The static potential in lattice QCD

Ph. de Forcrand[1,2]

[1]ETH Zürich

[2]CERN

Motivation

Confinement

$\updownarrow$

[linear] growth of static potential

- Definition and properties of static potential
- Understand confinement: what are the relevant d.o.f.?

$V \propto R \implies$ area law for Wilson loop, ie. disorder

confinement non-perturbative $\leftrightarrow$ non-perturbative d.o.f. $\leftrightarrow$ topological

Outline

What is the static potential ?

static potential = groundstate energy of static charges

("potential" energy of static charges)

- groundstate
- static charges

How to obtain the groundstate

Thanks to imaginary time $\tau = it$

Create $q\bar{q}$ pair with operator $\mathcal{O}$

Amplitude $C(t) \equiv < 0|\mathcal{O}^\dagger \exp(-itH)\mathcal{O}|0 >$

$$= (\sum_i < \psi_i|b_i^*)\exp(-itH)(\sum_j b_j|\psi_j >)$$

$$= \sum_i |b_i|^2 \exp(-itE_i) \text{ hopeless}$$

Rotate to imaginary time:

$$\rightarrow \sum_i |b_i|^2 \exp(-\tau E_i)$$

$$\tau \rightarrow \infty \Longrightarrow |b_0|^2 \exp(-\tau E_0)$$

i.e. $$E_0 = \lim_{\tau\to\infty} -\frac{1}{a}\log\frac{C(\tau)}{C(\tau-a)} \qquad \tau = Ta$$

Difficulty: $-\log\frac{C(T)}{C(T-1)} \approx aE_0(1 + \frac{|b_1|^2}{|b_0|^2}\exp(-Ta(E_1 - E_0))$

i.e. want $Ta \gg 1/(E_1 - E_0)$

Since (see later) $E_1 - E_0 \sim 1/R_{q\bar{q}}$, want elongated $R \times T$ loops

$$1 \ll \xi/a \ll R \ll T$$

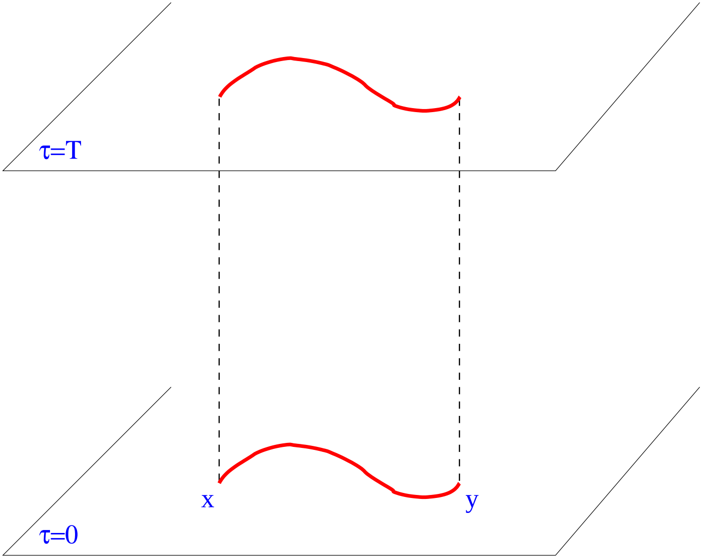
Creating static charges on the lattice

- Fix gauge to $A_0 = 0$
- At $\tau = 0$, create parallel transporter from x to y: $B_{xy} = (\prod_{x \to y} U)_r$
  Under local gauge transformation, $B_{xy} \longrightarrow \Omega_r(x)^\dagger B_{xy} \Omega_r(y)$
  identical to transformation of $(q_r(x), \bar{q}_r(y))$
  $\longrightarrow$ pair of opposite color charges in representation r
  (other possibilities, but pbc $\longrightarrow \sum q = 0 \mod N_c$)
- At $\tau = T$, create $B^\dagger_{xy} = B_{yx}$, ie. annihilate $q\bar{q}$ pair (same colors)
  $\longrightarrow$ amplitude $C(x,y,T) \equiv \sum_{ab} (B^\dagger_{xy})_{ab}(T)(B_{xy})_{ba}(0)$
- Average over gauge fields (in $A_0 = 0$ gauge):
  $\langle C(x,y,T)\rangle = (\sum_i \langle \psi_i | b_i^*) \exp(-TH)(\sum_j b_j |\psi_j\rangle)$
  $= \sum_i |b_i|^2 \exp(-TE_i)$
- $\lim_{T\to\infty} -\frac{1}{T} \log\langle C(x,y,T)\rangle = E_0$; E_0 is static potential
  Indep. of path B_{xy}; depends on $R = |y-x|$ and representation.
- $A_0 = 0$ gauge unnecessary: $C(x,y,T) = \mathrm{Tr}(\prod U)_r$ around loop

Wilson loop $\boxed{\langle W(R,T)\rangle = \frac{1}{N_c}\mathrm{Tr}\prod U}$ Potential $\boxed{V(R) = \lim_{T\to\infty} -\frac{1}{T}\log\langle W(R,T)\rangle}$

area law $\longleftrightarrow$ linear confinement

Creating a $q\bar{q}$ pair



The need to be clever: breakdown of naive approach

- Linear confinement: $V(R) \sim R/\xi \Longrightarrow \langle W(R,T)\rangle \sim \exp(-\frac{1}{\xi^2} R \times T)$

(ξ correlation length, $\sigma = 1/\xi^2$ string tension)

But for each measurement $i = 1..N$, $W_i(R,T) \sim \mathcal{O}(1)$:

signal goes down exponentially with R,T, but noise remains constant.

- Central limit theorem: after $N \gg 1$ measurements W_i,

$$\frac{1}{N}\sum_i W_i(R,T) = \langle W(R,T)\rangle + \mathcal{O}\left(\sqrt{\frac{\langle W(R,T)^2\rangle - \langle W(R,T)\rangle^2}{N}}\right)$$

Can only probe (R,T) such that $\langle W(R,T)\rangle \gtrsim \frac{1}{\sqrt{N}}$

$N = 10^6 \rightarrow \frac{R\times T}{\xi^2} \lesssim \log(10^3) \sim 7$

i.e. $R \lesssim 2\xi, T \lesssim 3\xi$ unsatisfactory:

  - $T >> R$ not satisfied $\rightarrow$ excited states bias
  - $R >> \xi$ not satisfied $\rightarrow$ not long-distance

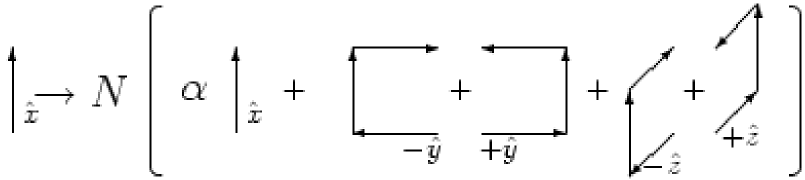
- Technical steps:
  - improve groundstate overlap
  - decrease variance

Improving groundstate overlap I: smearing

Idea 1: groundstate is string-like

$\rightarrow$ sum over many paths xy distributed like string profile

Spatial smearing: $U \rightarrow \mathrm{Proj}_{SU(N)}(\alpha U + \sum_4 \mathrm{staples})$



