Non-radial oscillations of stars in general relativity: a scattering problem.

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The problem of non-radial oscillations of stars can be formulated as a problem of resonant scattering of gravitational waves incident on the potential barrier generated by the spacetime curvature. This approach discloses some unespected correspondences between the theory of perturbations of stars and the theory of quantum mechanics. New relativistic effects are predicted, as the resonant behaviour of the axial modes in slowly rotating stars, due to the coupling with the polar modes induced by the Lense-Thirring effect.

1. Introduction

Non-radial oscillations of stars are manifested in a variety of astrophysical situations. For example, they are observed in the sun, and the corresponding frequencies, measured with very high accuracy, are used in modern heliosysmology to investigate the internal structure of the star. Moreover, non-radial pulsations are thought to be at the origin of the drifting subpulses and micropulses detected in some radio sources, and of the quasi-periodic variability seen in some X-ray burst sources and in a number of bright X-ray sources (McDermott, Van Horn & Hansen 1988). Due to their central role in astrophysics, oscillations of stars have been extensively studied both in the framework of the newtonian theory of gravity, and in general relativity. According to general relativity, a star vibrating into non radial modes emits gravitational waves, whereas gravitational waves do not exist in the newtonian theory. This difference is a substantial one, and it is the key point of a recent reformulation of the relativistic theory of stellar perturbations, whose main results we shall describe in this paper. This work has been developed in a series of papers, Chandrasekhar& Ferrari *a,b,c,d*, Chandrasekhar, Ferrari & Winston, 1991, Chandrasekhar& Ferrari *e*, to be referred to hereafter respectively as Paper I,II,III,IV,V and VI.

It is useful to clarify what are the specific questions to which one is addressed in formulating a theory of stellar oscillations. When a star is perturbed by some external agency, after a transient which depends on the cause of the perturbation, it will start to oscillate at some characteristic frequencies, that, as we have seen, appear to be coded in various radiative processes. Gravitational waves will also be emitted with these frequencies, and with some characteristic damping times which depend on the structure of the star. The determination of these characteristic frequencies is therefore one of the main objects of the theory. The new formulation of the problem of stellar oscillations presents several novelties with respect to the existing relativistic theory developed by Thorne and his collaborators (Thorne&Campolattaro 1967, Price & Thorne 1969, Thorne 1969). It leads to a different interpretation of the problem, which discloses some surprising and fascinating analogies with the theory of quantum mechanics. Moreover, it introduces a remarkable simplification of the problem, and allows a generalization of the theory to the case in which the star is slowly rotating. New phenomena, as the resonant behaviour of the axial modes, and the coupling between polar and axial modes induced by the Lense-Thirring effect, will emerge.

But in order to understand how the anticipated novelties are introduced by the new theory, we need to summarize and compare the newtonian theory and the previously formulated relativistic theory.

In the newtonian theory the equations that govern the adiabatic perturbations of a spherical star constitute a fourth-order linear differential system which couples the perturbation of the newtonian potential with the perturbations of the variables describing the fluid. All quantities are usually assumed to have a time dependence $\sim e^{i\sigma t}$, where σ is a constant frequency, an assumption which implies a Fourier decomposition of the modes of vibration. The system of equations must be integrated from the center to the surface of the star, with the boundary conditions that i) all physical quantities are regular at the origin, and ii) the perturbation of the pressure, δp, vanishes at the surface. These conditions are satisfied only for a specific set of *real* values of σ, $\{\sigma_n\}$, which are the frequencies of the *normal modes*. Thus the problem of finding the frequencies of the normal modes of a star in newtonian theory is *an eigenvalue problem*: one has to find the real values of σ such that the corresponding solution of the equations satisfies all the boundary conditions.

A relativistic theory of stellar perturbations can be constructed as a generalization of the newtonian theory. The resulting system of equations splits into two decoupled sets: the *polar modes*, (the even modes in Thorne's notation), that correspond to the tidal modes already present in the newtonian theory, and the axial modes (odd modes), whose effect is to induce a stationary rotation in the star, *but no pulsation in the fluid.* The axial modes do not have a counterpart in the newtonian theory, and since they do not induce any motion in the fluid, they have been disregarded as irrelevant in the literature. However, as we shall see in sections 4 and 9, under suitable circumstances they can exhibit very interesting properties.

Much more attention has been focused onto the polar modes, due to the fact that they do excite pulsations in the fluid. In the theory developed by Thorne and his collaborators it has been shown that the system of equations governing the polar perturbations can still be reduced, as in the newtonian case, to a fourth-order linear differential system *that couples the perturbations of the metric with the perturbations of the fluid.* (The reduction to a fourth order system has been accomplished by Lindblom& Detweiler 1983). This system describes the evolution of the perturbations inside the star. However, unlike the newtonian case, at this stage the description of the problem is not complete. The perturbations in the interior must be matched with the perturbations of the gravitational field in the exterior of the star, to properly take into account the emission of gravitational waves. In general relativity the frequencies of oscillation of a star are complex. The presence of an imaginary part derives from the fact that the mechanical energy of vibration is exponentially damped by the emission of gravitational waves. Consequently, the corresponding modes are called *quasi-normal* modes. They are defined as the solutions of the sistem of equations which govern the polar perturbations, both inside and outside the star, that satisfy the following boundary conditions: i) regularity of all functions at the center, (ii) $\delta p = 0$ at the surface, (iii) continuous matching of the interior and the exterior solution, and (iiii) at radial infinity the solution must reduce to a pure outgoing wave. In the approach we have described, the nature of the problem does not change substantially with respect to the newtonian theory: it is still an *eigenvalue problem associated to a system of equations which couples, in the interior of the star, the perturbations of the gravitational field with the perturbations of the fluid.*

The new relativistic theory of stellar perturbations has been constructed having as a guide the theory of perturbations of black holes rather than the newtonian theory. In order to describe the perturbed spacetime we have choosen the same gauge which has been used to study the perturbations of a Schwarzschild black hole (see *The mathematical theory of black holes*, Chandrasekhar 1983, this book will be referred in the sequel as *M.T.*).

This assumption, as remarked by Price & Ipser (Price & Ipser 1991), corresponds to an incomplete constraint on the coordinates. However this additional degree of freedom has no physical consequences because it is eliminated by the requirement that all perturbed quantities are well behaved at $r = 0$. Conversely, this choice is rich in consequences and implications. The first is that the resulting equations are particularly simple both for the polar and for the axial modes. A scrutiny of the structure of the equations for the polar modes immediately shows that it is possible to *decouple the equations describing the perturbations of the gravitational field from the equations describing the perturbations of the fluid.* As a consequence, the equations for the perturbed gravitational field can be solved with no reference to the motion that can be induced in the fluid. This is a relevant difference between our approach and the newtonian approach (or its previous relativistic generalization). In fact, due to this decoupling, the problem of finding the frequencies of the quasi-normal modes is transformed into a problem of *resonant scattering.* But in order to fully understand the physical content of the theory and its consequences, we now need to enter into the details of its mathematical formulation.

2. The equilibrium configuration

The metric for a static, spherically symmetric distribution of matter can be written in the standard form[1]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\mu_2}(dr)^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

Inside the star, the functions ν and μ_2 can be determined by solving Einstein's equations coupled to the equations of hydrostatic equilibrium. We shall assume that the star is

[1] We shall adopt the convenctions $G = c = 1$, $G_{ij} = 2T_{ij}$, and the Riemann tensor defined as in M.T. ch. 1

composed by a perfect fluid, whose energy-momentum tensor is given by

$$T^{\alpha\beta} = (p+\epsilon)u^\alpha u^\beta - pg^{\alpha\beta}, \tag{2}$$

where p and ϵ are respectively the pressure and the energy density, that are assumed to have an isotropical distribution, and u^α is the four-velocity of the fluid. By defining the mass contained inside a sphere of radius r as

$$m(r) = \int_0^r \varepsilon r^2 dr, \tag{3}$$

the relevant equations are

$$\nu_{,r} = -\frac{p_{,r}}{p+\epsilon}, \tag{4}$$

$$\left[1 - \frac{2m(r)}{r}\right] p_{,r} = -(\varepsilon + p)\left[pr + \frac{m(r)}{r^2}\right], \tag{5}$$

$$\text{and} \qquad e^{2\mu_2} = (1 - \frac{2m(r)}{r})^{-1}. \tag{6}$$

When the equation of state of the fluid is specified, eqs. (3) and (5) can be solved numerically and the distribution of pressure and energy-density through the star can be determined. Once ϵ and p are known, eq. (4) can be integrated

$$\nu = -\int_0^r \frac{p_{,r}}{(\epsilon+p)} dr + \nu_0. \tag{7}$$

The constant ν_0 is fixed by the condition that at the boundary of the star, $r = R$, the metric reduces to the Schwarzschild metric

$$(e^{2\nu})_{r=R} = (e^{-2\mu_2})_{r=R} = 1 - 2M/R, \tag{8}$$

where $M = m(R)$ is the total mass.

