Gravitational waves, stars and black holes

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I had the priviledge of collaborating with professor Chandrasekhar for twelve years during which we explored the General Theory of Relativity and developed a new formulation of the theory of stellar perturbations, the startling complexity and richness of which I will try to describe in this lecture.

In order to understand the basic ideas underlying our approach, we need to frame the problem in an hystorical perspective, and start describing some major results of the theory of perturbations of a Schwarzschild black hole, which is beautifully illustrated in Chandra's book *The mathematical theory of black holes"* [1].
In 1957 T. Regge and J.A. Wheeler [2] derived the equations governing the perturbations of a static, spherically symmetric black hole. The separation of variables was accomplished by expanding the perturbed metric tensor in tensorial spherical harmonics, and since these harmonics have a different behaviour under the angular transformation $\theta \rightarrow \pi - \theta, \quad \varphi \rightarrow \pi + \varphi$, the separated equations split in two sets: the *polar* or *even*, belonging to the parity $(-1)^{\ell}$, and the *axial* or *odd*, belonging to the parity $(-1)^{(\ell+1)}$. Regge and Wheeler reduced the equations describing the *axial* perturbations to a single Schroedinger-like equation

$$\frac{d^2 Z_\ell^-}{dr_*^2} + \left[\omega^2 - V_\ell(r)\right] Z_\ell^- = 0, \tag{1}$$
$$V_\ell^-(r) = \frac{1}{r^3}\left(1 - \frac{2M}{r}\right)[\ell(\ell+1)r - 6M],$$

where $r_* = r + 2M\log(\frac{r}{2M} - 1)$, M is the black hole mass, ω is the frequency and the perturbed functions have been Fourier-expanded. The theory of perturbations of black holes was born.

Due to the analytical complexity of the polar equations, only much later, in 1970, F. Zerilli [3] was able to derive also for the *polar* perturbations a single Schroedinger-like equation, but with a different potential barrier

$$\frac{d^2 Z_\ell^+}{dr_*^2} + [\omega^2 - V_\ell^+(r)] Z_\ell^+ = 0, \tag{2}$$

$$V_\ell^+(r) = \frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3],$$
$$n = \frac{1}{2}(\ell+1)(\ell-2).$$

Equations (1) and (2) show that the curvature generated by a point-like mass appears in the perturbed equations as a potential barrier which extends throughout spacetime. Consequently, the response of a black hole to a generic perturbation can be studied by investigating the manner in which a gravitational wave incident on the black hole is transmitted, absorbed and reflected by this barrier, a phenomenon with which we are familiar in elementary quantum theory.

1. The quasi-normal modes of a black hole

In 1970 Vishveshwara [4] had pointed out that the equations governing the perturbations of a Schwarzschild black hole should allow complex frequency solutions behaving at radial infinity as pure outgoing waves. W.H. Press [5] confirmed this idea by numerically integrating the equations, and by showing that an arbitrary initial perturbation ends in a ringing tail, which indicates that black holes possess some proper modes of vibration.

Since the oscillations must be damped by the emission of gravitational waves, these modes were called *quasi-normal modes*, and they were defined to be solutions of the perturbed equations belonging to complex eigenfrequencies $\omega = \omega_0 + i\omega_i$, and satisfying the boundary conditions of a pure outgoing wave at infinity and of a pure ingoing wave at the horizon. The first condition identifies physically acceptable modes, i.e. those that damp the star (provided $\omega_i > 0$). The latter is the requirement that nothing can escape from the horizon. It should be noted that in scattering theory these boundary conditions associated to a Schroedinger equation with a one-dimensional potential barrier identify the singularities of the scattering amplitude.

In 1975 S. Chandrasekhar and S. Detweiler [6] computed the complex eigenfrequencies of the quasi-normal modes of a Schwarzschild black hole. The first few values for $\ell = 2$ and $\ell = 3$ are given in Table 1.

The real part of the frequency is inversally proportional to the mass, while the damping is proportional to it. If the black hole mass is $M = nM_\odot$, the oscillation frequency and the damping of the modes can be computed by the

	$M\omega + iM\omega_i$		$M\omega + iM\omega_i$
$\ell = 2$	0.3737+i0.0890	$\ell = 3$	0.5994+i0.0927
	0.3467+i0.2739		0.5826+i0.2813
	0.3011+i0.4783		0.5517+i0.4791
	0.2515+i0.7051		0.5120+i0.6903

Table 1: *The complex characteristic frequencies of the quasi-normal modes of a Schwarzschild black hole.*

following formulae

$$\nu_0 = \frac{c}{2\pi n \cdot M_\odot (M\omega)} = \frac{32.26}{n}(M\omega) kHz, \quad \tau = \frac{nM_\odot}{(M\omega)c} = \frac{n \cdot 0.4937 \cdot 10^{-5}}{(M\omega)} s. \tag{3}$$

For example, the lowest $\ell = 2$ quasi-nomal mode of a black hole of one solar mass, and of a supermassive black hole of $10^6 M_\odot$ belong, respectively, to the following frequencies

$$\begin{array}{lll} M = 1M_\odot, & \nu_0 = 12.06 \ kHz, & \tau = 5.55 \cdot 10^{-5} s \\ M = 10^6 M_\odot, & \nu_0 = 1,21 \cdot 10^{-2} Hz, & \tau = 55.5 s. \end{array} \tag{4}$$

The frequencies of oscillation of a black hole depend exclusively on the parameters that identify the spacetime geometry: the mass, and the angular momentum or the charge if the black hole is rotating or charged.

In ref. [6] S. Chandrasekhar and S. Detweiler also showed that the transmission and the reflection coefficients associated respectively to the polar and to the axial potential barriers are equal. This equality can be explained in terms of a transformation theory which clarifies the relations that exist between potential barriers admitting the same reflection and absorption coefficients (this theory is extensively illustrated in ref. [1]). [1] However the physical reason why this happens is still unclear:

[1]The equality of the transmission and reflection coefficients can also be justified by the following considerations. The perturbations of a Schwarzschild black hole can be described in terms of the Bardeen-Press equation [7] written for the Weyl scalars Ψ_0 and Ψ_4, which represent the ingoing and outgoing radiative part of the gravitational field. The

"In spite of $V^{(+)}$ and $V^{(-)}$ appearing so very different, they are *iso-spectral* in the sense that the reflection and absorption coefficient for incident polar and axial gravitational waves are identically the same for all frequencies. In tracing the origin of this identity, one is led to a 'transformation theory' whose significance remains illusive"
(From S. Chandrasekhar "The series Paintings of Claude Monet and the Landscape of General Relativity" 1992 [8]).

Numerical integration of the wave equations (1) and (2) with different sources (see ref. [9] for an extensive bibliography) have shown that the gravitational signal emitted as a consequence of a generic perturbation will, during the last stages, decay as a superposition of the quasi-normal modes. In addition, a newborn black hole generated either by the gravitational collapse of a massive star or by the coalescence of two compact objects, will oscillate and emit gravitational waves until its residual mechanical energy is radiated away, and again the dominant contribution is expected to be due to the quasi-normal modes. Being the axial and polar perturbations isospectral, the gravitational radiation emitted in these processes will carry a definite signature on the nature of the emitting source; in fact, as we shall later discuss, the axial and polar perturbations of a star **are not** isospectral [10].