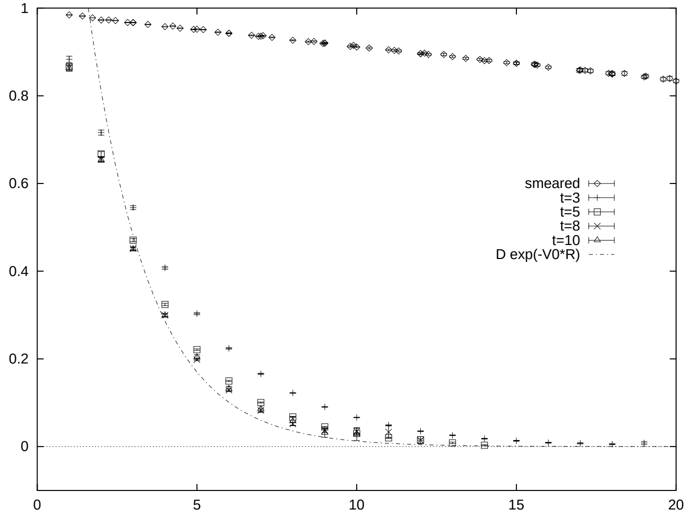
$\mathrm{Proj} = \frac{1}{\sqrt{\det}}$ for $SU(2)$

perform m smearing steps $\rightarrow\sim$ Gaussian smearing, width $\sim \sqrt{m}$

optimize $m(R)$ to maximize $|b_0|^2$ in $\langle W(R,T)\rangle = \sum_i |b_i|^2 \exp(-TE_i)$

$\rightarrow \quad m \propto \frac{1}{a^2}$

Improving groundstate overlap I: effect of smearing



$|b_0|^2$ vs R, with and without smearing: spectacular!

Improving groundstate overlap II: variational basis

Idea 2: variational basis of operators $\{B^k{}_{xy}\}$: vary path or nb. smearing steps

- Measure correlation matrix $M(T)$:

  $$M_{kl}(T) = \langle \prod_T U_4^\dagger(x) \cdot B^k{}_{xy}(T)^\dagger \cdot \prod_T U_4(y) \cdot B^l{}_{xy}(0) \rangle$$

- Solve generalized eigenvalue problem:

  $$M(T)v_i = \lambda_i M(T_0)v_i \quad \text{(same eigenvectors at } T_0 \text{ and } T\text{)}$$

  Solved by $M(T_0)^{-1/2}M(T)M(T_0)^{-1/2}(M(T_0)^{1/2}v_i) = \lambda_i(M(T_0)^{1/2}v_i)$

  $\Rightarrow$ take T_0 small enough that $M(T_0)$ positive definite (statistical noise)

Theorem (Lüscher & Wolff): $\lambda_i = \exp(-(T-T_0)E_i)(1+\mathcal{O}(\exp(-T(E_i-E_j))))$

Look for plateau in effective mass: $E_i^{eff}(T) = -\log(\frac{\lambda_i(T+1)}{\lambda_i(T)})$

- Control over systematic error:

  even a poor measurement of $(E_1 - E_0)$ is very useful:

  expect reliable measurement of E_0 only for $T \gtrsim \frac{1}{E_1 - E_0}$

- Very large/noisy matrix M: project onto significant modes first

Variance reduction I: multi-hit (Parisi)

Idea: perform Monte Carlo integration more thoroughly where it matters

Wilson loop: integrate over time-like links of loop itself

$U_4 \longrightarrow \bar{U}_4 = \frac{1}{n}\sum_{i=1}^{n} U_4^{(i)}$ 1-link MC, all others fixed

Or analytically: $\bar{U}_4 = \langle U_4 \rangle = \frac{\int dU_4 U_4 \exp(\beta \mathrm{Tr} U_4 X)}{\int dU_4 \exp(\beta \mathrm{Tr} U_4 X)}$ $(U(1), SU(2), SU(3))$

Substitute $U_4 \longrightarrow \bar{U}_4$ simultaneously on all independent links

Typically (Wilson action; $R > 1$):

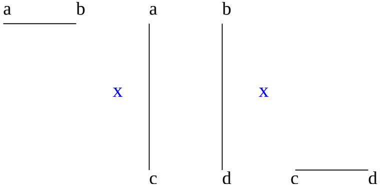
$\Rightarrow$ variance reduction $\sim \exp(c\ T)$ (C. Michael)

Best at strong coupling: $c \searrow$ as $\beta \nearrow$

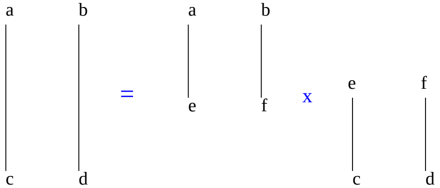
Variance reduction II: Lüscher-Weisz, aka multi-level

Idea: generalize multi-hit to link pairs $U_4(x,t)U_4^\dagger(y,t)$

$W(R,T) = \mathrm{Tr}\prod U =$



Factorize:



or more factors

$$W_4^{abcd}(0,T) = W_4^{abef}(\tfrac{T}{2},T)\cdot W_4^{efcd}(0,\tfrac{T}{2})$$

Take Monte Carlo average of each factor separately?

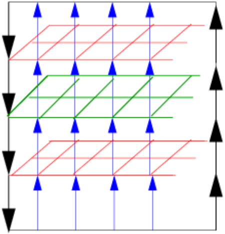
Possible, if the d.o.f in first and second factor do not interact

Obtained by freezing all spatial links at $\tau = 0, \frac{T}{2}, T$

Lüscher-Weisz algorithm: details

Perform sub-Monte Carlo organized into "slabs":

spatial links at boundaries (colored links) frozen



Generalizes in hierarchical way (update green links, then repeat) $\longrightarrow$ "multi-level"

Single level more efficient. Optimum slab "thickness" $\sim \frac{1}{2}\frac{1}{T_c}$ (Pepe)

Variance reduction $\sim \exp(cT)$ like multi-hit, but much larger c

Very large T become accessible $\longrightarrow$ control over groundstate

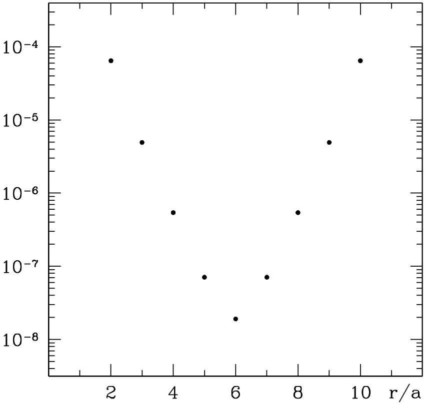
Gain so large that old calculation worth repeating

Watch out: storage (higher representation, $SU(N)$, baryon)

Need action local in time: improved actions? full QCD??

Lüscher-Weisz algorithm: illustration

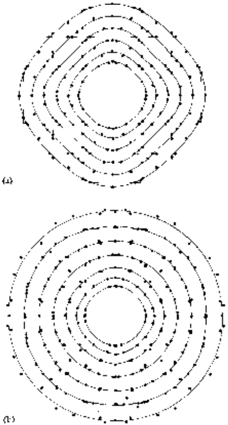
Correlation of 2 Polyakov loops vs R ($12^4, \beta = 5.7$):



Naive approach: $W \sim \mathcal{O}(1)$ for each measurement $\Rightarrow$ ca. 10^{16} measurements!

r_I: short-distance improvement (Sommer)

When R not $\gg a$, lattice distortions: how to correct?



Asymptotic freedom $\longrightarrow$ $V_{q\bar{q}} \sim$ Coulomb as $R \longrightarrow 0$

Include lattice Coulomb potential in global fit of $V(R)$?