Outside the star the metric is the Schwarzschild metric in its standard form

$$ds^2 = \left(1 - \frac{2M}{r}\right)(dt)^2 - \left(1 - \frac{2M}{r}\right)^{-1}(dr)^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{9}$$

3. The perturbed spacetime

We shall restrict our analysis to the study of axisymmetric perturbations of a star. This assumption implies no loss of generality, since, due to the spherical symmetry of the background, non-axisymmetric modes can be deduced from axisymmetric perturbations by a suitable rotation of the polar axes (see M.T. §4.23). A line-element appropriate to describe an axially symmetric, time-dependent spacetimes is

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2. \qquad (10)$$

In the following we shall project the equations onto an orthonormal tetrad

$$e^i_{(a)} e^j_{(b)} g_{ij} = \eta_{(a)(b)}, \qquad (11)$$

where $\eta_{(a)(b)} = (1, -1, -1, -1)$.

When a star is perturbed, each element of fluid suffers an infinitesimal displacement from its equilibrium position, identified by the lagrangian displacement $\vec{\xi}$. Consequently, the metric and the thermodinamical variables change by an infinitesimal amount with respect to their unperturbed values (indicated by a bar)

$$\begin{aligned} \nu &\longrightarrow \bar{\nu} + \delta\nu \qquad & \mu_2 &\longrightarrow \bar{\mu}_2 + \delta\mu_2 \qquad & \varepsilon &\longrightarrow \bar{\varepsilon} + \delta\varepsilon \\ \psi &\longrightarrow \bar{\psi} + \delta\psi \qquad & \mu_3 &\longrightarrow \bar{\mu}_3 + \delta\mu_3 \qquad & p &\longrightarrow \bar{p} + \delta p, \end{aligned} \qquad (12)$$

and

$$\omega \longrightarrow \delta\omega \quad , \qquad q_2 \longrightarrow \delta q_2 \quad , \qquad q_3 \longrightarrow \delta q_3. \qquad (13)$$

It should be recalled that ω, q_2 and q_3 are zero in the unperturbed state. All perturbed quantities depend on t, r and θ. If we now write Einstein's equations, the hydrodynamical equations and the conservation of barion number (see Paper II §§4 and 11) we find that they decouple into two sets: the *polar modes*, involving the variables given in eqs.

(13) and the lagrangian displacement $\vec{\xi}$, and the *axial modes*, involving the off-diagonal perturbations of the metric (13). The same decoupling into axial and polar modes also occurs when a Schwarzschild black hole is perturbed. However, as we shall see in the following, polar and axial perturbations of stars behave differently.

4. The axial modes

The axial modes do not have a newtonian counterpart. They are purely gravitational modes, since they do not produce any motion in the fluid except for a stationary rotation. The equations for the axial modes are the following.

$$\delta R_{(1)(2)} = 2\delta T_{(1)(2)} \qquad \rightarrow \qquad (e^{3\psi+\nu-\mu_2-\mu_3} Q_{23})_{,\theta} + e^{3\psi-\nu-\mu_2+\mu_3} Q_{02,t} = 0, \tag{14}$$

$$\delta R_{(1)(3)} = 2\delta T_{(1)(3)} \qquad \rightarrow \qquad (e^{3\psi+\nu-\mu_2-\mu_3} Q_{23})_{,r} - e^{3\psi-\nu+\mu_2-\mu_3} Q_{03,t} = 0, \tag{15}$$

where:

$$Q_{23} = \delta q_{2,\theta} - \delta q_{3,r}, \qquad Q_{02} = \delta\omega_{,r} - \delta q_{2,t}, \qquad Q_{03} = \delta\omega_{,\theta} - \delta q_{3,t}\ .$$

Assuming that all perturbed quantities have a time dependence $e^{i\sigma t}$, eqs. (14) and (15) can easily be reduced to the following second order equation

$$(e^{-3\psi+\nu-\mu_2+\mu_3} X_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3} X_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3} X = 0, \tag{16}$$

where we have put

$$e^{+3\psi+\nu-\mu_2-\mu_3} Q_{23} = X. \tag{17}$$

Equation (16) can be separated by expanding the function X in terms of the Gegenbauer polynomials $C_n^{\nu}(\theta)$, defined by the equation

$$\left[\frac{d}{d\theta}\sin^{2\nu}\theta\frac{d}{d\theta} + n(n+2\nu)\sin^{2\nu}\theta\right] C_n^{\nu}(\theta) = 0\ . \tag{18}$$

By introducing a new radial variable r_* defined as

$$r_* = \int_0^r e^{-\nu+\mu_2} dr, \tag{19}$$

and putting

$$X = rZ(r)C_{\ell+2}^{-\frac{3}{2}}(\theta), \tag{20}$$

eq. (16) reduces to the following radial equation

$$\frac{d^2Z}{dr_*^2} + [\sigma^2 - U(r)]Z = 0, \tag{21}$$

where

$$U(r) = \frac{e^{2\nu}}{r^3}[\ell(\ell+1)r + r^3(\varepsilon - p) - 6m(r)] \ . \tag{22}$$

Outside the star ϵ and p are zero and eq. (22) reduces to the Regge-Wheeler potential (Regge & Wheeler 1957).

Thus *the axial modes are completely described by the Schroedinger equation (21), valid from the center of the star to radial infinity, with a potential barrier (22), that depends on how the energy-density and the pressure are distributed in the interior of the unperturbed star.*

Given a model of star, solution of the equilibrium equations, eq. (21) can be integrated numerically. The solution free of singularities at the origin has the expansion

$$Z \sim r^{l+1} + \frac{1}{2(2l+3)}\{(l+2)[\frac{1}{3}(2l-1)\epsilon_0 - p_0] - \sigma_0^2\}r^{l+3} + \dots \ , \tag{23}$$

where ϵ_0 and p_0 are the values of the energy-density and of the pressure at the center of the star, and $\sigma_0 = e^{2\nu_0}\sigma$. The asymptotic behaviour of the function Z when $r_* \to \infty$ is

$$\begin{aligned} Z \quad \to \quad & + \quad \{\alpha - \beta\frac{n+1}{\sigma r} - \frac{1}{2\sigma^2}[n(n+1)\alpha - 3M\sigma\beta]\frac{1}{r^2} + \dots\} \cos\sigma r_* \\ & - \quad \{\beta + \alpha\frac{n+1}{\sigma r} - \frac{1}{2\sigma^2}[n(n+1)\alpha + 3M\sigma\beta]\frac{1}{r^2} + \dots\} \sin\sigma r_*. \end{aligned} \tag{24}$$

α and β, which will play a relevant role in the following development of the problem, are functions of σ , and can be determined by matching the solution obtained by numerical integration of eq. (21), with the asymptotic behaviour (24).

Since the axial modes are described by the Schroedinger equation (21), the problem of studying the axial perturbations of a spherical star is a problem of *pure scattering* in a spherically symmetric, static potential. Therefore we can apply the methods developed in the framework of quantum mechanics in the context of the classical theory of relativity. We can assume that the star is perturbed by an incident gravitational wave of arbitrary frequency, and study the response of the star by evaluating how much of the incident wave will be transmitted or reflected by the potential barrier, in the same way in which the properties of a nucleus described by a Schroedinger equation are investigated by scattering waves of different energy on its potential barrier.

A relevant question which emerges at this point is whether the scattering is, in our context, resonant. If it is not resonant, the star will simply behave as a center of elastic scattering for incident gravitational radiation. Conversely, if it is resonant, the star will be able to emit gravitational waves with frequencies equal to the characteristic resonance frequencies. An extensive answer to this question will be given in section 9.

5. The polar modes

The polar modes couple the perturbations of the diagonal part of the metric (10) $(\delta\nu, \delta\psi, \delta\mu_2, \delta\mu_3)$, with the perturbations of the energy density $\delta\epsilon$, of the pressure δp, and the lagrangian displacement $\vec{\xi}$. In contraxt with the case of the axial modes, polar modes do excite the motion of the fluid that composes the star.

We shall assume that the perturbations take place adiabatically, i.e., that the changes in the pressure and in the energy-density arise without dissipation.

