Parma

4

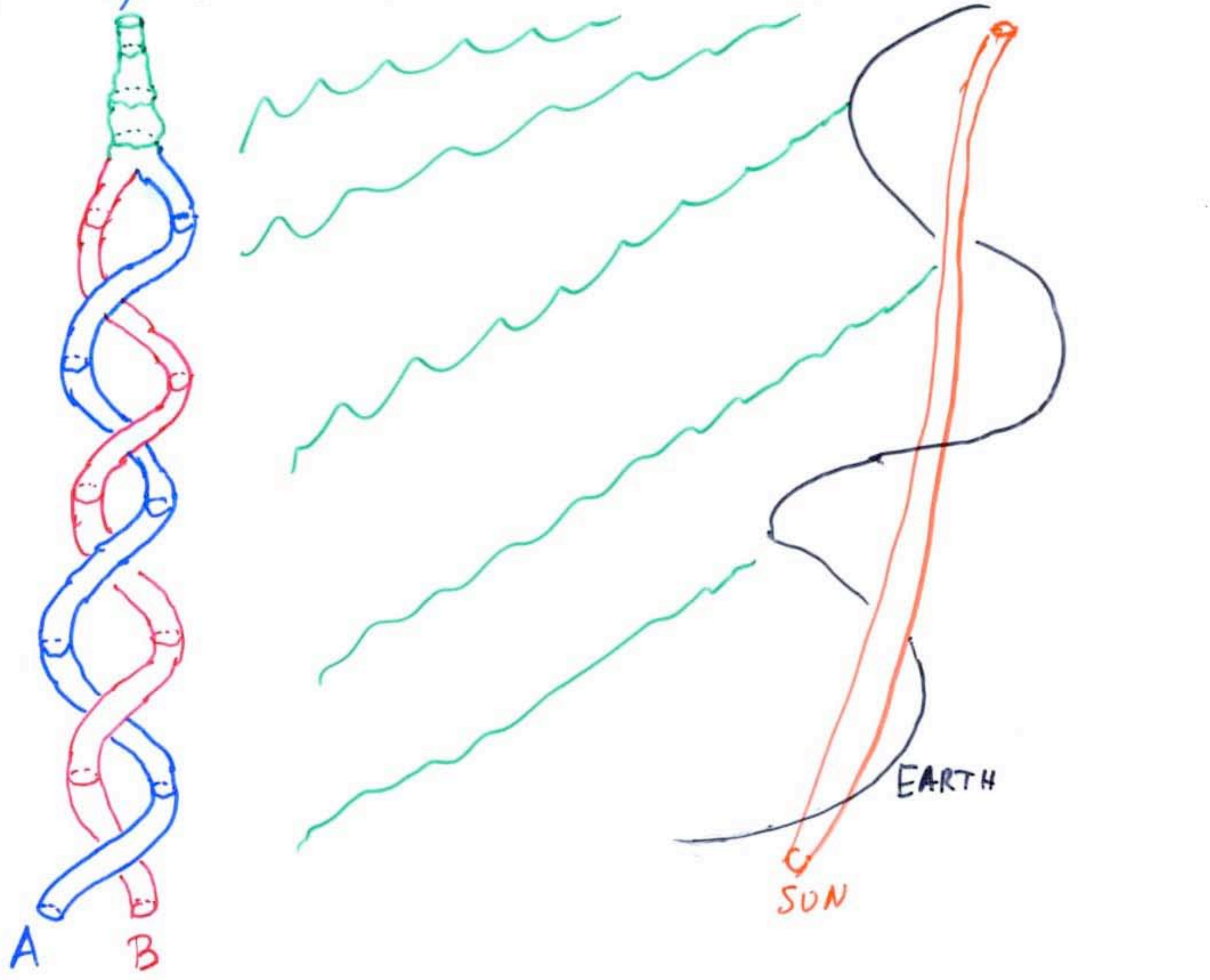
GRAVITATIONAL WAVES

FROM

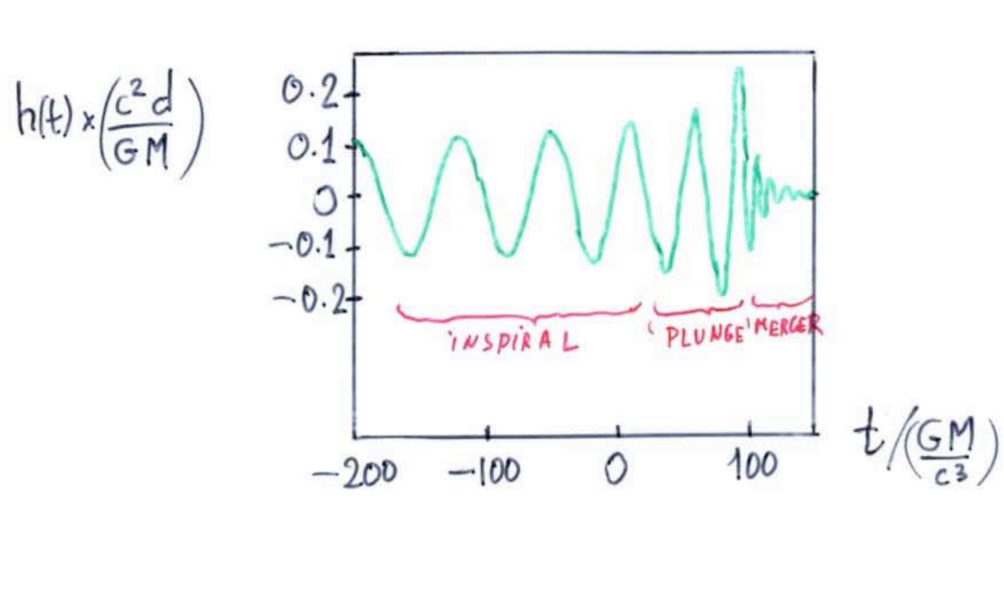
COALESCING BLACK HOLES

GRAVITATIONAL WAVES :

NOTABLY FROM COALESCING BINARY BLACK HOLES



GRAVITATIONAL WAVE SIGNAL EXPECTED ON EARTH



CHALLENGES

ANALYTICAL APPROACHES

- NEED EQS OF MOTION TO VERY HIGH PERTURBATION ORDER
- NEED GW GENERATION FORMALISM TO HIGH PERTURBATION ORDER
- NEED TO PACKAGE THIS PERTURBATIVE INFORMATION IN A FORM WHICH REMAINS VALID DURING BOTH INSPIRAL AND PLUNGE, AND WHICH CAN CONNECT WITH MERGER AND RINGDOWN
- (QUASI-) ANALYTICAL DESCRIPTION IS NEEDED FOR COVERING THE FULL PARAMETER SPACE :

$m_1, m_2, \vec{S}_1, \vec{S}_2$, ECCENTRICITY, ...

NUMERICAL APPROACHES

- NEED STABLE CODES
- NEED TO DESCRIBE B. HOLES
- NEED TO EXTRACT GW WAVEFORMS
- LIMITED TO A FEW ORBITS BEFORE MERGER
- LIMITED TO EXPLORING A FEW SAMPLE POINTS IN THE PARAMETER SPACE
- GIVES CRUCIAL NON PERTURBATIVE INFORMATION THAT GOES BEYOND WHAT CAN BE ANALYTICALLY COMPUTED

