Lattice Gauge Theory: A brief introduction

Ferdinando Gliozzi

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Dipartimento di Fisica Teorica, Università di Torino and
INFN, sezione di Torino, via P. Giuria, 1, I-10125 Torino, Italy.

e-mail: gliozzi@to.infn.it

Abstract

These are informal lecture notes on lattice gauge theory prepared for the *XIV Seminario Nazionale di Fisca Teorica*, Parma 29/8 - 10/9 2005.

1 QCD

There is little doubt that the physics of strong interactions is accurately described by Quantum Chromodynamics or QCD, a gauge theory where the elementary matter fields are the quarks, which fill the color triplet, i.e. the fundamental representation of the gauge or color group SU(3), while their interaction is mediated by the gluons, the gauge particles filling the color octet, i.e. the adjoint representation of SU(3).

The pure gauge part of the dynamics is described by the Yang-Mills action

$$S = -\frac{1}{4}\int d^4x \,\mathrm{tr}\,(F_{\mu\,\nu}F^{\mu\,\nu}) \;\;,\; F_{\mu\,\nu} = \sum_{a=1}^{a=8}\lambda_a F^a_{\mu\,\nu} \tag{1.1}$$

where the λ_a's are the infinitesimal generators of the Lie algebra of SU(3) and

$$F_{\mu\,\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\;, \tag{1.2}$$

where A_μ is the gauge potential and g the adimensional coupling constant.

By far the most important properties of QCD are

- *the asymptotic freedom* which tells us, roughly, that the forces between quarks become weak for small quark separations. This is a property that QCD shares with the four-dimensional YM theories with non-abelian Lie groups. It implies that the theory becomes a free field theory in the ultraviolet limit, hence its perturbative expansion can be put in a rigorous, sound basis. Because of this asymptotic freedom it is possible to carry out quantitative calculations of strong interaction observables which are sensitive to the short distance structure of the theory. The discovery of the asymptotic freedom led very quickly to the realization that QCD was the right theory of the strong interactions, and this was what really completed the Standard Model. It is one of the most important discoveries of 20th century physics. [1]

- *the quark confinement*, which tells us that the physical states of QCD are singlets of SU(3). This implies that the quarks are permanently confined in a hadron. More specifically, one says the a gauge theory is in a confined phase if the potential between point-like static sources increases linearly with the intersource distance:

  $$V(r) = \sigma\, r + c + O(1/r) \tag{1.3}$$

  where the physical constant σ is known as the string tension. Contrarily to what happens for the asymptotic freedom, this is an infrared property which is, strictly speaking, still at the conjectural stage. The debate on the confining mechanisms, started at the mid-1970's, is still open. The second week of the "Seminario nazionale di fisica teorica" 2005 is entirely dedicated to this subject. My lecture notes constitute a preamble to introduce the main concepts.

2 Need of non-perturbative methods

One consequence of asymptotic freedom is that there must be physical quantities which cannot be expanded in a perturbative series in the coupling constant g.

In a nutshell, the argument goes as follows . Quantising the YM theory requires regularising it by the introduction of a cut-off in order to control the UV divergences coming when two fields are evaluated at the same point. For instance, we can introduce a spatial cut-off a representing the minimal distance between two local fields[2]. Let m be a physical observable with the dimension of a mass (for instance it could be the mass of the lowest physical state). Its functional form is necessarily

$$m = m(a, g) = f(g)/a\ , \tag{2.1}$$

where f is, for the moment, an unknown function of g. Since the classical action (1.1) does no contain any dimensional parameter, the scaling dimension is necessarily due to

[1] The calculation of the Yang-Mills beta function was completed in 1973 about the same time by David Politzer (a student of Sidney Coleman's at Harvard) and David Gross working with his student Frank Wilczek at Princeton. Gross was actually trying to complete a proof that all Quantum Field Theories had bad ultraviolet behaviour; he still was suffering from the pre-QCD prejudice common to almost all physicists of that time, that the strong interactions could never be understood via QFT, that one needed instead to do S-matrix theory or string theory or something other than QFT. Gerald 't Hooft had done the beta function calculation one year earlier, but he didn't work out the experimental implications for deep inelastic scattering, which was what Gross, Politzer and Wilczek did. They were awarded the Nobel prize for Physics in 2004.