The equations which describe the polar perturbations are the Einstein equations, the hydrodynamical equations, and the conservation of barion number. Since we are mainly interested in showing the results of the new theory, we shall omit the esplicit derivation of these equations which can be found in Paper I and II. Here we only remark that the relevant equations can be separated by performing the following substitutions in terms of the Legendre polynomials, P_l, and their derivatives:

$$\delta\nu = N(r)P_\ell(\cos\vartheta) \qquad \delta\mu_2 = L(r)P_\ell(\cos\vartheta) \tag{25}$$
$$\delta\mu_3 = T(r)P_\ell + V(r)P_{\ell,\vartheta,\vartheta} \qquad \delta\psi = T(r)P_\ell + V(r)P_{\ell,\vartheta}\cot\vartheta \ ,$$

(cf. M.T. p. 147, eqs. (36)-(39) originally due to J.Friedman), and

$$\delta p = \Pi(r)P_\ell(\cos\vartheta) \qquad 2(\varepsilon + p)e^{\nu+\mu_2}\xi_2(r,\vartheta) = U(r)P_\ell$$
$$\delta\varepsilon = E(r)P_\ell(\cos\vartheta) \qquad 2(\varepsilon + p)e^{\nu+\mu_3}\xi_3(r,\vartheta) = W(r)P_{\ell,\vartheta} \ , \tag{26}$$

where ξ_2 and ξ_3 are respectively the r and θ tetrad-components of the lagrangian displacement. After the separation, we are left with a system of coupled equations involving the following variables: $N(r), L(r), T(r), V(r)$, which describe the radial part of the perturbation of the metric, and $U(r)$, $W(r)$, $\Pi(r)$, $E(r)$, which describe the radial part of the perturbation of the fluid. The resulting equations are:

$$\left[\frac{d}{dr}+\left(\frac{1}{r}-\nu_{,r}\right)\right](2T-kV)-\frac{2}{r}L = -U, \tag{27}$$

$$(T-V+N)_{,r}-\left(\frac{1}{r}-\nu_{,r}\right)N-\left(\frac{1}{r}+\nu_{,r}\right)L = 0, \tag{28}$$

$$\frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}N_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(2T-kV)_{,r}-\frac{2}{r}\left(\frac{1}{r}+2\nu_{,r}\right)L\right]+ \frac{1}{2}\left[-\frac{1}{r^2}(2nT+kN)+\sigma^2e^{-2\nu}(2T-kV)\right] = \Pi, \tag{29}$$

$$V_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)V_{,r}+\frac{e^{2\mu_2}}{r^2}(N+L)+\sigma^2e^{2\mu_2-2\nu}V = 0, \tag{30}$$

$$W = -(T-V+L), \tag{31}$$

$$\Pi = -\frac{1}{2}\sigma^2e^{-2\nu}W-(\varepsilon+p)N, \tag{32}$$

$$U = \frac{\left[\frac{1}{2}(\sigma^2e^{-2\nu}W)_{,r}+(Q+1)\nu_{,r}\frac{1}{2}(\sigma^2e^{-2\nu}W)+(\varepsilon_{,r}-Qp_{,r})N\right]}{\frac{1}{2}\left[\sigma^2e^{-2\nu}+\frac{e^{-2\mu_2}\nu_{,r}}{\varepsilon+p}(\varepsilon_{,r}-Qp_{,r})\right]}, \tag{33}$$

$$E = Q\Pi+\frac{e^{-2\mu_2}}{2(\varepsilon+p)}(\varepsilon_{,r}-Qp_{,r})U, \tag{34}$$

where

$$k=\ell(\ell+1), \qquad 2n=(\ell-1)(\ell+2)=k-2, \qquad Q=\frac{(\epsilon+p)}{\gamma p}, \tag{35}$$

and

$$\gamma=\frac{(\varepsilon+p)}{p}\left(\frac{\partial p}{\partial\varepsilon}\right)_{entropy=const}\ ; \tag{36}$$

is the adiabatic exponent (defined in Paper I, equation (106)).

One can immediately recognize that eqs. (31)-(34) give the fluid variables as a combination of the metric perturbations T, V, L, and N. Therefore, if we replace the expressions of U and Π given by eqs. (33) and (32) on the right-hand side of eqs. (27) and (29), we are left with a system of equations which involves only the perturbations of the metric functions (T, V, L, N)!

It should be stressed that the decoupling of the equations governing the metric perturbations from the equations governing the hydrodynamical variables is possible in general, and requires no assumptions on the equation of state of the fluid.

We are therefore in a situation totally different from the newtonian case: *we can solve the equations for the perturbations of the metric independently on the motion which is induced in the fluid.*

Outside the star the variables related to the fluid, Π and U, vanish and the system of equations (27)-(30) can be reduced to a single Schroedinger equation (the Zerilli equation (Zerilli 1972a,b)) with an associated potential barrier

$$\left(\frac{d^2}{dr_*^2}+\sigma^2\right) Z = VZ, \tag{37}$$

where the function Z is defined as

$$Z=\frac{r}{nr+3M}\left(\frac{3M}{n}X-rL\right) \ , \tag{38}$$

and

$$V(r)=\frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3+3Mn^2r^2+9M^2nr+9M^3] \ . \tag{39}$$

The radial variable r_* is the 'tortoise' coordinate

$$r_* = r+2M\log(\frac{r}{2M}-1). \tag{40}$$

We now want to integrate the perturbation equations both inside and outside the star.

(a) The integration of the equations

In order to numerically integrate the *decoupled* system for (T,V,L,N) in the interior of the star (we assume that the aforementioned substitution for U and Π in eqs. (27) and (29) has been performed), we need to find the behaviour of these functions near $r=0$.

We can seek a power series solution of the type

$$(T,V,L,N) \sim (T_0,V_0,L_0,N_0)r^x + O(r^{x+2}), \tag{41}$$

where x is an exponent to be determined, but if we substitute these expressions into the equations we discover that the system is linearly dependent near the origin. This difficulty can be circumvented by introducing a suitably defined new variable. For the sake of simplicity, in the following we shall restrict our consideration to the case when the fluid obeys a *barotropic* equation of state, i.e. when the pressure is a unique function of the energy density, $p = p(\epsilon)$. In this case $Q = \frac{\epsilon_{,r}}{p_{,r}}$, and the equations considerably simplify. We shall replace the variable T by the new variable G defined as

$$\begin{aligned} G = \nu_{,r}[\frac{n+1}{n}X - T]_{,r} \quad &+ \quad \frac{1}{r^2}(e^{2\mu_2}-1)[n(N+T)+N] + \frac{\nu_{,r}}{r}(N+L) - \\ &- \quad e^{2\mu_2}(\varepsilon+p)N + \frac{1}{2}\sigma^2 e^{2(\mu_2-\nu)}[L - T + \frac{2n+1}{n}X]\,, \end{aligned} \tag{42}$$

and the variable V by $X = nV$.

The final set of equations we shall integrate is

$$X_{,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)X_{,r} + \frac{n}{r^2}e^{2\mu_2}(N+L) + \sigma^2 e^{2(\mu_2-\nu)}X = 0, \tag{43}$$

$$(r^2G),r = n\nu_{,r}(N-L) + \frac{n}{r}(e^{2\mu_2}-1)(N+L) + r(\nu_{,r}-\mu_{2,r})X_{,r} + \sigma^2 e^{2(\mu_2-\nu)}rX\ , \tag{44}$$

$$\begin{aligned} -\nu_{,r}N_{,r} \quad &= -G + \nu_{,r}[X_{,r} + \nu_{,r}(N-L)] + + \frac{1}{r^2}(e^{2\mu_2}-1)(N - rX_{,r} - r^2G) \\ &-e^{2\mu_2}(\varepsilon+p)N + \frac{1}{2}\sigma^2 e^{2(\mu_2-\nu)}\left\{N + L + \frac{r^2}{n}G + \frac{1}{n}[rX_{,r} + (2n+1)X]\right\}, \end{aligned} \tag{45}$$

$$\begin{aligned} -L_{,r} = (N+2X)_{,r} \quad &+ \left(\frac{1}{r} - \nu_{,r}\right)(-N+3L+2X) + \\ &+ \left[\frac{2}{r} - (Q+1)\nu_{,r}\right]\left[N - L + \frac{r^2}{n}G + \frac{1}{n}(rX_{,r} + X)\right]\ . \end{aligned} \tag{46}$$

This is a fifth-order linear differential system. It has been shown (Price & Ipser 1991) that it can be reduced to a fourth order system, however we prefer not to use that reduction because our equations are considerably simpler.