Elegant solution

- Relevant quantity is force $F(R) = -\frac{dV}{dR}$, not $V(R)$ itself

$V(R)$ contains UV-divergent, arbitrary constant $\longrightarrow$ removed by differentiation

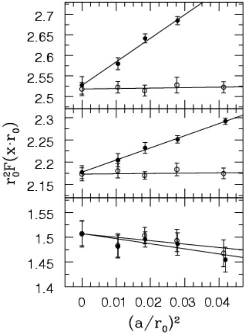
- On lattice: $\frac{-1}{a}(V(R+a) - V(R)) = F(R_I)$, $R_I = \frac{a}{2}(1 + \mathcal{O}(a^2))$

Take tree-level improved observable:

$R_I =$ exact solution when V is Coulomb potential

- Aside: "Sommer scale" r_0 solution of $r_0^2 F(r_0) = 1.65 \longrightarrow r_0 \sim 0.45$ fm

better than string tension (numerically & phenomenologically)



Lattice results: overview

• Prerequisite: continuum extrapolation $a \longrightarrow 0$

Bosonic theory:

$\langle W\rangle(a) = \langle W\rangle(a=0)(1 + c_2 a^2 + ..), \quad c_2 = \sum_{k=0} c_2^{(k)} g^{2k}$

Set to zero $c_2^{(k)}, k = 0, ..$ by perturbative (tree-level, 1-loop, ..)

or non-perturbative $\longrightarrow c_2 = 0$

improvement of action and/or observables (Symanzik)

Outline:

• Short distance: $\Lambda_{\overline{MS}}$
• Long distance: Lüscher term, string picture
• Flux tube
• String breaking
• Large-N_c
• Baryon potential
• State of the art

Running coupling and $\Lambda_{\overline{MS}}$

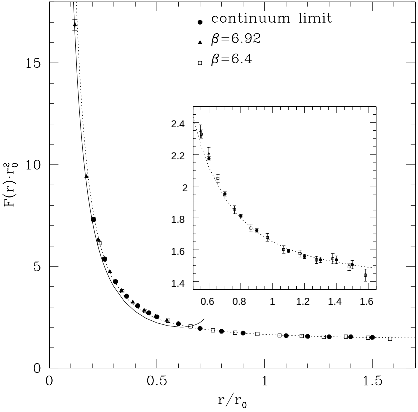
One-gluon exchange: $F(R) = C_F g^2 \frac{1}{4\pi R^2}$, $C_F = 4/3$.

Use as definition of running coupling $\bar{g}(R)$

RG eq.: $\beta(\bar{g}) = -\frac{d}{dR}\bar{g}(R) = -\sum_{\nu=0} b_\nu \bar{g}^{2\nu+3}$

b_0, b_1 universal, b_2 known

$\Lambda_{\overline{MS}} \times r_0$ known from previous work (or fit (Schilling et al.))



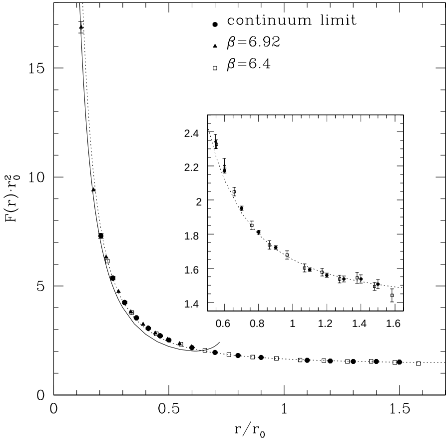
Necco & Sommer

Lüscher term

aka Casimir, bosonic string, "universal"

$R \to \infty, \quad V(R) \approx \sigma R - \frac{\pi}{24}(d-2)\frac{1}{R}$ in d dim. (Coulomb-like in $d=4$)

$$F(R) = \sigma + \frac{\pi}{24}(d-2)\frac{1}{R^2}$$



String picture

String forms $R \times T$ worldsheet; small transverse fluctuations $\vec{\phi}(x,\tau) \in \mathcal{R}^{d-2}$

Bosonic string: action is $\sim \sigma \times$ area $A(\vec{\phi})$; effect of fluctuations?

$Z(R,T) = \int d\phi \exp(-\sigma A(\vec{\phi}))$; $\quad V(R) = \lim T \to \infty - \frac{1}{T} \log Z(R,T)$

- Classical solution: minimize action $\to \vec{\phi} \equiv 0$, $V(R) = \sigma R$
- Quantum fluctuations (1-loop): gradient expansion of $A(\vec{\phi})$

$A(\vec{\phi}) = \int_0^R dx \int_0^T d\tau \sqrt{1 + |\frac{\partial\vec{\phi}}{\partial x}|^2 + |\frac{\partial\vec{\phi}}{\partial\tau}|^2}$ (Pythagoras); b.c. $\vec{\phi} = 0$ for Wilson loop

Expand: $A(\vec{\phi}) \approx \int_0^R dx \int_0^T d\tau (1 + \frac{1}{2}\sum_k^{d-2} |\nabla\vec{\phi}_k|^2)$

All $(d-2)$ components decoupled $\to$ overall factor $(d-2)$.

Take $\vec{\phi} = \phi_1$, integrate by parts: $A(\phi_1) \approx \int_0^R dx \int_0^T d\tau (1 - \frac{1}{2}\Delta\phi_1)$

Δ is unique $\mathcal{O}(m^2)$ term with rotation invariance $\to$ universal

In Fourier space: $\phi = \sum_{mn} c_{mn} \exp(i(k_m x + k_n \tau))$, $\; k_m = m\frac{\pi}{R}, k_n = n\frac{\pi}{T}$

$A(\phi_1) \approx RT(1 + \frac{\pi^2}{2}\sum_{mn} c_{mn}^2 (\frac{m^2}{R^2} + \frac{n^2}{T^2}))$

$Z(R,T) \to \prod$ Gaussian integrals over c_{mn}

$\propto \exp(-\sigma RT)\, \det'(-\Delta)^{-1/2}$ (zero-mode excluded)

String picture

Result (Dietz & Filk, 1983): $\det'(-\Delta) = \frac{1}{\sqrt{2R}}\eta(i\frac{T}{R})$

Dedekind η-function: $\eta(\tau) = \exp(i\frac{\pi}{12}\tau)\prod_{n=1}^{\infty}(1-\exp(2i\pi n\tau))$

Consequence for static potential: $T \gg R \to \eta(i\frac{T}{R}) \approx \exp(-\frac{\pi}{12}\frac{T}{R})$

$V(R) = \lim_{T\to\infty} -\frac{1}{T}\log Z(R,T) = \mathrm{const.} + \sigma R - \frac{\pi}{24}(d-2)\frac{1}{R}$

(factor $-1/2$ from $\det^{-1/2}$)

- "Universal": independent of higher-order terms in gradient expansion
  ie. of particular string action
- "Casimir": similar calculation; free modes; geometry-dependence
- Trick? Sum over Fourier modes is infinite $\to$ UV divergence.
  - ζ-function regularization
  - lattice regularization

ζ-function regularization

$$\zeta(s,-\Delta) \equiv \sum_n' \lambda_n^{-s} = \sum_n' e^{-s\log\lambda_n} \quad \rightarrow \quad \log\det(-\Delta) = -\frac{d}{ds}\zeta(s,-\Delta)|_{s=0}$$

Remove from ζ divergent piece $\propto \frac{1}{s}$; keep finite piece (Hawking)

cf. dimensional regularization

$$\zeta(s,-\Delta) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1} \sum_{mn}' \exp(-t\,\pi^2(\frac{m^2}{R^2}+\frac{n^2}{T^2}))$$
$$= \frac{1}{\Gamma(s)}(\frac{R^2}{\pi})^s \int_0^\infty dt\, t^{s-1}\,(\Omega(t)\Omega(\frac{T^2}{R^2}t)-1)$$

$\Omega(t) \equiv \sum_{n=-\infty}^{+\infty} \exp(-\pi n^2 t)$; -1 for zero-mode $m=n=0$

Symmetry: $\Omega(1/t) = \sqrt{t}\,\Omega(t)$ (elliptic $\theta_3(0,\exp(-\pi t))$)

