From confined to deconfined matter: The Phase Diagram of QCD

Massimo D'Elia

Parma, XIV Seminario Nazionale di Fisica Teorica,
07 settembre 2005

OUTLINE

- What is interesting in the Phase Diagram of QCD?
- Why are not we able to study it with standard lattice QCD simulations?
- How can we circumvent problems and what can we say?
- Conclusions and perspectives

The steps towards the idea of deconfinement

- **1965, Hagedorn**: The observed exponential growth of the density of new hadron resonances leads to an hypothesis on the existence of an ultimate limiting temperature $T_H \sim 200\, MeV$.

- **1975, Cabibbo and Parisi**: The Hagedorn temperature, instead of being an ultimate limiting temperature, could correspond to a phase transition leading to quark liberation. Indeed, in the high T phase the coupling should be small because of asymptotic freedom, perturbative theory should be at work: deconfinement.

- **1980, Kuti, Polonyi and Szlachanyi**: the theoretical answer comes from QCD. The deconfinement phase transition is observed for the first time in numerical simulations of lattice gauge theories.

- **???, ???**: Nature has still not given any definite answer. The experimental search for a new deconfined phase of matter at high temperatures (quark-gluon plasma, or QGP) is the goal of Heavy Ion Collision experiments: SPS, Cern; RHIC, Brookhaven; LHC, Cern.

Going to finite density

- Heavy Ion Collisions do not explore just the high T phase of QCD: in the hot bubble which is formed after the collision there is a net unbalance of matter and antimatter. That is actually a system at finite temperature and density. One is thus exploring the whole QCD phase diagram at finite (high) T and finite (small) chemical potential μ.

- The nature of compact astrophysical objects (like neutron stars) is also related to the structure of the $T - \mu$ phase diagram: they are described by the low T high μ region of the diagram.

- Increasing μ and $T \simeq 0$, one expects to encounter the phase transition corresponding to the onset of nuclear matter ($\mu_B \sim m_N$). By further increasing μ, the effective interaction, regarding only particles around the Fermi sphere, decreases because of asymptotic freedom. A new phase transition is thus expected. Perturbative calculations describe the new phase as a color superconductor.

Sketching the QCD phase diagram

- The high T phase transition and the high μ phase transition are naturally expected to be connected by a line of phase transitions: the nature of this line is of fundamental importance for the experimental search of the transition.

- Various models (NJL models, random matrix models, . . .) predict the high μ transition to be first order.

- At high T, $\mu = 0$ the answer is still unclear: true phase transition or crossover?

- First order at high μ and crossover at high T would mean a second order critical endpoint along the critical line: this critical endpoint would have very clear experimental signatures (Rajagopal, Shuryak, Stephanov 1998). This is the reason of the big excitement around this possibility.

- Definite answers can only come through a theoretical investigation of the phase diagram based on QCD first principles. The natural tool to do that are numerical lattice simulations, which however, as we will see, encounter some difficulties.

1 – QCD at finite temperature and density

The finite temperature QCD partition function, $Z(V,T) = \mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}}{T}}\right)$, can be written as a functional integral over euclidean space-time with finite temporal extent $\tau = 1/T$

$$Z = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\int_0^{1/T} dt \int d^3x \mathcal{L}_{\mathrm{QCD}}}$$

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{4g^2}G^a_{\mu\nu}G^a_{\mu\nu} + \bar{\psi}\left(\gamma^E_\mu(\partial_\mu + iA^a_\mu T^a) + m\right)\psi$$

In the time/temperature direction appropriate boundary condition have to be taken:

periodic b.c. for gauge fields

$U_\mu(t=0,\vec{x}) = U_\mu(t=1/T,\vec{x})$

antiperiodic b.c. for fermionic fields

$\psi(t=0,\vec{x}) = -\psi(t=1/T,\vec{x})$

The partition function can be easily discretized on a lattice:

$$Z = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-(S_G+S_F)} = \int \mathcal{D}U e^{-S_G} \det M[U]$$

$U_\mu(x) = \mathcal{P}\exp\left(i\int_x^{x+\hat{\mu}} dy_\nu A_\nu(y)\right)$ is the gauge link variable

$S_G = \beta\sum_{x,\mu<\nu}\left(1 - \frac{1}{N_c}\mathrm{Tr}\Pi_{\mu\nu}(x)\right)$ is the pure gauge action, $\beta = \frac{2N_c}{g^2}$

$\Pi_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)$ is the plaquette variable

$S_F = \bar{\psi}_i M_{ij}\psi_j$ is the fermionic action with M the fermionic matrix, naïvely

$S_f = \frac{1}{2}\sum_{x,\mu}\bar{\psi}(x)\gamma_\mu^E\left[U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\right] + \sum_x m\bar{\psi}(x)\psi(x)$

By integrating the quadratic fermion action over the fermion fields one obtains a partition function in terms of gauge variables only.

Dynamical fermion contributions are encoded in the fermion determinant $\det M[U]$ which appears after integration.

$$T = \frac{1}{\tau} = \frac{1}{N_t a(\beta, m)}$$

$a \to 0$ as $\beta \to \infty$, so T is a monotonic increasing function of β, and sometimes β is plotted in the phase diagram instead of T (but be careful around phase transitions!).

The thermal expectation value of a generic operator O is written as

$$\langle O \rangle = \frac{\int DU \, \det M[U] \, e^{-S_g[U]} \, O[U]}{\int DU \, \det M[U] \, e^{-S_g[U]}} \qquad (1)$$

if $\det M[U] \, e^{-S_g[U]} > 0$ this has a probabilistic interpretation and Monte Carlo methods can be applied to numerically determine it: only few gauge field configurations give sensible contribution to the functional integral and one looks for a good algorithm to sample them (importance sampling).

The reality of the fermion determinant is guaranteed by γ_5 hermiticity

$$\gamma_5 M^\dagger \gamma_5 = M \implies (\det M)^* = \det M$$

Finite density can be introduced in the grand canonical formalism by adding a finite chemical potential

$$Z(\mu) = \mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}-\mu N}{T}}\right)$$

where $N = \int d^3x\psi^\dagger\psi = \int d^3x\bar{\psi}\gamma_0\psi$ is the quark (baryonic) number operator. In the euclidean functional integral formulation the fermionic part of $\mathcal{L}_{\mathrm{QCD}}$ is modified as follows:

$$\bar{\psi}\left(\gamma_\mu(\partial_\mu + iA_\mu) + m\right)\psi \rightarrow \bar{\psi}\left(\gamma_\mu(\partial_\mu + iA_\mu) + m + \mu\gamma_0\right)\psi$$

The naïve discretization

$$\int d^4x\bar{\psi}\gamma_0\psi \rightarrow a^4\sum\bar{\psi}\gamma_0\psi$$

leads to an UV diverging contribution $\sim \frac{\mu^2}{a^2}$ for the internal energy density at $T = 0$.

The correct way to discretize $Z(\mu)$ on the lattice is to consider μ as being part of the covariant derivative, like the temporal component of a $U(1)$ imaginary background field

P. Hasenfratz F. Karsch, Phys. Lett. B125 (1983) 308

J.B. Kogut et al., Nucl. Phys. B225 (1983) 93

R. V. Gavai Phys. Rev. D32 (1985) 519

This is implemented on the lattice by modifying the temporal gauge links appearing in the fermion matrix:

$$U_t \rightarrow e^{a\mu} U_t \tag{2}$$

$$U_{-t} \rightarrow e^{-a\mu} U_{-t} \Longrightarrow (U_{-t})^\dagger \neq U_t \tag{3}$$

The hermiticity properties of the fermion matrix in general are lost and the residual surviving symmetry is

$$(\det M(\mu))^* = \det M(-\mu)$$

which means that $\det M$ is in general complex, so that Monte Carlo evaluations of thermal expectation values at finite μ are unfeasibile.

This is usually known as the **sign problem** and is common to several fermion systems with a net unbalance between particles and holes. In the case of gauge theories it can be easily understood in terms of the introduction of an imaginary $U(1)$ background field.

2 – Trying to evade the sign problem

The importance of studying the QCD phase diagram from first principles lattice gauge theory simulations has been worth a lot of efforts from different groups in trying to evade the sign problem.

Most methods try to extract information from simulations where the sign problem is absent.

The rest of this talk is devoted to a (surely incomplete) review of the different methods explored and of the results obtained up to now.

As we will see, the battle is still far to be won, but things have been moving a lot in the last few years, mostly in the region of small μ and high T, which is the one relevant for heavy ion collisions.