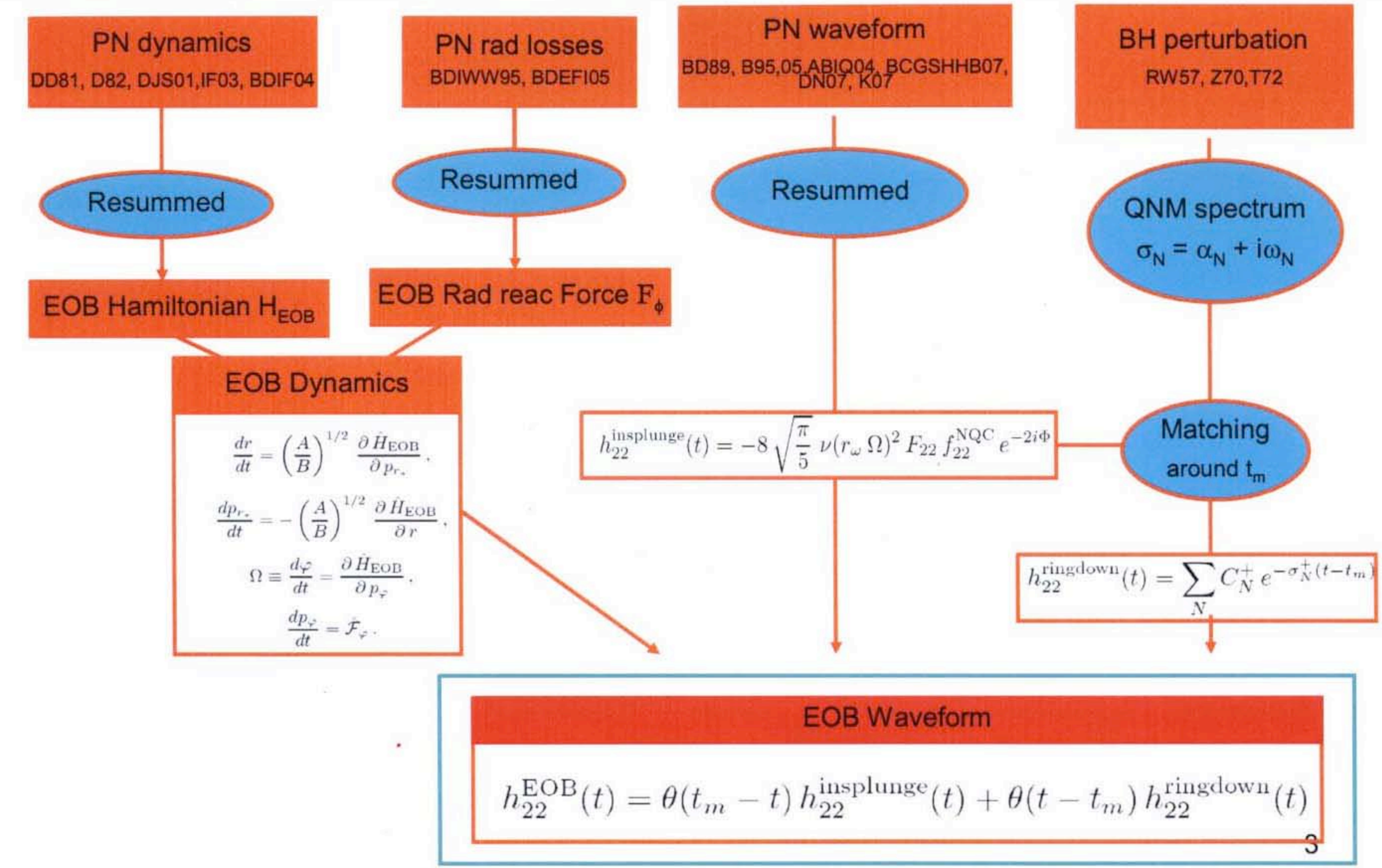
COMPLEMENTARITY

AIM: TAKE ADVANTAGE OF FLEXIBILITY OF ANALYTICAL METHODS, NOTABLY OF RESUMMED (EFFECTIVE ONE BODY + PADÉ) PN METHODS, TO CONSTRUCT GW TEMPLATES THAT HAVE A GOOD PHASING ALL OVER INSPIRAL + PLUNGE + MERGER + RINGDOWN

TOOL: COMPARISON BETWEEN EOB-LIKE TEMPLATES/PREDICTIONS AND NUMERICAL RESULTS

- $\nu \equiv \frac{m_1 m_2}{(m_1+m_2)^2} \sim \frac{1}{4}$ COMPARABLE MASSES
- OR $\nu \ll 1$ EXTREME MASS RATIO

Structure of EOB formalism



WHAT IS NEEDED FOR DESCRIBING COALESCING BINARIES

EoM $\quad \dfrac{d^2\vec{x}_a}{dt^2} = \vec{a}_a^{\,CONS} + \vec{a}_a^{\,RR}$ $\qquad r h_{ij}^{TT} = \left[U_{ij} + U_{ijk} \frac{N^k}{c} + \dots \right]^{TT}$

$a = 1,2$

NEED RADIATION REACTION TO HIGHEST POSSIBLE ACCURACY

$$a^{RR} = \frac{GM}{r^2}\left[\frac{v^5}{c^5} + \frac{v^7}{c^7} + \underbrace{\frac{v^8}{c^8} + \frac{v^9}{c^9} + \frac{v^{10}}{c^{10}} + \frac{v^{11}}{c^{11}} + \frac{v^{12}}{c^{12}}}_{\text{HEURISTICALLY OBTAINED BY ASSUMING BALANCE OF } E,\, \vec{J}} \right]$$

WAVE FORM :

$$U_{ij} = \mu\, x^i x^j \left\{ 1 + \frac{v^2}{c^2} + \frac{v^3}{c^3} + \frac{v^4}{c^4} + \frac{v^5}{c^5} + \frac{v^6}{c^6} + \frac{v^7}{c^7} \right\}$$

\+ NEED CONSERVATIVE DYNAMICS TO HIGHEST POSSIBLE ACCURACY

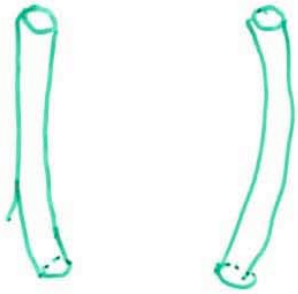
$$a^{CONS} = \frac{GM}{r^2}\left[1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} \right]$$

OR

$$H[x,p] = H_0 + \frac{1}{c^2} H_2 + \frac{1}{c^4} H_4 + \frac{1}{c^6} H_6$$

HIGH-PERTURBATION ORDER CALCULATIONS OF EQS. OF MOTION 6

- REPLACE



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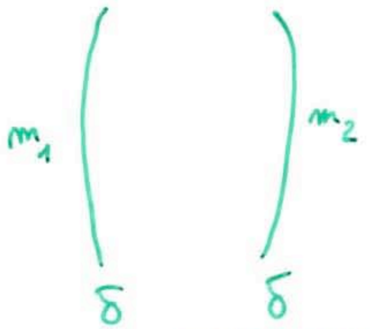


BECAUSE OF EFFACEMENT OF INTERNAL STRUCTURE UP TO $\left(\frac{v}{c}\right)^{10} \sim$ 5PN ORDER
(Damour, '82)