$\longrightarrow$ split integral into $(\int_0^1 + \int_1^\infty)$, and transform the first:

$$\int_0^1 .. = \int_1^\infty dt'\, t'^{-s}(\Omega(t')\Omega(\frac{R^2}{T^2}t')-1) + \frac{1}{s-1} - \frac{1}{s}$$

Pole at $s=0$ isolated; discard and differentiate regular piece:

$\longrightarrow$ Geometry-dependent part:

$$\int_1^\infty dt \left(\frac{1}{t}(\Omega(t)\Omega(\frac{T^2}{R^2}t)-1) + (\Omega(t)\Omega(\frac{R^2}{T^2}t)-1)\right)$$

Lattice regularization

Want $\log\det'(-\Delta) = \sum' \log(a^2\hat{p}^2)$, $\hat{p}^2 = \hat{p}_1^2 + \hat{p}_2^2$,

$$\hat{p}_1 = \frac{2}{a}\sin\frac{ap_1}{2},\ p_1 = \pi\frac{m}{R},\quad m = 0,..,R/a-1$$
$$\hat{p}_2 = \frac{2}{a}\sin\frac{ap_2}{2},\ p_2 = \pi\frac{n}{T},\quad n = 0,..,T/a-1;\quad a \to 0$$

Split $\sum'$ into 2 terms (cf. Bunk):

(1) $m = 0 \ \to\ \sum'_n \log(4\sin^2\frac{\pi n}{2T}) \to$ indep. of $R \to$ drops out

(2) $m \neq 0$: define $\varepsilon(m)$ via $\sinh\frac{\varepsilon(m)}{2} = \sin\frac{\pi m}{2R}$

$$\to \quad a^2\hat{p}^2 = 4\sinh^2\frac{\varepsilon(m)}{2} + 4\sin^2\frac{\pi n}{2T}$$
$$= \exp(\varepsilon(m))\,(1 - 2\cos\pi\tfrac{n}{T}\exp(-\varepsilon(m)) + \exp(-2\varepsilon(m)))$$

$\log\det'(-\Delta) =\ (1)\ ..$

$$..+\ \frac{T}{a}\sum'_m \varepsilon(m) + \sum'_m\sum_n \log(1 - 2\cos\pi\tfrac{n}{T}\exp(-\varepsilon(m)) + \exp(-2\varepsilon(m)))$$

- $\sum_n \log(..) = 2\log(1 - \exp(-\frac{T}{a}\varepsilon(m)))$, using $\prod_1^{n-1}(x^2 - 2\cos\frac{k\pi}{n} + 1) = \frac{x^{2n}-1}{x^2-1}$
  $T/a \to \infty \Rightarrow$ cf. $\prod_{n=1}^{\infty}(1 - \exp(2i\pi n\tau))$ in η-function, subleading
- $\frac{T}{a}\sum'_m \varepsilon(m)$ gives $\frac{RT}{a^2} - \frac{T}{R}\frac{\pi}{12} + \mathcal{O}(a^2)$, ie. Lüscher term
  Divergent term renormalizes string tension

Further beauty of Dedekind η-function

- What about $R \leftrightarrow T$ symmetry of Wilson loop?

Modular transformation $\eta(\frac{-1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$, with $-i\tau = \frac{T}{R}$ (cf. $\Omega(t)$)
Achieves $\frac{T}{R} \leftrightarrow \frac{R}{T}$

- Additional terms (beyond Lüscher) sizeable when $T \sim R$ (Gliozzi)

- Application to correlator of Polyakov loops (length T, separation R):

  fixed $\longrightarrow$ periodic b.c. in τ

  $\longrightarrow \phi = \sum_{mn} c_{mn} \exp(i(k_m x + k_n \tau)),\;\; k_m = m\frac{\pi}{R}, k_n = 2n\frac{\pi}{T}$

- $R \ll T$, zero-temp., $V(R) = \sigma R - \frac{\pi}{24}(d-2)\frac{1}{R}$ (T-b.c. irrelevant)

- $R \gg T$, $C(R,T) \approx \exp(-R(\sigma T - \frac{\pi}{6}(d-2)\frac{1}{T}))$

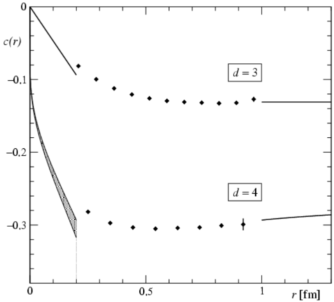
  ie. $\sigma_{eff} = \sigma - \frac{\pi}{6}(d-2)\frac{1}{T^2}$, $\longrightarrow$ deconfinement!

  transition sensitive to subleading terms(Caselle)

Dedicated measurement of Lüscher term + subleading

$$V(R) = V_0 + \sigma R + c_L \frac{1}{R} \to F(R) = \sigma - c_L \frac{1}{R^2} \to c_L = \frac{1}{2} R^3 \frac{d^2 V}{dR^2}$$

Measure $c_{eff}(R_I) = \frac{R_I^3}{2a^2}(V(R+a) + V(R-a) - 2V(R))$ (Lüscher-Weisz 02)



- Beautiful precision, clear convergence
- “Precocious scaling” from $R \sim 0.3$ fm ($\leftrightarrow$ Kuti)
- Discretization errors negligible ?
- Correction $\sim \frac{1}{R}$ in fact absent (Lüscher-Weisz 04) $\longrightarrow \frac{1}{R^2}$

Subleading term: open-closed string duality (Lüscher-Weisz)

- Consider correlator of 2 Polyakov loops of length T:

$$\langle P^*(0)P(R)\rangle = \exp(-\sigma RT - \mu T)\,\eta(q)^{2-d}$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \;\; q = \exp(-\pi\frac{T}{R})$$

Polyakov loops $\longrightarrow$ static charges present $\forall\tau$ $\longrightarrow$ sum over all $q\bar{q}(R)$ states:

$\langle P^*(0)P(R)\rangle = \mathbf{Z}_{q\bar{q}} = \sum_{n=0}^{\infty} w_n \exp(-E_n T)$, w_n integers

Identify with η expansion $\longrightarrow$ whole spectrum for open string of length R

$E_n = \sigma R + \mu + \frac{\pi}{R}(\frac{2-d}{24} + n)$, ie. splitting $\frac{\pi}{R}$

- Going beyond free (Gaussian) string action:
  - First term is boundary term $b\int d\tau(|\frac{d\phi}{dx}|^2_{x=0} + |\frac{d\phi}{dx}|^2_{x=R})$
  - Perturbation theory in $b \longrightarrow \langle P^*(0)P(R)\rangle = \mathbf{Z}_{q\bar{q}}$ to 1rst order
- Duality: the same worldsheet $\langle P^*(0)P(R)\rangle = \mathbf{Z}_{q\bar{q}}$ represents the propagation of a closed string of length T through Euclidean time R

$\longrightarrow$ Expand in energy-levels of closed strings (using modular transf. of η)

Cannot accomodate a piece $\propto b$ in energy-levels $\Rightarrow$ $b = 0$

Subleading term: pedestrian

Hand-waving: $T\ E(R) = T(\sigma R + \frac{c_L^o}{R} + \frac{b}{R^2})$

$R\ E(T) = R(\sigma T + \frac{c_L^c}{T} + \frac{b}{T^2})$

Modular transf. $\frac{R}{T} \leftrightarrow \frac{T}{R}$, but not $\frac{T}{R^2} \leftrightarrow \frac{R}{T^2}$