Reweighting

Gauge configurations sampled at $\mu = 0$ can in principle be used to obtain expectation values at $\mu > 0$, using the following identity:

$$\langle O \rangle = \frac{\int DU \, \det M(\mu) \, e^{-S_g[U]} \, O[U]}{\int DU \, \det M(\mu) \, e^{-S_g[U]}}$$

$$= \frac{\int DU \, \det M(0) \frac{\det M(\mu)}{\det M(0)} \, e^{-S_g[U]} \, O[U]}{\int DU \, \det M(0) \, \frac{\det M(\mu)}{\det M(0)} e^{-S_g[U]}} = \frac{\left\langle O \frac{\det(M(\mu))}{\det(M(0))} \right\rangle_{\mu=0}}{\left\langle \frac{\det(M(\mu))}{\det(M(0))} \right\rangle_{\mu=0}}$$

The method, proposed in the 80's by the Glasgow group, fails (the onset of nuclear matter at $T = 0$ is not observed at the correct value $\mu = m_N/3$). The failure can be related to two major problems:

- **Bad sampling:** configurations sampled ad $\mu = 0$ give poor sampling of the integral at $\mu \neq 0$. That means that the two statistical ensembles corresponding to $\mu \neq 0$ and $\mu = 0$, being related to different physical situations, may have very poor overlap.
- **Large errors coming from the oscillating factor $\frac{\det M(\mu)}{\det M(0)}$:** the sign problem comes back, $\langle \det M(\mu)/\det M(0) \rangle \sim 0$ at large μ's, the statistics required for a given accuracy grow exponentially with the volume

Multiparameter reweighting

An improvement consists in reweighting both in β and μ instead that only in μ.

$$Z(\mu,\beta) = \int \mathcal{D}U e^{-S_g(\beta_0)} \det M(0) \left\{ e^{-S_g(\beta)+S_g(\beta_0)} \frac{\det M(\mu)}{\det M(0)} \right\},$$

Configurations sampled at a transition point can be used to reconstruct the integral functional at a different transition point in the $T-\mu$ plane: overlap is greatly enhanced.
Residual problems at large volumes and large densities, where the overlap decreases and the sign problem worsens.

Taylor expansion

For small densities, physical quantities can be naturally expanded as a Taylor series in μ around $\mu = 0$. The coefficients of the series can obtained as expectation values of local operators at $\mu = 0$. For instance consider the quark density $n(\mu)$:

$$n(\mu) = n_1 \, \mu + n_3 \, \mu^3 + O(\mu^5)$$

the first coefficient n_1 is nothing but the quark number susceptibility computed at $\mu = 0$. The method is naturally well suited only for the region of small μ and high T.

Using an imaginary chemical potential

Consider a purely imaginary chemical potential, $\mu = i\mu_I$

$$U_t \to e^{ia\mu_I} U_t \qquad\qquad U_{-t} \to e^{-ia\mu_I} U_{-t} = (e^{ia\mu_I} U_t)^\dagger$$

this is like adding a constant and real $U(1)$ background field. $\det M[U] > 0$, Monte Carlo simulations are feasible.

Analytic continuation to real μ

Away from critical points $Z(T, \mu)$ is a regular function of μ^2. Results at μ_I can be used to fit the expected dependence, as continued from real values of μ. Actually this is a continuation from negative to positive values on the real μ^2 axis rather than a continuation in the complex plane. The critical line itself can be continued.

Reconstruction of the canonical partition function

$Z(i\mu_I)$ can be used to reconstruct the canonical partition function $Z(n)$ at fixed quark number n (Roberge, Weiss, 1986)

$$Z(n) = \mathrm{Tr}\left((e^{-\frac{H_{\mathrm{QCD}}}{T}}\delta(N-n)\right) =$$

$$\frac{1}{2\pi}\mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}}{T}}\int_0^{2\pi} \mathrm{d}\theta e^{i\theta(N-n)}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta e^{-i\theta n} Z(i\theta T)$$

where $\theta = \mu_I/T$. As n grows $e^{-i\theta n}$ oscillates more and more rapidly and the integration error grows exponentially.

Other theories without the sign problem

QCD at finite isospin density

Remember $(\det M(\mu))^* = \det M(-\mu)$, that means that a theory with a finite isospin density μ_{is} (i.e. $\mu_u = -\mu_d = \mu_{is}/2$) has no sign problem (assuming $m_u = m_d$)

$$\det M(\mu) \det M(-\mu) = \det M(\mu)(\det M(\mu))^* = |\det M(\mu)|^2 > 0$$

This is like ignoring the determinant phase, $\det M(\mu) = |\det M(\mu)| e^{i\theta}$ (phase quenched QCD). Of course it is expected to give interesting results as long as the phase is not relevant (small μ_{is}, indeed for $\mu_{is} > m_\pi/2$ unphysical pion condensation happens).

QCD with two colours

$N_c = 2$ is a special case of gauge theory: the fundamental representation for quarks is real ($\sigma_2 U_\mu \sigma_2 = U_\mu^*$).

$\Longrightarrow$ Traces over closed loops are always real

$\Longrightarrow$ Gauge invariant quantities like $\det M[U]$, being expressible in terms of traces over closed loops, are real.

Of course QCD with 2 colours is also very different from real QCD in many respects (baryons $\sim$ mesons, for instance ...)

The Density of States Method

This is a general method which has been proposed for several physical problems affected by a sign problem (like QCD with a θ-term)

Bhanot, Bitar, Salvador (1987), Karliner, Sharpe, Chang (1988), Azcoiti, Di Carlo, Grillo (1990), Luo (2001), Ambjorn *et al* (2002)

It consists in re-expressing $\langle O \rangle$ as

$$\langle O \rangle = \frac{\int dx \langle O[U] f[U] \rangle_x \rho(x)}{\int dx \langle f[U] \rangle_x \rho(x)}$$

where $\rho(x)$ is the density of states relative to some operator ϕ (plaquette, quark number density, . . .) constrained to x

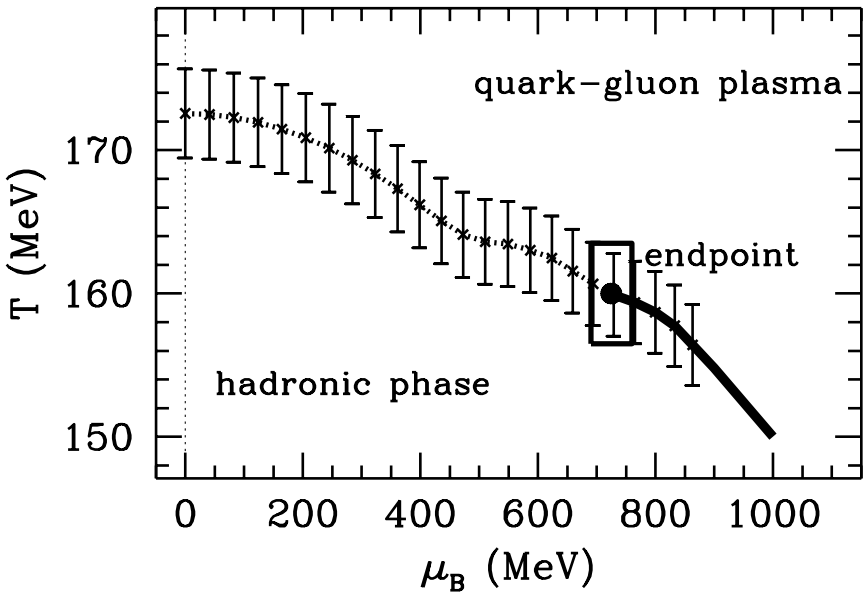
$$\rho(x) \equiv Z_\phi(x) = \int DU g[U] \delta(\phi - x)$$

For instance $g[U] = |\det M[U]| e^{-S_g[U]}$ and $f[U] = e^{i\theta}$. The method is well suited if in the range of x relevant for $\rho(x)$ the sign problem is mild (i.e. one has not $\langle e^{i\theta} \rangle_x \sim 0$): this is of course more and more difficult as the volume increases.

3 – Overview of numerical results

3.1 – Multiparameter reweighting

(Z. Fodor and S.D. Katz, 2002)



from Z. Fodor, S. D. Katz, JHEP 0203 (2002) 014

- The method has been used to determine the critical line in the $T-\mu$ plane and the equation of state for the case of $n_f = 2+1$ dynamical flavours.
- By analyzing the fi nite size scaling behaviour of the Lee-Yang zeroes information can be obtained about a possible critical point in the $T-\mu$ plane: Lee-Yang zeroes in the complex β plane approach the real axis for a phase transition, like $1/V$ for fi rst order.
  - numerically expensive, needs exact evaluation of $\det M[U]$ on each confi guration. Limited to small volumes
  - simulations with quark masses about 3 times the physical values and lattice sizes up to $8^3 \times 4$ indicate $\mu_E \sim 700$ MeV
  - simulations with quark masses around the physical values and lattice sizes up to $12^3 \times 4$ indicate $\mu_E \sim 360$ MeV

Doubts have been recently cast on the validity of this results from several points of view:

- Ejiri (2005):the Lee-Yang zeroes analysis is based on the effect of an oscillating term associated with a complex coupling in the partition function. In presence of another oscillating factor (the phase of the determinant) the analysis may be wrong. Ejiri suggests to look for the second zero associated to first order.

