TWO-LOOP Renormalization in the Making

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Part I

Prolegomena

Motivations

After LEP

After the end of the Lep era it became evident that including estimates of higher order radiative corrections into one-loop calculations for physical (pseudo-)observables could not, anymore, satisfy the need of precision required by the new generation of experiments.

ILC vs LHC

Admittedly, LHC is an arena for discovery physics, more than anything else: high precision is certainly not needed, at least in its first phase. According to some predestinate design hadron machines are alternating with electron-positron ones and, hopefully, ILC will come into operation; at that moment the highest available theoretical precision will play a fundamental role.

Possible landscapes

As a matter of fact, it is not clear – at this moment – what kind of scenario will follow after the first few months of running at LHC; any evidence of new pysics will favor a striking search for new theoretical models, for their Born predictions, and the hearthquake could be so strong to remove any interest in quantum effects of the standard model. On the contrary, after few months of running, we could be back to the familiar landscape: effects of new physics hidden inside loops.

si vis pacem para bellum

We decided to build the envirovment that allows for a complete two-loop analysis of a spontaneously broken gauge field theory. This construction requires several steps, so it is difficult to caractherize the approach with a single achronimus; there are a lot of analytical aspects in what we are doing, yet the final step (computing arbitrary two-loop diagrams) can only be done with ‘the numerical way’: we call it the algebraic - numerical approach.

What's new?

If one thinks for a while, everything is in the old papers of 't Hooft and Veltman; however, translating few formal properties into a working scheme is far from trivial; most of the times it is not a question of *how do I do it?*, rather it is a question of bookkeeping, namely *can I do it without exhausting the memory of my computer?*, or, *is there any practical way of presenting my results besides making my codes public?*.

layout

First we will deal with general aspects of a spontaneously broken gauge theory; the treatment of tadpoles, everybody knows how to do it, yet general results are never presented in a way that everybody can use them. Secondly, there is the need for a proper diagonalization, order-by-order, of the neutral sector of a theory of fundamental interactions: once again, we need a comprehensive collections of results which allows for practical applications.

Counterterms?

Then, there is the perennial question, with or without counter-terms? In a way, it is a fake question. The two approaches are fully equivalent and we will discuss the transition from bare parameters to renormalized ones. Finally we discuss the ultimate step in any renormalization procedure: the transition from renormalized parameters to a set of physical (pseudo-)observables.

Perhaps, one should try to make a clear vocabulary of renormalization in QFT; a renormalization procedure is designed to bring you from a Lagrangian to theoretical predictions; it includes,

- regularization (nowadays dimensional regularization is easy to understand),
- a renormalization scheme and
- an input parameter set.

Comments

- The scheme, being a transitory step, is almost irrelevant; it can be on-mass-shell or $\overline{MS}$ or complex poles, but unless you do something illegal (resummations that are not allowed or similar things) it really does not matter.
- One can define $\overline{MS}$ quantities as convenient landmarks but it is the last step that matters, at least as long as we have a convenient subtraction point (which we miss in QCD). Renormalized quantities should always be expressed in terms of a set of physical quantities.
- One may indulge to the introduction of an $\overline{MS}$ running e.m. coupling constant (importing from QCD to QED, which sounds strange anyway) but, finally, only cross sectios matter.

Steps

- All the Green functions of the theory have to be made finite, up to two-loops, by introduction of counter-terms and all counter-terms are of non logarithmic nature, to respect unitarity.
- Renormalized Ward-Slavnov-Taylor identities must be satisfied.
- All ultraviolet finte parts must be classified and an algorithm has to be designed for their evaluation at any scale.

Of course, there are preliminar steps – not always the easy ones – but it is only the full control on the multi-scale level that pays off.