- Next-order: prediction in $3d$,

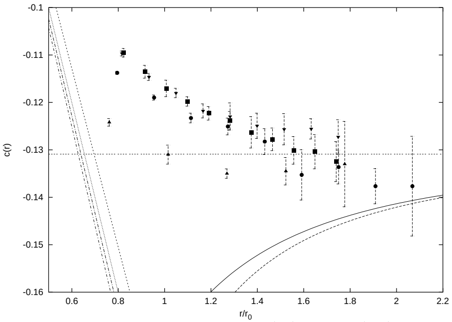
$$E_n = \sigma R + \mu + \frac{\pi}{R}(n - \frac{1}{24}) - \frac{\pi^2}{2\sigma}(n - \frac{1}{24})^2 \frac{1}{R^3} + \mathcal{O}(\frac{1}{R^4})$$

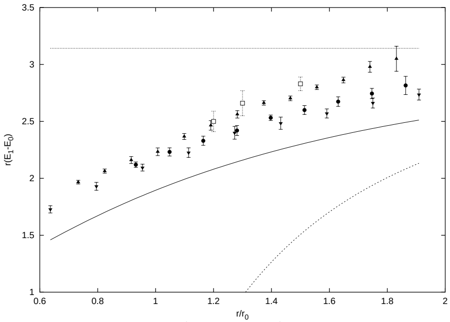
- Coincides with expansion of Nambu-Goto string (action $\propto$ area):

$$E_n = \sigma R \sqrt{1 + \frac{2\pi}{\sigma R^2}(n - \frac{1}{24}(d-2))} \quad \text{(Arvis)}$$

Work under way

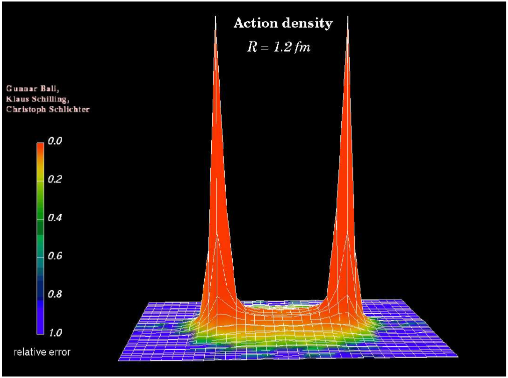
- Change gauge group ($SU(N)$ vs Abelian, discrete,..) $\longrightarrow$ universality
- Info on excited states:
  - directly (Wilson loop + variational basis) (Kuti et al., Majumdar)
  - indirectly (Polyakov loops at non-zero temperature) (Caselle et al.)
- Very challenging: high accuracy at large R/a + extrapolation $a \rightarrow 0$





Majumdar: 3d $SU(2)$, $c_{eff}(R)$ (compare w/LW 02); $(E_1 - E_0)$

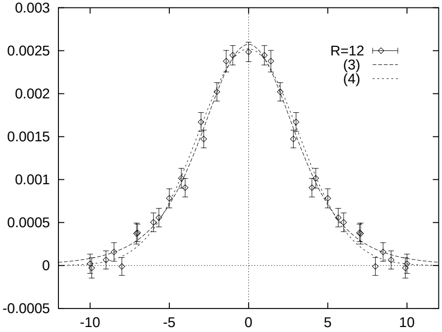
Flux tube



Measure shift in action density caused by $q\bar{q}$: $\frac{\langle q\bar{q}|F^2_{\mu\nu}|q\bar{q}\rangle}{\langle q\bar{q}|q\bar{q}\rangle} - \langle F^2_{\mu\nu}\rangle$

$$= \frac{\langle W F^2_{\mu\nu}\rangle}{\langle W\rangle} - \langle F^2_{\mu\nu}\rangle$$

Flux tube profile

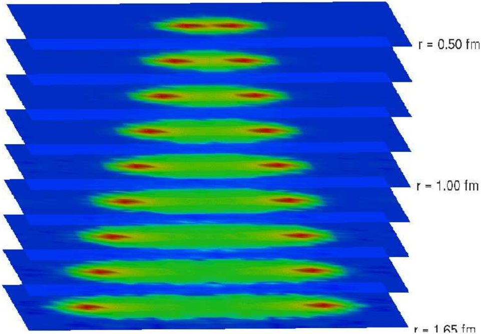


Confront with models, esp. dual superconductor (see other speakers)

Here, dipole $\sim \frac{1}{(\delta^2+|x_\perp|^2)^3}$ (pert. th.)

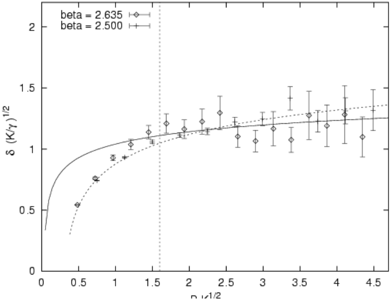
and Gaussian $\sim \exp(-\frac{|x_\perp|^2}{\delta^2})$ (bosonic string, Caselle et al.)

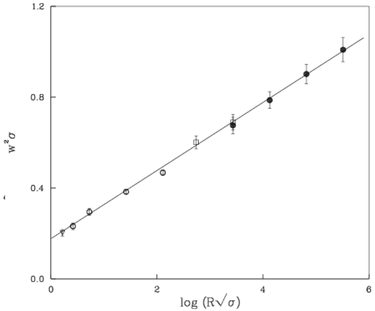
Flux tube width



Bosonic string: massless fluctuations $\longrightarrow$ string worldsheet is rough:
$\langle|\phi(x)-\phi(0)|^2\rangle \sim c\log|x|$, $c = \frac{1}{\pi\sigma}$ (Lüscher, Münster & Weisz)

Roughening?





Width vs R, c predicted (solid) or fitted (dotted)

Here $SU(2)$ (Bali); $3d$ Z_2 over distance scale ~ 100! (Gliozzi)

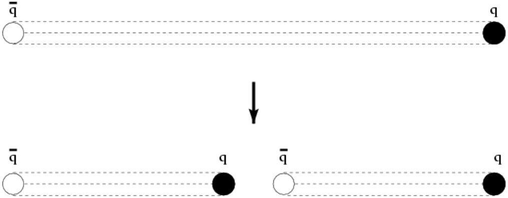
Expected from roughening transition:

- strong coupling expansion of Wilson loop: $\langle|\phi(x)-\phi(0)|^2\rangle$ finite
- $\beta > \beta_{rough}$: expansion ceases to converge (but free energy analytic)

String breaking

So far, quenched: no dynamical quarks

With dynamical quarks: string breaks by $q\bar{q}$ creation



Creation of 2 static-light mesons

- Same if dynamical fundamental charges are bosons (Higgs), if static charges are adjoint, ...
- Is the theory still “confining” ?

no order parameter, but no isolated colored states $\longrightarrow$ semantics

- Observe string breaking on the lattice?

Long history of failures until ...

2×2 basis

Use (again) variational basis:
(Philipsen & Wittig: $3d$ $SU(2)$+Higgs, hep-lat/9807020
Knechtli & Sommer: $4d$ $SU(2)$+Higgs, hep-lat/9807022)

- Include as trial states "broken" $|\mathcal{B}\rangle$ and "unbroken" $|\mathcal{U}\rangle$ string states
- Diagonalize 2×2 correlation matrix $\longrightarrow$ groundstate + excited state

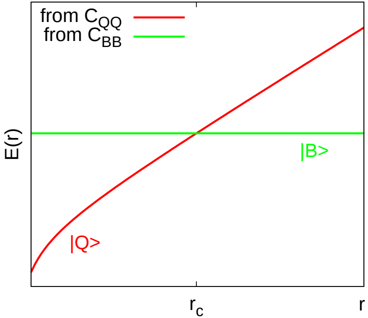
$$C(R,T) = \begin{pmatrix} \mathcal{U}\,\mathcal{U} & \mathcal{U}\,\mathcal{B} \\ \mathcal{B}\,\mathcal{U} & \mathcal{B}\,\mathcal{B} \end{pmatrix} = \mathcal{R}^{\dagger}_{2\times 2} \begin{pmatrix} \exp(-E_0(R)T) & 0 \\ 0 & \exp(-E_1(R)T) \end{pmatrix} \mathcal{R}_{2\times 2}$$

- Eigenstates are $\begin{pmatrix} \Phi_0 \\ \Phi_1 \end{pmatrix} = \mathcal{R}_{2\times 2} \begin{pmatrix} \mathcal{U} \\ \mathcal{B} \end{pmatrix}$, ie.