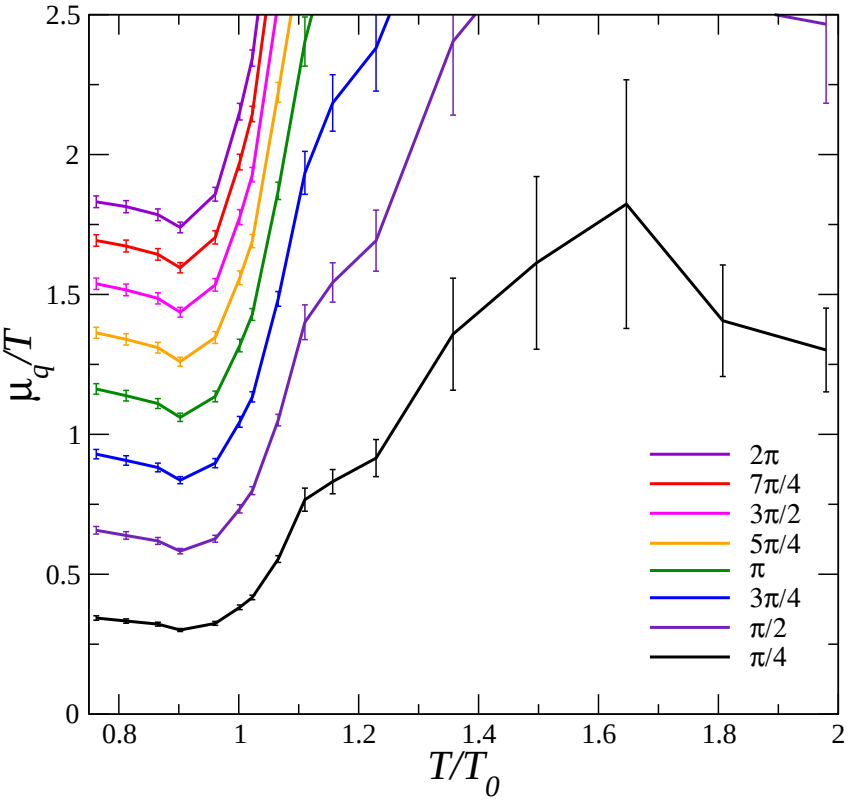
- Splittorf (2005): the critical points found by Fodor and Katz lie dangerously close to the pion condensation line for QCD at finite isospin density: that means in a region where the sign problem is surely relevant, i.e. $e^{i\theta}$ is strongly oscillating.

- Systematic effects related to finite volume should also be properly taken into account, the largest spatial sizes used are always $\sim 2m_\pi^{-1}$: is the infinite volume limit of Lee-Yang zeroes reliable? But going to larger volumes may be difficult.

3.2 – Taylor series

The Bielefeld-Swansea group has performed computations up to the 6th order in μ for the theory with 2 flavour, using a $16^3 \times 4$ lattice and the $p4 - improved$ action

- Below T_c various physical quantities (pressure, quark condensate, . . .) are in agreement with a hadron resonance gas (HRG) model (i.e. a model of Hagedorn like gas of non interacting resonances).
- They determine the pseudo-critical line by looking at susceptibilities and look for a critical point by studying the convergence radius of the series: $\rho = \lim_{n\to\infty} \rho_n \equiv \lim_{n\to\infty} \sqrt{\left|\frac{c_n}{c_{n+2}}\right|}$ They fi nd no clear evidence of it.
- They also determine the expansion for the variance of the phase of $\det M$, $\sigma(\theta)$, obtaining indications on which region of the phase diagram is seriously affected by the sign problem (that's where reweighting surely fails).



Gupta and Gavai have determined the baryon number susceptibility up to 6th order in the Taylor expansion for two flavor QCD. They use a standard staggered quark action and lattice sizes up to $24^3 \times 4$.

They search for a critical end-point by looking at the convergence radius of the series

$$\rho = \lim_{n\to\infty} \rho_n \equiv \lim_{n\to\infty} \sqrt{\left|\frac{c_n}{c_{n+2}}\right|}$$

.

They estimate $T^E/T_c \simeq 0.95(2)$, $\mu_B^E/T^E \simeq 1.1(2)$.

- strong volume dependence: more terms are needed to estimate ρ as V increases.

- enough terms in the series to estimate ρ?

3.3 – Imaginary chemical potential

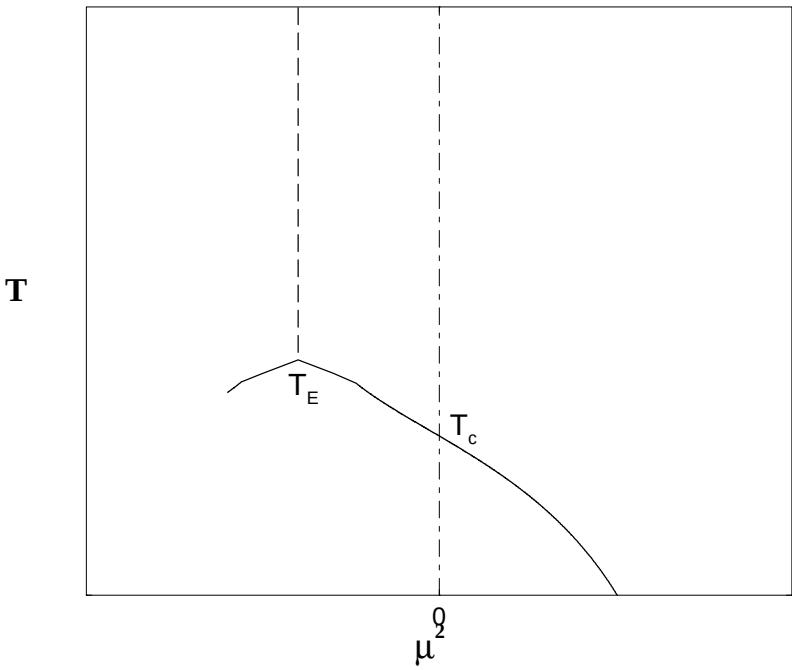
Phase structure in the $T - i\mu_I$ plane:

Roberge and Weiss have shown that $Z(i\mu_I)$ is always periodic in μ_I/T with period $2\pi/N_c$ and they have predicted

- smooth, analytic periodic behaviour at low T, as predicted from a weak coupling calculation
- non-analytic periodic behaviour at high T, as predicted from a weak coupling calculation, with phase transitions at $\frac{\mu_I}{T} = \frac{2\pi}{N_c}(k + 1/2)$

The high temperature RW phase transition can be understood as follows:

- in the quenched theory there is spontaneous breaking of the center Z_3 symmetry in the high T phase with 3 degenerate vacua
- fermions break Z_3 explicitly and select the true vacuum at high T
- μ_I couples to the Polyakov line thus rotating, like an external fi eld, the fermion contribution: which vacuum is selected depends on μ_I



Simulations with imaginary chemical potential have been performed using staggered fermions in the theory with 2 and 3 flavours (P. De Forcrand & O. Philipsen, 2002, 2003) and in theory with 4 flavours (M. D'E. & M.P. Lombardo, 2003, 2004). Recently results have been obtained also with Wilson fermions (H. S. Chen and X. Q. Luo, arXiv:hep-lat/0411023).

Simulations have also been done in the theory with gauge group $SU(2)$ (P. Giudice and A. Papa, Phys. Rev. D69 (2004) 094509): in this case both real μ and imaginary μ can be simulated $\Longrightarrow$ direct test of analytic continuation.

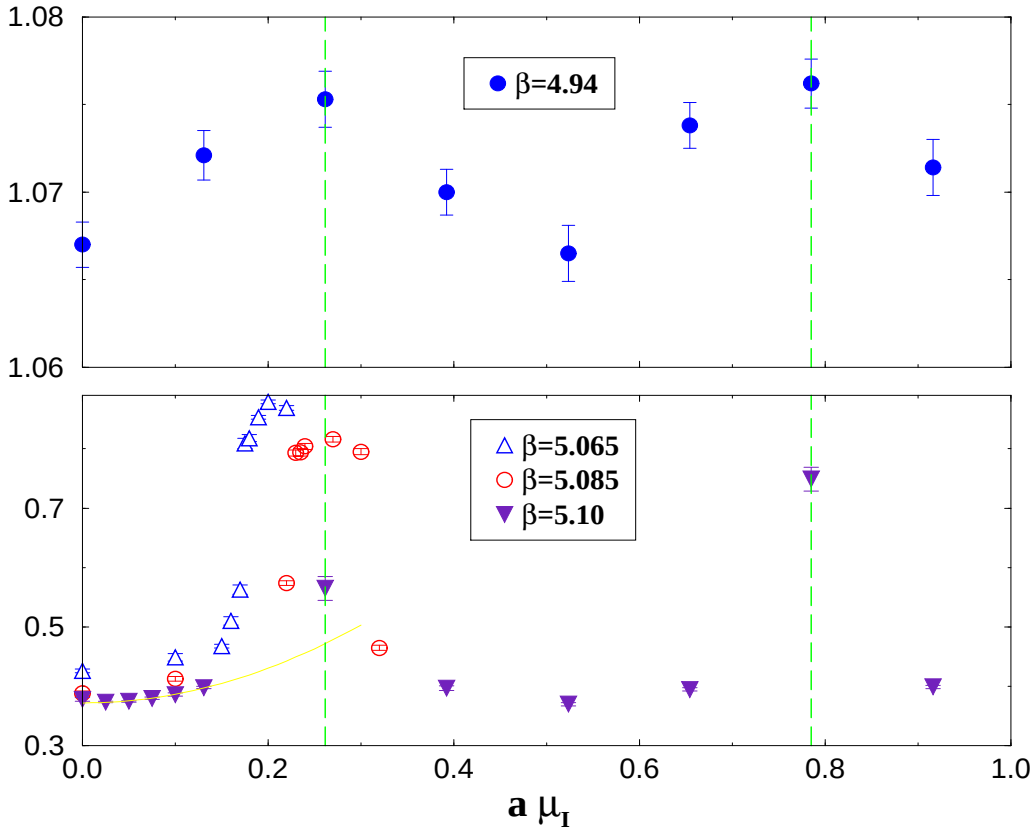
A generalization of the imaginary chemical potential method has been recently proposed (V. Azcoiti, G. Di Carlo, A. Galante, V. Laliena, JHEP 1204:010 and hep-lat/0503010).