Part II

Higgs tadpoles

the basics

The minimal Higgs sector of the Standard Model (SM) is given by the Lagrangian

$$\mathcal{L}_S = -(D_\mu K)^\dagger (D_\mu K) - \mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2, \tag{1}$$

where the covariant derivative is given by

$$D_\mu K = \left(\partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g' B^0_\mu\right) K, \tag{2}$$

$g'/g = -\sin\theta/\cos\theta$, τ^a are the standard Pauli matrices, B^a_μ is a triplet of vector gauge bosons and B^0_μ a singlet. For the theory to be stable we must require $\lambda > 0$. We choose $\mu^2 < 0$ in order to have spontaneous symmetry breaking (SSB). The scalar field in the minimal realization of the SM is

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta + i\phi_0 \\ -\phi_2 + i\phi_1 \end{pmatrix}, \tag{3}$$

The parameter v is *not* a new parameter of the model; its value must be fixed by the requirement that $\langle H \rangle_0 = 0$ [i.e. $\langle K \rangle_0 = (1/\sqrt{2})(v, 0)$], so that the vacuum doesn't absorb/create Higgs particles. To see how this works at the lowest order, consider the part of $\mathcal{L}_S$ containing the Higgs field:

$$-(1/2)(\partial_\mu H)^2 - (\mu^2/2)(H+v)^2 - (\lambda/8)(H+v)^4. \tag{4}$$

These terms generate vertices that imply absorption of H in the vacuum, namely those linear in H,

$$\left[-\mu^2 v - (\lambda/2)v^3\right] H, \tag{5}$$

which correspond to the vertex H  This vertex gives a non-zero value to the diagrams with one ingoing H line, and thus a non-zero VEV. We will set it to zero, i.e. $v = (-2\mu^2/\lambda)^{1/2}$ (or $v = 0$, but then, no SSB).

Definitions and Lagrangian

In h.o. of perturbation theory there are more complicated diagrams contributing to $\langle H \rangle_0$. The parameter v must then be readjusted to make $\langle H \rangle_0 = 0$. First of all, let's introduce

- the new *bare* parameters M (the W mass),
- $M_{_H}$, the mass of the physical Higgs particle and
- β_h (the tadpole constant) according to the following definitions:

$$\left\{ \begin{array}{lcl} M & = & gv/2 \\ M^2_{_H} & = & \lambda v^2 \\ \beta_h & = & \mu^2 + \frac{\lambda}{2} v^2 \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{lcl} v & = & 2M/g \\ \lambda & = & (gM_{_H}/2M)^2 \\ \mu^2 & = & \beta_h - \frac{1}{2} M^2_{_H} \end{array} \right. \qquad (6)$$

The new set of (bare) parameters is therefore

$$\{g, g', M, M_H, \beta_h\} \quad \text{(instead of } \{g, g', \mu, \lambda, v\}\text{)}. \tag{7}$$

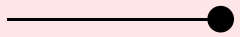
Remember that β_h (like v) is not an independent parameter. In terms of these parameters the interaction part of the scalar Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_S^I &= -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2 = -\beta_h \Big[\frac{2M^2}{g^2} + \frac{2M}{g} H \\
&+ \frac{1}{2}\left(H^2 + \phi_0^2 + 2\phi_+\phi_-\right)\Big] \\
&+ \frac{M_H^2 M^2}{2g^2} - \frac{1}{2} M_H^2 H^2 - g\frac{M_H^2}{4M} H \left(H^2 + \phi_0^2 + 2\phi_+\phi_-\right) \\
&- g^2 \frac{M_H^2}{32M^2}\left(H^2 + \phi_0^2 + 2\phi_+\phi_-\right)^2,
\end{aligned} \tag{8}$$

with $\phi_\pm = (\phi_1 \mp i\phi_2)/\sqrt{2}$.

β_h setting at the lowest order

Let's now set β_h such that the VEV of H remains zero to each order of PT. At the lowest order, the only diagram contributing to $\langle H \rangle_0$ is

H  (9)

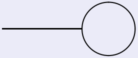
originated by the term in $\mathcal{L}_S^I$ linear in H, $-(2\beta_h M/g)H$. Therefore, at the lowest order we will simply set $\beta_h = 0$.

β_h setting up to one loop

Define

$$\beta_h = \beta_{h_0} + \beta_{h_1} g^2 + \beta_{h_2} g^4 + \cdots . \tag{10}$$

The lowest-order β_h setting of the previous section amounts to $\beta_{h_0} = 0$. At the one-loop level, two types of diagrams contribute to the Higgs VEV up to $\mathcal{O}(g)$:

T_0 :  + T_1 :  (11)

where the empty blob on the r.h.s. symbolically indicates all the one-loop diagrams containing a scalar field (H, $\phi_{\pm}$, ϕ_0), a gauge field (Z, $W_{\pm}$), a Faddeev–Popov ghost field (X_+, X_-, X_z), or a fermionic field.