The system of equations (43)-(46), which involves only the perturbations of the gravitational field, can now be integrated from the center to the surface of the star, in the following way. As before, we shall assume that, near the origin, the functions have the asymptotic expansion

$$(X, G, N, L) = (X_0, G_0, N_0, L_0) r^x + (X_2, G_2, N_2, L_2) r^{x+2} \ , \tag{47}$$

where both the exponent x and the coefficients of the expansion have to be determined by inserting eq. (47) into equations (43)-(46), and by setting to zero the coefficients of different powers of r. From the lower order terms we obtain a homogeneous algebraic system of four equations for the four coefficients (X_0, G_0, N_0, L_0)

$$\begin{aligned}
& x(x+1)X_0 + n(L_0 + N_0) = 0 \qquad (48)\\
& [(a-b)x + \sigma_0^2]X_0 - (x+2)G_0 + n(a+b)N_0 - n(a-b)L_0 = 0 \\
& \left[(a-b)x + \frac{\sigma_0^2}{2n}(x+2n+1)\right] X_0 - G_0 + a(x-1)N_0 + \frac{1}{2}\sigma_0^2 N_0 + \frac{1}{2}\sigma_0^2 L_0 = 0 \\
& 2\left[x\left(\frac{n+1}{n}\right) + 2\right] X_0 + (x+1)(N_0 + L_0) = 0,
\end{aligned}$$

where a and b are the coefficients of the expansion of the metric functions

$$e^{2\mu_2} \sim 1 + br^2 = 1 + (\frac{2}{3}\epsilon_0)r^2, \qquad e^{2\nu} \sim 1 + ar^2 = 1 + (p_0 + \frac{1}{3}\epsilon_0)r^2. \tag{49}$$

The system (48) admits a non-trivial solution only if the determinant is zero. This condition provides the indicial equation for the determination of x

$$na(x+1)(x-\ell)^2(x+\ell+1)^2 = 0 \ . \tag{50}$$

Surprisingly, we see that there are only two coincident values of x which correspond to regular solutions, i.e. $x = l$. That means that, although our original system is of order five, only two independent solutions are acceptable. This is a great simplification with respect to the old theory, where four independent solutions had to be integrated through the star and then matched in order to satisfy the boundary conditions. In selecting the admissible values of x, we eliminate the extra degree of freedom due to our incomplete gauge specification. A possible choice for the two independent solution is

$$1) \qquad L_0 = 0, \qquad N_0 = 1, \qquad X_0 = -\frac{n}{\ell(\ell+1)} N_0, \tag{51}$$
$$G_0 = +\frac{1}{2}\ell - 1) \left\{ a + b - \frac{1}{\ell(\ell+1)}[(a-b)\ell + \sigma_0^2] \right\} N_0 \ ,$$

$$2) \qquad N_0 = 0 \qquad L_0 = 1, \qquad X_0 = -\frac{n}{\ell(\ell+1)} L_0, \tag{52}$$
$$G_0 = -\frac{1}{2}\ell - 1) \left\{ a - b + \frac{1}{\ell(\ell+1)}[(a-b)\ell + \sigma_0^2] \right\} L_0 \ .$$

The coefficients (X_2, G_2, N_2, L_2) in the expansion (47), can be found by equating to zero the coefficients of the next power of x into the expanded equations.

We can now numerically integrate eqs. (43)-(46), with the initial conditions (51)-(52). It remains to be ascertained whether two independent solutions are sufficient to satisfy the boundary conditions required by the problem. As in the newtonian case, we need to impose that the perturbation of the pressure δp vanishes at the surface $r = R$, but in addition we need to impose that the interior solution joins continuously with the solution in the exterior of the star. In order to satisfy the continuity condition at $r = R$, eqs. (27)-(30), which are equivalent to eqs. (43)-(46), must reduce to those appropriate to the vacuum, and therefore it must be

$$\Pi = 0, \qquad \text{and} \qquad U = 0. \tag{53}$$

The vanishing of δp at the boundary is included in the first of eqs. (53), since Π is the radial part of δp (see eq. (26)). Since we are solving the barotropic case, from eqs. (32) and (33) it follows that

$$\Pi = -\frac{1}{2}\sigma^2 e^{-2\nu} W - (\varepsilon + p) N, \quad \text{and} \quad U = W_{,r} + (Q-1)\nu_{,r} W \ . \tag{54}$$

For a fluid star ϵ and p tend to zero at the boundary. Moreover

$$Q = \frac{\epsilon_{,r}}{p_{,r}}, \quad \rightarrow \quad \frac{Q_1}{(R-r)}, \quad \nu_{,r} \quad \rightarrow \quad \nu_1', \quad \text{and} \quad W \quad \rightarrow \quad (R-r) W_1 e^{\alpha(R-r)} \ , \tag{55}$$

where Q_1, ν_1', W_1 and α are constant. Since ϵ, p and W tend to zero, from equation (54) it follows that the first condition, $\Pi = 0$, is automatically satisfied by any independent solution! Conversely, from eqs. (55) it follows that U tends to a constant value

$$U \sim W_1 + \nu_1' Q_1 W_1 = \text{const} \ , \tag{56}$$

and we need to consider a linear combination of the two independent solutions in such a way that the remaining condition, $U = 0$, is satisfied at the boundary. Therefore the two degrees of freedom given by eq. (51) and (52) are precisely what we do need to match the interior and the exterior solution, and to satisfy the condition $\delta p = 0$.

Now the strategy of integration is clear: we integrate the two independent solutions of eqs. (43)-(46) for the metric perturbations, with the initial conditions (51) and (52). Then we linearly superimpose the two solutions in such a way that at the boundary $U = 0$. At this point we have the values of $X, L, X_{,r}$ and $L_{,r}$ at $r = R$, and we can construct the functions $Z(R)$ and $Z_{,r_*}(R)$ given by

$$Z(R) = \lim_{r \rightarrow R-0} \frac{r}{nr + 3M}\left(\frac{3M}{n} X - rL\right), \qquad Z_{,r_*}(R) = \left(1 - \frac{2M}{R}\right) \lim_{r \rightarrow R-0} Z_{,r}(r). \tag{57}$$

With these initial values, equation (37) can be integrated. The asymptotic behaviour of the function Z for large r is

$$Z \rightarrow \ + \ \left\{\alpha - \frac{n+1}{\sigma}\frac{\beta}{r} - \frac{1}{2\sigma^2}\left[n(n+1)\alpha + -\frac{3}{2} M\sigma \left(1 + \frac{2}{n}\right)\beta\right]\frac{1}{r^2} + ...\right\} \cos \sigma r_* \tag{58}$$

$$- \quad \left\{ \beta + \frac{n+1}{\sigma} \frac{\alpha}{r} - \frac{1}{2\sigma^2} \left[n(n+1)\beta + + \frac{3}{2} M\sigma \left(1 + \frac{2}{n}\right) \alpha \right] \frac{1}{r^2} + ... \right\} \sin \sigma r_* \ ,$$

where, as in the axial case, α and β are functions of σ to be determined by matching the integrated solution with the asymptotic behaviour (58). The solution is now complete.

The consequences of the decoupling

In this section we have shown how to construct the solution for the polar modes by solving a system of equations that do not involve the variables which describe the perturbed fluid: they can be found, if required, from eqs. (31)-(34) in terms of the metric perturbations by simple algebraic relations. We therefore concentrate our attention on the perturbations of the gravitational field with no reference to the motion of the fluid, and, again, we can treat the problem as a scattering problem. This is a relevant result that does not have a counterpart in the newtonian theory.

A counterpart has to be found in the theory of perturbations of a Schwarzschild black hole. In that case, both the polar and the axial modes are governed by a Schroedinger equation, and the problem is manifestely a scattering problem: incident gravitational waves are scattered by the curvature of the spacetime. The analogy is immediate in the case of the stellar axial modes which, as we have seen, are also described by a unique Schroedinger equation. In that case the potential barrier is generated by the curvature of the spacetime produced by the particular distribution of energy density and pressure inside the star.

In the case of the polar modes we do not have a simple problem in potential scattering, as it was in the case of the axial modes. Here a Schroedinger equation holds only in the exterior of the star, and a much more complicated fifth-order system must be solved in the interior. However we can still imagine that the perturbation is originated by an incident polar gravitational wave, and that the incoming wave drives the fluid pulsations which emit the scattered component of the wave.