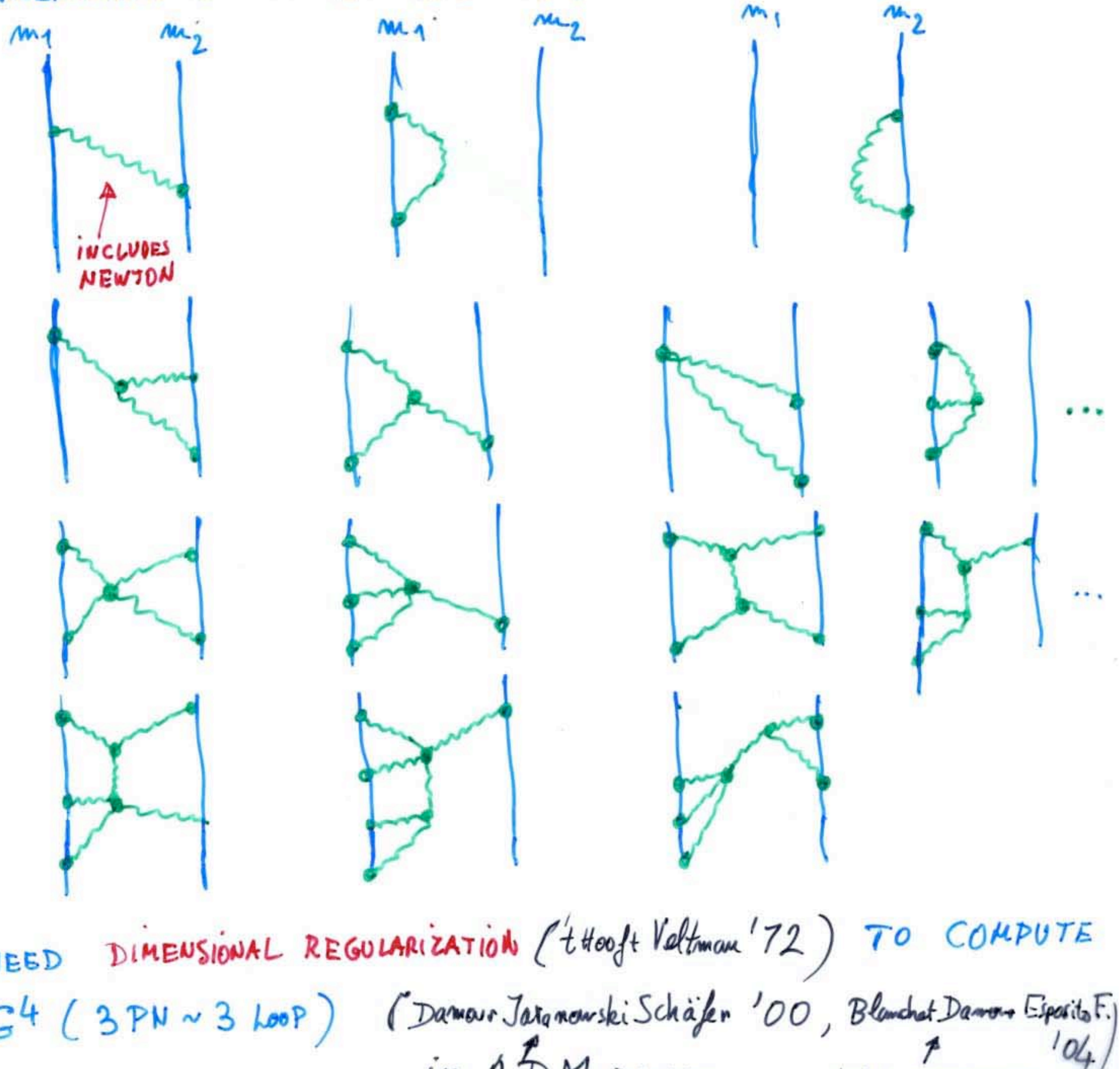
- CONSERVATIVE PART OF HAMILTONIAN :

G^1

G^2

G^3

G^4



- NEED DIMENSIONAL REGULARIZATION ('t Hooft Veltman '72) TO COMPUTE
G^4 (3PN ~ 3 LOOP) (Damour Jaranowski Schäfer '00, Blanchet Damour Esposito-F. '04)
in ADM FORMALISM   HARMONIC GAUGE

2-body Taylor-expanded 3PN Hamiltonian [JS98, DJS00,01]

$$H(\mathbf{x}_a,\mathbf{p}_a)=\sum_a m_a c^2 + H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)+\frac{1}{c^2}H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)+\frac{1}{c^4}H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)+\frac{1}{c^6}H_{\rm 3PN}(\mathbf{x}_a,\mathbf{p}_a)+\mathcal{O}\left(\frac{1}{c^8}\right)$$

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)=\sum_a\frac{\mathbf{p}_a^2}{2\,m_a}-\frac{1}{2}\sum_a\sum_{b\neq a}\frac{G\,m_a\,m_b}{r_{ab}}$$

$$H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\longleftrightarrow 2).$$

1PN

$$\begin{aligned}H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)=&\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.\\&\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]\\&+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\&-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\longleftrightarrow 2).\end{aligned}$$

2PN

$$\begin{aligned}H_{\rm 3PN}^{\rm reg}(\mathbf{x}_a,\mathbf{p}_a)=&-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left[-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{\left((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2\right)\mathbf{p}_1^2}{m_1^4m_2^2}+\frac{\left(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\right)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\right.\\&-10\frac{\left(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\right)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\&+\frac{\left(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2\right)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\\&\left.+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right]\\&+\frac{G^2m_1m_2}{r_{12}^2}\left[\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\,\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}\right.\\&+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}-\frac{1}{8}m_1\frac{\left(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)\right)(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\&-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\&\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]+\frac{G^3m_1m_2}{r_{12}^3}\left[-\frac{1}{48}\left(466\,m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\&+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(61\,m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\&\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right]+(1\longleftrightarrow 2).\end{aligned}\tag{12}$$

3PN

Defining H_{EOB} by thinking quantum-mechanically (Wheeler)

Real 2-body system (m_1, m_2) (in the c.o.m. frame) ⟺ **an effective particle of mass μ in some effective metric $g_{\mu\nu}^{eff}(M)$**

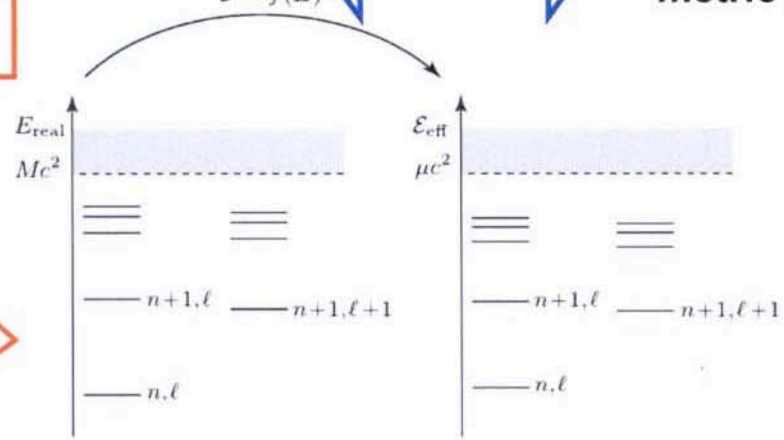


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

Sommerfeld "Old Quantum Mechanics":

$$\boxed{\begin{aligned} J &= \ell\hbar = \frac{1}{2\pi}\oint p_\varphi\, d\varphi \\ N &= n\hbar = I_r + J \\ I_r &= \frac{1}{2\pi}\oint p_r\, dr = I_r + J \end{aligned}}$$

$\boxed{H^{classical}(q,p)}$ — $\boxed{H^{classical}(I_a)}$ → $\boxed{H^{quantum}(I_a = n_a h)}$

Damour, Schäfer '88
Damour, Jaranowski, Schäfer, '00

SOMMERFELD-TYPE (DELAUNAY!) HAMILTONIAN

$\vec{q} = \vec{q}_1 - \vec{q}_2 \,;\; \vec{p} = \vec{p}_1 = -\vec{p}_2$

- CENTER-OF-MASS FRAME $H(\vec{q}_1 - \vec{q}_2, \vec{p}_1, \vec{p}_2) \rightarrow H(\vec{q}, \vec{p}, -\vec{p})$
- $H^{CoM}(\vec{q}, \vec{p}) = H_0(\vec{q}, \vec{p}) + \frac{1}{c^2} H_2(qp) + \frac{1}{c^4} H_4(qp) + \frac{1}{c^6} H_6(q,p)$

$$H_0(qp) = \frac{\vec{p}^{\,2}}{2\mu} + \frac{GM\mu}{|\vec{q}|}$$

SEPARATION OF VARIABLES $(\varphi, p_\varphi)\,;\,(\theta, p_\theta)\,;\,(r, p_r)$

ACTION VARIABLES (e.g. for $\theta = \frac{\pi}{2}$)

$$J = \ell \hbar = \frac{1}{2\pi} \oint p_\varphi \, d\varphi$$

$$I_r = \frac{1}{2\pi} \oint p_r \, dr$$

$$N = n\hbar = I_r + J$$

$\alpha \equiv \frac{G m_1 m_2}{\hbar}$

$$\Rightarrow \boxed{H_0(n,\ell) = -\frac{1}{2}\mu \left(\frac{GM\mu}{I_r + J}\right)^2 = -\frac{1}{2}\mu \frac{\alpha^2}{n^2}}$$

PRINCIPAL QUANTUM NUMBER

WHEN CONSIDERING $H_0 + \frac{1}{c^2} H_2 + \frac{1}{c^4} H_4 + \frac{1}{c^6} H_6$, MORE COMPLICATED CALCULATIONS, BUT SAME LOGIC:

$$\boxed{H^{2\text{-BODY}}(n,\ell) = -\frac{1}{2}\mu \frac{\alpha^2}{n^2} \left[1 + \frac{\alpha^2}{c^2}\left(\frac{c_{11}}{n\ell} + \frac{c_{20}}{n^2}\right) + \frac{\alpha^4}{c^4}\left(\frac{c_{13}}{n\ell^3} + \frac{c_{22}}{n^2\ell^2} + \frac{c_{31}}{n^3\ell} + \frac{c_{40}}{n^4}\right) + \dots\right]}$$

