Proceedings of The International Conference
On Gravitational Waves: Sources and Detectors
Cascina (Pisa), March 19-23, 1996
World Scientific Publishing Company
Invited Talk, in press

THE QUASI-NORMAL MODES OF STARS AND BLACK HOLES

V. FERRARI

ICRA (International Center for Relativistic Astrophysics) Dipartimento di Fisica "G.Marconi", Università di Roma, Rome, Italy

Non-radial oscillations of stars excited by external perturbations, are associated to the emission of gravitational waves. The characteristic eigenfrequencies of these oscillations, computed by using the relativistic theory of stellar perturbations, will be compared with those of black holes.

1 Introduction

The study of stellar oscillations started at the beginning of this century, when Shapley[1] (1914) and Eddington[2] (1918) suggested that the variability observed in some stars is due to periodic pulsations. The subsequent study of this phenomenon, carried out in the framework of the newtonian theory of gravity, has been a powerful tool in the investigation of stellar structure. In General Relativity, the interest in the theory of stellar pulsations is enhanced by the fact that a pulsating star emits gravitational waves with frequencies and damping times each belonging to characteristic "quasi-normal" modes. Since the fluid composing the star and the gravitational field are coupled, the emitted radiation carries information on the structure of the star, and also on the manner in which the gravitational field couples to matter. Conversely, for black holes the quasi-normal modes are purely gravitational, and the corresponding eigenfrequencies depend only on the parameters that identify the spacetime geometry: mass, charge and angular momentum. In sections 2, 3 and 4 of this lecture, I shall introduce the basic equations of the theory of stellar perturbations which has been developed in collaboration with S. Chandrasekhar[3,9], under the assumption of no rotation. In section 5 the characteristics of the spectrum of the quasi-normal modes of stars and black holes, and the information it gives on the nature and the structure of the source will be discussed.

2 The perturbed spacetime

As a consequence of a perturbation, all metric functions change by an infinitesimal amount with respect to their unperturbed values, and, if we are dealing with a star, each element of fluid suffers an infinitesimal displacement from its equilibrium position, identified by the lagrangian displacement $\vec{\xi}$. Consequently, the thermodynamical variables ϵ and p, respectively the energy-density and the pressure, also change by an infinitesimal amount. Our analysis will presently be restricted to the study of adiabatic, axisymmetric perturbations of stars composed by a perfect fluid, and we shall assume that all perturbed quantities have a time dependence $e^{i\sigma t}$. The perturbed quantities are determined by solving Einstein's equations coupled to the hydrodynamical equations for a star, while for a black hole only Einstein's equations for the metric perturbations need to be considered. In order to separate the variables, all tensors can be expanded in tensorial spherical harmonics, and the azimuthal number m can be set to zero (axisymmetic perturbations). These harmonics belong to two different classes depending on the way they transform under the parity transformation $\theta \rightarrow \pi - \theta$ and $\varphi \rightarrow \pi + \varphi$. In particular those that transform like $(-1)^{(\ell+1)}$ are said to be *axial*, and those that transform like $(-1)^{(\ell)}$ are said to be *polar*. Consequently, the perturbed equations split into two distinct sets the *axial* and the *polar*, each belonging to different parities. If we choose the following line-element, appropriate to describe an axially symmetric, time-dependent spacetimes, [a]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \quad (1)$$

we find that the *axial* equations involve the perturbations of the off-diagonal components of the metric, i.e. $\{\delta\omega, \delta q_2$ and $\delta q_3\}$, and that the *polar* equations involve the diagonal part of the metric $\{\delta\nu, \delta\mu_2, \delta\psi, \delta\mu_3\}$, coupled to the thermodynamical variables $\{\delta\epsilon, \delta p, \vec{\xi}\}$ in the case of stars.