2. A conservation law for the scattering of gravitational waves by a black hole

One of the major problems in General Relativity is that an energy conservation law governing the scattering of gravitational waves by black holes does not exist in the framework of the exact non linear theory. However, such law can be derived in perturbation theory both for Schwarzschild , Kerr and Reissner-Nordstrom black holes. We shall now derive the conservation law for a Schwarzschild black hole, by following the procedure adopted in ref. [1].

Due to the short-range character of the potential barriers of eqs. (1) and (2), the asymptotic behaviour of the solution Z at $r_* = \pm\infty$ is, in

Bardeen-Press equation admits solutions which satisfy the boundary conditions of the quasi-normal modes, and since both Ψ_0 and Ψ_4 can be expressed as a combination of the Regge-Wheeler and of the Zerilli functions and their first derivatives [1], the axial and the polar perturbations must be isospectral.

general, a superposition of outgoing and ingoing waves

$$Z_{out} \sim e^{-i\omega r_*}, \qquad \text{and} \qquad Z_{in} \sim e^{+i\omega r_*}. \tag{5}$$

Consider two solutions of the wave equations, say Z_1 and Z_2, satisfying respectively the following boundary conditions

$$\begin{array}{lll} r_* \to +\infty & Z_1 \to e^{-i\omega r_*}, & \text{pure outgoing wave} \\ r_* \to -\infty & Z_2 \to e^{+i\omega r_*} & \text{pure ingoing wave.} \end{array} \tag{6}$$

The pairs (Z_1, Z_1^*) and (Z_2, Z_2^*), where the $*$ indicates complex conjugation and ω is assumed to be real, will be pairs of independent solutions of the wave equation, since their Wronskians are different from zero. In fact, by a direct evaluation, for example, at $\pm\infty$, one finds [2]

$$r_* \to +\infty, \quad [Z_1, Z_1^*]_{r_*} = -2i\omega, \qquad r_* \to -\infty, \quad [Z_2, Z_2^*]_{r_*} = +2i\omega. \tag{7}$$

Therefore, we can write Z_1 as a linear combination of (Z_2, Z_2^*) and viceversa:

$$\begin{aligned} Z_1 &= A(\omega)Z_2 + B(\omega)Z_2^*, \\ Z_2 &= C(\omega)Z_1 + D(\omega)Z_1^*. \end{aligned} \tag{8}$$

We now divide Z_2 by $D(\omega)$ and define

$$Z_R = \frac{Z_2}{D(\omega)} = R_1(\omega)Z_1 + Z_1^*, \tag{9}$$

where $R_1(\omega) = \frac{C(\omega)}{D(\omega)}$, and similarly

$$Z_L = \frac{Z_1}{B(\omega)} = R_2(\omega)Z_2 + Z_2^*, \tag{10}$$

where $R_2(\omega) = \frac{A(\omega)}{B(\omega)}$. Z_R and Z_L have the following asymptotic behaviour

$$Z_R \to \begin{cases} T_1(\omega)e^{+i\omega r_*} & r_* \to -\infty \\ e^{+i\omega r_*} + R_1(\omega)e^{-i\omega r_*} & r_* \to +\infty \end{cases} \tag{11}$$

[2] $[A, B]_r = A_{,r} \cdot B - A \cdot B_{,r}$

$$Z_L \to \begin{cases} e^{-i\omega r_*} + R_2(\omega)e^{+i\omega r_*} & r_* \to -\infty \\ T_2(\omega)e^{-i\omega r_*} & r_* \to +\infty \end{cases}$$

where we have set $T_1(\omega) = \frac{1}{D(\omega)}$, and $T_2(\omega) = \frac{1}{B(\omega)}$.

Thus, Z_R represents a wave of unitary amplitude incident on the potential barrier from $+\infty$ which gives rise to a reflected wave of amplitude $R_1(\omega)$ and to a transmitted wave of amplitude $T_1(\omega)$. Conversely, Z_L is a unitary wave incident from $-\infty$ which is partially reflected $(R_2(\omega))$ and partially transmitted $(T_2(\omega))$. Furthermore, by computing the Wronskian of the two solutions at $\pm\infty$ it is easy to verify that

$$\begin{aligned} [Z_L, Z_R]_{r_*} &= -2i\omega T_2(\omega) && r_* \to +\infty \\ [Z_L, Z_R]_{r_*} &= -2i\omega T_1(\omega) && r_* \to -\infty, \end{aligned} \tag{12}$$

and since the Wronskian is constant, it follows that

$$T_1(\omega) = T_2(\omega) = T(\omega). \tag{13}$$

Similarly

$$\begin{aligned} [Z_L, Z_L^*]_{r_*} &= 2i\omega(|R_2(\omega)|^2 - 1) && r_* \to -\infty \\ [Z_L, Z_L^*]_{r_*} &= -2i\omega |T_2(\omega)|^2 && r_* \to +\infty, \end{aligned} \tag{14}$$

and consequently

$$|R_2|^2 + |T_2|^2 = 1. \tag{15}$$

By a similar procedure applied to Z_R we easily find

$$|R_1|^2 + |T_1|^2 = 1. \tag{16}$$

This means that R_1 and R_2 can differ only by a phase factor and that

$$|R|^2 + |T|^2 = 1 \tag{17}$$

holds in general. This equation establishes the symmetry and the unitarity of the S-matrix, and it expresses the conservation of energy because it says that if a wave of unitary amplitude is incident on one side of the potential barrier, it splits into a reflected and a transmitted wave such that the sum of the square of their amplitudes is still one.

The existence of conservation laws for the scattering of gravitational waves by a black hole raised an interesting question: is it possible to establish a similar conservation law for the polar perturbations of a static, spherically symmetric spacetime generated either by an electromagnetic source or by a non rotating star? That such law should exist was known on a theoretical ground: A. Ashtekar, J. Friedmann, R.Sorkin and R. Wald had told us that the existence of a *conserved simplectic current* can in principle be inferred for any field theory derived from a suitably defined Lagrangian action. However, Chandra wanted to derive the conserved current by using a procedure similar to that used for Schwarzschild black holes. In that case, the central point of the derivation was to show that the Wronskian of two independent solutions of the wave equations is a constant. Conversely, the equations for the polar perturbations of a star are a fourth order linear differential system: what would be the role played by a Wronskian in this context? The solution of the problem required a considerable amount of hard work on the equations, but at the end the result was rewarding: we found that there exists a vector $\vec{\mathbf{E}}$ which satisfies the following equation [11]

$$\frac{\partial}{\partial x^\alpha} E^\alpha = 0, \qquad \alpha = (x^2 = r, x^3 = \vartheta). \tag{18}$$

The vanishing of the ordinary divergence implies that, by Gauss's theorem, the flux of $\vec{\mathbf{E}}$ across a closed surface surrounding the star is a constant.