SOMMERFELD HAMILTONIAN FOR 'EFFECTIVE ONEBODY' PROBLEM

TO START WITH, CONSIDER ONE BODY OF MASS μ MOVING IN SOME 'EXTERNAL' METRIC

$$g^{ext}_{\mu\nu}(x)\, dx^\mu dx^\nu = -A(R)\,c^2 dT^2 + B(R)\,dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

UNKNOWN FUNCTIONS, TO BE DETERMINED

WRITE

$$A(R) = 1 + a_1 \frac{GM}{c^2R} + a_2\left(\frac{GM}{c^2R}\right)^2 + a_3\left(\frac{GM}{c^2R}\right)^3 + \dots$$

$$B(R) = 1 + b_1\left(\frac{GM}{c^2R}\right) + b_2\left(\frac{GM}{c^2R}\right)^2 + \dots$$

THEN CONSIDER HAMILTON-JACOBI EQ FOR EOB DYNAMICS

$$g^{\mu\nu}_{eff} \frac{\partial S_{eff}}{\partial x^\mu}\frac{\partial S_{eff}}{\partial x^\nu} + \mu^2c^2 = 0$$

- SEPARATE VARIABLES

$$S_{eff} = -\mathcal{E}_{eff} T + J_{eff}\,\varphi + S_{eff}(R)$$

- COMPUTE $I_R = \frac{1}{2\pi}\oint P_R\, dR$ AND THEN $H_{eff}(J_{eff}, N_{eff})$

$$\boxed{H^{eff}(n_{eff}, l_{eff}) = \mu c^2 - \frac{1}{2}\mu\frac{\alpha^2}{n_{eff}^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{c_{11}^{eff}}{n_{eff}\, l_{eff}} + \frac{c_{20}^{eff}}{n_{eff}^2}\right) + \frac{\alpha^4}{c^4}(\dots) + \dots\right]}$$

DICTIONARY BETWEEN REAL AND EFFECTIVE DYNAMICS

$$H^{real}(n, \ell) \longleftrightarrow H^{eff}(n_{eff}, \ell_{eff})$$

- IDENTIFY QUANTIZED INTEGERS : $\boxed{\begin{array}{l} n = n_{eff} \\ \ell = \ell_{eff} \end{array}}$

- LOOK FOR ANOTHER FUNCTION REALIZING ENERGY CORRESPONDENCE

$$\frac{E_{eff}}{\mu c^2} - 1 = f\left(\frac{E_{real}}{\mu c^2}\right) = \frac{E_{real}}{\mu c^2}\left(1 + \alpha_1 \frac{E_{real}}{\mu c^2} + \alpha_2 \left(\frac{E_{real}}{\mu c^2}\right)^2 + \dots\right)$$

MANY EQUATIONS FOR MANY UNKNOWNS $(a_1, a_2, a_3 \dots, b_1, b_2 \dots, \alpha_1, \alpha_2 \dots)$

- IF ONE ASSUMES $a_1 = -2$, $b_1 = 2$ (LINEARIZED SCHWARZSCHILD), ONE FINDS A UNIQUE SOLUTION FOR $a_2, a_3, b_2, \alpha_1, \alpha_2$ AT 2PN ($1/c^4$) WHICH CAN THEN BE EXTENDED @ 3PN IF ONE ADDS SOME $\vec{p}^{\,4}$ TERMS IN HAMILTON-JACOBI EQ.

VERY SIMPLE f

$$\boxed{\frac{E_{eff}}{\mu} = \frac{\mathcal{E}_{real}^2 - m_1^2 - m_2^2}{2 m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2 m_1 m_2}}$$

Explicit form of the effective metric

The effective metric at 3PN + a 4PN correction

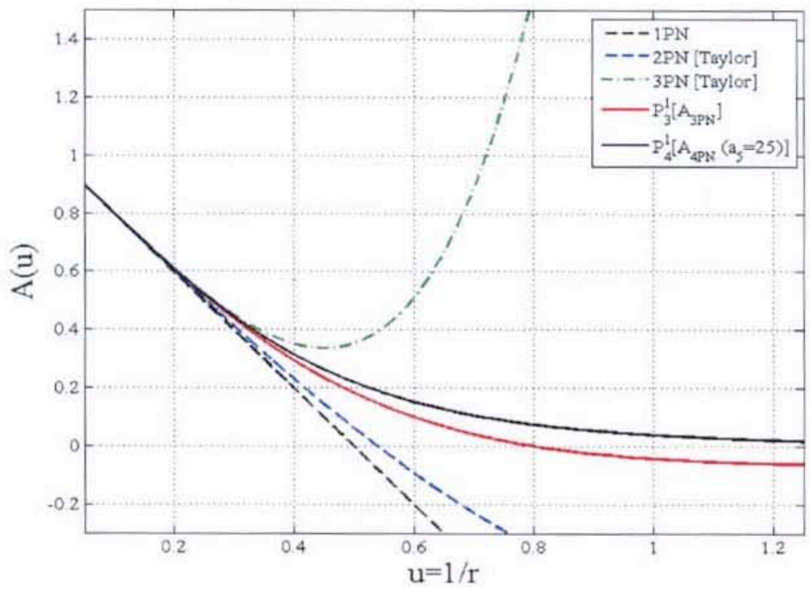
$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

where the coefficients are a ν-dependent "deformation" of the Schwarzschild ones:

$$(BA)^{3\text{PN}}(r) \equiv D^{3\text{PN}}(r) \equiv 1 - \frac{6\nu}{r^2} + 2(3\nu - 26)\frac{\nu}{r^3}.$$

$$A^{\text{Taylor}}(u) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu u^4 + a_5 \nu u^5 + \mathcal{O}(\nu u^6), \qquad u = 1/r$$

- Extremely compact representation of PN dynamics
- Bad behaviour at 3PN. Padé resummation of A(r) is needed to ensure that an effective horizon exists.
- Impose, by continuity with the Schwarzschild case, that A(r) has a simple zero at r~2.
- The a_5 constant parametrizes (yet) uncalculated 4PN corrections



The EOB Hamiltonian

The effective Hamiltonian (+quartic-in-momenta non-geodesic contribution)

$$\hat{H}_{\rm eff} \equiv \sqrt{p_{r_*}^2 + A\left(1 + \frac{p_\varphi^2}{r^2} + \boxed{z_3 \frac{p_{r_*}^4}{r^2}}\right)}.$$

$$\frac{dr_*}{dr} = \sqrt{\frac{B}{A}}$$

$$p_{r_*} = \left(\frac{A}{B}\right)^{1/2} p_r$$

The real EOB Hamiltonian of the binary system (from the energy map)

$$\hat{H}^{\rm EOB} = \frac{1}{\nu}\sqrt{1 + 2\nu\left(\hat{H}_{\rm eff} - 1\right)}$$

The Hamiltonian (and the related dynamics) depends, through the “potential” A(u), on the 4PN parameter a_5.
a_5 is a “free”parameter that needs to be fixed via comparisons with NR simulations.

$$M = m_1 + m_2$$

$$\nu = m_1 m_2/(m_1 + m_2)^2$$

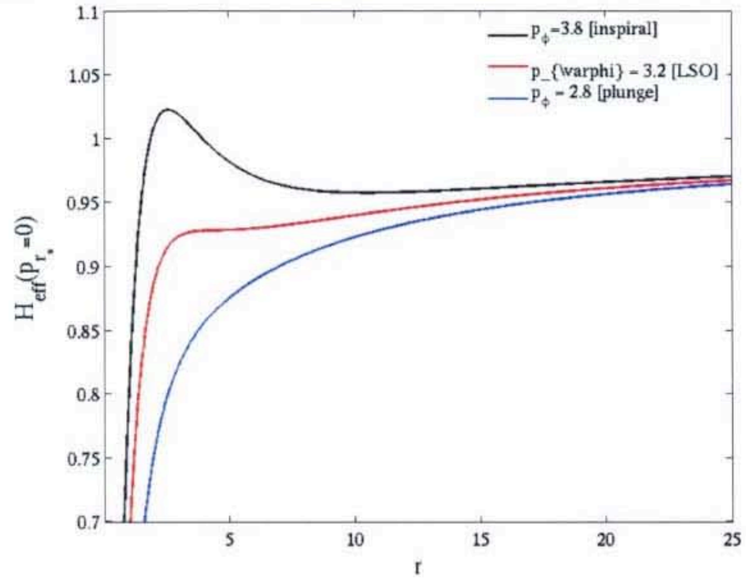
Hamilton's equation + radiation reaction