3 A Schroedinger equation for the axial perturbations

The equations for the axial perturbations can be considerably simplified by introducing, after separating the variables, a new function $Z_\ell(r)$, constructed from the radial part of the axial metric components, and which satisfies the

[a] It may be noted that with our choice of the gauge the number of free functions is seven. The extra degree of freedom which we allow will be eliminated by imposing boundary conditions suitable to the problem on hand.

following Schroedinger-like equation

$$\frac{d^2 Z_\ell^{ax}}{dr_*^2} + [\sigma^2 - V_\ell(r)] Z_\ell^{ax} = 0, \tag{2}$$

where $r_* = \int_0^r e^{-\nu+\mu_2} dr$. For a black hole[10]

$$V_{\ell BH}(r) = \frac{e^{2\nu}}{r^3}[l(l+1)r - 6Mr], \qquad \text{and} \qquad e^{2\nu} = 1 - \frac{2M}{r}, \tag{3}$$

and for a star[4]

$$V_{\ell Star}(r) = \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)], \qquad \nu_{,r} = -\frac{p_{,r}}{\epsilon + p}. \tag{4}$$

Outside the star ϵ and p are zero and eq. (4) reduces to eq. (3), also known as the Regge-Wheeler potential.

Thus the axial perturbations of black holes and stars are fully described by a Schroedinger-like equation with a potential barrier that depends, respectively, on the black hole mass, and on how the energy-density and the pressure are distributed inside the star in its equilibrium configuration. It should be stressed that the axial perturbations of stars are not coupled to any fluid pulsation: *they are pure gravitational perturbations, and do not have a newtonian counterpart.*

4 The polar perturbations

The expansion in tensorial spherical harmonics (with $m = 0$) shows that the polar metric functions and the thermodynamical variables have the following angular dependence

$$\begin{array}{ll} \delta\nu = N_\ell(r) P_l(\cos\theta) e^{i\sigma t} & \delta\mu_2 = L_\ell(r) P_l(\cos\theta) e^{i\sigma t} \\ \delta\mu_3 = [T_\ell(r) P_l + V_\ell(r) P_{l,\theta,\theta}] e^{i\sigma t} & \delta\psi = [T_\ell(r) P_l + V_\ell(r) P_{l,\theta} \cot\theta] e^{i\sigma t}, \\ \delta p = \Pi_\ell(r) P_l(\cos\theta) e^{i\sigma t} & 2(\epsilon + p) e^{\nu+\mu_2} \xi_r(r,\theta) e^{i\sigma t} = U_\ell(r) P_l e^{i\sigma t} \\ \delta\epsilon = E_\ell(r) P_l(\cos\theta) e^{i\sigma t} & 2(\epsilon + p) e^{\nu+\mu_3} \xi_\theta(r,\theta) e^{i\sigma t} = W_\ell(r) P_{l,\theta} e^{i\sigma t}, \end{array} \tag{5}$$

where $P_l(\cos\theta)$ are the Legendre polynomials. After separating the variables the relevant Einstein's equations become

$$\begin{cases} (T_\ell - V_\ell + N_\ell)_{,r} - \left(\frac{1}{r} - \nu_{,r}\right) N_\ell - \left(\frac{1}{r} + \nu_{,r}\right) L_\ell = 0, \\ V_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) V_{\ell,r} + \frac{e^{2\mu_2}}{r^2}(N_\ell + L_\ell) + \sigma^2 e^{2\mu_2 - 2\nu} V_\ell = 0, \end{cases} \tag{6}$$

$$\begin{cases} -(T_\ell - V_\ell + L_\ell) = W_\ell & (=0 \quad \text{for B.H.}), \\ \left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2T_\ell - kV_\ell) - \frac{2}{r}L_\ell = -U_\ell & (=0 \quad \text{for B.H.}), \\ \frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}N_{\ell,r} + \left(\frac{1}{r}+\nu_{,r}\right)(2T_\ell - kV_\ell)_{,r} - \frac{2}{r}\left(\frac{1}{r}+2\nu_{,r}\right)L_\ell\right] + & \\ \frac{1}{2}\left[-\frac{1}{r^2}(2nT_\ell + kN_\ell) + \sigma^2 e^{-2\nu}(2T_\ell - kV_\ell)\right] = \Pi_\ell & (=0 \quad \text{for B.H.}), \end{cases} \tag{7}$$

where $k = l(l+1)$, and $2n = (l-1)(l+2)$. After some manipulation, the hydrodynamical equations and the conservation of barion number give the following expression for the hydrodynamical quantities