The following results regard four staggered flavours of bare mass $a \cdot m = 0.05$ on a $16^4 \times 4$ lattice. $T = 1/(N_t a)$, with $N_t = 4$ in our case, $\Longrightarrow$ $\theta = \mu_I/T = 4a\mu_I$.

At $\mu = 0$, a strong first order phase transition is present at $\beta_c \simeq 5.04$. A first order transition line is therefore expected in the whole $T - \mu$ plane.

Despite being unphysical, $N_f = 4$ is a test ground for comparison among several methods, since an exact algorithm exists.

Chiral condensate



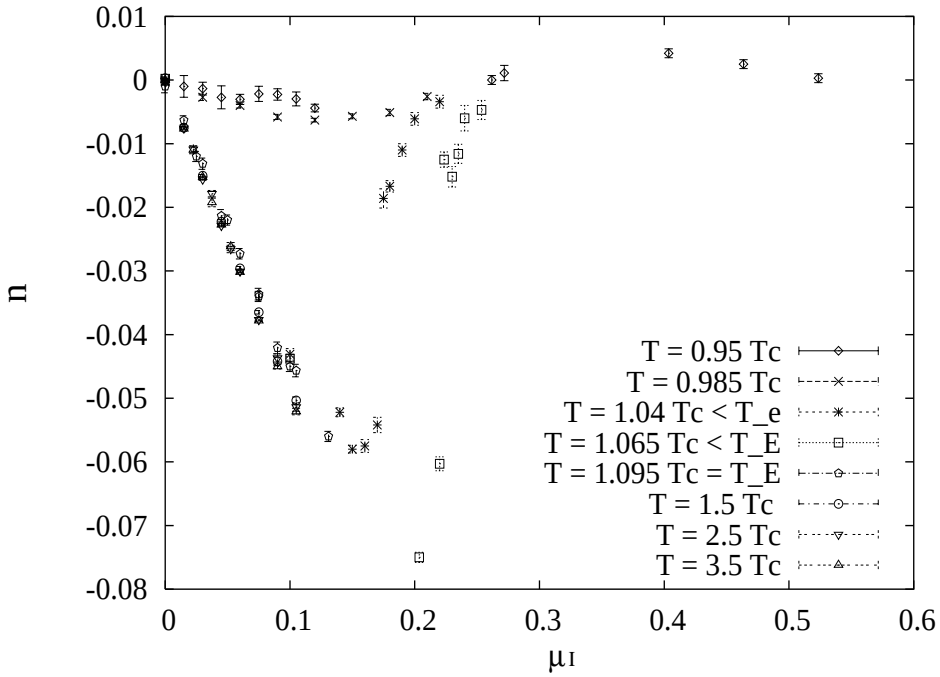
We display in the fi gure our results for the chiral condensate. The green vertical dashed lines correspond to $\theta = \frac{\pi}{3}$ and $\theta = \pi$.

Symmetries of $\langle\bar{\psi}\psi\rangle$: it must be periodic $\frac{2\pi}{3}$ in $\theta = N_t a\mu_I$ and, like the partition function, an even function of μ_I $\longrightarrow$ we expect symmetry around all points $\theta = n\pi/3$. This is verifi ed in our data.

For $\beta < \beta_c$, $\langle\bar{\psi}\psi\rangle$ is continuous in μ_I. At $\beta = 5.065$ there is a critical value $a\mu_I \simeq 0.17$ above which the theory has a transition to a spontaneously broken chiral symmetry phase: we are crossing the chiral critical line. At $\beta = 5.085$ this happens at $a\mu_I \simeq 0.22$ and as we increase μ_I, we observe a transition back to the chirally restored phase at $a\mu_I \simeq 0.30$, which is the symmetric point with respect to $\theta = \frac{\pi}{3}$.

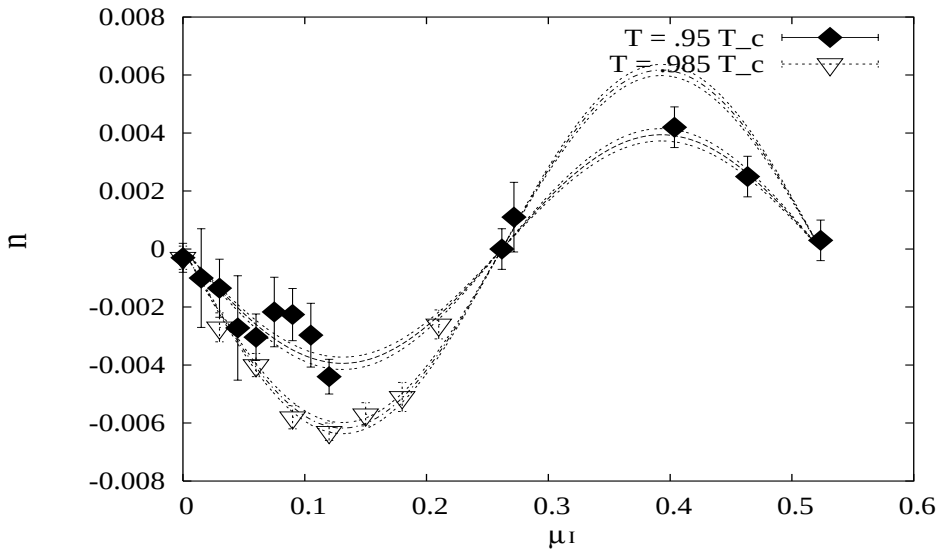
At $\beta = 5.10$ we do not cross, when changing μ_I, the chiral critical line, but the RW critical lines.

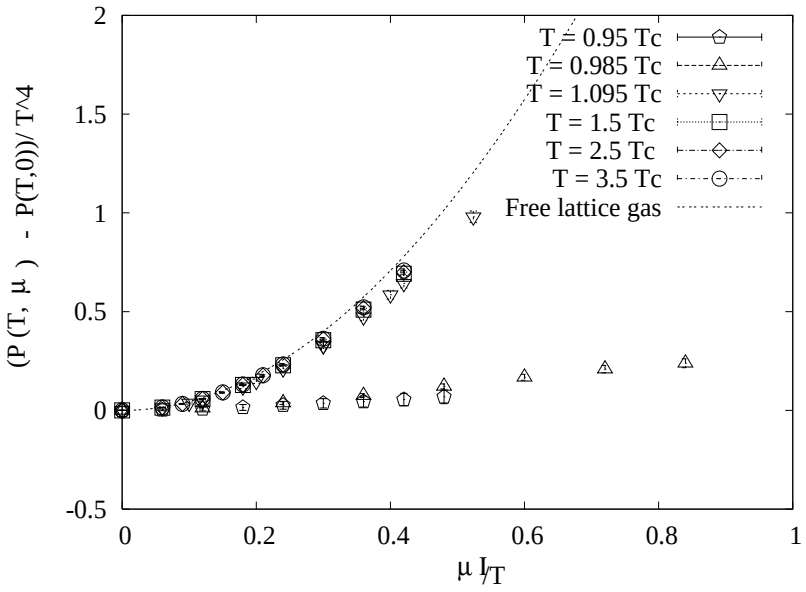
Imaginary part of the baryon density



Overview of the results for the baryon density as a function of μ_I: the behavior is smooth in the hadronic phase, shows the expected discontinuity associated with the chiral/deconfi ning transition in the intermediate region $T_c < T < T_E$, and increases rapidly in the quark gluon plasma phase.

Below: a fi t of the results for $T < T_C$ according to the hadron resonance gas (HRG) model.