The consequences of this new viewpoint will be manifest in the next sections where we shall develop a very simple algorithm to find the frequencies of the quasi-normal modes, and a method to evaluate how the gravitational energy flows through the star and in the exterior. Another element of interest in this theory is the remarkable simplification of the problem: only two independent solutions are needed to find the complete solution and satisfy the boundary conditions.

6. An algorithm to find the frequencies of the quasi-normal modes

In Paper V we have developed a method to determine the complex characteristic frequencies of the quasi-normal modes, which is based on the scattering nature of the problem. We shall now formulate the theory in general, and then specify how it can be applied to the axial and the polar modes. Let us consider a Schroedinger equation

$$\frac{d^2 Z_c}{dr_*^2} + (\sigma^2 - V) Z_c = 0, \tag{59}$$

where V is a spherically symmetric, short-range potential barrier , i.e. $V < o(r_*^{-1})$ for $r_* \to \infty$. We want to find the complex values of the frequency such that the corresponding solution of equation (59), regular at $r_* = 0$, behaves as a pure outgoing wave at radial infinity, i.e.

$$Z_c \sim e^{-i\sigma_c r_*}, \qquad \text{when} \qquad r_* \to \infty, \tag{60}$$

where $\sigma_c = \sigma + i\sigma_i$ and $Z_c = Z + iZ_i$. By separating the real and the imaginary part in eq. (59), we find

$$\frac{d^2 Z}{dr_*^2} - VZ + (\sigma^2 - \sigma_i^2) Z - 2\sigma\sigma_i Z_i = 0, \tag{61}$$

$$\frac{d^2 Z_i}{dr_*^2} - VZ_i + (\sigma^2 - \sigma_i^2) Z_i - 2\sigma\sigma_i Z = 0. \tag{62}$$

We shall assume that $\sigma_i \ll \sigma$. In our context this condition implies that the decay time of the emission of gravitational waves, $\tau = \frac{1}{\sigma_i}$, is much longer than the real part of the frequency σ, a condition which is always satisfied for stars (only for black holes σ_i is comparable with σ). If we now put $Z_i = \sigma_i Y$, and neglect the terms of order $O(\sigma_i^2)$ in eqs. (61) and (62), they become

$$\frac{d^2Z}{dr_*^2} + (\sigma^2 - V)Z = 0, \tag{63}$$

$$\frac{d^2Y}{dr_*^2} + (\sigma^2 - V)Y + 2\sigma Z = 0\ . \tag{64}$$

From eq. (64) it follows that

$$Y(r_*, \sigma) = \frac{\partial}{\partial\sigma} Z(r_*, \sigma)\ , \tag{65}$$

and consequently

$$Z_c(r_*, \sigma_c) = Z(r_*, \sigma) + i\sigma_i \left[\frac{\partial}{\partial\sigma} Z(r_*, \sigma)\right]. \tag{66}$$

Therefore when $\sigma_i << \sigma$, we can construct the complex solution Z_c corresponding to a complex value of the frequency σ_c, by integrating only equation (63) *for the real part Z, and for real values of the frequency σ.*

The asymptotic behaviour of Z_c

When $r_* \to \infty$, the potential V tends to zero and eq. (63) admits two linearly independent solutions Z_1 and Z_2 which have the following asymptotic behaviour

$$Z_1 \to \cos\sigma r_* + O(r_*^{-1}), \qquad Z_2 \to \sin\sigma r_* + O(r_*^{-1})\ .$$

Thus the general real solution Z is

$$Z(r_*, \sigma) = \alpha(\sigma) Z_1(r_*, \sigma) - \beta(\sigma) Z_2(r_*, \sigma), \tag{67}$$

where $\alpha(\sigma)$ and $\beta(\sigma)$ are functions to be determined by matching eq. (67) with the integrated solution of eq. (63) for different initially assigned values of real σ. From eq. (65) and (66) it follows that the complete solution for Z_c, up to terms of order $O(\sigma_i^2)$ is

$$Z_c = Z + i\sigma_i \frac{\partial Z}{\partial \sigma} = \alpha(\sigma) Z_1 - \beta(\sigma) Z_2 - i\sigma_i [\alpha'(\sigma) Z_1 - \beta'(\sigma) Z_2 + \alpha(\sigma) Z_1' - \beta(\sigma) Z_2'] \ , \quad (68)$$

where the prime indicates differentiation with respect to σ. For sufficiently large values of r_* the behaviour of Z_c is

$$Z_c \rightarrow (\alpha + i\sigma_i\alpha' - i\sigma_i\beta r_*) \cos \sigma r_* - (\beta + i\sigma_i\beta' - i\sigma_i\alpha r_*) \sin \sigma r_*. \quad (69)$$

It is clear that the terms proportional to r_* would eventually diverge if $r_* \rightarrow \infty$. However, in the limit $\sigma_i << \sigma$, the asymptotic behaviour (67) that we use to determine α and β, is established long before these terms begin to dominate. Therefore, if the value of r_* where we start to match the integrated real solution Z with the asymptotic behaviour, is large enough that eq. (67) can be applied, but not so far that the exponential growth has taken over in eq. (69), the diverging terms can be neglected, and the asymptotic form of Z_c can be written as

$$\begin{aligned} Z_c \rightarrow & \frac{1}{2}[(\alpha - \sigma_i\beta') + i(\beta + \sigma_i\alpha')]e^{i\sigma r_*} + \frac{1}{2}[(\alpha + \sigma_i\beta') - i(\beta - \sigma_i\alpha')]e^{-i\sigma r_*} \\ & = I(\sigma)e^{+i\sigma r_*} + O(\sigma)e^{-i\sigma r_*}. \end{aligned} \quad (70)$$

(That such value of r_* does indeed exist has been shown be a direct verification in Paper V). We now impose the outgoing wave condition, by setting to zero the coefficient of the ingoing wave, $I(\sigma)$, in eq. (70)

$$\alpha - \sigma_i\beta' = 0, \qquad \text{and} \qquad \beta + \sigma_i\alpha' = 0 \ . \quad (71)$$

Eliminating σ_i we finally find

$$\alpha\alpha' + \beta\beta' = 0 \ . \quad (72)$$

This equation says that if there exists a value of real σ, say $\sigma = \sigma_0$, where the function $(\alpha^2 + \beta^2)$ has a minimum, then the solution Z_c at infinity will represent a pure outgoing wave. Therefore σ_0 is the real part of the complex characteristic frequency belonging to a quasi-normal mode. The imaginary part can be obtained from eqs. (71) evaluated at $\sigma = \sigma_0$

$$\sigma_i = i\,\frac{\alpha}{\beta'}\bigg|_{(\sigma=\sigma_0)} = -\frac{\beta}{\alpha'}\bigg|_{(\sigma=\sigma_0)} . \tag{73}$$

Equation (72) suggests an alternative method to find σ_i. Since the function $(\alpha^2 + \beta^2)$ has a minimum when $\sigma = \sigma_0$, in the region $\sigma \sim \sigma_0$ it can be approximated by a parabola

$$\alpha^2 + \beta^2 = const\left[(\sigma - \sigma_0)^2 + \sigma_i^2\right] \tag{74}$$

and σ_i can be determined by matching the values of $(\alpha^2 + \beta^2)$ obtained by numerical integration, with eq. (74).

The application of the algorithm we have described to the axial modes is straightforward. We integrate the Schroedinger equation (21) with the initial conditions (23) for different values of *real* σ. For sufficiently large r_*, we match the integrated solution with the asymptotic behaviour of Z given in eq. (24) and determine the values of α and β. Then we find the values of $\sigma = \sigma_0$ where the resonance curve $(\alpha^2 + \beta^2)$ has a minimum (if they exist): σ_0 will be the real part of the eigenfrequency. The imaginary part will be found from eq. (73), or alternatively, by fitting the resonance curve with the parabola (74). The same procedure can be applied in the case of the polar modes. The difference with respect to the axial case is that inside the star we need to integrate the system of equations (43)-(46) in the manner described in section 5. The purpose is to find the initial values for the function Z at the boundary of the star, which are needed to integrate the Schroedinger equation (37) outside the star. At sufficiently large values of r_*, the integrated solution will be matched with the asymptotic behaviour (58), and α and β will be determined. We shall then proceed as in the axial case.

To conclude this section we would like to stress the basic difference that exist between the newtonian and the relativistic approach to the problem of finding the frequencies of the normal (quasi-normal in the relativistic case) modes. In the newtonian theory one has to solve an eigenvalue problem associated to a system of equations *which couple the perturbations of the fluid with the perturbations of the gravitational field.* In the relativistic theory we solve a problem of *resonant scattering of gravitational waves by a potential barrier.* The implications of the analogy with resonant scattering in quantum mechanics will be further discussed in the next sections.