$\Phi_0 = \cos\theta\ \mathcal{U} + \sin\theta\ \mathcal{B}$
$\Phi_1 = -sin\theta\ \mathcal{U} + cos\theta\ \mathcal{B}$

- Crucial role of off-diagonal elements $\longrightarrow$ mixing
- Monitor E_0, E_1, θ as a function of R $\longrightarrow$ quick 90° rotation at $R = R_b$ but not too quick!

No mixing



QCD: $\mathcal{U}$ is $Q\bar{Q}$ (static); $\mathcal{B}$ is $Q\bar{q} + q\bar{Q}$, ie. pair of static-light mesons

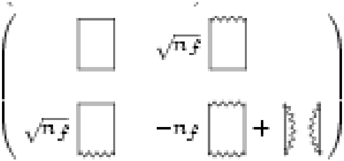
No mixing $\Longrightarrow \theta(R)$ jumps from 0 to 90° at $R = R_b$

Small mixing $\rightarrow$ rapid variation $\rightarrow$ need good resolution $\delta R \ll a$ in R

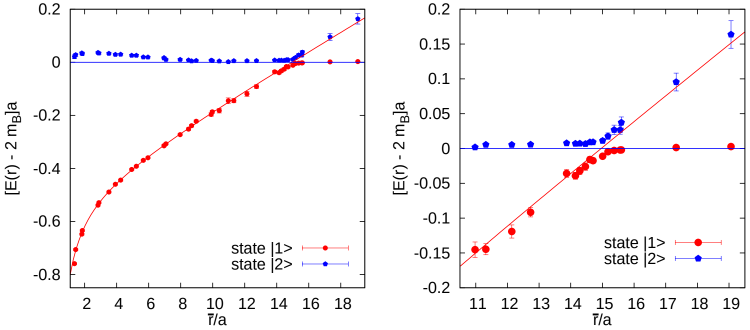
$\rightarrow$ off-axis correlations

Heroic efforts

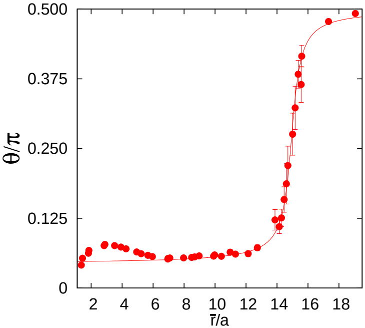
$$C(t) = \begin{pmatrix} C_{QQ}(t) & C_{QB}(t) \\ C_{BQ}(t) & C_{BB}(t) \end{pmatrix} = e^{-2m_Q t}$$

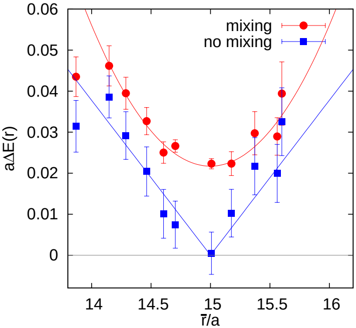


(Michael & Pennanen: off-diagonal elt; MILC; Bali, Schilling et al.)



Results





Good resolution for mixing angle $\theta(R)$

Quadratic dependence $\Delta E(R) \sim \Delta E_c + c(R - R_b)^2 \quad \leftrightarrow \quad$ no mixing

$R_b \approx 1.25$ fm; $\Delta E_c \approx 50$ MeV

Systematics: - Reduce quark mass ($m_\pi \sim 640$ MeV)

- Take continuum limit

$\Delta E_c(a = 0) = 0$?

Mixing "put in by hand" ?

String breaking from Wilson loops only

Wilson loop: $|\mathcal{U}\rangle = \cos\theta\,|\Phi_0\rangle - \sin\theta\,|\Phi_1\rangle$

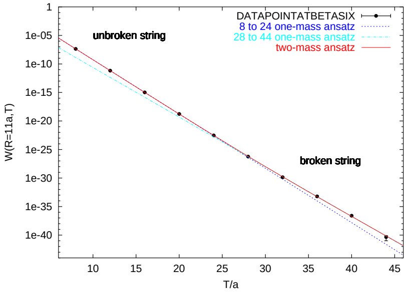
$R < R_b$: $\theta \approx 0 \;\rightarrow\; |\mathcal{U}\rangle \approx |\Phi_0\rangle$: easy

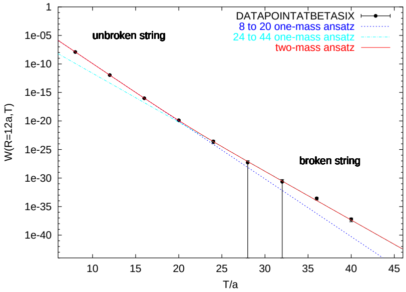
$R > R_b$: $\theta \approx \frac{\pi}{2} \;\rightarrow\; |\mathcal{U}\rangle \approx |\Phi_1\rangle$: how to retrieve $(|\Phi_0\rangle, E_0)$?

$\langle\mathcal{U}|\exp(-HT)|\mathcal{U}\rangle = \cos^2\theta\exp(-E_0T) + \sin^2\theta\exp(-E_1T)$

When $\theta \approx \frac{\pi}{2}$: 2nd term dominant

Solution: increase T until $\exp(-(E_1 - E_0)T) << \frac{\cos^2\theta}{\sin^2\theta}$

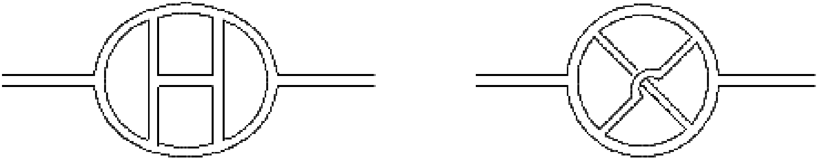




Only $SU(2)$ adjoint (Kratochvila & PdF) and Z_2+ Higgs(Gliozzi & Rago) in $3d$

Large N_c: motivation

- The $N_c \to \infty$ theory is simpler, but still non-perturbative
- Define 't Hooft coupling $\lambda \equiv g^2 N_c$. Fix λ while $N_c \to \infty$.
- Pert. expansion re-ordered into topological expansion ('t Hooft)



genus-h diagram $\sim (\frac{1}{N_c^2})^h$

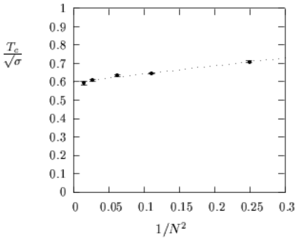
$N_c \to \infty \implies$ planar diagrams only

planar diagrams $\sim \exp(\#$ vertices) "only" (solved in $2d$)

Fermions: N_f fixed $\to \mathcal{O}(\frac{1}{N_c})$; $\implies$ quenched

Study $N_c \to \infty$ limit of $SU(N)$ Yang Mills on the lattice

$\lim_{N_c\to\infty} SU(N_c)$ on lattice (Teper et al., Del Debbio et al.)



- Good extrapolation in $\frac{1}{N_c^2}$ from $N_c = 2$ (coeff $\lesssim 1$)

T_c, latent heat, interface tension, glueball spectrum, topology, ...