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi},$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi .$$



Angular momentum loss due to GW emission: start from the PN expression for *radiation reaction* that is explicitly known during the quasi-circular adiabatic inspiral (3.5PN + 4PN correction)

$$\hat{\mathcal{F}}_\varphi^{\rm Taylor} = -\frac{32}{5}\,\nu\,\Omega^5\,r_\omega^4\,\hat{F}^{\rm Taylor}(v_\varphi)$$

$$\begin{aligned}\hat{F}^{\rm Taylor}(v) &= 1 + A_2(\nu)\,v^2 + A_3(\nu)\,v^3 + A_4(\nu)\,v^4 + A_5(\nu)\,v^5 \\ &\quad + A_6(\nu, \log v)\,v^6 + A_7(\nu)\,v^7 + A_8(\nu = 0, \log v)\,v^8\end{aligned}$$

Needs resummation of energy flux!

The PN expansions are non-uniformly and non-monotonically convergent in the strong-field regime. One needs to "resum" them in some form in order to extend their validity during the late-inspiral and plunge

- Factorize a simple pole in the GW energy flux
- Resum using near-diagonal Padé approximants (DIS98)

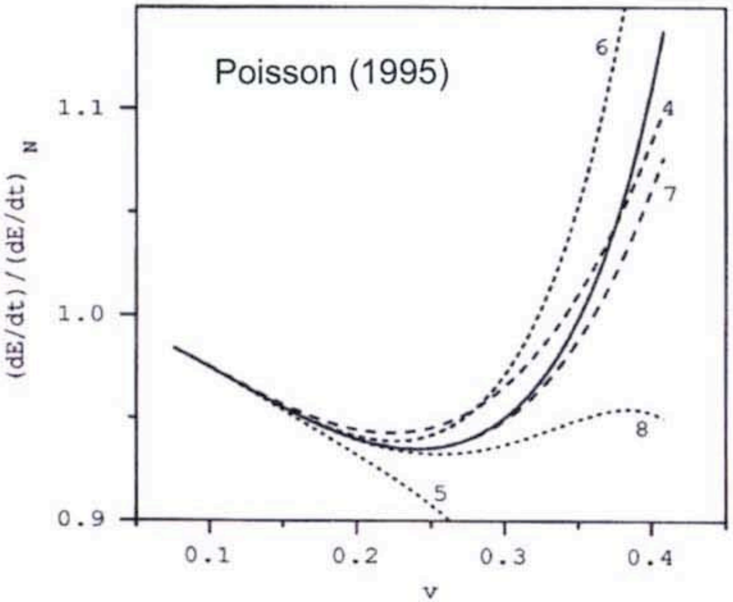


FIG. 1. Various representations of $(dE/dt)/(dE/dt)_N$ as a function of orbital velocity $v = (M/r)^{1/2} = (\pi M f)^{1/3}$. The solid curve represents the exact result $P(v)$, as calculated numerically. The various broken curves represent the post-Newtonian approximations $P_n(v)$, for $n = \{4, 5, 6, 7, 8\}$. The smallest value of v corresponds to an orbital radius r of $175M$; the largest value of v corresponds to $r = 6M$, the innermost stable circular orbit.

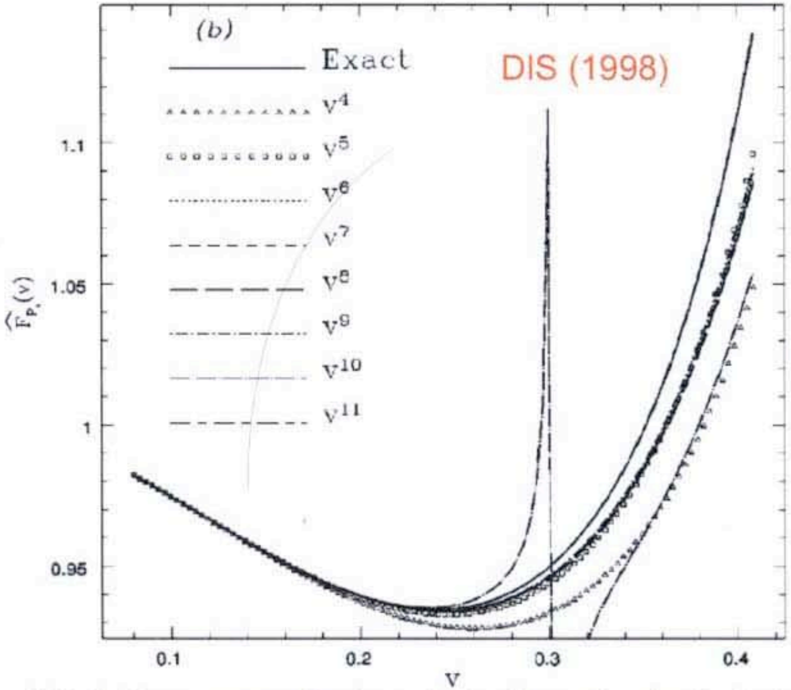


FIG. 3. Newton-normalized gravitational wave luminosity in the test particle limit: (a) T-approximants and (b) P-approximants.

Resumming radiation reaction

Padé resummation of $\hat{\mathcal{F}}_\varphi^{\rm Taylor} = -\frac{32}{5}\,\nu\,\Omega^5\,r_\omega^4\,\hat{F}^{\rm Taylor}(v_\varphi)$

Note prefactor
à la DG06

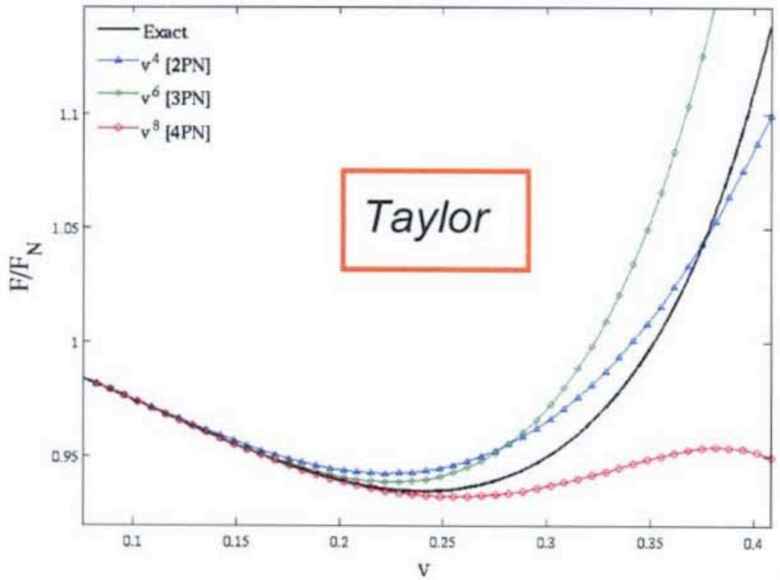
- factorize a pole parametrized by v_{pole}
- consider logarithms as coefficients
- use comparable-mass 3.5PN+test-mass 4PN flux
- choose P^4_4 which has no spurious poles
- add non-quasi-circular correction parametrized by a^{RR} [with ε=0.12]
- choose argument $v=r\Omega\psi^{1/3}$