In order to write explicitely the components of the vector $\vec{\mathbf{E}}$ (I shall omit the details of its derivation) we write the metric of a generic static, spherically symmetric spacetime in the following form

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}d\theta^2 + e^{2\psi}d\varphi^2, \tag{19}$$

where the metric functions depend only on r and ϑ. We shall restrict to the case when this metric represents the spacetime generated by an unperturbed star composed by a perfect fluid, though in ref. [11] we derived a similar conservation law also for charged solutions of Einstein's equations. The axisymmetric perturbations of the spacetime (19) can be described by the line-element

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - q_2 dr^2 - q_3 d\theta - \omega dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2. \tag{20}$$

It should be noted that the number of unknown functions in eq. (20) is seven, one more than needed. However, this extra degree of freedom disappears when the boundary conditions of the problem are fixed. As a consequence of a generic perturbation, the metric functions will experience small changes with respect to their unperturbed values, which we assume to be known

$$\begin{aligned} &\nu \longrightarrow \nu + \delta\nu, \qquad && \mu_2 \longrightarrow \mu_2 + \delta\mu_2, \\ &\psi \longrightarrow \psi + \delta\psi, && \mu_3 \longrightarrow \mu_3 + \delta\mu_3, \\ &\omega \longrightarrow \delta\omega, && q_2 \longrightarrow \delta q_2 \quad , \qquad q_3 \longrightarrow \delta q_3. \end{aligned} \tag{21}$$

Since each element of fluid in the interior of the star undergoes an infinitesimal displacement from its equilibrium position, identified by the lagrangian displacement $\vec{\xi}$, the energy density and the pressure will change by an infinitesimal amount

$$\epsilon \longrightarrow \epsilon + \delta\epsilon, \qquad\qquad p \longrightarrow p + \delta p. \tag{22}$$

Under the assumption of axisymmetric perturbation, all perturbed quantities depend on t, r and θ. If we now write Einstein's equations supplemented by the hydrodynamical equations and the conservation of barion number, expand all tensors in tensorial spherical harmonics and Fourier-expand the time dependent quantities, we find that, as for black holes, the equations decouple into two sets, the *polar* and the *axial*, but with a major difference: the *polar* perturbations involve the same metric variables $(\delta\nu, \delta\mu_2, \delta\psi, \delta\mu_3)$ as for black holes, but now they are coupled to the thermodynamical variables

$$\begin{cases} \delta\nu \\ \delta\mu_2 \\ \delta\psi \\ \delta\mu_3 \end{cases} \longrightarrow \begin{cases} \delta\epsilon \\ \delta p \\ \xi_r \\ \xi_\vartheta \end{cases}. \tag{23}$$

Conversely the *axial* perturbations $[\delta\omega, \delta q_2, \delta q_3]$ do not induce motion in the fluid except for a stationary rotation. However, we shall see that the fluid plays a role, though different from that played in the polar case. In terms of the perturbed metric and fluid variables the E_2-component of the polar vector $\vec{\mathbf{E}}$ is

$$\begin{aligned} E_2 = r^2 e^{\nu-\mu_2} \sin\theta \{ [\delta\mu_3, \delta\mu_3^*]_2 + [\delta\psi, \delta\psi^*]_2 - [\delta\nu_{,2}\delta(\psi+\mu_3)^* - c.c] + \\ +[\delta\mu_2 \delta(\psi+\mu_3)^*_{,2} - c.c] + \Big[2[(\epsilon+p)\delta(\psi+\mu_3-\mu_2)^* - \delta p] e^{\nu+\mu_2} \xi_2 - c.c \Big] \}, \end{aligned} \tag{24}$$

and the E_3-component can be obtained by interchanging 2 with 3.

Equation (24) includes, as expected, Wronskians of the polar functions $[\delta\mu_3, \delta\mu_3^*]_2$ and $[\delta\psi, \delta\psi^*]_2$, and it reduces to the Wronskian of the solutions of the Zerilli equation as indicated in section 2, when the source terms ϵ and p are zero. We derived a similar expression for $\vec{\mathbf{E}}$ when the source is an electromagnetic field. G. Burnett and R. Wald [12] subsequently showed that in the Einstein-Maxwell case our conservation law can be obtained by constructing a simplectic current associated to the perturbed equations derived from a Lagrangian variational principle.

The conserved current $\vec{\mathbf{E}}$ represents the flux of gravitational energy which develops through the stars and propagates outside. Indeed, it can also be derived from the second variation of the Einstein pseudo-tensor $t_E^{\mu\nu}$ [13], [14]. The reason for choosing the Einstein pseudo-tensor is that among the infinite number of pseudo-tensors that can be defined for the gravitational field, all differing by a divergenceless term, $t_E^{\mu\nu}$ is the only one the second variation of which retains the divergence-free property, provided only the equations governing the static spacetime and its linear perturbations are satisfied. This property derives from the fact that the Einstein pseudo-tensor is a Noether operator for the gravitational field.

In addition, Raphael Sorkin pointed out that the contribution of the source should be introduced not by adding the second variation of the stress-energy tensor of the source $T^{\mu\nu}$, but through a suitably defined Noether operator, the form of which he derived for the electromagnetic case. This operator does not coincide with $T^{\mu\nu}$, but it gives the same conserved quantities. It should be mentioned that the Noether operator to be added to the Einstein pseudo-tensor when the source is a fluid has been derived only much recently by Vivek Iyer [15].

The existence of a conservation law for a spacetime with a perfect fluid source suggested to Chandra that the non-radial oscillations of stars should be reformulated as a problem in scattering theory.

"In general relativity, any distribution of matter (or more generally energy of any sort) induces a curvature of the spacetime – a potential well. Matter implies gravity and gravity implies matter. Therefore, instead of picturing the non-radial oscillations of a star as caused by some unspecified external perturbation, we can picture them as caused by incident gravitational radiation. Viewed in this manner, the reflection and absorption of incident gravitational waves by black holes and the non-radial oscillations of stars, become different aspects of the same basic theory. But how different – as we shall see!"

After completing the first paper on the flux integral, Chandra and I started to work on the perturbed equations, and reduced them to an interesting form [16], fairly different from that obtained by Thorne and his collaborators, who first developed the theory of stellar perturbations in general relativity in 1967 [17].

3. The polar equations

If one expands the perturbed metric tensor and the stress-energy tensor of the fluid in tensorial spherical harmonics, under the hypothesis of axisymmetric perturbations the polar metric functions and the thermodynamical variables turn out to have the following angular dependence

$$\begin{aligned} \delta\nu &= N_\ell(r)P_l(\cos\theta)e^{i\omega t} & \delta\mu_2 &= L_\ell(r)P_l(\cos\theta)e^{i\omega t} \\ \delta\mu_3 &= [T_\ell(r)P_l + V_\ell(r)P_{l,\theta,\theta}]e^{i\omega t} & \delta\psi &= [T_\ell(r)P_l + V_\ell(r)P_{l,\theta}\cot\theta]e^{i\omega t}, \\ \delta p &= \Pi_\ell(r)P_l(\cos\theta)e^{i\omega t} & 2(\epsilon+p)e^{\nu+\mu_2}\xi_r(r,\theta)e^{i\omega t} &= U_\ell(r)P_l e^{i\omega t} \\ \delta\epsilon &= E_\ell(r)P_l(\cos\theta)e^{i\omega t} & 2(\epsilon+p)e^{\nu+\mu_3}\xi_\theta(r,\theta)e^{i\omega t} &= W_\ell(r)P_{l,\theta}e^{i\omega t}, \end{aligned} \tag{25}$$

where $P_l(\cos\theta)$ are the Legendre polynomials. After separating the variables the relevant Einstein's equations become