As an example, consider only the r.h.s. diagram containing the H field: if this were the only T_1 diagram, in order to have $\langle H \rangle_0 = 0$ it should cancel with thel.h.s. one (T_0), i.e.

$$(2\pi)^4 i \left(-\beta_h \frac{2M}{g} \right) - g \frac{3M_H^2}{4M} i\pi^2 A_0(M_H) = 0, \qquad (12)$$

where $i\pi^2 A_0(m) = \mu^{4-n} \int d^n q/(q^2 + m^2 - i\epsilon)$. The solution of this equation is $\beta_{h_0} = 0$ and

$$\beta_{h_1}^{(H)} = \frac{1}{(2\pi)^4 i} \left(\frac{T_1}{2Mg} \right) = -\frac{1}{16\pi^2} \left[\frac{3M_H^2}{8M^2} A_0(M_H) \right]. \qquad (13)$$

Of course, $\beta_{h_1}^{(H)}$ is just the contribution to β_{h_1} arising from the one-loop tadpole diagram containing the H field.

The complete expression for β_{h_1} in the R_ξ gauge is

$$\beta_{h_1} = -\frac{1}{16\pi^2}\Big[\,\frac{3}{2}A_0(M) + \frac{3}{4c^2}A_0(M_0) + M^2 + \frac{M_0^2}{2c^2} + \\ + \frac{M_H^2}{8M^2}\Big(A_0(\xi_Z M_0) + 2A_0(\xi_W M)\Big) + + \frac{3M_H^2}{8M^2}A_0(M_H) \\ - \sum_f \frac{m_f^2}{M^2}A_0(m_f)\Big], \qquad (14)$$

where $M_0 = M/c$ and m_f are the Z and fermion masses, and $c = \cos\theta$.

β_h vertices in one-loop calculations

Beyond the lowest order, β_h is *not* zero and the Lagrangian $\mathcal{L}^I_S$ contains the following vertices involving a β_h factor:

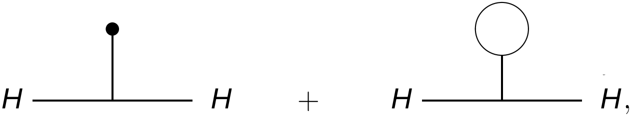
H		$(2\pi)^4 i\,(-2M\beta_h/g)$	(15)
H	H	$(2\pi)^4 i\,(-\beta_h)$	(16)
ϕ_0	ϕ_0	$(2\pi)^4 i\,(-\beta_h)$	(17)
ϕ_+	ϕ_-	$(2\pi)^4 i\,(-\beta_h)$	(18)

(as usual, the combinatorial factorials for identical fields are included.

Note that only scalar fields appear in the β_h vertices. These β_h vertices must be included in the relevant one-loop calculations. Consider, for example, the Higgs self-energy at the one-loop level. The diagrams contributing to this $\mathcal{O}(g^2)$ quantity are

(19)

where the empty blob on the r.h.s. represents all the one-loop contributions (two possible topologies). The l.h.s. diagram containing a two-leg β_h vertex shouldn't be forgotten and plays an important role in the Ward identities (see later). One should also include diagrams containing tadpoles:

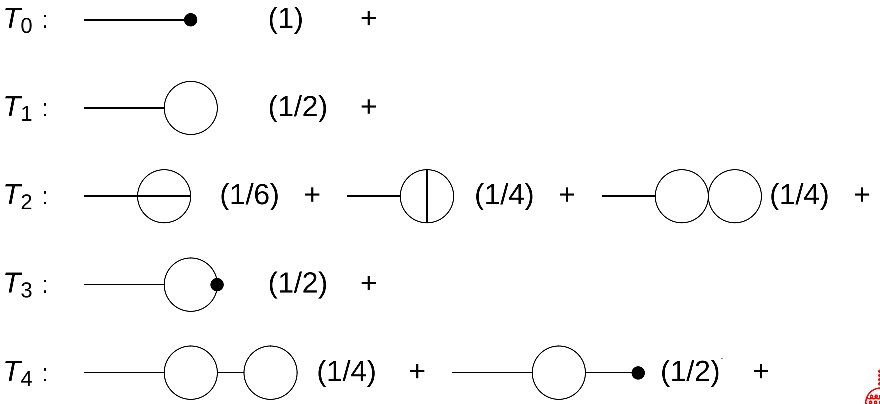


(20)