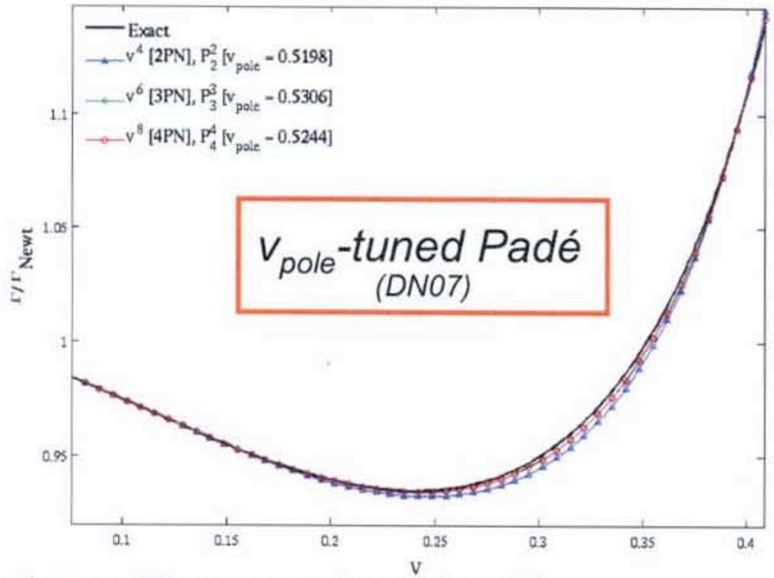
$$\boxed{\hat{F}^{\rm resummed}(v_\varphi) = \left(1-\frac{v_\varphi}{v_{\rm pole}}\right)^{-1} P^4_4\left[\left(1-\frac{v_\varphi}{v_{\rm pole}}\right)\hat{F}^{\rm Taylor}(v_\varphi)\right]\left(1+\bar{a}^{\rm RR}\frac{p_{r_*}^2}{(r\Omega)^2+\epsilon}\right)^{-1}}$$

3.5PN + test-mass 4PN contribution

v_{pole} , a^{RR} are (in addition to a5 in H_{EOB}) "free" parameters that need to be fixed via comparisons with NR simulations.

Comparing Taylor and (tuned) Padé in test-mass case





- Maximum difference on interval v<0.4:

Taylor(2PN): 0.039	Padé(2PN): 0.0069
Taylor(3PN): 0.130	Padé(3PN): 0.0033
Taylor(4PN): 0.189	Padé(4PN): 0.0035

Henri Padé, 1863-1953

Which one is the most "effective" ?

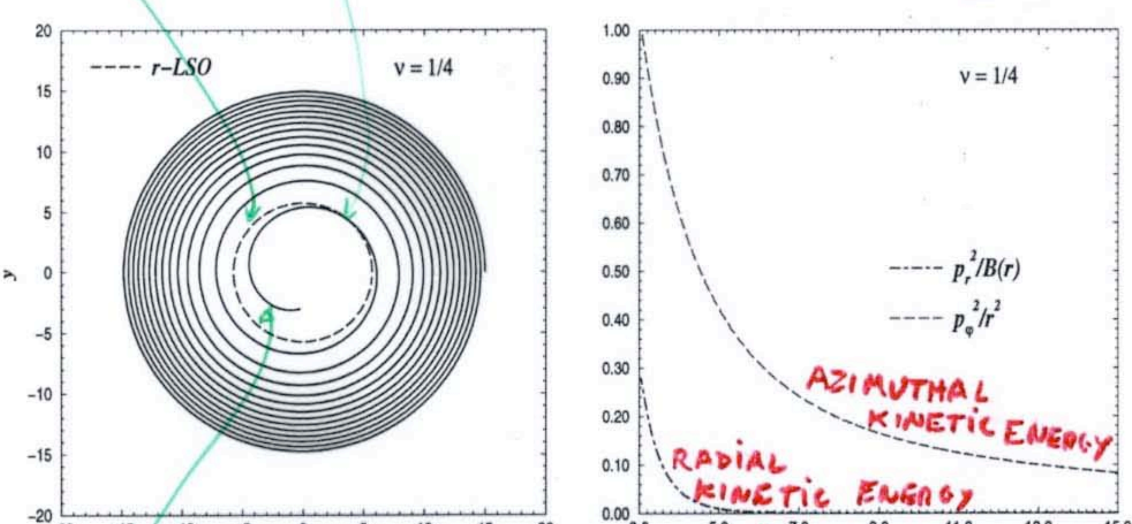
(RESUMMED) EFFECTIVE ONE BODY DYNAMICS

+ RESUMMED RADIATION REACTION (QUASI-CIRCULAR ORBITS)

(1) YIELDS INITIAL DYNAMICAL DATA $(q_1 q_2 P_1 P_2)$
AT BEGINNING OF PLUNGE: 0.6 ORBIT LEFT

GOOD IN VIEW OF STATE OF THE ART NUMERICAL SIMULATIONS (Pretorius 05)

TRANSITION INSPIRAL → PLUNGE WITH ARBITRARY MASS RATIO



LIGHT-RING

LSO

REMAINS QUASI-CIRCULAR DURING THE WHOLE PLUNGE

(2) FIRST ESTIMATE OF FULL WAVEFORM:
"6M" → "3M" ≃ MERGER

Resummed EOB *metric* gravitational waveform: inspiral+plunge

- Zerilli-Moncrief normalized (even-parity) waveform (*Real part gives h_+ & imaginary part gives h_x*).
- Multipolar decomposition (expansion on spin-weighted spherical harmonics) here, *l=m=2*.

$$\left(\frac{c^2}{GM}\right)\Psi_{22}^{\rm insplunge}(t) = -4\sqrt{\frac{\pi}{30}}\nu(r_\omega\Omega)^2 f_{22}^{\rm NQC} F_{22} e^{-2i\Phi}$$

New *PN-resummed (3^{+2}PN) correction factor* (*DN07a, 07b*): 3PN comparable mass + up to 5PN test-mass

$$F_{22}(t) = \hat{H}_{\rm eff} T_{22} f_{22}(x(t)) e^{i\delta_{22}(t)}$$

- H_{eff}: resums an infinite number of binding energy contributions
- $$T_{\ell m} = \frac{\Gamma(\ell+1-2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi\hat{\hat{k}}} e^{2i\hat{\hat{k}}\log(2kr_0)}$$ *resums* an infinite number of leading logarithms in tail effects (both amplitude and phase) obtained from exact solution of Coulomb wave problem
- $$f_{22}(x;\,\nu) = P_2^3\left[f_{22}^{\rm Taylor}(x;\,\nu)\right]$$ Padé-resummed remaining PN-corrected amplitude [flexibility in choice of argument *x(t)*]
- δ_{22}: computed at 3.5PN

Non-quasi-circular corrections to waveform amplitude and phase:

$$f_{22}^{\rm NQC} = \left[1 + a\frac{p_{r_*}^2}{(r\Omega)^2+\epsilon}\right]\exp\left(+ib\frac{p_{r_*}}{r\Omega}\right)$$

b=0; a is fixed by requiring that the maximum of the modulus of the waveform coincides with the maximum of the orbital frequency

EOB *metric* gravitational waveform: merger and ringdown

EOB approximate representation of the merger (DRT1972 inspired) :

- sudden change of description around the "EOB light-ring" $t=t_m$ (maximum of orbital frequency)
- "match" the insplunge waveform to a superposition of QNMs of the final Kerr black hole
- matching on a 5-teeth comb (*found efficient in the test-mass limit, DN07a*)
- comb of width 7M centered on the "EOB light-ring"
- use 5 positive frequency QNMs (found to be near-optimal in the test-mass limit)
- Final BH mass and angular momentum are computed from a fit to NR ringdown (*5 eqs for 5 unknowns*)

$$\Psi_{22}^{\text{ringdown}}(t) = \sum_N C_N^{+} e^{-\sigma_N^{+} t}$$

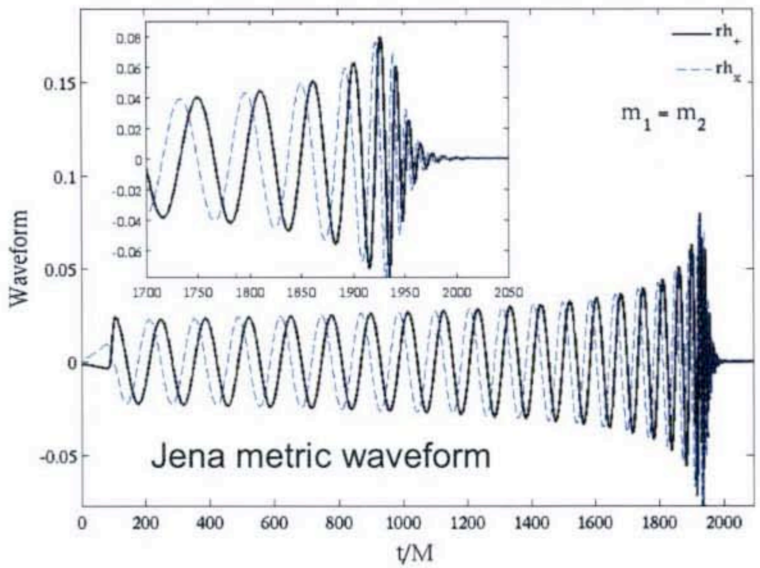
Total EOB waveform covering inspiral-merger and ringdown

$$h_{22}^{\text{EOB}}(t) = \theta(t_m - t)\, h_{22}^{\text{insplunge}}(t) + \theta(t - t_m)\, h_{22}^{\text{ringdown}}(t)$$

