$\mathcal{H}(x) \approx 0$, $\mathcal{H}_i(x) \approx 0$, ...

∞ # OF ∂_x^k CONDNS

?

HAMILTONIAN $\Leftrightarrow$ NULL GEODESIC

$$\mathcal{H} \approx 0 \leftrightarrow \langle \mathcal{P}, \mathcal{P} \rangle \approx 0$$

OK MODULO SOME $\ell=3$ TERMS

? OTHER CONSTRAINTS

CONSTRAINTS AND THE E_{10} COSET MODEL

TD, Kleinschmidt, Nicolai / 07

GRAVITY BOSONIC CONSTRAINTS

$$e_i(t,x_0)\,,\; e^{ij}(t,x_0)\,,\; e_{i_1\cdots i_5}(t,x_0)\,,\; e^{i_0}_{[ijk]}(t,x_0)\,,\;\cdots$$

$$\updownarrow$$

COSET BOSONIC CONSTRAINTS (REDEFINED)

$$\overset{(-3)}{\mathcal{L}}{}^{m_1\cdots m_9} = 28\,\overset{(-1)}{J}{}^{m_1m_2m_3}\,\overset{(-2)}{J}{}^{m_4\cdots m_9} + \overset{(0)}{J}{}^{m_1}{}_{p}\,\overset{(-3)}{J}{}^{p|m_2\cdots m_9}$$

$$\overset{(-4)}{\mathcal{L}}{}^{m_1\cdots m_{12}} = \overset{(-2)}{J}\cdots\overset{(-2)}{J}\cdots + \overset{(-1)}{J}\cdots\overset{(-3)}{J}\cdots$$

$$\overset{(-5)}{\mathcal{L}}{}^{m_1\cdots m_{15}} = \overset{(-2)}{J}\cdots\overset{(-3)}{J}\cdots\cdots$$

$$\overset{(-6)}{\mathcal{L}}{}^{m_1\cdots m_{18}} = \overset{(-3)}{J}\cdots\cdots\overset{(-3)}{J}\cdots\cdots$$

SUGAWARA-LIKE $\boxed{\overset{(-\ell)}{\mathcal{L}}{}^{n_1\cdots n_{3\ell}} = \sum_m \overset{(-\ell+m)}{J}\cdots\overset{(-m)}{J}\cdots}$

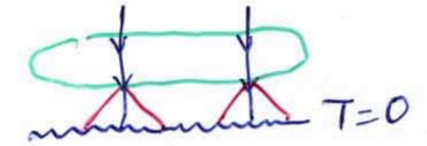
WHERE $\boxed{J = g^{-1}Pg \in Lie(E_{10})}$

IS THE CONSERVED 1-d E_{10} NOETHER CURRENT (OR CHARGE)

- $\overset{(-\ell)}{\mathcal{L}}$ ARE STRONGLY CONSERVED
- $\overset{(-\ell)}{\mathcal{L}}$ IS A LINEAR REPRESENTATION OF E_{10}^{+} ← PARABOLIC SUBGROUP $GL_{10} \oplus E^{\alpha>0}$
- GENERALIZES SUGAWARA CONSTR. $\underset{\text{VIRASORO}}{L_m} \propto \sum_n k_{ab}\, \underset{\uparrow\text{ AFFINE KAC-MOODY}}{J^a_{m-n} J^b_n}$
- $\overset{(-\ell)}{\mathcal{L}}$ **IS** NOT A REPRESENTATION OF FULL E_{10}

CONCLUSIONS

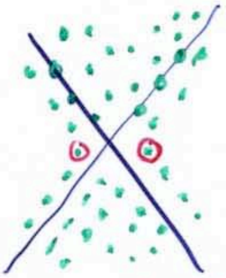
- THE 'NEAR COSMOLOGICAL SINGULARITY LIMIT' SUGGESTS ∃ HIDDEN $E_{10}(\mathbb{R})$ SYMMETRY OF $SUGRA_{11}$ (AND M-THEORY) [+ $AE_m(\mathbb{R})$ FOR GR_{m+1} ?]