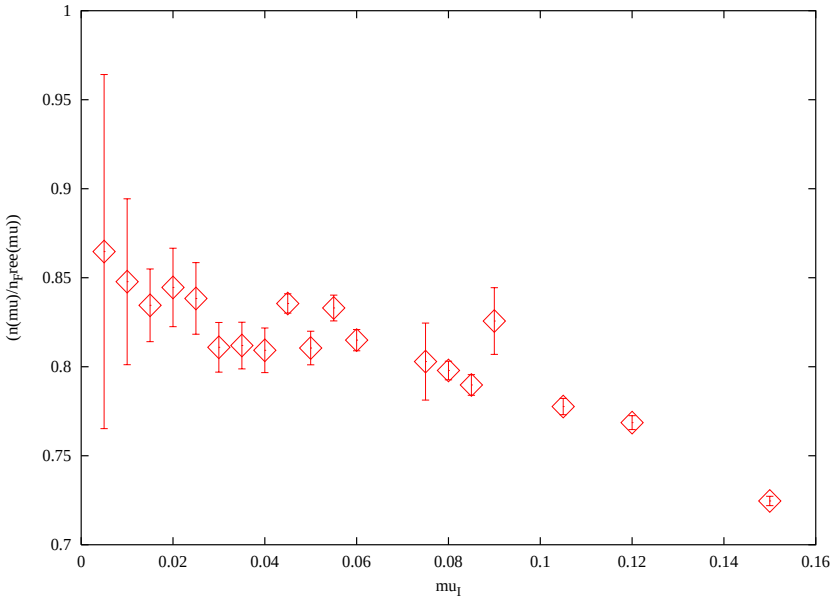


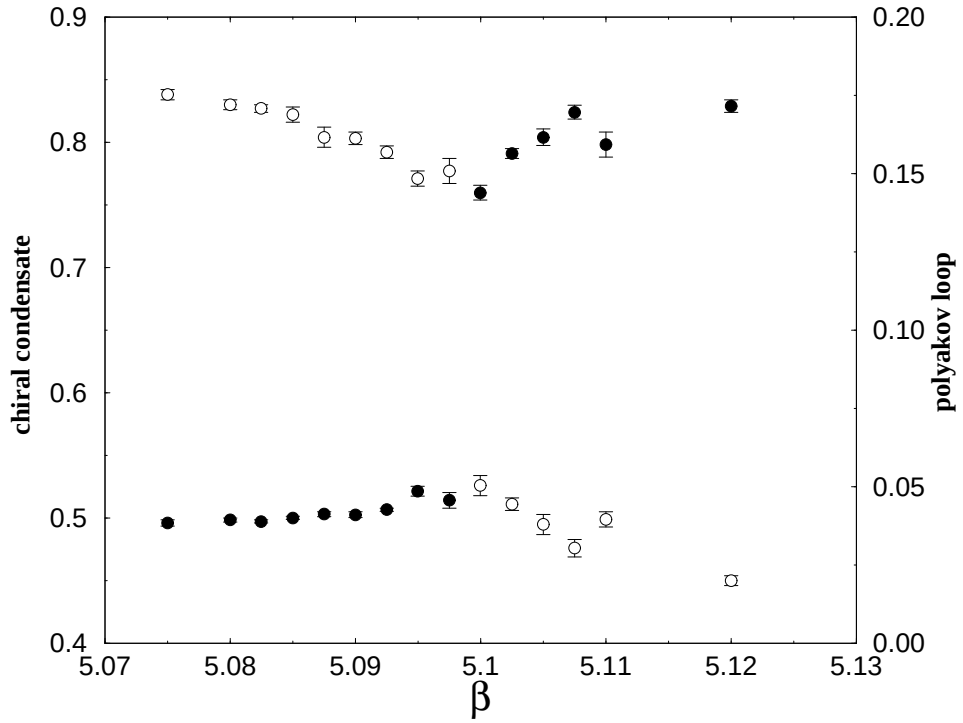
Above: Overview of the results for $\Delta P(T, \mu_I, m_q)/T^4$

$$(P(T, \mu, m_q) - P(T, \mu = 0, m_q))/T^4 = N_t^4 \int n(\mu) d\mu$$

Results at high T are in nice agreement with free lattice gas results.

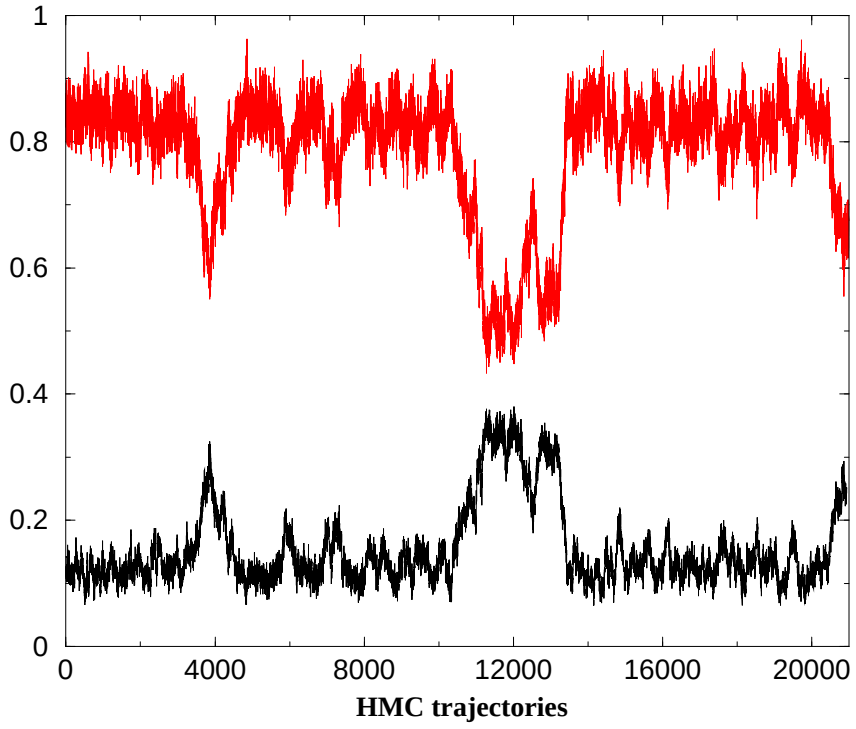
Below: testing the strongly interacting Quark-Gluon plasma (sQGP) right above T_c ($T = 1.095 T_c$)



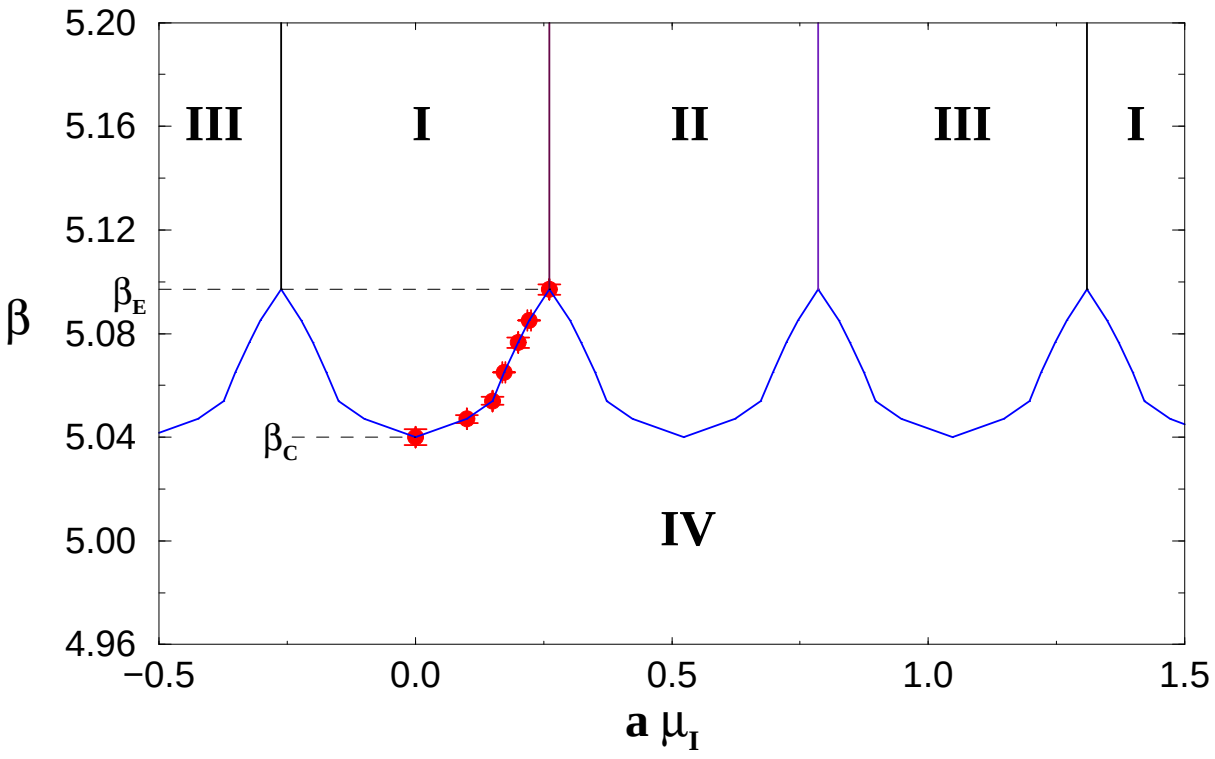


correlation between the chiral condensate and the Polyakov loop at the chiral phase transition at $\mu_I = 0.15$.

the chiral and the deconfi nement phase transition are coincident also at $\mu \neq 0$



Phase diagram



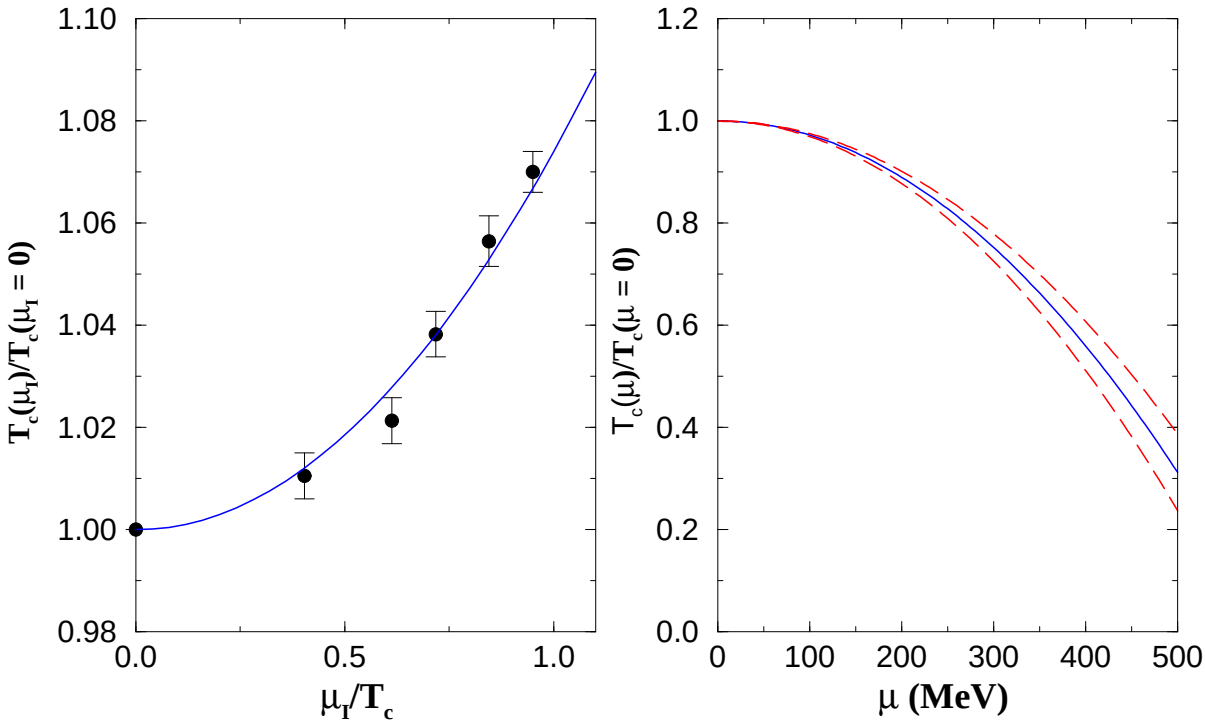
This is a sketch of the phase diagram in the β–μ_I plane, as emerges from our data.