$$\begin{aligned} &a) \quad (T_\ell - V_\ell + L_\ell) = -W_\ell \\ &b) \quad \left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2T_\ell - kV_\ell) - \frac{2}{r}L_\ell = -U_\ell \\ &c) \quad \frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}N_{\ell,r} + \left(\frac{1}{r} + \nu_{,r}\right)(2T_\ell - kV_\ell)_{,r} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,r}\right)L_\ell\right] + \\ &\qquad \frac{1}{2}\left[-\frac{1}{r^2}(2nT_\ell + kN_\ell) + \omega^2 e^{-2\nu}(2T_\ell - kV_\ell)\right] = \Pi_\ell \\ &d) \quad (T_\ell - V_\ell + N_\ell)_{,r} - \left(\frac{1}{r} - \nu_{,r}\right)N_\ell - \left(\frac{1}{r} + \nu_{,r}\right)L_\ell = 0, \\ &e) \quad V_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)V_{\ell,r} + \frac{e^{2\mu_2}}{r^2}(N_\ell + L_\ell) + \omega^2 e^{2\mu_2 - 2\nu}V_\ell = 0, \end{aligned} \tag{26}$$

where $k = l(l+1)$, and $2n = (l-1)(l+2)$. After some reduction, the hydrodynamical equations and the conservation of barion number provide the following expressions for the fluid variable [3]

$$\Pi_\ell = -\frac{1}{2}\omega^2 e^{-2\nu} W_\ell - (\epsilon + p) N_\ell, \qquad E_\ell = Q\Pi_\ell + \frac{e^{-2\mu_2}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r})U_\ell, \tag{27}$$

$$U_\ell = \frac{[(\omega^2 e^{-2\nu} W_\ell)_{,r} + (Q+1)\nu_{,r}(\omega^2 e^{-2\nu} W_\ell) + 2(\epsilon_{,r} - Qp_{,r})N_\ell](\epsilon+p)}{[\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})]}, \tag{28}$$

where

$$Q = \frac{(\epsilon+p)}{\gamma p}, \qquad \gamma = \frac{(\epsilon+p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const} \tag{29}$$

and γ is the adiabatic exponent.

Outside the star, the source vanishes and the polar equations can be reduced to the Zerilli equation (2), with the following identification

$$Z^+_\ell(r) = \frac{r}{nr + 3M}\left(3MV_\ell(r) - rL_\ell(r)\right). \tag{30}$$

A remarkable simplification of eqs. (26) is possible. Equation 26a) and eqs. (27) show that the fluid variables $[W_\ell, U_\ell, E_\ell, \Pi_\ell]$ can be expressed as a combination of the metric perturbations $[T_\ell, V_\ell, L_\ell, N_\ell]$ and their first derivatives. Therefore, after their direct substitution on the right hand side of the last four eqs. (26) a set of new equations which involves exclusively the perturbations of the metric functions $[T_\ell, V_\ell, L_\ell, N_\ell]$ can be derived. The final set is

$$\begin{cases} X_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) X_{\ell,r} + \frac{n}{r^2} e^{2\mu_2}(N_\ell + L_\ell) + \omega^2 e^{2(\mu_2-\nu)} X_\ell = 0, \\ (r^2 G_\ell)_{,r} = n\nu_{,r}(N_\ell - L_\ell) + \frac{n}{r}(e^{2\mu_2} - 1)(N_\ell + L_\ell) + r(\nu_{,r} - \mu_{2,r})X_{\ell,r} + \omega^2 e^{2(\mu_2-\nu)} r X_\ell, \\ -\nu_{,r} N_{\ell,r} = -G_\ell + \nu_{,r}[X_{\ell,r} + \nu_{,r}(N_\ell - L_\ell)] + \frac{1}{r^2}(e^{2\mu_2} - 1)(N_\ell - rX_{\ell,r} - r^2 G_\ell) \\ -e^{2\mu_2}(\epsilon+p)N_\ell + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}\left\{N_\ell + L_\ell + \frac{r^2}{n}G_\ell + \frac{1}{n}[rX_{\ell,r} + (2n+1)X_\ell]\right\}, \\ L_{\ell,r}(1-D) + L_\ell\left[\left(\frac{2}{r} - \nu_{,r}\right) - \left(\frac{1}{r} + \nu_{,r}\right)D\right] + X_{\ell,r} + X_\ell\left(\frac{1}{r} - \nu_{,r}\right) + DN_{\ell,r} + \\ +N_\ell\left(D\nu_{,r} - \frac{D}{r} - F\right) + \left(\frac{1}{r} + E\nu_{,r}\right)\left[N_\ell - L_\ell + \frac{r^2}{n}G_\ell + \frac{1}{n}(rX_{\ell,r} + X_\ell)\right] = 0, \end{cases} \tag{31}$$

where

$$\begin{cases} A = \frac{1}{2}\omega^2 e^{-2\nu}, \qquad B = \frac{e^{-2\mu_2}\nu_{,r}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r}), \\ D = 1 - \frac{A}{2(A+B)} = 1 - \frac{\omega^2 e^{-2\nu}(\epsilon+p)}{\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \\ E = D(Q-1) - Q, \\ F = \frac{\epsilon_{,r} - Qp_{,r}}{2(A+B)} = \frac{2[\epsilon_{,r} - Qp_{,r}](\epsilon+p)}{2\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \end{cases} \tag{32}$$

[3] We restrict our analysis to adiabatic perturbations of fluid stars.

and V_ℓ and T_ℓ have been replaced by X_ℓ and G_ℓ defined as

$$\begin{cases} X_\ell = nV_\ell \\ G_\ell = \nu_{,r}[\frac{n+1}{n}X_\ell - T_\ell]_{,r} + \frac{1}{r^2}(e^{2\mu_2}-1)[n(N_\ell+T_\ell)+N_\ell] \\ +\frac{\nu_{,r}}{r}(N_\ell+L_\ell) - e^{2\mu_2}(\epsilon+p)N_\ell + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}[L_\ell - T_\ell + \frac{2n+1}{n}X_\ell]. \end{cases} \qquad (33)$$

Equations (31) describe the perturbations of the gravitational field in the interior of the star, with no reference to the motion of the fluid.
Once these equations have been solved, the fluid variables can be obtained in terms of the metric functions from eqs. 26a) and eqs. (27). This fact is remarkable: it shows that all the information on the dynamical evolution of a perturbed star is encoded in the gravitational field, a result which expresses the physical content of Einstein's theory of gravity. Moreover, it should be stressed that the decoupling of the equations governing the metric perturbations from those governing the hydrodynamical variables is possible in general, and requires no assumptions on the equation of state of the fluid. Thus, if one is interested exclusively in the study of the emitted gravitational radiation, one can solve the system (31) and disregard the fluid behaviour.

Equations (31) have to be integrated for each value of the frequency from $r = 0$, where all functions must be regular, up to the boundary of the star. There, the spacetime becomes vacuum and sperically symmetric, and the perturbed metric functions and their first derivatives must be matched continuously with the functions that describe the polar perturbations of a Schwarzschild black hole (for a detailed discussion of the boundary conditions see refs. [16] and [18]).

It was subsequently shown by J.R.Ipser and R.H.Price [19] that the equations describing the polar gravitational perturbations decoupled from the fluid variables can be reduced to a fourth-order system.