7. Some further analogies between oscillations of stars and resonant scattering in quantum mechanics

There is clearly a strong resemblance between eq. (74) and the Breit-Wigner formula

$$\text{cross-section} = \frac{const}{(E - E_0)^2 + \frac{1}{4}\Gamma^2} \tag{75}$$

(see for example Landau & Lifschitz 1977, pp.603-611) used in atomic and nuclear physics, and it is interesting to clarify this analogy. In the context of quantum mechanics, resonant scattering occurs when a system is in a quasi-stationary state that decays, as for example a radioactive nucleus which emits an α-particle with energy E_0 and lifetime $\tau = \frac{\hbar}{\Gamma}$. The Schroedinger equation appropriate to that problem is

$$\frac{d^2 Z}{dr_*^2} + (E - V)Z = 0, \tag{76}$$

and one assumes that Z is *an analytic function of the complex energy* E. (In the notation of this paper $\sigma = const\sqrt{E}$, and the constant of proportionality is real and positive.) The complex plane is cut along the positive real E-axis in order to make Z a single valued function. The asymptotic solution for large values of r_* is of the form

$$Z(E) \sim I(E)e^{+i\sqrt{E}r_*} + O(E)e^{-i\sqrt{E}r_*}, \tag{77}$$

and if E is real and positive $O(E) = I^*(E)$, and $Z(E)$ is real. The scattering amplitude follows in the usual way

$$S_l = e^{2i\delta_l} = (-1)^{l+1}(I^*/I), \tag{78}$$

where l is the angular momentum associated to the order of the Legendre polynomial, and δ_l is the phase-shift. A quasi-stationary state corresponds to a zero of the function $I(E)$ (or to a pole of the scattering amplitude S_l), where the corresponding asymptotic wave function (77) reduces to a pure outgoing wave. In order to obtain the Breit-Wigner formula, one *postulates* the existence of a pole lying close to the positive real axis, at some *complex energy* $E = E_0 - \frac{1}{2}i\Gamma$, and by expanding $I(E)$ in the vicinity of the zero

$$I(E) \sim const(E - E_0 + \frac{1}{2}i\Gamma), \tag{79}$$

the cross-section (75) immediately follows.

Let us now see what is the connection between this approach and the algorithm developed in section 6. We have shown that if the function $(\alpha^2 + \beta^2)$ has a minimum for a value of *real* E (real σ), then the amplitude of the ingoing part of the asymptotic wavefunction $I(E)$ is zero, provided $\Gamma << E$ ($\sigma_i << \sigma$). Therefore, for such value $E = E_0$, $|I(E)|^2$ must also have a minimum and

$$II^{*\prime} + I^*I' = 0 \qquad \text{or} \qquad I'/I = -(I^{*\prime}/I^*), \tag{80}$$

where the prime indicates differentiation with respect to E. Thus, apart from the trivial case $I' = 0$, (I'/I) is imaginary at E_0, say $-2i/\Gamma$. Since the logarithmic derivative is purely imaginary at E_0, we may analytically continue the function I in the complex plane and expand in the vicinity of E_0

$$I(E) \sim I(E_0)[1 + (I'/I)_{E=E_0}(E - E_0)] \sim I(E_0)[1 + 2i(E - E_0)/\Gamma]. \tag{81}$$

A comparison with eq. (79) shows that the Breit Wigner formula can now be derived by the usual procedure. Thus our approach also leads to the Breit-Wigner formula, but we have focused the attention on the amplitude of the standing wave prevailing at infinity

$$A(\sigma) = < 2Z^2 >_{av}^{\frac{1}{2}} = \alpha + i\beta, \tag{82}$$

rather than on the amplitude of the ingoing part of the wave $I(\sigma)$. It should be stressed however that, while in quantum mechanics the existence of a resonance is *postulated*, and the values of E_0 and Γ are known from experiments, in our context we provide a method to evaluate both σ_0 and σ_i.

The analogies between the theory of oscillations of stars and quantum mechanics do not end here. We shall see in the next section that a suitable generalization of the Regge theory allows to define the flow of gravitational energy through the star.

8. The flow of gravitational energy, an application of the Regge theory

The Regge theory (Alfaro & Regge 1963) is applicable to the problem of potential scattering when the wave equation is separable, and the wave function can be written in terms of a radial function and a Legendre polynomial $P_l(\cos\theta)$. In that case the radial wave equation can be written by separating explicitely the 'centrifugal' part of the potential barrier

$$\frac{d^2Z}{dr^2} + \left[\sigma^2 - \frac{l(l+1)}{r^2} - U(r)\right] Z = 0, \tag{83}$$

and $U(r)$ is a short range, central potential. The amplitude of the standing wave at infinity is now considered as a function of the frequency and of the angular momentum l

$$A(\sigma, l) = \alpha(\sigma, l) + i\beta(\sigma, l), \tag{84}$$

and it is assumed to be an analytic function in the variables σ and l, which are both assumed to be *complex*. Further, to any given pole $(\sigma_0 + i\sigma_i; l_0)$, corresponding to a fixed integral value of the angular momentum l_0, there exists a Regge pole in the complex l-plane, $(\sigma_0; l_0 + il_i)$, belonging to the same quasi-stationary state. Consequently, in the neighbourhood of (σ_0, l_0), the amplitude A can be analytically continued either in the complex σ-plane

$$A(\sigma) \sim \left[\frac{\partial A(\sigma)}{\partial \sigma}\right]_{\sigma=\sigma_0} [\sigma - (\sigma_0 + i\sigma_i)] \tag{85}$$

and

$$|A(\sigma)|^2 = \alpha^2 + \beta^2 \sim \left[\frac{\partial A(\sigma)}{\partial \sigma}\right]^2_{\sigma=\sigma_0} [(\sigma - \sigma_0)^2 + \sigma_i^2]. \tag{86}$$

or in the complex l-plane

$$A(\sigma) \sim \left[\frac{\partial A(\sigma_0; l)}{\partial l}\right]_{l=l_0} [l - (l_0 + il_i)], \tag{87}$$

and

$$(\alpha^2 + \beta^2) \sim \left[\frac{\partial A(\sigma_0; l)}{\partial l}\right]^2_{l=l_0} [(l - l_0)^2 + l_i^2]. \tag{88}$$

It is now clear that we can generalize the algorithm developed in section 6 to find the resonance in the complex σ-plane, to determine the corresponding resonances in the complex l-plane. We shall assume that $\sigma = \sigma_0$ is known and fixed, and that the angular momentum is complex

$$l_c = l + il_i. \tag{89}$$

If we assume that $|l_i| << l$, in analogy with eq. (66) the corresponding complex solution $Z_c = Z + iZ_i$, where now $Z = Z(r; \sigma_0, l)$, can be written as

$$Z_c(r; \sigma_0, l + il_i) = Z(r; \sigma_0, l) + il_i \left[\frac{\partial}{\partial l} Z(r; \sigma_0, l)\right], \tag{90}$$

and the complete complex solution Z_c can be derived from the only knowledge of the real solution $Z(r; \sigma_0, l)$. The procedure to find l_0 and l_i is therefore the same as that described in section 6 (eqs. (72), (73) and (74)), with the only difference that now the square amplitude of the standing wave at infinity ($\alpha^2+\beta^2$) has to be considered a function of real l.

Once l_0 and l_i are known, they can be substituted explicitely into eq. (83), that becomes

$$\frac{d^2 Z_c}{dr^2} + \left[\sigma^2 - \frac{l_0(l_0+1)}{r^2} - U(r)\right] Z_c = i l_i \frac{(2l_0+1)}{r^2} Z_c + O(l_i^2). \tag{91}$$

Multiplying equation (91) by Z_c^* and subtracting from the resulting equation its complex conjugate (complex conjugation is taken with respect to l_c), we find that

$$\frac{d}{dr}[Z_c, Z_c^*]_r = 2 i l_i \frac{(2l_0+1)}{r^2} |Z_c|^2, \tag{92}$$

where

$$[Z_c, Z_c^*]_r = Z_{c,r} Z_c^* - Z_{c,r}^* Z_c \tag{93}$$

is the wronskian. Since Z_i is of order l_i (cfr. eq.(90)), up to terms of order $\mathrm{O}(l_i^2)$ $|Z_c|^2 = Z^2$, and from equation (92) it follows that

$$[Z_c, Z_c^*]_r = 2 i l_i (2l_0+1) \int_0^r \frac{dr}{r^2} Z^2. \tag{94}$$

The integral on the right-hand side converges for $r \to \infty$ and it is positive definite. In quantum mechanics the non constancy of the wronskian exhibited in eq. (94) is interpreted as the emission of a new particle in the field volume. (see for example Landau & Lisfshitz 1977, bottom of page 588). The knowledge of the pole (l_0, l_i) is therefore essential to evaluate eq. (94).