$\Rightarrow$ any $N_c = \infty$ theoretical prediction applies to real world within $\sim 10\%$

- Cf. Nambu-Goto closed string $E_n = \sigma R\sqrt{1 + \frac{8\pi}{\sigma R^2}(n - \frac{1}{24}(d-2))}$

$\rightarrow \frac{T_c}{\sqrt{\sigma}} = \sqrt{\frac{3}{\pi(d-2)}}$ (2^{nd} order, mean field) (Pisarski-Alvarez 82; Olesen 85)

- $d = 3$: 0.977 vs $SU(2) : 1.12(1)$ and $SU(3) : 0.98(2)$
- $d = 4$: 0.691 vs $SU(2) : 0.709(4)$ and $SU(3) : 0.646(3)$

- In $d = 3$, $\frac{\sqrt{\sigma}}{g^2}$ very close to $\sqrt{\frac{N_c^2-1}{8\pi}}$ (Nair)

k-strings

- In $SU(N)$, N values for color charge ("N-ality") $\rightarrow$ $(N-1)$ string tensions σ_k
  Focus on ratios $\sigma_k/\sigma_1,\ k = 1..\mathrm{int}(N/2)$, with 2 theoretical guesses:
  - Casimir: $\frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1}$ $\leftarrow$ pert. th.
  - sine: $\frac{\sigma_k}{\sigma_1} = \frac{\sin\ k\pi/N}{\sin\ \pi/N}$ $\leftarrow$ supersymmetry (Douglas & Shenker)
    M-theory (Hanany, Strassler & Zaffaroni)
    but (Herzog & Klebanov)

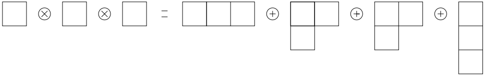
- Construct correlator of 0-momentum spatial Polyakov loops in representation r:

$$C_r(t) = \sum_{x_1,x_2} \langle \chi_r[P(0;0)]\chi_r[P(x_1,x_2;t)]\rangle, \quad P(x_1,x_2;t) = \prod_{x_3=1}^{L_3} U_3(x_1,x_2,x_3;t)$$

$$\chi_f[P] = \mathrm{Tr}P;$$

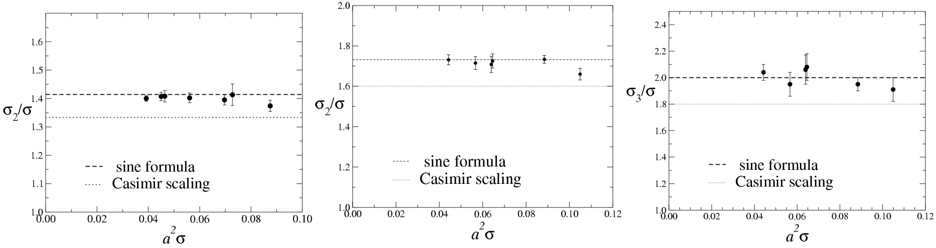
$$\chi_{k=2}[P] = \mathrm{Tr}P^2 - (\mathrm{Tr}P)^2;$$

$$\chi_{k=3}[P] = 2\mathrm{Tr}P^3 - 3\mathrm{Tr}P^2\mathrm{Tr}P + (\mathrm{Tr}P)^3 \quad \text{(antisymmetric)}$$



Extract σ_k with fit $E_k(L_3) = \sigma_k L_3 - \frac{\pi}{3L_3}$ + continuum extrapolation

k-strings: lattice results $SU(4)$ and $SU(6)$ (Del Debbio et al.)



Teper et al.: larger errors; in-between Casimir & sine law

- No reason to expect either at finite (small) N
- Real issue (Shifman): corrections $\mathcal{O}(\frac{1}{N})$ or $\mathcal{O}(\frac{1}{N^2})$?

Keep k fixed, increase N

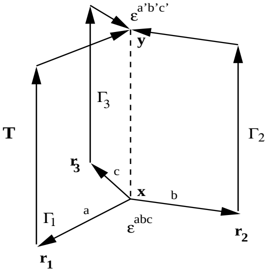
- Casimir: $\frac{k(N-k)}{N-1} \longrightarrow k(1-\frac{k-1}{N})$
- Sine law: $\frac{\sin\ k\pi/N}{\sin\ \pi/N} \longrightarrow k(1-\frac{(k^2-1)\pi^2}{6N^2})$

Topological expansion in $\frac{1}{N^2} \longrightarrow$ Sine law

Static baryon potential

Form color singlet from N fundamental charges $\longrightarrow$ baryon. ($N = 3$ mostly)

Repeat $q\bar{q}$ construction $\Rightarrow$ Y-shaped lines and baryon junction



$$W_{3q} = \frac{1}{3!}\varepsilon^{abc}\varepsilon^{a'b'c'}U(x,y,1)^{aa'}U(x,y,2)^{bb'}U(x,y,3)^{cc'}$$

- Junction needed for gauge invariance; positions at $\tau = 0$ and T irrelevant
- Form of potential V_{3q} ?

  short-distance $\longrightarrow$ pert. th.: $V_{3q}(r_1,r_2,r_3) \approx \frac{1}{2}\sum_3 V_{q\bar{q}}(|r_{ij}|)$

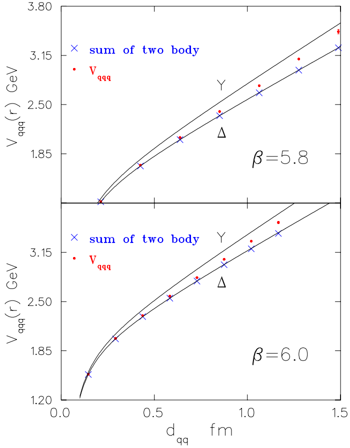
  long-distance $\longrightarrow$ Y-shaped flux tubes: $V_{3q}(r_1,r_2,r_3) \approx \sigma_{q\bar{q}} L_Y$?

  ie. evidence for 3-body interactions

Y-law vs Δ-law

- Δ-law: 2-body forces only. $V_{3q} = \frac{1}{2}\sum_3 V_{q\bar{q}}(|r_{ij}|) \rightarrow \sigma_{q\bar{q}}\frac{\text{perimeter}}{2}$ (Cornwall)
- Y-law: minimize total string length $\rightarrow$ Steiner point $\rightarrow \sigma_{q\bar{q}} L_Y$ (Isgur & Paton)

Numerically: $1 \leq \frac{2L_Y}{\text{perimeter}} \leq 1.15..$ (equilateral) (Suganuma..; Alexandrou..)



3-body force inside a proton?

Baryonic Lüscher term (Jahn & PdF)

Baryonic Wilson loop made of 3 worldsheets

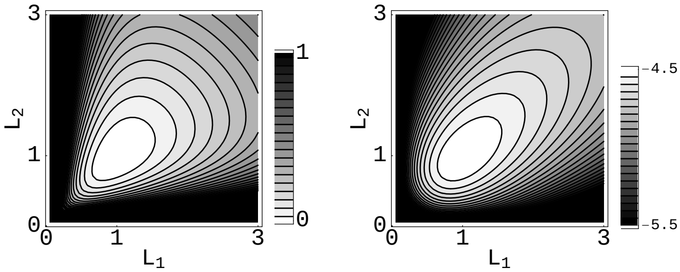
$\longrightarrow$ integrate over Gaussian fluctuations?