4. A Schroedinger equation for the axial perturbations

The equations for the axial perturbations are much simpler than the polar ones. Their radial behaviour is completely described by a function $Z_\ell(r)$, which satisfies the following Schroedinger-like equation

$$\frac{d^2 Z_\ell}{dr_*^2} + [\omega^2 - V_\ell(r)]Z_\ell = 0, \qquad (34)$$

where $r_* = \int_0^r e^{-\nu+\mu_2} dr$, and

$$V_\ell(r) = \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)], \qquad \nu_{,r} = -\frac{p_{,r}}{\epsilon + p}. \tag{35}$$

Outside the star ϵ and p vanish and eq. (35) reduces to the Regge-Wheeler potential barrier (1). It should be stressed that the potential depends on how the energy-density and the pressure are distributed inside the star in its equilibrium configuration.

Since an axial gravitational wave incident on a star does not induce fluid motion, for a long time these perturbations have been considered as trivial. But this is not true if we adopt the scattering approach: the absence of fluid motion simply means that an incident axial wave experiences a potential scattering as it does in the case of a Schwarzschild black hole. There is however an important difference. The Schwarzschild potential vanishes at the black hole horizon, and it has a maximum at $r_{max} \sim 3M$. Conversely, due to the centrifugal contribution $\frac{\ell(\ell+1)}{r^2}$ the potential barrier of a perturbed star tends to infinity at $r = 0$. In addition, for a Schwarzschild black hole the Schroedinger-like equation describes a problem of scattering by a one-dimensional potential barrier, whereas in the case of a star it describes the scattering by a central potential.

Being the axial perturbations described by a Schroedinger equation, the axial component of the energy flux can be derived from the Wronskians of independent solutions, as in the black hole case. However, due to the different boundary conditions, the evaluation of this flux requires the application of the Regge theory of potential scattering in a central field. This theory can be generalized to be applicable also to the polar perturbations, and to explicitely compute the energy flux associated to the vector $\vec{\mathbf{E}}$ [20].

5. The quasi normal modes of a star

In our approach the non-radial oscillations of stars are thought to be induced by the incidence of polar or axial gravitational waves on the spacetime curvature generated by the star. In this view, a resonant scattering occurs when the star is in a quasi-stationary state that decays, i.e. when it oscillates in a quasi-normal mode.

The quasi-normal modes are solutions of the axial and polar equations that satisfy the following boundary conditions. As in the black hole case, at radial infinity only pure outgoing waves must prevail, whereas the pure ingoing wave condition at the black hole horizon is replaced by the requirement that all perturbed functions have a regular behaviour at $r = 0$. Furthermore, they must match continuously with the exterior perturbation at the surface of the star. Both the polar and the axial quasi-normal modes satisfy the same boundary conditions, but the underlying scattering problem is much different in the two cases. In fact, since a polar perturbation excites the fluid motion, the amount of radiation which leaks out of the star depends on the exchange of energy between the fluid and the gravitational field, whereas the scattering of axial gravitational waves is a pure scattering by a spherically symmetric, static potential.

In studying the theory of stellar perturbations in the framework of General Relativity, one encounters new phenomena that do not have a newtonian counterpart. A first example is the existence of new families of modes of vibration, which are modes of the radiative field. They appear because the spacetime is not symply a medium in which gravitational waves propagate: it has its own dynamics and spectrum, as it is clearly shown by the existence of the quasi-normal modes of black holes. Spacetime modes exist also for stars but, due to the different boundary conditions, their spectrum will be much different from that of black holes. One of these new families are the highly damped polar and axial **w**-modes, discovered by K.Kokkotas and B.Schutz [21]. Actually it was later shown that there exist two families of such modes [22], but we shall not go into such a detail in the present context. The **w**-modes are modes of vibration in which the motion of the fluid is barely excited, if not excited at all as in the axial case. In an article appeared in Physics World in 1991, Bernard Schutz makes an interesting analogy that vividly illustrates the nature of these modes [23]:

"Consider a violin played in an infinitely large room. The air by itself does not have conventional outgoing-wave modes: any sound waves are coming in from somewhere and going out somewhere else. But put a violin string in the room, and there appears a family of modes with purely outgoing sound waves that exchange a small amount of energy with the string, and die away very fast. These modes are strongly damped, and the weaker the coupling of the string to the air, the faster they damp away, so that in the limit of a vacuum around the string, they go away entirely."

Typical values of the lowest **w**-mode range between $\approx 8 - 12 kHz$, (the

$\frac{R}{M}$	ν_0 in kHz	τ in s	$\frac{R}{M}$	ν_0 in kHz	τ in s
2.4	8.6293	$1.52 \cdot 10^{-3}$	2.28	4.4333	10.8
	-	-		6.0168	$2.50 \cdot 10^{-1}$
	-	-		7.5462	$1.44 \cdot 10^{-2}$
	-	-		8.9891	$1.83 \cdot 10^{-3}$
2.3	5.6153	0.54	2.26	2.6041	$5.38 \cdot 10^{3}$
	7.5566	$1.16 \cdot 10^{-2}$		3.5427	$1.69 \cdot 10^{2}$
	9.3319	$1.02 \cdot 10^{-3}$		4.4802	$1.22 \cdot 10^{1}$
	-	-		5.4127	$1.37 \cdot 10^{-1}$

Table 2: *The characteristic frequencies and damping times of the* $\ell = 2$ **s**-*modes of homogenoeus stars, with* $M = 1.35 M_\odot$ *and increasing compactness.*

frequency of the **w**-modes increases with the order of the mode), and the corresponding damping times are $\approx 0.02 - 0.1 ms$.

Chandra and I brought to light a further family of spacetime modes [24]. Contrary to the **w**-modes they are slowly damped, and therefore I shall call them the **s**-modes. They exist only for the axial perturbations and their appearence is related to the depth of the potential well inside the star, as the following illustrative example shows. Let us compare the shape of the axial potential barriers generated by homogeneous stars of increasing compactness, i.e. of decreasing ratio $\frac{R}{M}$. It should be reminded that homogeneous stars can exist only if their radius R exceeds $\frac{9}{8} R_s$, or equivalently, if $\frac{R}{M} > 2.25$. In figure 1 it is shown how the potential well inside the star becomes deeper as the value of $\frac{R}{M}$ decreases and the star shrinks. In the exterior the potential coincides with the Regge-Wheeler potential that has a maximum at $r \approx 3M$. When $(R/M) < 2.6$ the potential well in the interior becomes deep enough to allow the existence of one or more quasi-normal modes. In table 2 the characteristic frequencies and damping times of the $\ell = 2$ **s**-modes of homogenoeus stars with $M = 1.35 M_\odot$ and different values of R/M are listed.

It should be stressed that the modes that one finds when the radius of the star approaches the limiting value, are not related to the quasi normal modes of a Schwarzschild black hole, because both the boundary conditions and the underlying scattering process are different. Moreover, the progressive increasing of the damping time for these modes means that they are more effectively trapped by the curvature of the star.

The existence of the **s**-modes was proved by using homogeneous stars as a model, and we have seen that they appear only if $\frac{R}{M}$ is sufficiently close to the limiting value 2.25. It would be interesting to understand whether this constraint on $\frac{R}{M}$ derives from the particular choice of the model we have used, or whether it could be relaxed by the use of a different equation of state. And further, is the existence of the **s**-modes related to some characteristic property of the equation of state, as, for example, on how stiff this equation is?