Since the temperature $T = \frac{1}{N_t a(\beta)}$ is a monotonic increasing function of β, this is analogous to the T–μ_I phase diagram.

The red points correspond to our determinations of the chiral critical line, obtained either changing β at fi xed μ_I or changing μ_I at fi xed β.

The rest of the chiral critical line has been obtained by interpolating our determinations and by extending to the whole range of μ_I by exploiting the symmetries of the partition function (periodicity and symmetry under $\mu_I \rightarrow -\mu_I$).

Analytic continuation of the critical line



On the left we show our determinations of $T_c(\mu_I)/T_c(0)$ for the chiral critical line. We have tried a quadratic fi t to our data. Our best fi t is

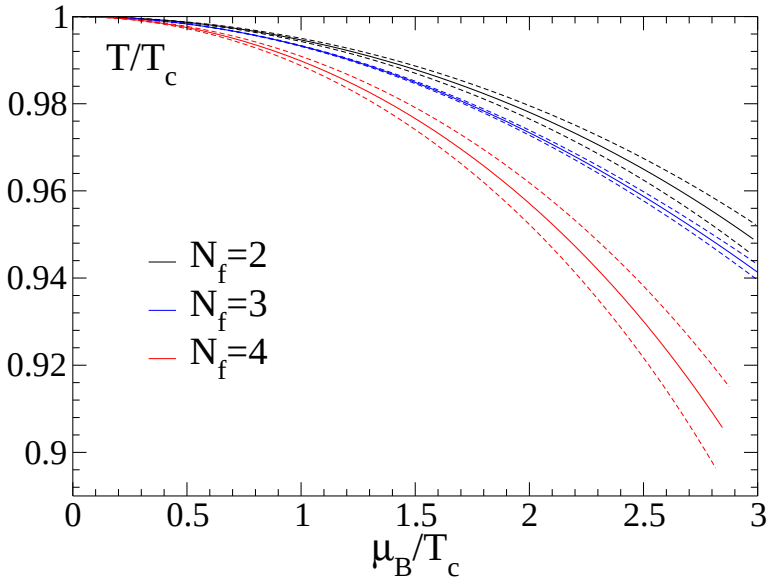
$$\frac{T_c(\mu_I)}{T_c(0)} = 1 + 0.074(5)\left(\frac{\mu_I}{T_c}\right)^2 \qquad (4)$$

with a χ^2/d.o.f. = 0.9. Our data are not precise enough to permit a determination of the quartic and higher order terms.

The analytic continuation of our fi tted chiral critical line into the T – real chemical potential plane is shown on the right hand side.

The critical temperature $T_c(\mu)$ to be expected in Heavy Ion Experiments ($\mu/T \leq 0.1$) is shifted by less than 0.1% with respect to the zero density value $T_c(\mu = 0)$

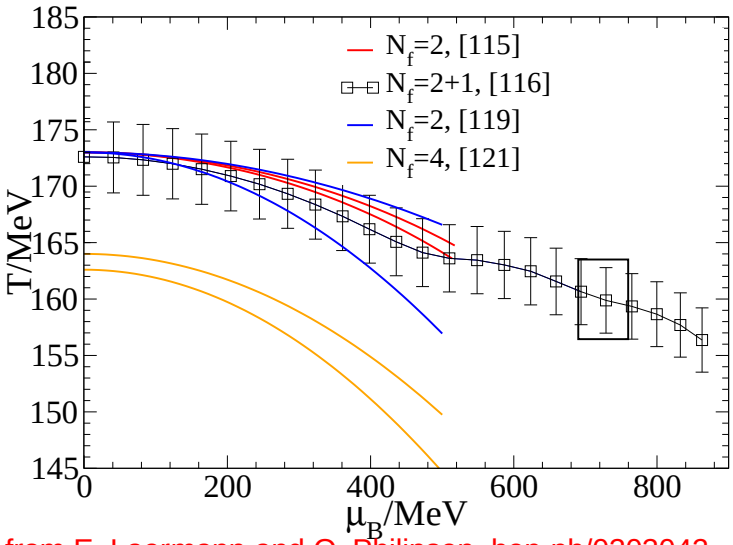
De Forcrand and Philipsen have simulated both $N_f = 2$ (2002) and $N_f = 3$ (2003)



from Ph. de Forcrand and O. Philipsen, hep-lat/0307020

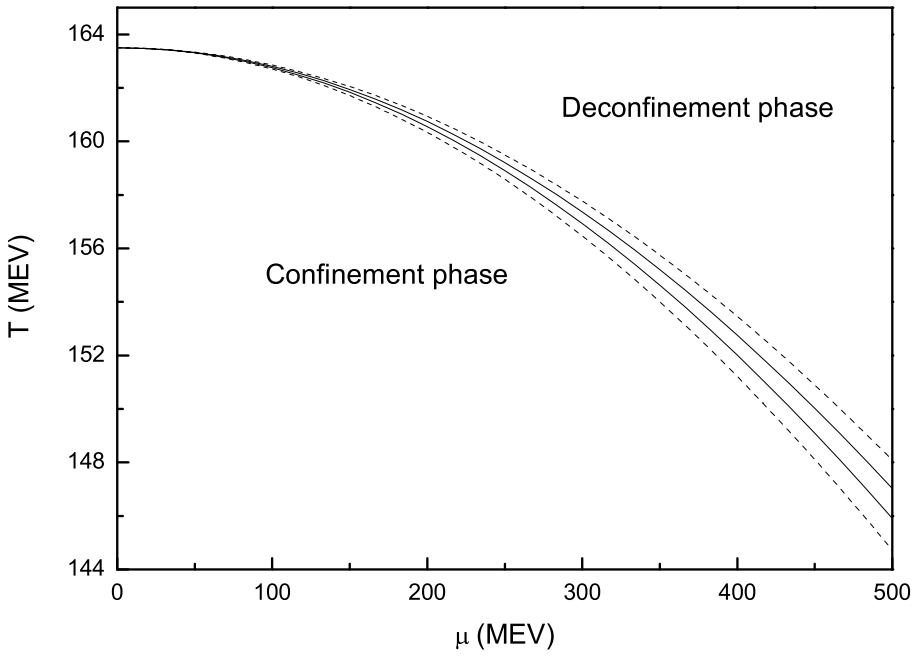
In the case $N_f = 3$ they have also tried to locate a possible critical endpoint in the $T - \mu$ plane by looking at the dependence of the critical mass m_c on the imaginary chemical potential μ_I.

However recent numerical evidence (talks at latest summer conferences) with an exact algorithm (RHMC) indicates absence of a critical end point (see also results at fi nite isospin density by Kogut and Sinclair).



from E. Laermann and O. Philipsen, hep-ph/0303042

Recently results have been obtained also with 4 flavours of Wilson fermions by H. S. Chen and X. Q. Luo, arXiv:hep-lat/0411023



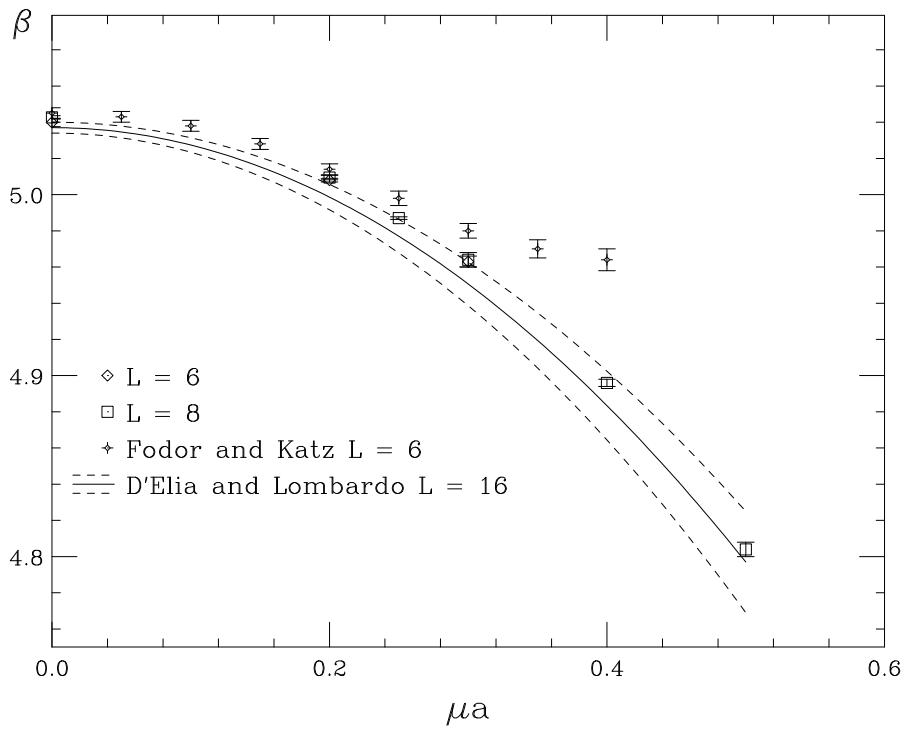
A good agreement emerges between the Wilson fermions critical line (solid band) and the staggered fermions results (dotted line, by M. D'E., Lombardo), , even if the coincidence of the two lines at $\mu = 0$ has been put in by hand.