[2] In the subsequent sections the role of such a spatial cut-off will be played by the lattice spacing; for the moment this further assumption is not necessary

the cut-off a. On the other end if,m is a physical quantity, it should not depend on the cut-off, which has to be regarded as a computational artifice to get finite results.Thus

$$\frac{d}{d\,a} m = 0 \ . \tag{2.2}$$

As a consequence, if a is varied, also g must change in order to keep m constant. Therefore $g = g(a)$. But a is a dimensionful quantity, while g is adimensional, hence

$$g = g(a\Lambda) \ , \tag{2.3}$$

where Λ has the dimension of a mass and is independent of the cut-off, hence it is a physical quantity. As we shall see, it sets the scale of the strong interactions, because it turns out that any physical mass can be expressed as a numerical constant times Λ.

Eq. (2.2) yields

$$f'(g) a \frac{d}{d\,a} g = f(g) \ . \tag{2.4}$$

Introducing the beta function of Callan-Symanzik, defined as

$$a \frac{d}{d\,a} g = \beta(g) \ , \tag{2.5}$$

we can write the differential equation

$$\frac{d\,f}{f} = \frac{d\,g}{\beta(g)} \ , \tag{2.6}$$

i.e.

$$f(g) = f(g_o)\, e^{\int_{g_o}^{g} \frac{d\,g}{\beta(g)}} \ . \tag{2.7}$$

In SU(N) YM theories $\beta(g)$ can be evaluated perturbatively and gives

$$\beta(g) = \frac{11N}{3} \frac{g^3}{(4\pi)^2} + \frac{34N^2}{3} \frac{g^5}{(4\pi)^4} + \dots \tag{2.8}$$

where the positive sign of the first term encodes the defining property of the asymptotic freedom. Taking into account, for sake of simplicity, only the first perturbative term of Eq.(2.8) we get

$$f(g) = C\, e^{-\frac{24\pi^2}{11N\, g^2}} \ , \tag{2.9}$$

which shows that any physical mass has the same functional dependence on the coupling constant, therefore the ratios of different masses is a numerical constant. The other important property is that the above function cannot evidently be expanded perturbatively around $g = 0$, because of the essential singularity of Eq.(2.9).

In conclusion, the perturbative methods cannot give any information about the dimensional physical quantities of the theory, thus some non-perturbative approach is needed.

3 A crucial step: Statistical Field Theory

The Statistical Field Theory (SFT) is a formulation of QFT which uses the methods and the language of Classical Statistical Mechanics (CSM). More precisely, it is a theory which develops tools useful both for CSM and QFT. One discovers that QFT's and CSM of critical systems are exactly the same theory: it is possible to translate a QF model into a critical system of the statistical mechanics and vice-versa. For instance it can be shown that the Ising model at criticality in any dimension is, when translated in the language of QFT, the φ^4 theory with real φ.

The starting point is the resemblance of the generating functional of the Feynman diagrams in the functional approach of QFT [3]

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-i\frac{S}{\hbar}} \tag{3.1}$$

with the canonical partition function Z which encodes all the statistical properties of a classical system described by an Hamiltonian H and in thermal equilibrium with a thermostat at a temperature T:

$$Z(T,V) = \sum_{\text{configurations}} e^{-H/\kappa T} \; . \tag{3.2}$$

Clearly

$$\int \mathcal{D}\varphi \; \leftrightarrow \sum_{\text{configurations}} \; , \tag{3.3}$$

but at this level there are still many differences between the two approaches: the former has only a formal meaning away from the gaussian (or perturbative) limit and it is mainly used to get the right multiplicity of the Feynman diagrams. The latter is a finite, well-defined, quantity and there are analytic or numerical methods which allow to extract at least some estimate of many physical quantities related to Z.

It is then natural to try to modify the functional approach in order to apply these powerful methods. First one performs a Wick rotation $t \to i\tau$ which implies $-iS \to -S_E$, where S_E is the Euclidean action. In this way any field configuration has a real Boltzmann weight.