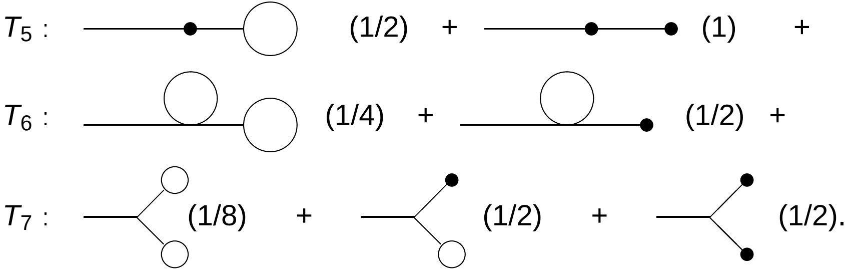
but these diagrams add up to zero as a consequence of our choice for β_h.

β_h setting up to two loops

Up to terms of $\mathcal{O}(g^3)$, $\langle H \rangle_0$ gets contributions from the following diagrams:



(21)


T_5 : (1/2) + (1) +
T_6 : (1/4) + (1/2) +
T_7 : (1/8) + (1/2) + (1/2).

The coefficients in parentheses indicate the combinatorial factors of each diagram when all fields are identical. By virtue of our previous choice for β_{h_0} and β_{h_1}, all the *reducible* diagrams add up to zero: $T_4 = T_5 = T_6 = T_7 = 0$. The equation

$$\sum_{i=0}^{3} T_i = 0 \tag{22}$$

provides then β_{h_2}:

$$\beta_{h_2} = \frac{1}{(2\pi)^4 i} \left(\frac{T_2 + T_3}{2Mg^3} \right) . \tag{23}$$

β_h vertices in two-loop calculations

As we described for calculations at the one-loop level, two-leg β_h vertices Eq.(16), Eq.(17),Eq.(18)) should be included in all the appropriate diagrams at the two-loop level, while all graphs (up to two loops) containing tadpoles will add up to zero as a consequence of our choice for β_{h_0}, β_{h_1} and β_{h_2}. Note that two-leg β_h vertices will also appear in $\mathcal{O}(g^4)$ self-energies of fields which do not belong to the Higgs sector; for example, in diagrams like these

which are representative of the only two irreducible $\mathcal{O}(g^4)$ Z self-energy topologies containing β_h vertices (excluding tadpoles, of course).

Definitions and Lagrangian

We will now consider a slightly different strategy to set the Higgs VEV to zero. Instead of using Eq.(6), the "β_h scheme", we will define the new bare parameters M' (the W mass), M'_H (the mass of the physical Higgs particle) and β_t (the tadpole constant) according to the following "β_t scheme":

$$\left\{\begin{array}{lcl} M'(1+\beta_t) & = & gv/2 \\ (M'_H)^2 & = & \lambda\,(2M'/g)^2 \\ 0 & = & \mu^2 + \frac{\lambda}{2}\,(2M'/g)^2 \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} v & = & 2M'(1+\beta_t)/g \\ \lambda & = & \left(gM'_H/2M'\right)^2 \\ \mu^2 & = & -\frac{1}{2}(M'_H)^2 \end{array}\right. \qquad (24)$$

The new set of bare parameters is therefore

$$\{g, g', M', M'_H, \beta_t\} \quad \text{instead of} \{g, g', \mu, \lambda, v\}. \tag{25}$$

Remember that β_t (like v and β_h) is not an independent parameter.
Note that, contrary to β_h, the parameter β_t appears in the Higgs doublet K via $\zeta = H + v$, with $v = 2M'(1+\beta_t)/g$ [Eq.(24)].
As a consequence, all three terms of the scalar Lagrangian $\mathcal{L}_S$ [Eq.(1)] depend on it. In particular, the interaction part of the scalar Lagrangian becomes