To answer these questions, in collaboration with Maria Alessandra Papa [25] we have studied the quasi-normal modes of polytropic stars having at the center a very small core with the equation of state of stiff matter $\epsilon = p$.

We chose this model because, as firstly suggested by Zeldovich [26], the equation $\epsilon = p$ represents the most extreme equation of state for high density matter compatible with the requirements of special relativity. For example, the Tsuruta-Cameron [27] equation of state has this asymptotic behaviour near the center of the star. Furthermore, we wanted to understand whether the presence of a stiff core would give any particular signature to the spectrum of the gravitational waves the star emits.

We determined the equilibrium configurations of such stars, and the range in which the radius of the stiff core can vary in order the star to be stable. The main characteristics of the models we have studied are summarized in tables 3 and 4. It should be noted that since the core is extremely small, neither the mass nor the radius change significantly as a function of R_{core} (they change at most by a few percents), when it varies in the stability range. Typical values of R and M for these stars are given in table 4. From table 3 we see that as the polytropic index of the envelope decreases, the core is allowed to occupy a larger fraction of the star. Moreover the ratio $\frac{R}{M}$ decreases and the star becomes smaller and more compact. We did not consider values of n lower than 0.5 because the star would become too small and the stiff core too big, and we did not want to deal with extreme situations.

Contrary to our expectations, we found that the depth of the potential

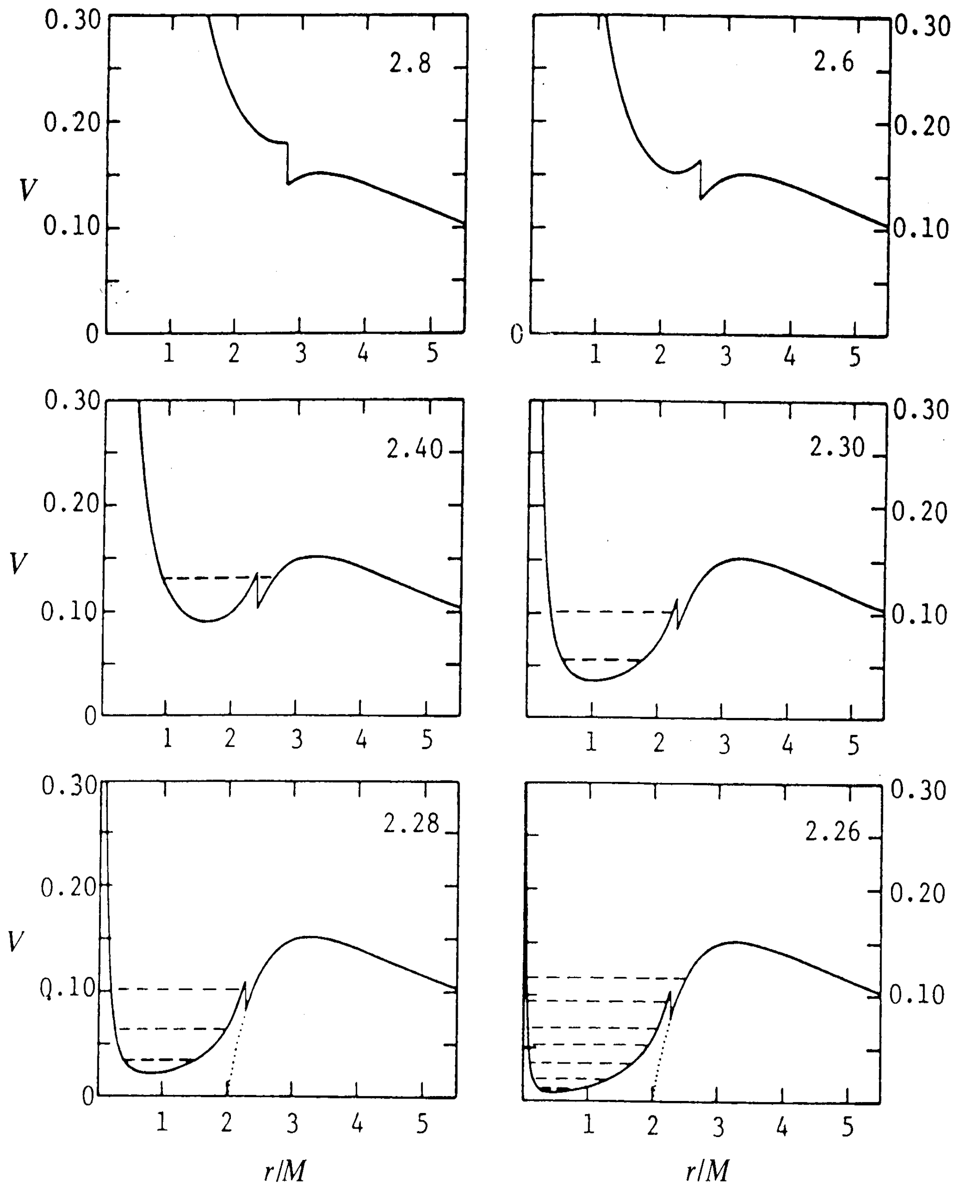

Figure 1: *The potential barrier of the axial perturbations of homogeneous stars is plotted for different values of the ratio* $\frac{R}{M}$ *ranging from 2.8 to 2.26.*

n	$\frac{R_{core}}{R}$	$\frac{R}{M}$
1.5	0.25% - 0.50%	4.21167 -4.21170
1.0	1.29% - 2.78%	3.2649 - 3.2645
0.5	5.68% - 11.22%	2.843 - 2.836

Table 3: *Parameters of the structure of a polytropic star with an* $\epsilon = p$ *core and different values of the polytropic index in the envelope. In order the star to be stable, the radius of the core must range in the interval given in column 2. In column 3 the corresponding range of variation of* $\frac{R}{M}$ *is given.*

	$\rho_c = 5 \cdot 10^{15} g/cm^3$		$\rho_c = 10^{16} g/cm^3$	
n	$\frac{M}{M_\odot}$	R in km	$\frac{M}{M_\odot}$	R in km
1.5	2.5	15.8	1.8	11.2
1.0	2.0	9.9	1.5	7.0
0.5	1.6	6.9	1.2	4.9

Table 4: *Typical values of mass and radius for two values of the central density. As the core radius varies in the allowed range given in Table 3,* M *and* R *change by at most a few percents.*

well inside the star does not significantly increase as the ratio $\frac{R}{M}$ decreases, and that these stars do not possess axial slowly damped modes. This result suggests that the appearence of the **s**-modes in the spectrum of the axial perturbations is more likely to be due to the incompressibility of the equation of state rather than to its stiffness. However, this point remains to be clarified, as well as whether it is the core or the envelope which play a fundamental role in this respect.