A generalization of the imaginary chemical potential method has been proposed by

V. Azcoiti, G. Di Carlo, A. Galante, V. Laliena, JHEP 1204:010 and hep-lat/0503010

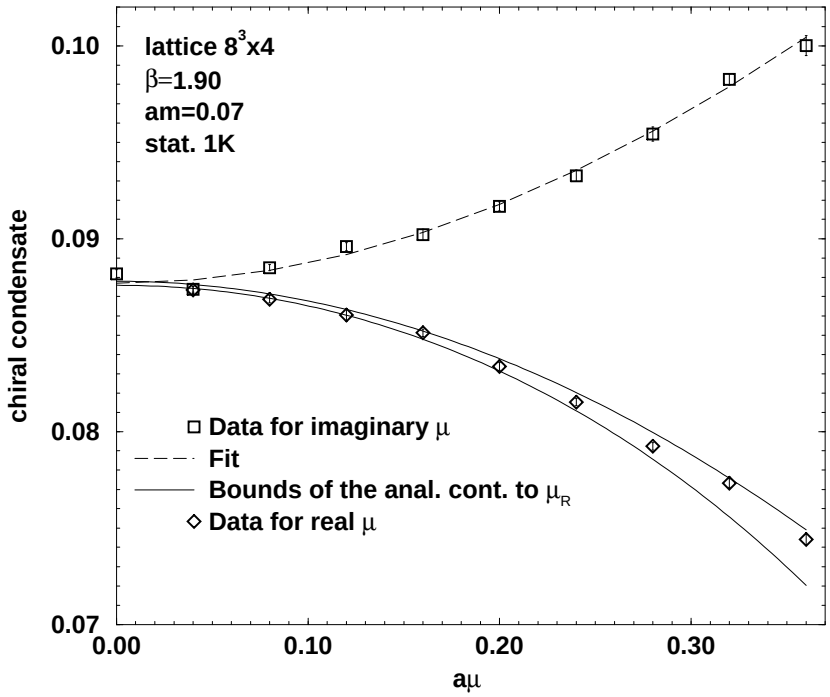
Apart from an imaginary chemical potential, they also put a variable prefactor in front of the temporal covariant derivative in the fermion matrix. The determinant is still real and positive. An extension to lower temperatures should be possible.

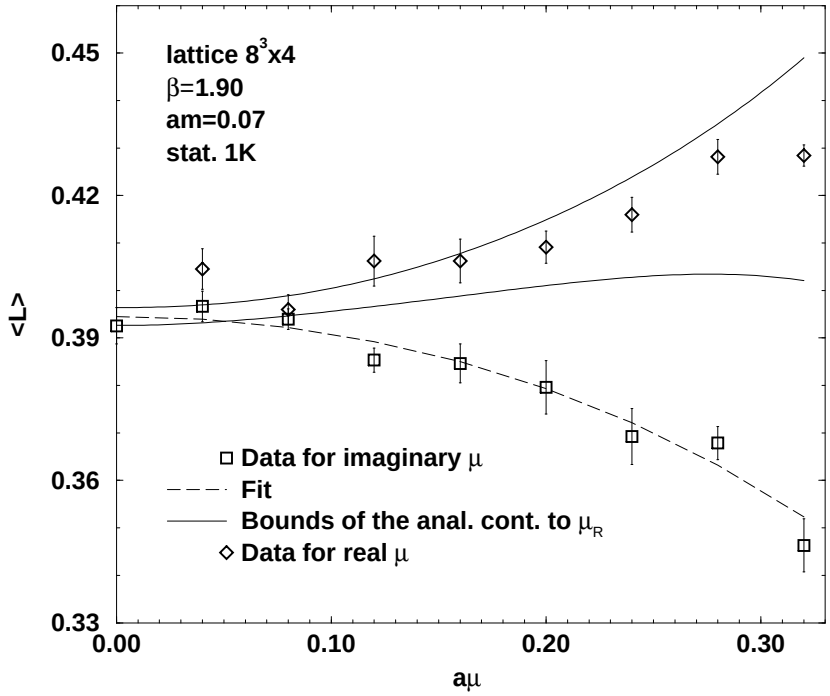
Much more numerically expensive (2 parameters)



P. Giudice and A. Papa (2004) have tested the imaginary chemical potential method with $N_c = 2$.

Values for the chiral condensate and for the Polyakov line measured at real μ agree with those extrapolated from imaginary μ.



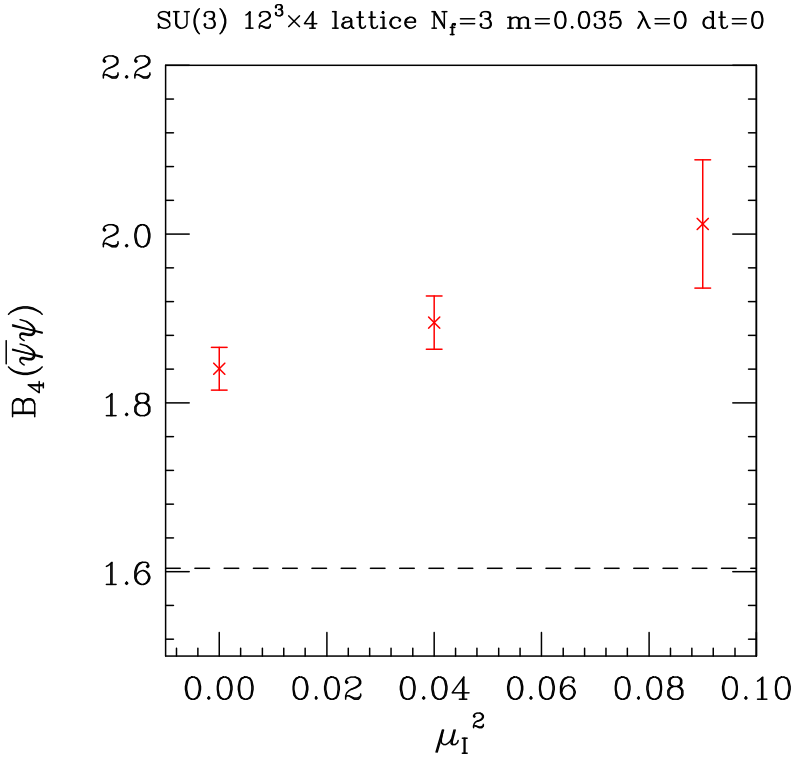


3.4 – Results at finite isospin density

Kogut and Sinclair with $N_f = 2$ (2002) and $N_f = 3$ staggered fermions. They use the standard non exact molecular dynamics algorithm.

They obtain nice agreement with other methods for the critical line at small chemical potential.

In the three flavor case, after a careful extrapolation in the finite step of molecular dynamics, they obtain no evidence for a critical endpoint



This result is being confirmed by De Forcrand and Philipsen by using an exact algorithm

3.5 – Canonical simulations

$$Z(n) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta e^{-i\theta n} Z(i\theta T)$$

The method has been applied to the 2D Hubbard model

Dagotto et al. (1990), Alford et al. (1999)

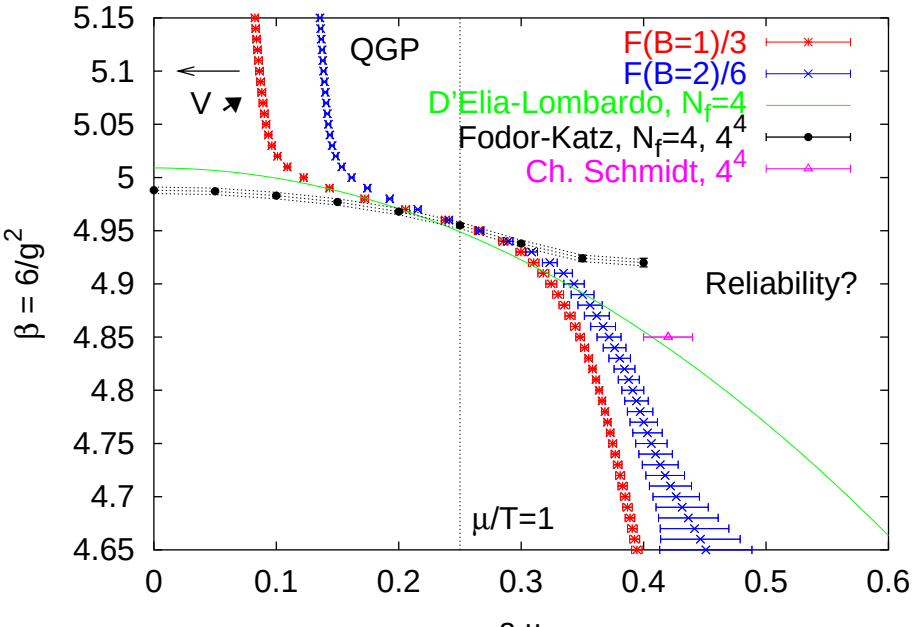
Earlier simulations for QCD were tried by

A. Hasenfratz and D. Toussaint (1992)

Recently renowed interest:

de Forcrand, Kratochvila with staggered fermions

Alexandru et al with Wilson fermions



From de Forcrand, Kratochvila, hep-lat/0409072

Numerical difficulties increase as n is increased: reasonable thermodynamical limit still possible?