$$
\mathcal{L}_S^I = -\mu^2 K^\dagger K - (\lambda/2)(K^\dagger K)^2 \tag{26}
$$

$$
\begin{aligned}
&= (1+\beta_t)^2 \Big(1 - \beta_t\,(2+\beta_t)\Big)\frac{M_H'^2 M'^2}{2g^2} \\
&- \beta_t\,(\beta_t+1)\,(\beta_t+2)\,\frac{M_H'^2 M'}{g} H \\
&\quad -\frac{1}{2} M_H'^2 H^2 - \frac{1}{4} M_H'^2 \beta_t\,(\beta_t+2)\,\Big(3H^2 + \phi_0^2 + 2\phi_+\phi_-\Big) \\
&\quad -g\,(1+\beta_t)\,\frac{M_H'^2}{4M'} H \Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big) \\
&- g^2 \frac{M_H'^2}{32M'^2}\Big(H^2 + \phi_0^2 + 2\phi_+\phi_-\Big)^2,
\end{aligned} \tag{27}
$$

while the term involving the covariant derivatives, $-(D_\mu K)^\dagger (D_\mu K)$, results in the same (lengthy) β_t-*independent* expression of the β_h scheme *plus* the following β_t-*dependent* terms

$$
\begin{aligned}
\beta_t \quad \times \quad & \Big[igsM' \left(\phi^- W^+_\mu - \phi^+ W^-_\mu\right) \left(A_\mu - \frac{s}{c} Z_\mu\right) \\
& -\frac{gM'}{2} H \left(2 W^+_\mu W^-_\mu + \frac{Z_\mu Z_\mu}{c^2} \right) \\
- \quad & \frac{M'^2}{2} (\beta_t + 2) \left(2 W^+_\mu W^-_\mu + \frac{Z_\mu Z_\mu}{c^2} \right) + \frac{M'}{c} Z_\mu \partial_\mu \phi_0 \\
+ \quad & M' W^+_\mu \partial_\mu \phi_- + M' W^-_\mu \partial_\mu \phi_+ \Big],
\end{aligned}
\tag{28}
$$

where, as usual, $W^\pm_\mu = (B^1_\mu \mp i B^2_\mu)/\sqrt{2}$, $s = \sin\theta$, $c = \cos\theta$, and

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} B^3_\mu \\ B^0_\mu \end{pmatrix}. \tag{29}$$

Where else, in the SM Lagrangian, does the parameter β_t appear? Wherever v does — as it can be readily seen from Eq.(24). Let's quickly discuss the other sectors of the SM: Yang–Mills, fermionic, Faddeev–Popov (FP) and gauge-fixing. The pure Yang–Mills Lagrangian obviously contains no β_t terms.

The gauge-fixing part of the Lagrangian, $\mathcal{L}_{gf}$, cancels in the R_ξ gauges the gauge–scalar mixing terms Z–ϕ_0 and $W^\pm$–$\phi^\pm$ contained in the scalar Lagrangian $\mathcal{L}_S$. These terms are proportional to $gv/2$, i.e., in the β_t scheme, to $M'(1+\beta_t)$.

gauge-fixing

The gauge-fixing Lagrangian $\mathcal{L}_{gf}$ is matter of choice: we adopt the usual definition

$$\mathcal{L}_{gf} = -\mathcal{C}_{+}\mathcal{C}_{-} - \frac{1}{2}\mathcal{C}_{Z}^{2} - \frac{1}{2}\mathcal{C}_{A}^{2}, \tag{30}$$

$$\mathcal{C}_{A} = -\frac{1}{\xi_{A}}\partial_{\mu}A_{\mu}, \quad \mathcal{C}_{Z} = -\frac{1}{\xi_{Z}}\partial_{\mu}Z^{0}_{\mu} + \xi_{Z}\frac{M'}{c}\phi_{0}, \quad \mathcal{C}_{\pm} = -\frac{1}{\xi_{W}}\partial_{\mu}W^{\pm}_{\mu} + \xi_{W}M'\phi_{\pm} \tag{31}$$

(that is, no β_t terms), thus canceling the $\mathcal{L}_S$ g-independent gauge–scalar mixing terms proportional to M', but not those proportional to $M'\beta_t$ [appearing at the end of Eq.(28)], which are of $\mathcal{O}(g^2)$. Clearly, this gauge fixing Lagrangian is *different* from the usual one of the β_h scheme because M and M' are not the same [$M = M'(1 + \beta_t)$].