The influence of a small stiff core on the spectrum of the polar modes is much more significant. In order to locate the frequencies of the quasi-normal modes, one usually plots a "resonance curve" $[\alpha^2(\omega) + \beta^2(\omega)]$, that represents the amplitude of the standing wave at radial infinity obtained by numerically integrating the perturbed equations for real frequency. It can be shown that the values of frequency at which this curve exhibits a sharp minimum correspond to the real part of a quasi-normal modes, provided the imaginary part of the corresponding eigenfrequency is small enough ($\omega_i << \omega_0$.) The damping time associated to a mode is related to the curvature of the parabola that fits the curve near a minimum: smoother minima correspond to shorter damping times [28]. It should be noted that this algorithms is designed to determine essentially the slowly damped modes. In figure 2 the resonance curve is shown for an $n = 1.5$ polytropic star with $\frac{R}{M} = 4.2$, as a function of the frequency. By analyzing the behaviour of the thermodynamical variables in correspondence of the frequencies of the quasi-normal modes, one can easily identify the **g**-, **f**-and **p**-modes that one defines in newtonian theory according to the Cowling classification [29], [30]. In figure 3 we plot the resonance curve for an $n = 1.5$ polytropic star having in its center a very tiny stiff core extending only up to the 0.30% of the total radius, and with the same ratio $\frac{R}{M} = 4.2$.

Compared to the case illustrated in figure 2, the structure of the spectrum becomes incredibly rich, and in particular a large number of **g**-modes appear that were not present in the fully polytropic star. In addition, smoother mimima are present, indicating that the composite star possesses both slowly-damped and highly-damped modes. This example powerfully illustrate how the spectrum of the quasi-normal modes of a star carry relevant information on its internal structure and on the manner in which the fluid and the gravitational field couple at supernuclear regimes.

There are further information that one can derive from the knowledge of the frequencies and damping times of the quasi-normal. In newtonian theory the frequency of the **f**-mode scales with the mean density of the star. In geometric units

$$\omega_f = \sqrt{\frac{2\ell(\ell+1)}{2\ell+1}\left(\frac{M}{R^3}\right)}. \tag{36}$$

This relation has been generalized by N. Andersson and K. Kokkotas [31] who have determined both the frequency and the damping time of the **f**-mode for several equations of state proposed in the literature for neutron stars. They find the following relations

$$\omega_f = 0.39 + 44.45\sqrt{\left(\frac{M}{R^3}\right)} \tag{37}$$

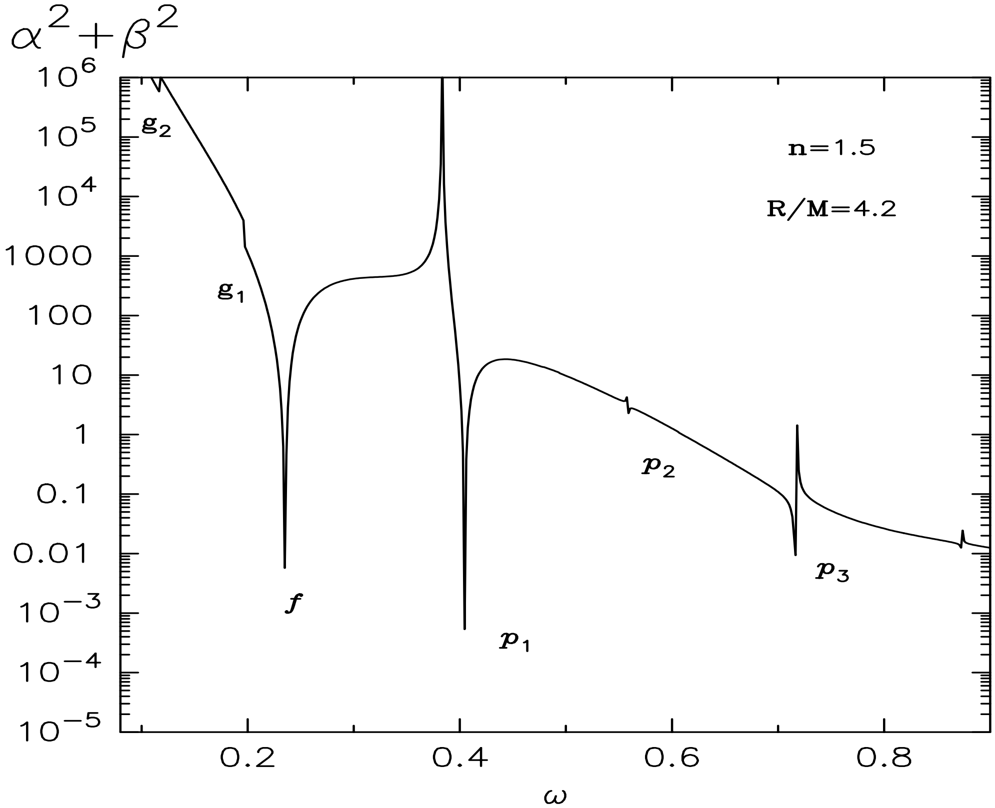

Figure 2: The resonance curve $\alpha^2 + \beta^2$ of a fully polytropic star, is plotted versus the real frequency ω, for $\ell = 2$. ω is measured in unities of $\epsilon_0^{-1/2}$.

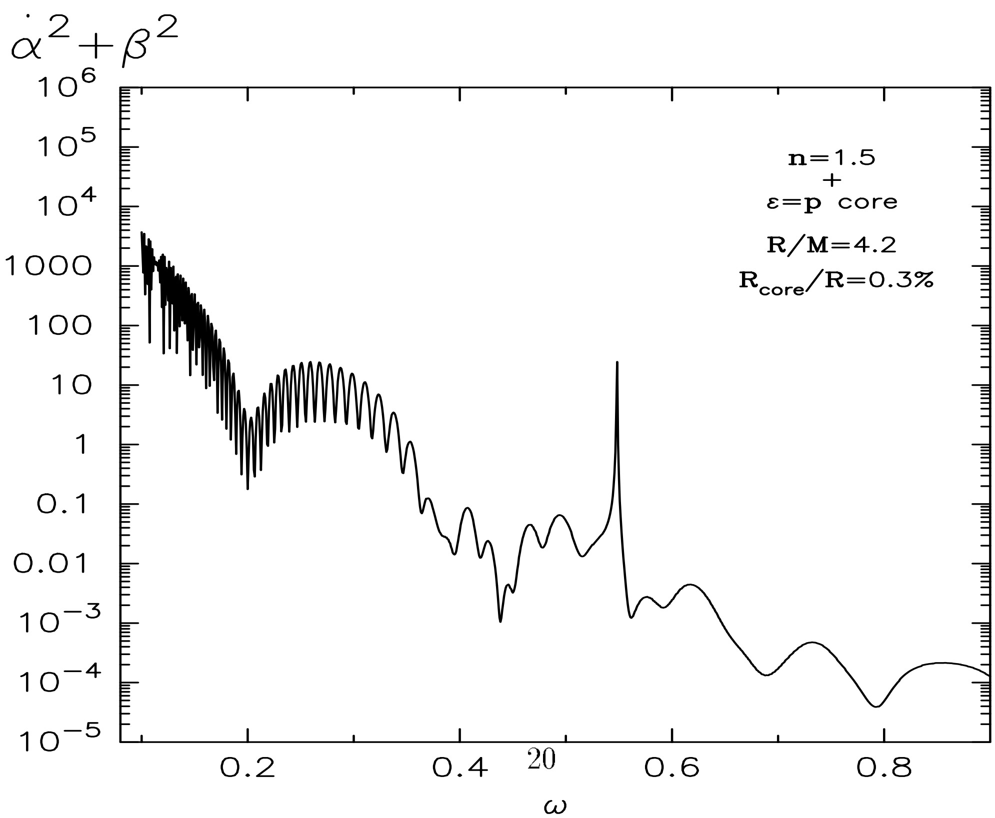

Figure 3: The resonance curve of the same polytropic star with a stiff core in its center.