The theory now described can be immediately applied to the axial modes, provided they are resonant. The fact that the radial wave equation (21) is obtained by expanding the wave-function in Gegenbauer polynomials $C_{l+2}^{-\frac{3}{2}}$, instead of Legendre polynomial $P_l(\cos\theta)$, does not affect any conclusion we have reached so far. The radial equation (21) can be rewritten in a form analogous to eq. (83):

$$\frac{d^2 Z_c}{dr_*^2} + \left[\sigma^2 - \frac{e^{2\nu}}{r^2} l(l+1) - U(r)\right] Z_c = 0, \tag{95}$$

where

$$U(r) = e^{2\nu}\left[(\epsilon - p) - \frac{6M}{r^3}\right]. \tag{96}$$

If we now assume that $\sigma = \sigma_0$ is the real part of the frequency of a quasi-normal mode previously determined, and $l = l_0 + il_i$, is the corresponding pole in the complex l-plane, equation (95) can be written in a form equivalent to eq. (91)

$$\frac{d^2 Z_c}{dr_*^2} + \left[\sigma_0^2 - \frac{e^{2\nu}}{r^2} l_0(l_0+1) - U(r)\right] Z_c = il_i(2l_0+1)\frac{e^{2\nu}}{r^2} Z_c, \tag{97}$$

where $Z_c = Z_c(r_*, \sigma_0, l_0 + il_i)$. Multiplying by Z_c^* and subtracting from the complex conjugate equation we find

$$[Z_c, Z_c^*]_{r_*} = 2il_i(2l_0+1)\int_0^{r_*} \frac{e^{2\nu}}{r^2} Z^2 dr_*. \tag{98}$$

In analogy with the interpretation of equation (94) given in the context of quantum mechanics, we can interpret the right-hand side of eq. (98) as the a measure of gravitational energy which crosses a sphere of radius r_*.

It should be stressed that in order to define the flow of energy through and outside the star *we do need* to use the Regge theory. One may ask why didn't we try to evaluate the flux by assuming l real, σ complex, and operating on eq. (95) with the function Z_c^* complex conjugate to Z_c with respect to σ. The result in that case would be

$$\frac{d}{dr_*}[Z_c, Z_c^*]_{r_*} = -4i\sigma_0\sigma_i |Z_c|^2, \tag{99}$$

where now $Z_c = Z_c(r_0, \sigma_0 + i\sigma_i, l_0)$. Consequently

$$[Z_c, Z_c^*]_{r_*} = -4i\sigma_0\sigma_i \int_0^{r_*} |Z_c|^2 dr_*. \tag{100}$$

But in this case, if we require that the solution is exponentially damped in time, i.e. $\sigma_i > 0$, the asymptotic behaviour of $|Z_c|^2$ would be

$$|Z_c|^2 \sim e^{-2\sigma_i t} e^{2\sigma_i r_*}, \qquad (101)$$

and the integral would explode. *The application of the Regge theory is therefore essential to circumvent the obstacle of the divergent integral.* The question now is whether this theory can be applied to the polar modes.

The resonant scattering of polar gravitational waves is not a conventional potential scattering. Inside the star we have to integrate a fifht-order differential system whose solution must be properly matched with the solution of the Schroedinger equation which governs the perturbations of the gravitational field outside the star. Thus the Regge theory cannot be applied in its standard form. However a generalization is possible. In Part II of Paper I (eqs. (132)-(134)) it was shown that the polar perturbations allow a conservation equation of the form

$$E_{2,2} + E_{3,3} = 0, \qquad (102)$$

where $\mathbf{E}$ is a vector we shall define, $x^2 = r$ and $x^3 = \mu = \cos\theta$. By Gauss's theorem, it follows that, if C_1 and C_2 are any two closed contours, one inside the other, in the (x^2, x^3)-plane

$$\int_{C_1} (E_2 dx^3 - E_3 dx^2) = \int_{C_2} (E_2 dx^3 - E_3 dx^2), \qquad (103)$$

provided $\mathbf{E}$ is not singular inside the area included between C_1 and C_2. If we now assume that the closed contour is a circle of radius r eq. (103) becomes

$$< E_2 > = \int_0^\pi r^2 E_2 \sin\theta d\theta = const, \qquad (104)$$

which expresses the conservation of the flux of the vector $\mathbf{E}$ across a spherical surface of radius r surrounding the star. We shall now write explicitely E_2, which is the only component of **E** relevant to our problem,

$$\begin{aligned} E_2 &= r^2 e^{\nu-\mu_2}\sin\theta\{[\delta\mu_3,\delta\mu_3^*]_2 + [\delta\psi,\delta\psi^*]_2 - [\delta\nu_{,2}\delta(\psi+\mu_3)^* - c.c] + \\ &+ [\delta\mu_2\delta(\psi+\mu_3)^*_{,2} - c.c] + \Big[2[(\epsilon+p)\delta(\psi+\mu_3-\mu_2)^* - \delta p]e^{\nu+\mu_2}\xi_2 - c.c\Big]\}. \end{aligned} \quad (105)$$

Separating the variables as in eqs. (25), (26), after some reduction we find

$$\begin{aligned} \frac{1}{4}(2l+1) &< E_2 >= -(n+1)\int_0^r e^{\nu+\mu_2}[(N+L)X^* - (N+L)^*X]dr \\ &+ e^{\nu+\mu_2}\left\{(n+1)r[(N+X)F^* - (N+X)^*F] + r^3(\Pi F^* - \Pi^* F)\right\} \\ &+ r^2 e^{\nu-\mu_2}\left\{\frac{1}{2}r\nu_{,r}(UF^* - U^*F) + \frac{1}{2(\epsilon+p)}(\Pi U^* - \Pi^* U)\right\}, \end{aligned} \quad (106)$$

where we have defined $F = L+X+W$. Equation (106) has been formally derived from the equations describing the polar perturbations, by considering a solution $(\delta\psi, \delta\mu_2, \delta\mu_3, \delta\nu, \xi_2)$, and the complex conjugate solution of the same equations, for *real* σ *and real* l. Under these conditions, according to eq. (104), $< E_2 >$ has the meaning of a conserved quantity. But the boundary conditions of the problem discussed in section 5 do not allow a physically meaningful complex conjugate solution for real σs and real ls. Therefore equation (106) cannot be used as it stands. However, we can assume, as we did for the axial modes, that the polar perturbations are described by functions that are analytic in the complex l-plane, and we can extend all perturbations as in eq. (88). For example, we can assume

$$N_c = N(r;\sigma_0,l_0) + il_i\left[\frac{\partial}{\partial l}N(r;\sigma_0,l)\right], \quad X_c = X(r;\sigma_0,l_0) + il_i\left[\frac{\partial}{\partial l}X(r;\sigma_0,l)\right], \quad (107)$$

and similarly for the other functions, where $N(r;\sigma_0,l)$, $X(r;\sigma_0,l)$, etc. are solutions of eqs. (43)-(46) corresponding to real $\sigma = \sigma_0$, and real l. In Paper VI we have shown that this extension is indeed possible, and that *to any pole* (σ_0, σ_i) *there exists a corresponding pole* (l_0, l_i) *belonging to the same quasi-normal mode.* Under these premises, the analytic extension of $< E_2 >$ in the complex l-plane can also be performed, and the right-hand side of equation (106) can be evaluated in terms of the extended solution (107) to give the flux of gravitational energy through the star.

9. Some consequences of the new relativistic theory

One of the major novelties introduced by the relativistic theory that we have described in the preceeding sections is that both the polar and the axial perturbations of a spherical star can be studied as a problem of scattering of incident gravitational waves by the curvature of the spacetime. However the two classes of modes differ in one important respect: the incidence of polar gravitational waves induces oscillations in the fluid, the incidence of axial gravitational waves does not. It is known from the newtonian theory that the polar modes are resonant, and frequencies of oscillation have been measured in several astrophysical contexts. Thus stars are expected to emit polar gravitational waves with these characteristic frequencies. The question now is whether the axial modes can be resonant, and, in that case, what are the frequencies of the emitted gravitational axial waves. We shall answer this question in two different context: *(a)* very compact stars, and *(b)* slowly rotating stars.