The other useful transformation is to put the theory on a lattice, for instance an hypercubic one, that we denote with Λ. In four space-time dimensions each node P of the lattice is selected by four integer numbers $P = (n_x a, n_y a, n_z a, n_t a)$ $(n \in \mathbb{Z})$, where a is the lattice spacing. Putting the QFT on the lattice means simply the following:

- Associate to each node an arbitrary value of the field φ:

$$P \;\mapsto\; \varphi(P) \tag{3.4}$$

[3]For the sake of simplicity we temporarily assume that the quantum system is described by an action $S[\varphi]$ which depends only on a scalar field φ.

➪ Evaluate $\mathcal{Z}$ by summing over all these possible values according to

$$\prod_{P\in\Lambda}\int d\,\varphi(P)e^{-S_E[\varphi]}\ . \tag{3.5}$$

[Exercise: verify that the SU(N) invariant field model defined by the action

$$S=\int d^dx[\frac{1}{2}\partial_\mu\varphi_i^*\partial^\mu\varphi^i-V(\varphi_i^*\varphi^i)]\ ,\quad (i=1,\ldots,N)$$

becomes on the hypercubic lattice a statistical system described by the Hamiltonian

$$H=a^d\sum_{P\in\Lambda}\left[\frac{1}{2}\sum_{\mu=1}^{d}[\nabla_\mu\varphi_i^*(P)\nabla_\mu\varphi^i(P)]|^2+V(\varphi^i(P)\varphi_i(P))\right] \tag{3.6}$$

with $\nabla_\mu\varphi(P)=\frac{\varphi(P+a\hat{\mu})-\varphi(P)}{a}$*]*

In this way there is a finite number of degrees of freedom per unit volume, therefore the functional integration $\int\mathcal{D}\varphi$ has now a precise meaning and the lattice spacing a acts as a spatial cut-off, eliminating all the UV divergences. In this way the correspondence $QFT\leftrightarrow CSM$ becomes exact, and we can build up a dictionary which translates typical terms of one formulation in the other:

QFT	*CSM*
functional integration	sum over configurations
d spatial dimensions	$D=d+1$ spatial dimensions
Euclidean action S_E	Hamiltonian H
$\hbar$	κT
Transition Amplitude	Partition Function
Vacuum expectation value of the T-product	Correlation function
Energy of the ground state	Free energy
Mass of the lightest particle	Inverse of the correlation length

3.1 The price of lattice regularisation

The price of transforming a QFT in an Euclidean lattice field theory is rather high: the Lorentz invariance is completely lost and replaced by the symmetry of the lattice, which has nothing to do with the physical properties of the original model. Moreover the transformed model, although calculable, depends in a crucial way on the lattice spacing a, while the true physical properties of the model should not depend on a, of course. One cannot put simply $a\to 0$ in S_E[4], because on one hand, as shown in §2, also the coupling constants depend on a and , on the other hand, the multiple integral (3.5) loses any meaning in this limit.

The way-out is to resort to Renormalization Group *à la* Wilson, which provides us with the rules to extract from the lattice regularised theory the physical, cut-off independent, properties. It turns out that a true continuum limit of the Euclidean QFT

[4]This is called the *naive limit*, since it does not take into account the other places where a has an important role in the functional integration.

exists near a continuous transition, where the correlation length ξ goes to infinity, so that the lattice details are negligible and the rotational invariance is restored. In other terms QFT's correspond to the universality classes of phase transitions of classical statistical systems [1].

4 Gauge invariance

Lat us start by considering the model defined in Eq.(3.6). It is invariant (i.e. $\delta H = 0$) under rigid SU(N) transformations

$$\phi(P) \to V\phi(P)\ ,\ \ \phi^\dagger \to \phi^\dagger V^\dagger\ ,\ \ \ V \in SU(N)\ ,\ \ \phi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ . \\ . \\ \varphi^N \end{pmatrix} \tag{4.1}$$

If the transformation V becomes a *local transformation*, i.e. depends on the nodes P in an arbitrary way, $V \to V(P)$ $\in$SU(N), the Hamiltonian (3.6) is no longer invariant, because of the contributions of the mixed terms in the kinetic part:

$$\phi^\dagger(P)\phi(Q) \to \phi^\dagger(P)V^\dagger(P)V(Q)\phi(Q)\ ,\ \ Q = P + a\hat{\mu}\ , \tag{4.2}$$

hence $\delta H \neq 0$.