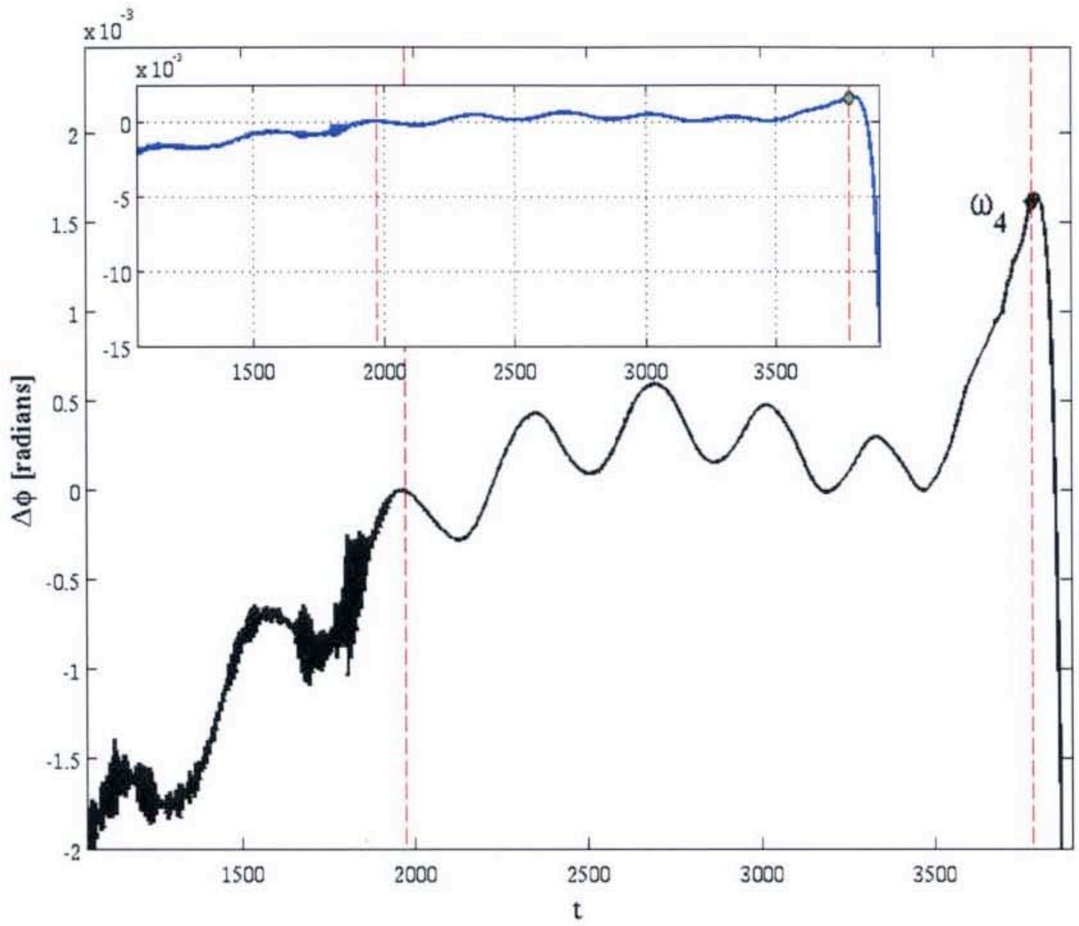
Accurate EOB-NR comparisons (and calibration)

NR, reduced eccentricity data used (non-spinning black holes only):

- very accurate *inspiral only* data ($m_1=m_2$), 30 GW-cycles *$r\psi_4$ curvature waveform* Caltech-Cornell [Boyle et al. 07] used up to GW frequency 0.1
- Albert-Einstein-Institute, 12 GW-cycles *metric (Zerilli) waveform*, inspiral+merger data ($m_1=m_2$) [DNDPR,08]
- Jena, about 20 GW-cycles *$r\psi_4$ curvature waveform*, inspiral+merger data ($m_1=m_2$; $m_1=2m_2$; $m_1=4m_2$) [DNHHB,08]
- Getting the *metric waveform* by twice integrating the curvature waveform and subtracting linear floors.

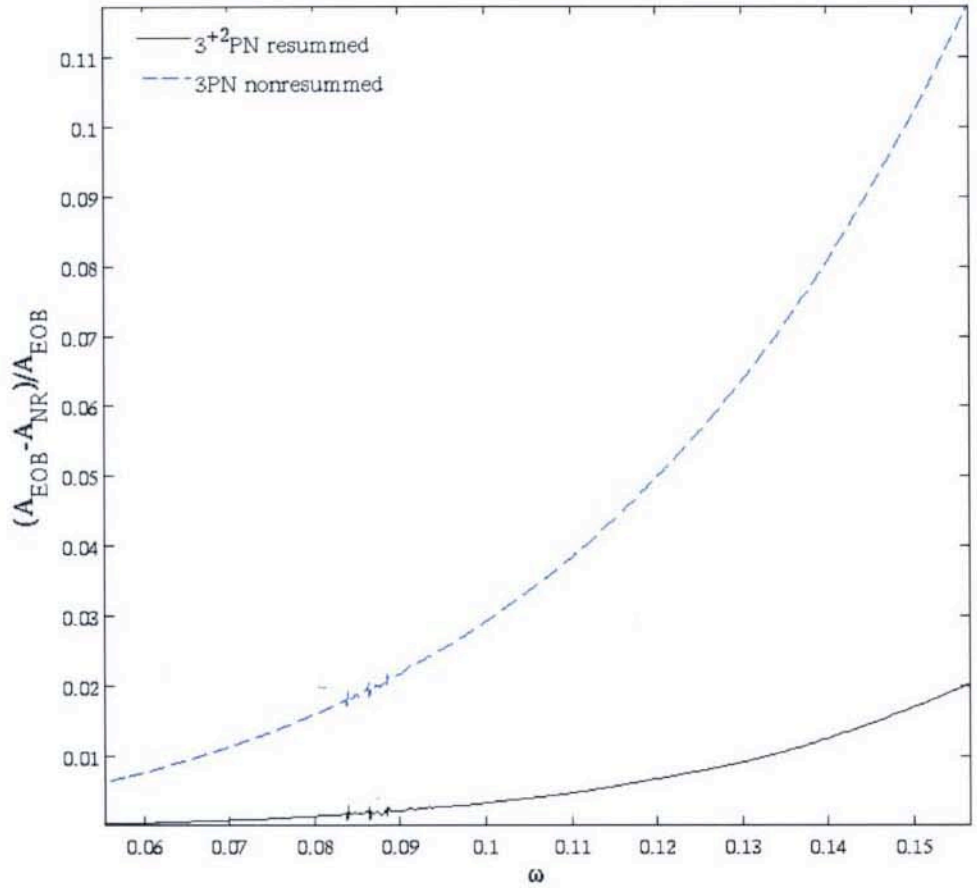


Curvature-waveform phase difference EOB-CC (actual data) for a_5=25



- Close up (black): maximum phase difference 0.002 radians up to GW frequency 0.1 (OK with DN07b, which used published data)
- Full range (blue, inset): maximum phase difference 0.015 radians accumulated between frequency 0.1 and 0.156
- Two "pinching-times" indicated