Alternatively, one could choose $M'(1 + \beta_t)$ instead of M' in eq. (31), thus canceling all $\mathcal{L}_S$ gauge–scalar mixing terms, both proportional to M' and $M'\beta_t$, but introducing then other new two-leg β_t vertices. In this latter case, the gauge fixing Lagrangian is indeed identical to the one of the β_h scheme. We will not follow this latter approach. Of course it's only matter of choice, but the explicit form of $\mathcal{L}_{gf}$ determines the FP ghost Lagrangian.

The parameter β_t shows up also in the FP ghost sector. The FP Lagrangian depends on the gauge variations of the chosen gauge-fixing functions $\mathcal{C}_A$, $\mathcal{C}_Z$ and $\mathcal{C}_\pm$. If, under gauge transformations, the functions $\mathcal{C}_i$ transform as

$$\mathcal{C}_i \to \mathcal{C}_i + \left(M_{ij} + g L_{ij}\right) \Lambda_j , \tag{32}$$

with $i = A, Z, \pm$, FP ghost Lagrangian is given by

$$\mathcal{L}_{FP} = \bar{X}_i \left(M_{ij} + g L_{ij}\right) X_j . \tag{33}$$

With the choice for $\mathcal{L}_{gf}$ given in eq. (30) [and the relation $gv/2 = M'(1+\beta_t)$] it's easy to check that the FP ghost Lagrangian contains the β_t terms

$$\mathcal{L}_{FP} = -\left(M'\right)^2 \beta_t \left(\xi_W \bar{X}^+ X^+ + \xi_W \bar{X}^- X^- + \xi_Z \bar{X}_Z X_Z / c^2\right) + \cdots, \qquad (34)$$

where the dots indicate the usual β_t–independent terms. Had we chosen $\mathcal{L}_{gf}$ with $M'(1+\beta_t)$ instead of M' in eq. (31), additional β_t terms would now arise in the FP Lagrangian.

In the fermionic sector, the parameter β_t appears in the mass terms:

$$\frac{v}{\sqrt{2}}\left(-\alpha\bar{u}u+\beta\bar{d}d\right)=-\left(1+\beta_t\right)\left(m_u\bar{u}u+m_d\bar{d}d\right) \tag{35}$$

[$v=2M'(1+\beta_t)/g$], α and β are the Yukawa couplings, and m_u, m_d are the masses of the fermions. The rest of the fermion Lagrangian does not contain β_t, as it doesn't depend on v. In the β_t scheme, contrary to the β_h one, we have (many) two- and three-leg β_t vertices containing also fields outside the scalar sector. Note that three-leg β_t vertices introduce a fourth irreducible topology for $\mathcal{O}(g^4)$ self-energy diagrams containing β_t vertices, namely:

β_t up to one loop

Define

$$\beta_t = \beta_{t_0} + \beta_{t_1} g^2 + \beta_{t_2} g^4 + \cdots . \quad (36)$$

As we did for β_h, we will now set the parameter β_t such that the VEV of the Higgs field H remains zero to each order of perturbation theory.

At the lowest order, the only diagram contributing to $\langle H \rangle_0$ is the same one depicted in (Eq.(9)), originated by the term in $\mathcal{L}_S^I$ linear in H, $-\beta_t(\beta_t + 1)(\beta_t + 2)(M_H'^2 M'/g)H$. Therefore, at the lowest order we can simply set $\beta_t = 0$, i.e. $\beta_{t_0} = 0$.

Up to one loop, the diagrams T_0' and T_1' contributing to the Higgs VEV are analogous to T_0 and T_1 appearing in (Eq.(11)), so that β_{t_1} can be set in analogy with β_{h_1}:

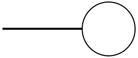
$$\beta_{t_1} = \frac{1}{(2\pi)^4 i} \left(\frac{T_1'}{2M' g M_H'^2} \right) . \quad (37)$$

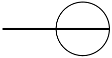
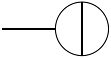
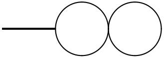
Note that T_1' and T_1 have the same functional form, but depend on different mass parameters; moreover, one gets $\beta_{t_1} = \beta_{h_1}/M_H^2 + \mathcal{O}(g^2)$.