In the continuum it is well known since the time of the Yang-Mills work (1952) a recipe to get a locally invariant theory: one has to introduce the gauge fields A_μ associated to the infinitesimal generators of SU(N). In particular, to transform $\phi^\dagger(P)\phi(Q)$ into a locally invariant quantity one introduces an arbitrary path γ connecting P to Q and build up the SU(N) path ordered product

$$U_\gamma(P,Q) = \mathcal{P}e^{ig\int_P^Q A_\mu(x)dx^\mu} \tag{4.3}$$

which under a local transformation becomes

$$U_\gamma(P,Q) \to V(P)\,U_\gamma(P,Q)\,V^\dagger(Q)\ \ , \tag{4.4}$$

thus it is evident that now $\phi^\dagger(P)U_\gamma(P,Q)\phi(Q)$ is invariant. It is also evident how to extend this construction to the lattice: we associate to each oriented link[5] $(P,\hat{\mu})$ an arbitrary element of SU(N):

$$(P,\hat{\mu}) \mapsto\ U_\mu(P)\ ,\ \ (Q = P + a\hat{\mu}, -\hat{\mu}) \mapsto U_{-\mu}(Q) \equiv U_\mu^\dagger(P)\ . \tag{4.5}$$

The ordered product of these link variables allows to construct lattice path operators in analogy with Eq.(4.3).

To modify the model under study in such a way to have a locally invariant Hamiltonian $\mathcal{H}$ it suffices replacing the mixed terms with

$$\phi^\dagger(P)\phi(Q)\ \to\ \phi^\dagger(P)U_\mu(P)\phi(Q)\ , \tag{4.6}$$

[5] An oriented link is the segment connecting two neighbouring nodes P and $Q = P + a\hat{\mu}$. It is uniquely selected by the pair $(P,\hat{\mu})$

thus the model is invariant under the joined local transformations

$$\phi(P) \to V(P)\,\phi(P)\,, \quad \forall P \in \Lambda \tag{4.7}$$

$$U_\mu(P) \to V(P) U_\mu(P) V^\dagger(Q)\,, \tag{4.8}$$

which constitute a *lattice gauge transformation* of SU(N).

[Exercise: Show that in the naive continuum limit the kinetic term of the gauge invariant version of the model under study can be expressed in the usual way in terms of the covariant derivatives.]

5 LGT

We have just seen that there is a simple recipe to transform a lattice field theory invariant under global SU(N) transformations into a gauge invariant theory: it suffices replacing the mixed term coming from the kinetic part with a term with a link variable as shown in Eq.(4.6), which is equivalent, in the continuum limit, to replace normal derivatives with covariant derivatives. Like in the continuum limit, the part of the action describing the gauge degrees of freedom is an independent gauge invariant. In the lattice models the role of the gauge fields is played by the link variables (4.5); how to construct a lattice analog of Eq.(1.1)?

Let Γ be any closed path on the lattice, made with the sequence of links

$$\Gamma = (P_1, \hat{\mu}_1)(P_2 = P_1 + a\hat{\mu}_1, \hat{\mu}_2)\dots(P_M = P_1 - a\hat{\mu}_M, \hat{\mu}_M) \tag{5.1}$$

an construct the corresponding ordered product of link variables

$$U_\Gamma = U_{\mu_1}(P_1) U_{\mu_2}(P_2)\dots U_{\mu_M}(P_M)\,, \tag{5.2}$$

which transforms as

$$U_\Gamma \to\ V(P_1) U_\Gamma\, V^\dagger(P_1)\,. \tag{5.3}$$

Hence $\mathrm{tr}\, U_\Gamma$ is gauge invariant. Such an observation might be used to construct a lattice analog of the YM action in many different ways. The simplest choice is the *Wilson action*, which is written as the sum of the trace in the fundamental representation of all the minimal loops one can draw in the lattice, i.e. the smallest squares made with four contiguous links, called *plaquettes*[6]

$$S_{\mathrm{W}} = -\sum_{plaquettes} \frac{\beta}{N} \Re e\,(\mathrm{tr}\, U_{plaq}) \tag{5.4}$$

with

$$U_{plaq} = U_\mu(P)\, U_\nu(P + a\hat{\mu})\, U^\dagger_\mu(P + a\hat{\nu})\, U^\dagger_\nu(P) \tag{5.5}$$