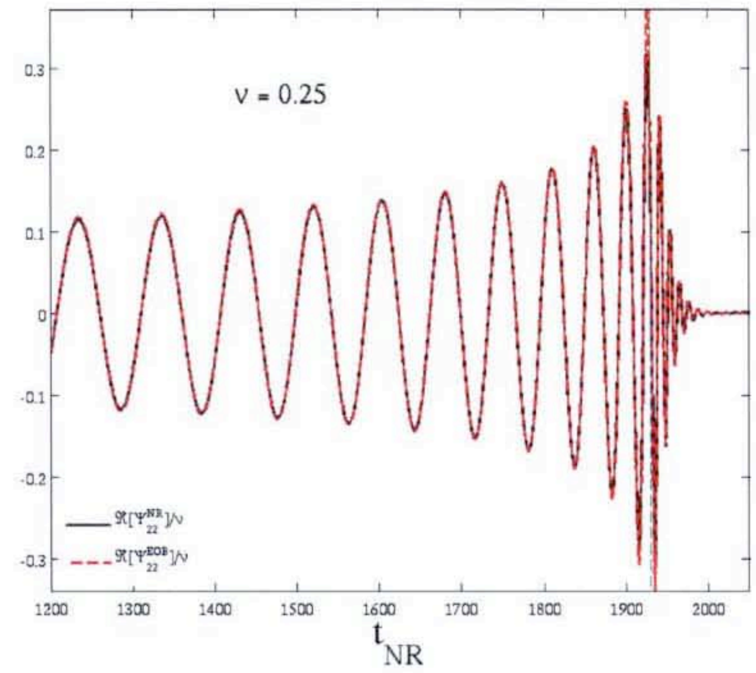
(Fractional) curvature amplitude difference EOB-CC (actual data) for $a_5=25$

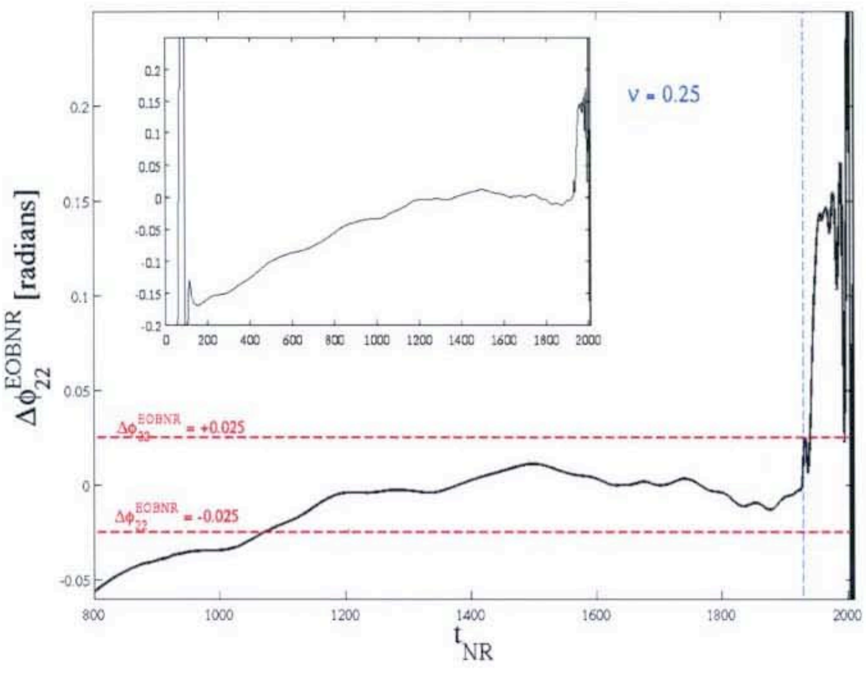


- Nonresummed: fractional differences start at the 1% level and build up to more than 10%
- New resummed EOB amplitude: fractional differences start at the 0.04% level and build up to only 2%
- *Resummation: factor ~20 improvement!*

Which one is the most "effective"?

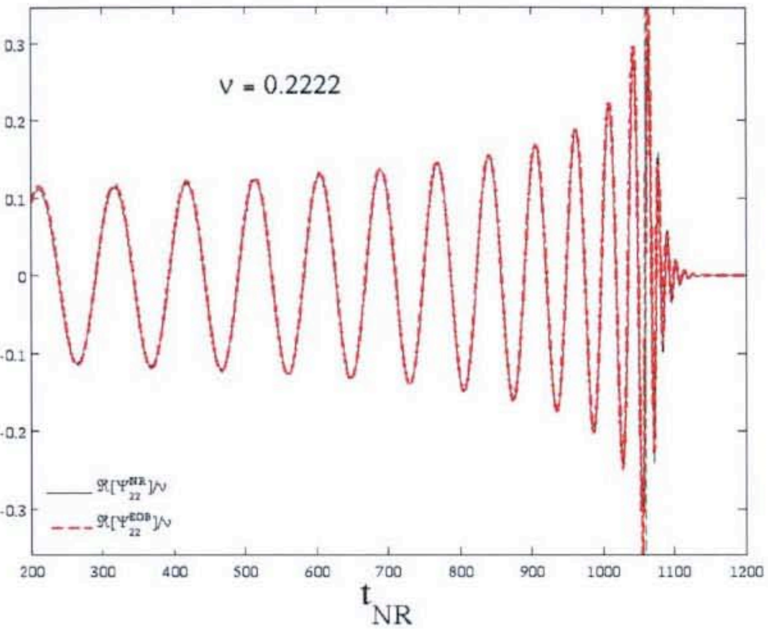
Comparing EOB-NR *metric* waveforms 1:1 case: Jena data

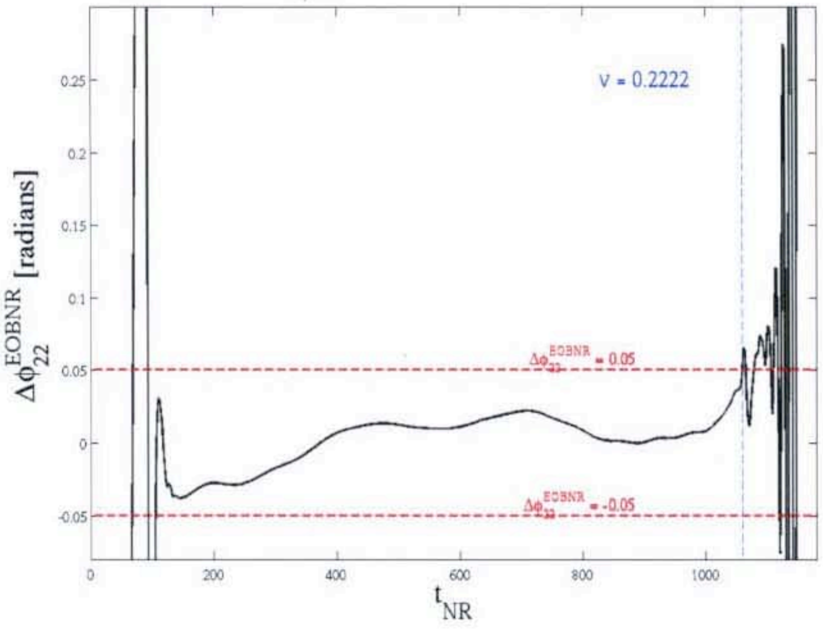




- Metric waveforms from double time-integration of NR curvature waveforms
- $-0.025<\Delta\phi_{22}<+0.025$ radians (=0.004 GW cycles) over 730 M [1200M-1930M]
- At merger, phase jump of only 0.15 radians [=0.02 GW cycles].
- We use the same values of flexibility parameters for CC and Jena data: consistency achieved!

Comparing EOB-NR *metric* waveforms 2:1 case: Jena data





- Metric waveforms from double time-integration of NR curvature waveforms
- $-0.05<\Delta\phi_{22}<+0.05$ radians (=0.008 GW cycles) over 957M [143M-1100M]
- At merger, phase jump of only 0.06 radians [= 0.009 GW cycles].
- We use the same values of flexibility parameters for CC and Jena data: consistency achieved!