4 – Some other open issues

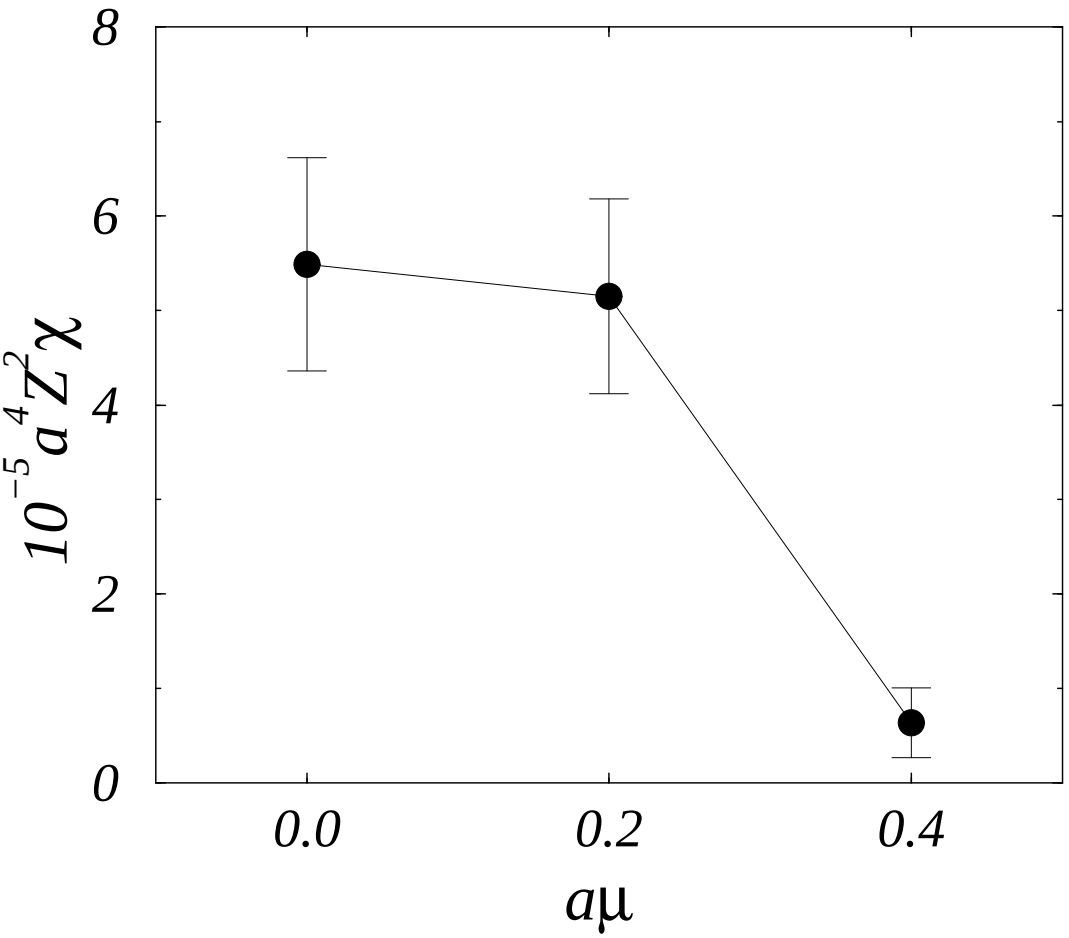
How the dynamics of the high density phase transition ($T \simeq 0$) resemble those of the well studied high temperature phase transition ($\mu = 0$)?

- what is the fate of topological excitations at high densities?

- what is the fate of the candidate confinement mechanism through the high density phase transition?

The issue of topology has been addressed for the theory with $N_c = 2$ obtaining preliminary results which indicate a suppression of the topological susceptibility $\chi = \langle Q^2 \rangle / V$ at the high density phase transition, analogous to the suppression taking place through the high T transition.

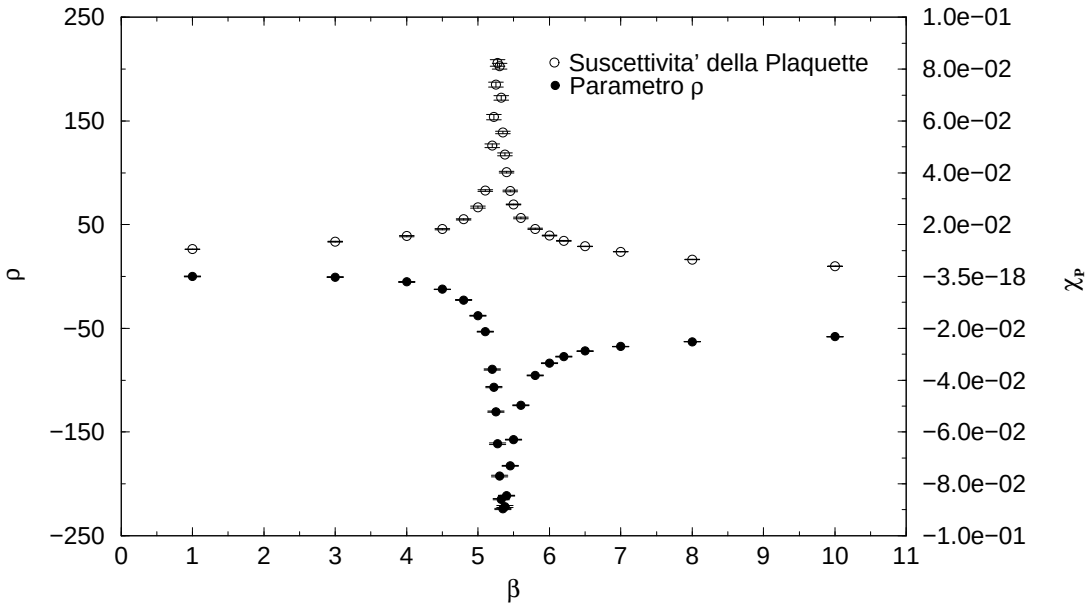
B. Allés, M. D'E., M.P. Lombardo, M. Pepe, Nucl.Phys.Proc.Suppl.94 (2001) 441



Results obtained with 8 staggered flavours on a $14^3 \times 6$ lattice at $\beta = 1.5$, $am = 0.07$.

The issue of the fate of the dual superconductivity mechanism through the high density phase transition has been addressed, very preliminarly, in the crude approximation of static quarks, i.e. in the double limit $m_q \to \infty$, $\mu \to \infty$ with $\exp(\mu)/m_q$ fixed. M. D'E. and M. Ferrari, in progress

If dual superconductivity disappears at high density through a mechanism analogous to what happens in usual superconductors when a strong magnetic fi eld is forced through the medium, then the static quark approximation should not harm too much.



In the fi gure the parameter $\rho = \frac{\partial}{\partial\beta} \ln\langle\mu\rangle$ is plotted together with the plaquette susceptibility. $\langle\mu\rangle$ is the expectation value of a magnetically charged operator developed in Pisa that detects dual superconductivity. The sharp negative peak of ρ hints at the possible disappearance of dual superconductivity in coincidence with the phase transition (peak of the plaquette susceptibility).

5 – Conclusions and perspectives

- The study of the QCD phase diagram is of fundamental importance in several fields (Early Universe, H.I. Collisions, Neutron Stars, . . .)

  Lattice QCD simulations are the candidate first principles tool, but the **sign problem** makes life difficult.

- Recent efforts by several groups using different methods are giving consistent information on the phase diagram, starting from the region high T and low μ, which is the one relevant for the experimental search for deconfined states of matter in Heavy Ion Experiments.

  Progress is not very fast, but steady.

- Is there any critical end point in the $T - \mu$ plane?
  - The assumption behind the ansatz for a critical end point is that at $\mu = 0$ there is a crossover instead of a phase transition. This however is not a well established fact and the presence of a true phase transition (first order?) is not excluded (D'Elia, Di Giacomo, Pica 2005, see Di Giacomo's lectures at this School).
  - Evidence for a critical endpoint in $N_f = 2+1$ obtained by multiparameter reweighting has been recently questioned (Ejiri, Splittorf)

- Evidence with $N_f = 2$ obtained by Taylor expansion is unclear (Gavai and Gupta see the critical point, the Bielefeld and Swansea groups do not)
- Recent data for $N_f = 3$ seem to disprove the presence of a critical endpoint in that case (Kogut and Sinclair at finite isospin density, de Forcrand and Philipsen within the imaginary chemical potential method)

- A lot of other important issues (fate of confinement mechanisms and topology at high density) are still open.

- We are still waiting for a real breakthrough opening the possibility of exploring by numerical simulations the whole $T - \mu$ plane.