[Exercise: Show that in the naive continuum limit $\Re e(\mathrm{tr}\, U_{plaq.}) = N - g^2 a^4 \mathrm{tr}\,(F^2_{\mu\,\nu}) + O(a^5)$ *. Hint: use the exponential map* $U_\mu(P) = e^{iagA_\mu(P)}$ *and Taylor expand about the center of the plaquette.]*

[6] An oriented plaquette passing trough the point P is uniquely selected by the triple $(P, \hat{\mu}, \hat{\nu})$ with $\hat{\mu} \neq \hat{\nu}$.

Comparing the result of the above exercise with Eq.(1.1) we can read off

$$\beta = \frac{2N}{g^2} \tag{5.6}$$

which relates the β parameter of the lattice with the gauge coupling constant of the continuum theory.

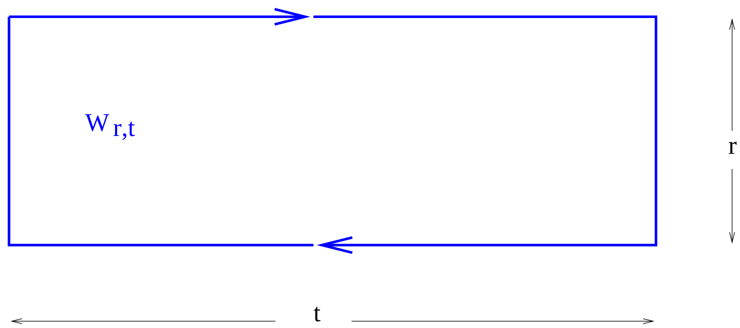
For the quantum YM theory we have to specify how to do functional integrals. The sum over all the gauge configurations on the lattice amounts to integrate over all link variables. So, the SU(N) Yang-Mills theory on the lattice is described by the partition function

$$Z = \int \prod_{P\in\Lambda} \prod_{\mu=1,\dots 4} dU\mu(P)\, e^{-S_{\rm W}} \tag{5.7}$$

where dU is the invariant measure of the SU(N) group. Since SU(N), as any other compact group, has a finite volume, we can always normalise to have $\int dU = 1$, thus Eq.(5.7), like its obvious generalisations to whatever compact group, is a perfectly well-defined expression which is finite and in principle calculable, at least approximately, thus there is no need to fix whatever gauge: this is a great advantage with respect to the continuum quantum formulations, where the zero modes of the kinetic part of Eq.(1.1) force the choice of gauge fixing terms and the the introduction of the corresponding Fadeev-Popov ghosts.

In the present lattice regularised theory the vacuum expectation value of any observable $\mathcal{O}$ is defined as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{\rm links} dU_\mu(P) \mathcal{O} e^{-S_{\rm W}} \ . \tag{5.8}$$



Of particular interest are the gauge invariant operators $W_\Gamma \equiv {\rm tr}\, U_\Gamma$, ($U_\Gamma$ is defined in (5.2)) called *Wilson loops*. In particular a rectangular Wilson loop $W_{r,t}$ (see Figure) can be interpreted as the contribution to the action of a pair of point-like sources in the representation f and $\bar{f}$ respectively, which are created at a time $t = 0$ and placed at a distance r and then annihilated at the time t. The vacuum expectation value of these operators in the $t \to \infty$ limit allows to define the static intersource potential:

$$V(r) = - \lim_{t\to\infty} \frac{1}{t} \log\langle W_{r,t} \rangle \ . \tag{5.9}$$

Comparing this expression with Eq.(1.3) we see that a gauge theory is confining if the vacuum expectation value of large Wilson loops drop off exponentially with the minimal area encircled by Γ.

References

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