$$\tau_f = 0.1 - \left(\frac{M}{R}\right) + 2.69\left(\frac{M}{R}\right)^2,$$

where M and R are expressed in km, ω_f in kHz and τ_f in ms. These two relations provide an estimate both for M and R, good within 5% if compared with the true values. It should be noted that the frequency of the **f**-mode ranges in the interval $\approx 1 - 2kHz$, and the damping time is $\approx 0.1 - 0.5s$. A further relation is provided by the damping time of the lowest **w**-mode computed for the same models

$$\frac{1}{\tau_{w_0}} = 0.104 - 0.063\left(\frac{M}{R}\right). \tag{38}$$

Andersson and Kokkotas have also studied how the axial quasi-normal modes are excited when an initial Gaussian pulse is scattered by the potential barrier of the axial perturbations of homogeneous stars. Their simulation shows that, in principle, the various modes can be excited. However, it would be interesting to know how the different modes are excited in some realistic situations. For example during the last stages of the gravitational collapse, when the newborn star wildly oscillate releasing gravitational waves, or when a mass, smaller than the star mass, is scattered or captured by the big one [32], [33].

6. Slowly rotating stars

The theory of stellar perturbations developed for static stars can be generalized to the case when the star is rotating so slowly that the distortion from spherical symmetry is quadratic in the angular velocity Ω, and may be ignored [34]. The unperturbed configuration is described by the following metric [35],[36]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2, \tag{39}$$

where ν, ψ, μ_2, μ_3 differ from those of a static star by quantities of order Ω^2, while ω (that is zero in the non-rotating case) is a first order quantity in Ω. The equations governing ν, ψ, μ_2, μ_3 are given in sections 3. The equation for ω is

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} - (\mu_2 + \nu)_{,r}\left(\varpi_{,r} + \frac{4}{r}\varpi\right) = 0, \tag{40}$$

where

$$\varpi = \Omega - \omega. \tag{41}$$

In the vacuum outside the star, $\mu_2 + \nu = 0$ and the solution of eq. (40) reduces to $\varpi = \Omega - 2Jr^{-3}$, where J is the angular momentum of the star. In ref. [34] we showed that the axial perturbations of a slowly rotating star couple to the polar perturbations, and viceversa.

The way this coupling works for the axial perturbations is illustrated by the following equation [4]

$$\sum_{l=2}^{\infty} \left\{ \frac{d^2 Z_l^1}{dr_*^2} + \omega^2 Z_l^1 - \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)]Z_l^1 \right\} C_{l+2}^{-\frac{3}{2}}(\mu) = re^{2\nu-2\mu_2}(1-\mu^2)^2 \sum_{l=2}^{\infty} S_l^0(r,\mu), \tag{42}$$

where

$$S_l^0 = \varpi_{,r}[(2W_l^0 + N_l^0 + 5L_l^0 + 2nV_l^0 P_{l,\mu} + 2\mu V_l^0 P_{l,\mu,\mu}] + 2\varpi W_l^0 (Q-1)\nu_{,r} P_{l,\mu}, \tag{43}$$

and Q has been defined in eq. (29). $\mu = \cos\theta$, and $C_{l+2}^{-\frac{3}{2}}(\mu)$ and $P_l(\mu)$ are respectively the Gegenbauer and the Legendre polynomials.

Eq. (42) holds from the center of the star up to radial infinity, provided outside the star ϵ, p and W are set to zero. As described in previous sections, if the star does not rotate the axial and the polar perturbations are described by two distinct sets of equations: eqs. (31) for the polar variables $N_l^0, L_l^0 V_l^0, W_l^0$ etc., and eqs. (34) for the axial function Z_ℓ^0. If the rotation is switched on ($\varpi \neq 0$), the axial function of first order in Ω, Z_ℓ^1, couple as indicated in eq. (42) with the polar functions $(W_l^0, N_l^0, L_l^0 V_l^0)$ of zero order in Ω, i.e. evaluated in the case of no rotation.
It should be noted that the coupling function is the quantity ϖ which is responsible for the dragging of inertial frames in the Lense-Thirring effect. Thus, rotating stars extert not only a dragging of the bodies, but also of the waves, and consequently an incoming polar gravitational wave can convert, through the fluid oscillations it excites, some of its energy into outgoing axial waves.

[4] The equations describing the coupling of the polar with the axial perturbations were subsequently determined by Y.Kojima [37].

I would like to stress that this phenomenon is a purely relativistic effect with no counterpart in newtonian theory.

Equation (42) is not yet separated. When the angular dependence is removed, one finds that the axial and the polar perturbations couple according to the following rules:
- *The Laporte rule* - the polar modes belonging to *even* ℓ can couple only with the axial modes belonging to *odd* ℓ, and conversely.
- *The selection rule* - $l = m + 1$, or $l = m - 1$.
- *The propensity rule [38]* - the transition $l \rightarrow l + 1$ is strongly favoured over the transition $l \rightarrow l - 1$. This derives from the manner in which the behaviour of the axial function is affected by the polar source near the origin.

As a consequence of this coupling, new families of modes are likely to emerge. For example, in ref. [34] we studied the axial perturbations of a slowly rotating polytropic star with polytropic index $n = 1.5$, and we showed that if one scatters an $\ell = 2$ polar gravitational waves on the potential barrier of eq. (42), for some value of the frequency of the incident wave the $m = 3$ axial perturbation induced by the coupling behaves as a pure outgoing wave at radial infinity. These "induced" axial resonances are characterized by damping times considerably longer than those of the polar modes of order zero in Ω (up to hundred times).

7. Concluding remarks

The existence of an energy conservation law governing the non-radial oscillations of a spherical star, which was derived in analogy with the conservation law governing the scattering of gravitational waves by a Schwarzschild black hole, provides an additional constraint to the theory and allows to recast the problem of stellar perturbations as a problem in scattering theory. The scattering approach proves extremely powerfull in enlighthening some aspects of the theory that where obscured in previous formulations. The existence of the slowly-damped axial modes in ultra-compact stars, the coupling between the polar and axial perturbations in slowly-rotating stars and the resonances induced by this coupling naturally emerge in this framework, though they could have also been discovered by other approaches.

The scattering approach is applicable also when the star is newtonian, i.e. when its equilibrium configuration is built in the Newtonian framework and the curvature it generates is very shallow. Indeed we showed that the frequencies of oscillation of a newtonian star can be determined by integrating the polar equations in the limit of small curvature, under the condition that no radiation emerges, as in the case of the dipole oscillations [39].

At the end of this lecture I would like to add to the scientific illustration of my work with Chandra some personal recollection on our collaboration. It developed over twelve years, and it was certainly based on reciprocal respect, esteem, trust and common scientific interests. But the real engine was Chandra's genuine enthusiasm for science which he was able to communicate to me by making me feel that, no matter how difficult a problem was, together we could make it. I am grateful to Chandra for his precious gift of sharing with me his patrimony of knowledge, experience, craftsmanship, fruits of a life entirely dedicated to science.

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