$$\Pi_\ell = -\frac{1}{2}\sigma^2 e^{-2\nu}W_\ell - (\epsilon+p)N_\ell, \qquad E_\ell = Q\Pi_\ell + \frac{e^{-2\mu_2}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r})U_\ell, \tag{8}$$

$$U_\ell = \frac{[(\sigma^2 e^{-2\nu}W_\ell)_{,r} + (Q+1)\nu_{,r}(\sigma^2 e^{-2\nu}W_\ell) + 2(\epsilon_{,r} - Qp_{,r})N_\ell](\epsilon+p)}{[\sigma^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})]}, \tag{9}$$

where

$$Q = \frac{(\epsilon+p)}{\gamma p}, \qquad \gamma = \frac{(\epsilon+p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const} \tag{10}$$

and γ is the adiabatic exponent (defined in ref. [3], equation (106)).

For a black hole, a suitable reduction of eqs. (6) and (7), with W_ℓ, U_ℓ, Π_ℓ set equal zero, shows that the new function

$$Z_\ell^{pol}(r) = \frac{r}{nr+3M}\left(3MV_\ell(r) - rL_\ell(r)\right), \tag{11}$$

satisfies the following wave equation

$$\frac{d^2 Z_\ell^{pol}(r)}{dr_*^2} + [\sigma^2 - V_{BH}]Z_\ell^{pol}(r) = 0, \tag{12}$$

where

$$V_{BH}(r) = \frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3]. \tag{13}$$

Thus, as for the axial perturbations, the equations for the polar perturbations of a Schwarzschild black hole reduce to a single Schroedinger-like equation, but with a different potential barrier. Equation (12) with the potential (13) is known as the Zerilli equation[11], and it will also governe the metric perturbations in the exterior of a non-rotating star. The functions Z_ℓ^{ax} and Z_ℓ^{pol} contain all information on the gravitational waves emerging at infinity. In fact, it has been shown that the imaginary and the real part of the Weyl scalar Ψ_0, which represents the outgoing part of the radiative field (cfr. [12] eqs. 345 and 353), can be expressed in terms of Z_ℓ^{ax} and Z_ℓ^{pol}, respectively.

It is now interesting to see how eqs. (6)-(7) and the hydrodynamical equations (8,9) can be reduced if the perturbed object is a star. One may try to operate on these equations in a way similar to that used to find equation (12), hoping to find again a Schroedinger-like equation, possibly with some source in terms of the fluid variables. Unfortunately this is not possible, since the Schroedinger equation for black holes arises by virtue of the equilibrium equations, that are very different in the case of a star. In addition, this fact was to be expected, as already in newtonian theory the equations for the polar perturbations are described by a fourth order linear differential system. However a remarkable simplification is still possible. The first of eqs. (7) and eqs. (8,9) show that the fluid variables $[W_\ell, U_\ell, E_\ell, \Pi_\ell]$ can be expressed as a combination of the metric perturbations $[T_\ell, V_\ell, L_\ell, N_\ell]$ and their first derivatives. Therefore, after their direct substitution on the right hand side of the last three eqs. (7) we obtain a set of new equations which involves only the perturbations of the metric functions $[T_\ell, V_\ell, L_\ell, N_\ell]$. The final set is

$$\begin{cases} X_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) X_{\ell,r} + \frac{n}{r^2} e^{2\mu_2}(N_\ell + L_\ell) + \sigma^2 e^{2(\mu_2-\nu)} X_\ell = 0, \\ (r^2 G)_{\ell,r} = n\nu_{,r}(N_\ell - L_\ell) + \frac{n}{r}(e^{2\mu_2} - 1)(N_\ell + L_\ell) + r(\nu_{,r} - \mu_{2,r})X_{\ell,r} + \sigma^2 e^{2(\mu_2-\nu)} r X_\ell, \\ -\nu_{,r} N_{\ell,r} = -G_\ell + \nu_{,r}[X_{\ell,r} + \nu_{,r}(N_\ell - L_\ell)] + \frac{1}{r^2}(e^{2\mu_2} - 1)(N_\ell - rX_{\ell,r} - r^2 G_\ell) \\ -e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\sigma^2 e^{2(\mu_2-\nu)} \left\{ N_\ell + L_\ell + \frac{r^2}{n} G_\ell + \frac{1}{n}[rX_{\ell,r} + (2n+1)X_\ell] \right\}, \\ L_{\ell,r}(1 - D) + L_\ell \left[\left(\frac{2}{r} - \nu_{,r}\right) - \left(\frac{1}{r} + \nu_{,r}\right) D \right] + X_{\ell,r} + X_\ell \left(\frac{1}{r} - \nu_{,r}\right) + DN_{\ell,r} + \\ +N_\ell \left(D\nu_{,r} - \frac{D}{r} - F\right) + \left(\frac{1}{r} + E\nu_{,r}\right) \left[N_\ell - L_\ell + \frac{r^2}{n} G_\ell + \frac{1}{n}(rX_{\ell,r} + X_\ell) \right] = 0, \end{cases} \tag{14}$$