- 

BKL GRADIENT EXPANSION

$$\partial_{x^1}^{k_1}\partial_{x^2}^{k_2}\cdots\partial_{x^{10}}^{k_{10}} \ll \partial_T^{k_1+k_2+\cdots+k_{10}}$$

⟷



HEIGHT EXPANSION IN HYPERBOLIC KAC-MOODY ALGEBRA

ROOT: $\alpha = m_0\alpha_0 + m_1\alpha_1 + \cdots + n_9\alpha_9$

- SUGGESTS ∃ A 'HOLOGRAPHIC' (OR 'PHOTOGRAPHIC' (POLYAKOV)) CORRESPONDENCE GRAVITY IN $D=11$ DYNAMICS ⟷ $D=1$ PARTICLE DYNAMICS ON (∞-DIM) COSET SPACE

- OPTIMISTICALLY
  - → BACKGROUND-INDEPENDENT FORMULATION OF (A SECTOR OF) M-THEORY
  - → ? NEW DESCRIPTION OF THE (QUANTUM) NATURE OF SPACE-(TIME) AT PLANCK SCALE VIA A 'DE-EMERGENCE' OF SPACE NEAR A SINGULARITY

'SPACE' I.E. $\left\{\begin{matrix} G_{\mu\nu}(t,x) \\ A_{\mu\nu\lambda}(t,x) \\ \psi_\mu(t,x) \end{matrix}\right\} \longrightarrow$ INFINITE TOWER OF LIE-ALGEBRAIC VARIABLES $\left\{\begin{matrix} g(t) \in E_{10}/K(E_{10}) \\ \underline{\Psi}(t) \end{matrix}\right.$

OPEN ISSUES

- HOW TO GO BEYOND HEIGHT 29 IN TESTING THE CONJECTURE ?
- SUGAWARA-LIKE CONSTRAINTS $\mathcal{L}^{(-n)} \sim J \otimes J$?

ARE THEY DESCRIBING THE 'GAUGE SYMMETRY OF M-THEORY'

AS A VAST GENERALIZATION OF THE USUAL SUGAWARA CONSTRUCTION:

$$L_m \sim J_{AFFINE} \otimes J_{AFFINE} \longleftrightarrow \text{CONFORMAL SYMMETRY}$$

- MEANING OF BREAKING $E_{10} \to E_{10}^{+}$?

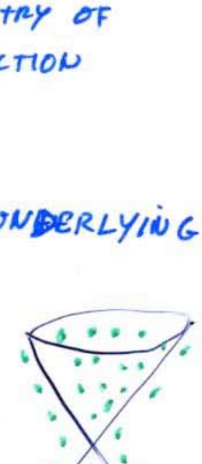
? SIMILAR TO STRING: $$\underset{\uparrow}{L_n} = \frac{1}{2} \sum_m \eta_{\mu\nu} \underset{\uparrow}{J^{\mu}_{n-m}} J^{\nu}_m$$

SPECTRUM GENERATING ALGEBRA

AUXILIARY SYMMETRY OF GAUGE-FIXED ACTION

MAYBE E_{10} COSET MODEL = GAUGE-FIXED VERSION OF AN UNDERLYING GAUGE-INVARIANT ACTION

- INFINITE # OF CONSTRAINTS $\mathcal{L}^{(-\ell)} = 0$ WELCOME FOR REDUCING THE # DOF TO ≃ SUGRA (OR M-THY?)



- QUANTIZED COSET ACTION ~ $\Box \Psi(g) = 0$

+ TOROIDAL COMPACTIFICATION $[\Rightarrow E_{10}(\mathbb{Z})]$ (Hull, Townsend '95)

$\Rightarrow$ MODULAR FORM OVER $E_{10}(\mathbb{Z}) \backslash E_{10}(\mathbb{R}) / K(E_{10}(\mathbb{R}))$?

(Ganor '99, Brown, Ganor, Helfgott '04)