Boundary cond. each sheet: - Dirichlet at static quark

- continuity + $\frac{2\pi}{3}$balance of forces at junction

$\longrightarrow$ Integrate over Gaussian fluctuations, then over junction worldlines

Junction fluctuations inside qqq plane, and $\perp$ qqq plane ($d > 3$)

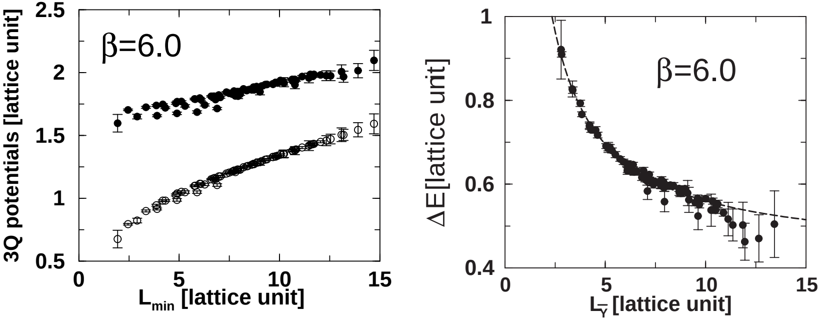


Coefficient of the universal Lüscher-like term $\frac{\pi}{24}\frac{1}{L_Y}$ as a function of the 3 relative string lengths $(L_1, L_2, L_3 = 1)$, in $d = 3$ (left) and 4 (right).

In $d = 3$, $C > 0$, so that V_{qqq} has an inflexion point.

Excited baryon potentials

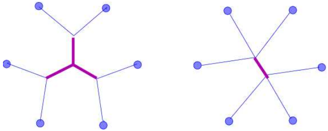
$\Delta V = (V_1 - V_0)$ calculable in string picture $\longrightarrow \frac{\pi}{L_Y}$
Lattice measurement (Suganuma et al.): $\Delta V(L_Y \sim 1\text{fm}) \sim$ 1 GeV (?)
$\lim_{L_Y \to \infty} \Delta V \sim$ 600 MeV (??)
$\longrightarrow$ constituent gluon mass (Cornwall)



Same group: tetraquark, pentaquark static potentials

Baryons and k-strings

$N > 3$, large distance: how do the flux tubes arrange?

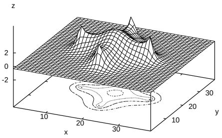


Minimize $\sum_k L_k \sigma_k \Rightarrow$ probe σ_k/σ_1

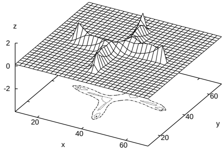
Geometrically: balance of forces $\rightarrow$ angles at each junction

- $SU(4)$: 1 or 2 junctions? inconclusive ($\sim$ 2-body forces only) (Alexandrou et al.)
- Sine-law threshold: take N quarks on regular polygon:

if $\exists k$ s.t. $\sigma_k < \sigma_1 \frac{\sin k\pi/N}{\sin \pi/N}$, Z_N symmetry broken (Gliozzi)



$\sigma_2 = 2\sigma_1$



$\sigma_2 = \sigma_1$

State of the art: simulation of the dual model

- Surprise: dual observables easier to measure than direct observables

't Hooft loops	Wilson loops
interface free energy	spin correlations

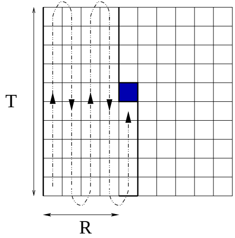
Why? $\langle \tilde{W} \rangle = \frac{Z_{\rm disordered}}{Z_0} = \frac{\int dU e^{-S_{\rm disordered}}}{\int dU e^{-S}}$ $\qquad \langle W \rangle = \frac{\int dU\ W\ e^{-S}}{\int dU e^{-S}}$

all integrands > 0 $\qquad W(U)$ can be > 0 or < 0

- Exploit positivity with "snake" algorithm (PdF, D'Elia & Pepe)

$R \times T$ interface: $\langle \tilde{W} \rangle = \frac{Z_{R\times T}}{Z_0}$

Factorize: $= \frac{Z_1}{Z_0} \times \frac{Z_2}{Z_1} \times .. \times \frac{Z_{R\times T}}{Z_{R\times T-1}}$ ie. 1 plaquette at a time

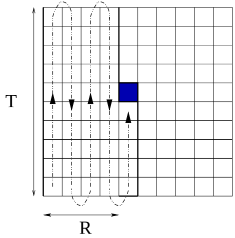


Each factor $O(1)$; estimated by separate Monte Carlo (positivity)

The magic virtue of the snake algorithm

$$\langle \tilde{W}_{R\times T}\rangle = \frac{Z_1}{Z_0}\times\frac{Z_2}{Z_1}\times .. \times \frac{Z_{R\times T}}{Z_{R\times T-1}}$$

$$\langle \tilde{W}_{R+1\times T}\rangle = \langle \tilde{W}_{R\times T}\rangle \times (T \text{ more factors}) \qquad (\text{strip } T\times 1)$$



Last T factors give $\frac{\langle \tilde{W}_{R+1\times T}\rangle}{\langle \tilde{W}_{R\times T}\rangle} = \exp(-T(\tilde{V}(R+1)-\tilde{V}(R))) \to$ force $F(R+1/2)$

same statistics $\Rightarrow$ same accuracy $\forall R$ (Hasenbusch)

Revolution under way for measurements of interface tensions, 't Hooft loops, ..

Using the snake algorithm for Wilson loops

- How to measure Wilson loops?

Perform duality transformation, then measure dual observable in dual model!

Example: $3d$ Z_2 gauge theory $\longrightarrow$ $3d$ Ising model (duality transformation)

Measure Ising interface free energies $\longleftrightarrow$ Z_2 Wilson loops

$H_{\text{Ising}} = \sum_{\langle ij \rangle} J\sigma_i\sigma_j$

$H_{\text{interface}}$: $J \longrightarrow -J$ for each link piercing interface

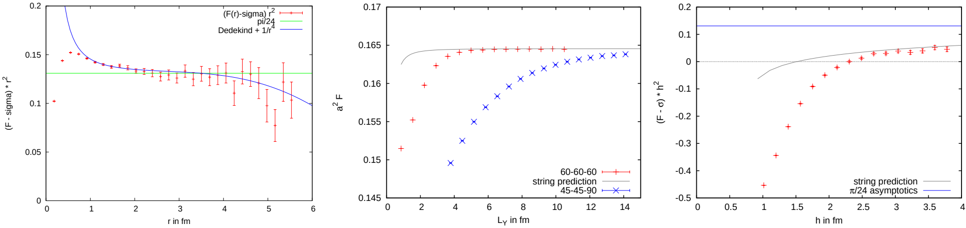
$$\frac{\sum_{\{\sigma\}} \exp(-\beta H_{R\times T\,\text{interface}})}{\sum_{\{\sigma\}} \exp(-\beta H_{\text{Ising}})} = \langle W_{R\times T}\rangle_{Z_2 \text{ gauge th.}}$$

- Need simple action after duality transformation $\longrightarrow$ restricted class

- $3d$: Z_N gauge $\longrightarrow$ N-states spin (eg. Z_2) $\longrightarrow$ bonus: cluster algorithm
- $4d$: $U(1)$ (Panero)
- Non-Abelian??

Illustrative results

3d Z_3 gauge theory $\longrightarrow$ 3d $q = 3$ Potts model (PdF & Jahn)



- $q\bar{q}$ Lüscher coeff. $c_L(R)$, with $\frac{1}{R^2}$ subleading term ($\sim 3\times$ prediction)
- F_{qqq} for 60-60-60 (no Lüscher term predicted) & 45-45-90 geometries
- Baryonic Lüscher coeff. for q_1&q_2 fixed ($\longrightarrow$ diquark): approaching $\frac{\pi}{24}$

Note amazing distances reached.

Conclusion

- Giant step in numerical studies:

  Lüscher-Weisz ‘multilevel’ and ’snake’ algorithms

- Enables theoretical progress + fundamental question:

  What is the effective string theory for QCD ?

- Explain confinement?