where

$$\begin{cases} A = \frac{1}{2}\sigma^2 e^{-2\nu}, \qquad B = \frac{e^{-2\mu_2}\nu_{,r}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r}), \\ D = 1 - \frac{A}{2(A+B)} = 1 - \frac{\sigma^2 e^{-2\nu}(\epsilon+p)}{\sigma^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \\ E = D(Q-1) - Q, \\ F = \frac{\epsilon_{,r} - Qp_{,r}}{2(A+B)} = \frac{2[\epsilon_{,r} - Qp_{,r}](\epsilon+p)}{2\sigma^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \end{cases} \tag{15}$$

and V_ℓ and T_ℓ have been replaced by X_ℓ and G_ℓ defined as

$$\begin{cases} X_\ell = nV_\ell \\ G_\ell = \nu_{,r}[\frac{n+1}{n}X_\ell - T_\ell]_{,r} + \frac{1}{r^2}(e^{2\mu_2} - 1)[n(N_\ell + T_\ell) + N_\ell] \\ +\frac{\nu_{,r}}{r}(N_\ell + L_\ell) - e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\sigma^2 e^{2(\mu_2-\nu)}[L_\ell - T_\ell + \frac{2n+1}{n}X_\ell]. \end{cases} \tag{16}$$

Equations (14) describe the perturbations of the gravitational field in the interior of the star, with no reference to the motion of the fluid. Once these equations have been solved, the fluid variables can be obtained in terms of the metric functions from the first of eqs. (7) and eqs. (8,9). This fact is remarkable: it shows that all the information on the dynamical evolution of a physical system is encoded in the gravitational field, a result which expresses the physical content of Einstein's theory of gravity. Moreover, it should be stressed that the decoupling of the equations governing the metric perturbations from the equations governing the hydrodynamical variables is possible in general, and *requires no assumptions on the equation of state of the fluid.* Thus, if we are interested exclusively in the study of the emitted gravitational radiation, we can solve the system (14) and disregard the fluid behaviour.

Equations (14) have to be integrated for each value of the frequency from $r = 0,$ up to the boundary of the star. There the spacetime becomes a vacuum sperically symmetric spacetime, and the perturbed metric functions match continuously with the metric functions that describe the polar perturbations of a Schwarzschild black hole, i.e. eqs (11,12,13). Thus the boundary conditions appropriate to the problem are

$i)$ all functions are regular at $r = 0,$ (17)

$ii)$ $\delta p = 0$ at the boundary of the star

$iii)$ all functions and their first derivatives are continuous at the boundary of the star.

5 The characteristic frequencies of the quasi-normal modes

The concept of quasi-normal modes plays a central role in the theory of perturbations of stars and black holes. In newtonian theory the oscillations of a perturbed star can be decomposed into normal modes, i.e. solutions of the perturbed equations that satisfy the boundary conditions (17) i), ii), and that correspond to a discrete set of real eigenfrequencies. Their relativistic generalization are the quasi-normal modes, and in this case the characteristic frequencies are complex, since the imaginary part is the inverse of the damping time associated to the emission of gravitational waves. Although the completeness of the quasi-normal modes has never been proved, numerical simulations show that an initial perturbation will, during the very last stages, decay as a superposition of these pure modes, and that a large fraction of the radiation will be emitted at the corresponding frequencies. The boundary conditions that identify the quasi-normal modes of a star are that, in addition to (17), at radial infinity only pure outgoing waves must prevail. The role of the equations in the interior of the star is that of providing the initial conditions for the integration of the Zerilli or the Regge-Wheeler equation in the exterior. Since a polar perturbation excites the fluid motion, the amount of energy which leaks out of

Table 1: The complex characteristic frequencies of the quasi-normal modes of a Schwarzschild black hole.