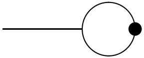
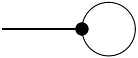
β_t up to two loops

The two-loop β_t fixing slightly differs from the β_h one. Up to terms of $\mathcal{O}(g^3)$, $\langle H \rangle_0$ gets contributions from the following diagrams:

T'_0 :  (1) +

T'_1 :  (1/2) +

T'_2 :  (1/6) +  (1/4) +  (1/4) +

T'_3 :  (1/2) +  (1/2),

plus *reducible* diagrams (analogous to those appearing in T_4–T_7 of section 2.4) which add up to zero because of our choice for β_{t_0} and β_{t_1}.

Note the new diagrams in T'_3, with three-leg β_t vertices, not present in the β_h case (T_3). The parameter β_{t_2} can be set in the usual manner, requiring

$$\sum_{i=0}^{3} T'_i = 0, \qquad \Longrightarrow \qquad \beta_{t_2} = \frac{1}{(2\pi)^4 i} \left(\frac{T'_2 + T'_3}{2M' g^3 M'^2_H} \right) - \frac{3}{2} \beta^2_{t_1}. \tag{38}$$

Note that $T'_{1,2}$ and $T_{1,2}$ have the same functional form (but depend on different mass parameters) while T'_3 and T_3 are different also in form.

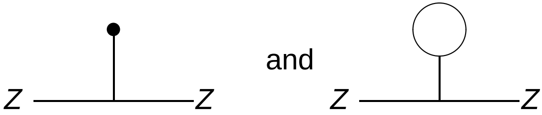
A comment on WST identities and mass renormalization

Consider the (doubly-contracted) WST identity relating the Z self-energy $\Pi_{\mu\nu,zz}(p)$, the ϕ_0 self-energy $\Pi_{\phi_o\phi_o}(p)$, and the Z–ϕ_0 transition $\Pi_{\mu,z\phi_o}(p)$:

$$p_\mu p_\nu \Pi_{\mu\nu,zz}(p) + M_0^2 \Pi_{\phi_o\phi_o}(p) + 2ip_\mu M_0 \Pi_{\mu,z\phi_o}(p) = 0. \qquad (39)$$

Both in β_h and β_t schemes, each of the three terms in Eq.(39) contains tadpoles diagrams, but they add up to zero, within each term.

For example, at the one-loop level, the first term in Eq.(39) contains the tadpoles diagrams



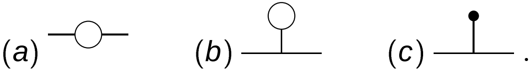
(40)

which cancel each other. In the β_h scheme at the one-loop level, only the second term of the identity (Eq.(39)) includes a diagram with a two-leg β_h vertex (Eq.(17)); in higher orders, two-leg β_h vertices will appear in all three terms. In the β_t scheme, all three terms of Eq.(39) contain the two-leg β_t vertices already at the one-loop level. Similar comments are valid for the WST identity involving the W self-energy.

Renormalization

Concerning renormalization, the constraint imposed on β_h (or β_t) in the previous sections is the renormalization condition to insure that $\langle 0|H|0\rangle = 0$, also in the presence of radiative corrections. In particular, the renormalized $\beta_{h,t}$ parameters are $\beta^{(R)}_{h,t} = \beta_{h,t} + \delta\beta_{h,t} = 0$. The equivalent of Eq.(6)) and Eq.(24) for the renormalized parameters are just the same equations with $\beta^{(R)}_h = \beta^{(R)}_t = 0$.

In the β_h scheme, the one-loop renormalization of the W and Z masses involves the diagrams

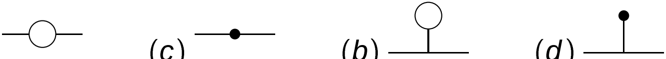


(41)

(Diagrams (a) have two possible loop topologies.)

Both (a) and (b) are gauge-dependent, but their sum is gauge-independent on-shell. However, as we choose the β_h tadpole (c) to cancel (b), the mass counterterm contains only (a) and is therefore gauge-dependent.

On the contrary, in the β_t scheme, the one-loop renormalization of the W and Z masses involves the diagrams



$$(a) \qquad (c) \qquad (b) \qquad (d) \quad . \tag{42}$$

Once again, both (a) and (b) diagrams are gauge-dependent, their sum is gauge-independent on-shell, and the β_t tadpole (d) is chosen to cancel (b). But, the mass counterterm is now gauge-independent, as it contains both (a) and the two-leg β_t vertex diagram (c) (which is missing in the β_h case).