	$M\sigma_0 + iM\sigma_i$		$M\sigma_0 + iM\sigma_i$
$\ell = 2$	0.3737+i0.0890	$\ell = 3$	0.5994+i0.0927
	0.3467+i0.2739		0.5826+i0.2813
	0.3011+i0.4783		0.5517+i0.4791
	0.2515+i0.7051		0.5120+i0.6903

the star in the form of gravitational waves depends on the exchange of energy between the fluid and the gravitational field. Conversely, an axial perturbation does not excite any fluid motion, and the boundary conditions depend only on the shape of the potential of the wave-equation, i.e. on how the energy-density and the pressure are distributed in the equilibrium configuration. Thus, the eigenfrequencies of the axial quasi-normal modes carry information essentially on the structure of the star, and the polar, in addition, elucidate the manner in which the fluid and the gravitational field couple at supernuclear regimes.

For a black hole, the quasi-normal modes are defined to be solutions of the wave-equations that satisfy the boundary conditions of a *pure outgoing wave at infinity* and of a *pure ingoing wave at the horizon* (no radiation can emerge from the horizon). The corresponding frequencies are characteristic of many different processes involving the dynamical perturbations of black holes, and are the same both for the polar and for the axial perturbations, i.e. *the two potential barriers (3) and (13) are isospectral.* In 1975 Chandrasekhar and Detweiler[13] computed the first few eigenfrequencies of a Schwarzschild black hole, and subsequently Leaver [14] determined the next values with very high accuracy. He showed that, for a given ℓ, $M\sigma_0$ decreases with the order of the mode, and approaches a non-zero constant value, while $M\sigma_i$ increases, i.e. the damping time decreases. In Table 1 we show the first four values, respectively for $\ell = 2$ and $\ell = 3$. For example, remembering that $1M_\odot = 1.48 \cdot 10^5 cm$ and assuming that the black hole mass is $M = nM_\odot$, the conversion to physical unities gives the following values of the frequency and damping time

$$\nu_0 = \frac{c}{2\pi n \cdot M_\odot (M\sigma_0)} = \frac{32.26}{n}(M\sigma_0) kHz, \quad \tau = \frac{nM_\odot}{(M\sigma_i)c} = \frac{n \cdot 0.4937 \cdot 10^{-5}}{(M\sigma_i)} s. \tag{18}$$

In order to compare the frequencies at which black holes and stars emit gravitational waves, we shall first consider, as an example, the polar perturba-

Table 2: Parameters of the three models of polytropic stars used to compute the polar eigenfrequencies

ρ in gr/cm^3	$\frac{M}{M_\odot}$	R in km	$\frac{2M}{R}$
$3 \cdot 10^{15}$	1.266	8.861	0.422
$6 \cdot 10^{15}$	1.35	7.413	0.538
10^{16}	1.3	6.465	0.594

Table 3: The characteristic frequencies and damping times of the $\ell = 2$ s and **w** polar modes of polytropic stars, compared with the first three eigenfrequencies of a Schwarzschild black hole with the same mass

	s-modes		**w**-modes		black hole	
$\frac{2M}{R}$	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s
0.422	3.0366	0.076	13.1556	$2.42 \cdot 10^{-5}$	9.5226	$7.02 \cdot 10^{-5}$
	6.7384	5.642	22.3438	$1.83 \cdot 10^{-5}$	8.8346	$2.28 \cdot 10^{-5}$
	10.1980	0.077	31.2207	$1.26 \cdot 10^{-5}$	7.6726	$1.31 \cdot 10^{-5}$
0.538	3.9166	0.060	12.4960	$3.65 \cdot 10^{-5}$	8.9300	$7.49 \cdot 10^{-5}$
	7.9610	0.623	19.4390	$2.30 \cdot 10^{-5}$	8.2848	$2.43 \cdot 10^{-5}$
	11.8669	0.035	26.3559	$1.94 \cdot 10^{-5}$	7.1952	$1.39 \cdot 10^{-5}$
0.594	4.5310	0.061	10.8420	$6.20 \cdot 10^{-5}$	9.2735	$7.21 \cdot 10^{-5}$
	8.7109	0.151	16.9960	$3.27 \cdot 10^{-5}$	8.6035	$2.34 \cdot 10^{-5}$
	12.7429	0.035	22.5540	$2.59 \cdot 10^{-5}$	7.4719	$1.34 \cdot 10^{-5}$

tions of three models of star with a polytropic equation of state

$$p = K\rho^{1+\frac{1}{m}}, \qquad m = 1, \qquad K = 100 \quad km, \tag{19}$$

identified by different values of the central density. The corresponding mass, radius and surface gravity are given in Table 2. The polar quasi-normal modes of a star belong essentially to two different classes

i) slowly-damped modes, or **s**-modes,

ii) highly-damped modes, or **w**-modes,

and the values of the first three eigenfrequencies of the $\ell = 2$ **s**-[15] and **w**-modes[16] are shown in Table 3, compared with the polar eigenfrequencies of a Schwarzschild black hole having the same mass.

The damping time τ indicates how fast the energy is dissipated in the form of gravitational waves, and since the τ's associated to the **w**-modes are of the same order of magnitude both for stars and black holes, (note also that they both decrease with the order of mode), it is natural to interpret the **w**-modes as being essentially modes of the gravitational field. However, since the boundary conditions to be imposed at the surface of the star and at the black hole horizon are different, the real part of the eigenfrequency, ν_0, will, in general, be different: higher for a star than for a black hole with the same mass, and increasing with the order of mode rather than decreasing. The **s**-polar modes have a different physical origin. They are essentially fluid pulsations whose energy is dissipated in the form of gravitational radiation at a rate which depends on how strong is the coupling between the fluid and the gravitational field. Thus, the values of the damping times are considerably longer that those of the **w**-modes. The frequency of the fundamental mode is smaller than that of a black hole with the same mass, and increases with the compactness of the star, because the time scale of these processes is related to the speed of acoustic waves in the fluid.

Let us now consider the axial perturbations. Since they do not excite any motion in the fluid, one may expect that slowly damped axial quasi-normal modes should not exist. However, this is not the case for the following reason. The slowly damped quasi-normal modes associated to the Schroedinger-like equation (2) with the potential barrier (4), are the equivalent of the quasi-stationary states that one encounters in quantum mechanics in the study of the emission of α-particles by a radioactive nucleus, also described by a Schroedinger equation. In that case σ^2 is replaced by the energy E and the potential barrier is suitable for the problem on hand. The boundary conditions for the two problems are the same: regularity of the wave function at the center, and pure outgoing waves emerging at infinity. In a quasi-stationary state E is allowed to be complex: $\Re E$ is the energy of the α-particle, and $\Im E$ is the inverse of the mean life-time (Γ) of the particle (the inverse of the damping time in our context). It is known from atomic physics that a quasi-stationary state will exist if the potential barrier has a minimum followed by a maximum, and if the potential well is sufficiently deep. For a star, the potential barrier should be considered in two regions: the interior $r < r_1$, where it depends on ϵ and p, and the exterior $r > r_1$, where it reduces to the barrier of a Schwarzschild black hole which has a maximum at $r = 3M$. If the radius of the star is smaller than $3M$ and if the star is very compact, the potential well in the interior may be deep enough to allow the existence of one or more quasi-normal **s**-mode. This conjecture can easily be proved, and in Table 4 we show the eigenfrequencies of the first four **s**-[6] and **w**-modes[17] computed for the very simple models of homogenous stars with decreasing values of the ratio R/M, i.e. increasing compactness. It emerges that if $R/M > 2.4$ the depth

Table 4: The characteristic frequencies and damping times of the first four $\ell = 2$, s and **w** axial modes of homogenoeus stars, with $M = 1.35 M_{\odot}$, and different values of R/M. The data are compared with the eigenfrequencies of a black hole with the same mass.

	s-modes		**w**-modes		black hole	
$\frac{R}{M}$	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s
2.4	8.6293	$1.52 \cdot 10^{-3}$	11.1738	$1.70 \cdot 10^{-4}$	8.9300	$7.49 \cdot 10^{-5}$
	–	–	14.2757	$8.03 \cdot 10^{-5}$	8.2848	$2.43 \cdot 10^{-5}$
	–	–	18.2232	$5.70 \cdot 10^{-5}$	7.1952	$1.39 \cdot 10^{-5}$
	–	–	22.6669	$4.88 \cdot 10^{-5}$	6.0099	$0.95 \cdot 10^{-5}$
2.3	5.6153	0.54	11.1084	$3.02 \cdot 10^{-4}$		
	7.5566	$1.16 \cdot 10^{-2}$	13.0403	$1.73 \cdot 10^{-4}$		
	9.3319	$1.02 \cdot 10^{-3}$	15.1512	$1.28 \cdot 10^{-4}$		
	–	–	17.4412	$1.06 \cdot 10^{-4}$		
2.28	4.4333	10.8	10.4128	$5.45 \cdot 10^{-4}$		
	6.0168	$2.50 \cdot 10^{-1}$	11.9074	$2.91 \cdot 10^{-4}$		
	7.5462	$1.44 \cdot 10^{-2}$	13.4813	$2.07 \cdot 10^{-4}$		
	8.9891	$1.83 \cdot 10^{-3}$	15.1428	$1.67 \cdot 10^{-4}$		
2.26	2.6041	$5.38 \cdot 10^{3}$	10.7852	$7.60 \cdot 10^{-4}$		
	3.5427	$1.69 \cdot 10^{2}$	11.6922	$5.34 \cdot 10^{-4}$		
	4.4802	$1.22 \cdot 10^{1}$	12.6138	$4.22 \cdot 10^{-4}$		
	5.4127	$1.37 \cdot 10^{-1}$	13.5512	$3.56 \cdot 10^{-4}$		

of the potential well in the interior is not sufficient to allow the existence of an **s**-mode, and only the **w**-modes survive. However, if $R/M < 2.4$, the **s**-modes appear, and their number is finite and increases with the compactness of the star, as well as the damping times.

The spectrum of the quasi-normal modes, whose main properties we have described, gives important information on the nature of the perturbed source: 1) If the axial and the polar spectra coincide, the source is a black hole. *This is a very strong signature.* In a suitably choosen TT-gauge the axial and the polar part of the metric tensor are respectively

$$h^{ax}_{\mu\nu} = \begin{pmatrix} (t) & (r) & (\varphi) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h^{ax}_{\vartheta\vartheta} & h^{ax}_{\vartheta\varphi} \\ 0 & 0 & h^{ax}_{\varphi\vartheta} & h^{ax}_{\varphi\varphi} \end{pmatrix}, \qquad h^{pol}_{\mu\nu} = \begin{pmatrix} (t) & (r) & (\varphi) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & h^{pol}_{rr} & h^{pol}_{r\vartheta} & 0 \\ 0 & h^{pol}_{\vartheta r} & h^{pol}_{\vartheta\vartheta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

thus, the detection of these two components of the emitted radiation will provide a direct evidence of the existence of black holes.
2) If the source is a star, the presence of the **s**-modes in the axial spectrum indicates that the star has a very compact core, while their number is directly related to the value of the ratio R/M. The question whether stars with a core compact enough to allow the existence of axial **s**-modes can exist in nature is open, and it will probably receive an answer when axial gravitational waves will be observed.

Can the quasi-normal modes be excited? In the case of black holes we know they can in a variety of situations, for example when a gravitational wave-packet is scattered on the potential barrier, or when a mass $m_0 << M$ is captured by the black hole. In this case, the integration of the Zerilli and the Regge-Wheeler equations with the source term given by the stress-energy tensor of the infalling mass allows to compute the waveform and the energy emitted in these processes. It has been shown (see ref. [18] for an extensive bibliography on the subject) that the burst of gravitational waves ends in a ringing tail emitted when the particle coaleshes into the black hole ($2 < \frac{r}{M} < 4.5$). This part of the signal can be fitted with a linear superposition of quasi-normal modes. For a particle falling radially the total radiated energy is $\Delta E \sim 0.01 \left(\frac{m_0^2}{M}\right)$, which can be increased by up to a factor of 50 if the particle has an initial angular momentum.

In the case of a star it has been shown (see K. Kokkotas' paper in this volume) that both the **s**- and the **w**-axial modes can be excited if a gravitational wave-packet is scattered by the potential barrier, but much remains to be done in more realistic situations like the capture of infalling masses. For a star, these kind of calculations are complicated by the fact that we do not know how the mass m_0 interacts with the fluid composing the star after it crosses the surface. A preliminary integration of the axial[19] and the polar equations[15] with a source due to an infalling mass, and performed by truncating the integration when m_0 reaches the surface of the star, shows that indeed both the **s**- and the **w**-axial modes are excited, and that a considerable fraction of the emitted energy goes into the **w**-modes. Further work on this subject is in progress.

I would like to conclude this lecture by stressing an interesting aspect of the theory of perturbations: although it is based on the simplifying assumption that the perturbations of the physical quantities are small with respect to their unperturbed values, nevertheless, the results that one obtains by using this assumption are, to some extent, general. For example, in 1985 Stark and Piran[20,21] computed the energy spectrum emitted when an axisymmetric distribution of rotating polytropic fluid collapses to form a black hole, and they showed that it is very similar to that one obtains by integrating the Zerilli or the Regge-Wheeler equations when a mass falls in. In particular, the relevant contribution to the emitted energy is given at those frequencies at which the newborn black hole oscillate, namely at the frequencies of the quasi-normal modes.

References

1. H. ShapleyAp. J. **40**, 448 (1914)
2. A.S. EddingtonM.N.R.A.S. **79**, 2 (1918)
3. S.Chandrasekhar, V. Ferrari *Proc. R. Soc. Lond.* **A428**, 325 (1990)
4. S.Chandrasekhar, V. Ferrari *Proc. R. Soc. Lond.* **A432**, 247 (1991)
5. S.Chandrasekhar, V. Ferrari *Proc. R. Soc. Lond.* **A433**, 423 (1991)
6. S.Chandrasekhar, V. Ferrari *Proc. R. Soc. Lond.* **A434**, 449 (1991)
7. S.Chandrasekhar, V. Ferrari, R. Winston *Proc. R. Soc. Lond.* **A434**, 635 (1991)
8. S.Chandrasekhar, V. Ferrari *Proc. R. Soc. Lond.* **A437**, 133 (1992)
9. V. Ferrari *Phil. Trans. R. Soc. Lond.* **A340**, 423 (1992)
10. T.Regge, J.A. Wheeler *Phys. Rev.* **108**, 1063 (1957)
11. F.J. Zerilli*Phys. Rev.* **D2**, 2141 (1970)
12. S.Chandrasekhar*The mathematical theory of black holes*Oxford: Claredon Press (1983)
13. S.Chandrasekhar, S.L.Detweiler*Proc. R. Soc. Lond.* **A344**, 441 (1975)
14. E.W. Leaver*Proc. R. Soc. Lond.* **A402**, 285 (1985)
15. V. Ferrari, F. Perrotta *in preparation*
16. K.D. Kokkotas, B.F. Schutz *Proc. Mon. Not. R. Astron.Soc.* **255**, 119 (1992)
17. K.D. Kokkotas*Mon. Not. R. Astron. Soc.* **268**, 1015 (1994)
18. V.Ferrari *Proceedings of the 7th Marcel Grossmann Meeting* ed. by Ruffini R. & Kaiser M., World Scientific Publishing Co Pte Ltd, 1995
19. A. Borrelli, V. Ferrari *in preparation*
20. R.F. Stark, T. Piran*Phys. Rev. Lett.* **55 n. 8**, 891 (1985)
21. R.F. Stark, T. Piran *Proceedings of the 4th Marcel Grossmann Meeting*

ed. by R. Ruffini Elsevier Science Publishers B.V. 327 1986