The phenomena which we are going to describe do not have any counterpart in the newtonian theory since they derive from purely general relativistic effects.

(a) The resonant behaviour of the axial modes

In order to ascertain whether the axial modes can be resonant, in Paper IV we have applied the method to find the complex eigenfrequencies developed in section 6 to a model of star with uniform energy density distribution. This model, although clearly unrealistic, presents several advantages. The equilibrium configuration is known as an exact solution of Einstein's equations (the Schwarzschild solution). Moreover, this assumption enables us to study the axial modes in a regime where the effects of general relativity are as strong as they can ever become under conditions of hydrostatic equilibrium. The unperturbed configuration (c.f.r. S.Chandrasekhar & J.C. Miller, 1974), is

$$\epsilon = constant, \qquad m(r) = \frac{\epsilon r^3}{3}, \qquad p = \frac{\epsilon(y - y_1)}{(3y_1 - y)}, \tag{108}$$
$$e^{2\nu} = \tfrac{(3y_1 - y)^2}{4}, \qquad e^{-2\mu_2} = (1 - \frac{2\epsilon r^2}{3}),$$
$$y = (1 - \tfrac{2\epsilon r^2}{3})^{\frac{1}{2}}, \qquad y_1 = (1 - \frac{2\epsilon {r_R}^2}{3})^{\frac{1}{2}}.$$

At the boundary of the star $r = R$,

$$e^{2\nu}_{(r=R)} = e^{-2\mu_2}_{(r=R)} = 1 - 2M/R, \tag{109}$$

and the metric exterior to the star reduces to the Schwarzschild metric.

Homogeneous stars can exist only if their radius R exceeds 9/8 times the Schwarzschild radius R_s, or $R/M > 2.25$. The models we shall consider in the following will be labelled by the parameter (R/M). For values of $(R/M) > 2.6$, we find that the axial modes are not resonant. The reason can be understood by plotting the potential barrier (22), computed for the model of star described in eqs. (108), as a function of r/M for different values of (R/M), as shown in fig. 1. It is known from atomic physics, that scattering by a potential barrier will exhibit resonances if the potential has a minimun followed by a maximum, and if the potential well is sufficiently deep to ensure the occurrence of quasi-stationary states. In our present context we see that only when $(R/M) < 2.6$, namely when the star becomes very compact, this condition is satisfied, and the axial modes *do become resonant.* In Table 1 it is also shown that the imaginary part of frequency dramatically tends to zero as we approch the limit $(R/M) = 2.25$. Therefore, the more compact is the star, the longer will be the time needed to damp the axial oscillation.

It is interesting to note that the axial quasi-normal modes that we have found for

Table 1: *The $l = 2$ axial resonances for homogeneous star with $\epsilon = 1$*

(M and σ are measured in the units $\epsilon^{\frac{1}{2}}$ and $\epsilon^{-\frac{1}{2}}$)

$(\frac{R}{M})$	M	σ_0	σ_i
2.26	0.509798	0.213863874	$0.23 \cdot 10^{-8}$
2.28	0.503105	0.3689962	$0.12 \cdot 10^{-5}$
2.30	0.496557	0.473525	$0.26 \cdot 10^{-4}$
2.40	0.465848	0.7767	$0.92 \cdot 10^{-2}$.

homogeneous stellar models with radii approaching the limiting radius, are not related to the Schwarzschild quasi-normal modes. We might have expected that, when the star tends to the limiting configuration, the frequencies of the quasi-normal modes would tend to those of a Schwarzschild black hole of the same mass. But, as on e can see from Table 1, this is not the case. For example, for a star with $R/M = 2.26$ we find $\tau \sim 4 \times 10^8$, while a Schwarzschild black hole of the same mass would have $\tau = 5.73$! The reason is that the nature of the scattering in the two cases (a compact star and a black hole) is different, and different are the boundary conditions associated to the problem. In the case of a star, we require that at $r = 0$ the solution is free of singularity, and that at $r = R$ the metric functions and their derivatives are continuous, *with no restrictions on the direction of the flow of radiation.* In contraxt, in the case of a black hole the only boundary condition is that at the horizon there cannot be an *outward directed wave*, and only *inward radiation* can be present. Consequently, a black hole will be characterized by a reflection and an absorbtion coefficient, while a star will behave as a center of *elastic* scattering for incident radiation. The progressive increasing of the damping time τ as the star tends to the limiting configuration means that the lowest quasi-stationary state is effectively trapped, and the star cannot radiate in that resonance frequency. In conclusion, we have shown that in extremely compact stars axial modes can become resonant. Since neutron stars are likely to have radii in the range $4 < R/M < 6$, resonant scattering of axial gravitational waves by neutron stars is not to be expected. However it is possible that these modes may be excited as transients during the gravitational collapse.

(b) The coupling of the axial and polar modes in slowly rotating stars.

The theory of non-radial oscillations of stars has been developed by assuming that the unperturbed star is static and spherically symmetric. However, all celestial objects are known to be rotating, and a generalization of the theory is needed to describe realistic situations. In Paper III we have considered the case of a star that rotates with an agular velocity Ω so slow that the distortion of its figure from spherical symmetry is of order Ω^2, and can be ignored. For compact objects, small angular velocity means

$$\Omega R << 1, \tag{110}$$

a condition which is satisfied by most realistic neutron star models. We have restricted our analysis to the axial modes of slowly rotating stars.

The metric for the unperturbed spacetime is (Hartle 1967, Chandrasekhar & Miller 1974)

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \tag{111}$$

where ν, ψ, μ_2, μ_3 differ from those of a spherical non-rotating star by quantities of order Ω^2, and ω (that is zero in the non-rotating case) is now of order Ω. The equations governing ν, ψ, μ_2, μ_3 to order zero in Ω are given in section 4. The equation determining ω is

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} - (\mu_2 + \nu)_{,r}\left(\varpi_{,r} + \frac{4}{r}\varpi\right) = 0, \tag{112}$$

where we have defined

$$\varpi = \Omega - \omega. \tag{113}$$

In the vacuum outside the star, $\mu_2 + \nu = 0$ and the solution of eq. (112) can be written as

$$\varpi = \Omega - 2Jr^{-3}, \tag{114}$$

where J is the angular momentum of the star. Both inside and outside the star ϖ is a function of r only, and the continuity of ϖ at the boundary requires that $(\varpi)_{r=R} = 6JR^{-4}$. It should be noted that the function ϖ is responsible for the dragging of inertial frames predicted by the Lense-Thirring effect.

The equations governing the perturbations of a slowly rotating star can be derived by assuming that the metric appropriate to describe the phenomenon has the same form as eq. (10). We retain the hypothesis of axisymmetric perturbations because the distortion of the unperturbed configuration from spherical symmetry due to the rotation is only of order Ω^2. However, there will be relevant changes with respect to the equations that we have derived in section 4 for the non-rotating case, since now the *unperturbed* fluid is in slow rotation with a velocity

$$v^{(\alpha)} = 0, \qquad (\alpha = 2, 3), \qquad v^{(1)} = V = e^{\psi-\nu}(\Omega - \omega) = e^{\psi-\nu}\varpi, \tag{115}$$

where $v^i = x^i_{,t}$, and $v^{(i)}$ are the tetrad components. The basic equation appropriate to describe the axial modes in the present context is

$$\begin{aligned}(e^{-3\psi+\nu-\mu_2+\mu_3}X_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3}X_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3}X \qquad (116)\\ = \varpi_{,r}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3)_{,\theta} - 4[(\epsilon + p)e^{\nu+\mu_2}\xi_2\varpi]_{,\theta} + 4[(\epsilon + p)e^{\nu+\mu_3}\xi_3\varpi]_{,r}.\end{aligned}$$

where we have made the assumption that all perturbed quantities have the *same* time-dependence $e^{i\sigma t}$, and that X is the same function defined in eq. (17). Equation (116) should be compared with eq. (16) valid in the non-rotating case. The difference is that on the right-hand side of eq. (116) in place of zero we have a combination of the perturbations $(\delta\psi, \delta\nu, \delta\mu_2, \delta\mu_3, \xi_2, \xi_3)$, that describe the *polar* modes, multiplied by ϖ and $\varpi_{,r}$.

Thus, if a star is slowly rotating the polar and the axial modes are no longer independent: they couple through the 'coupling function' ϖ that is responsible for the dragging of inertial frames.

In order to further clarify the nature of this coupling, we may expand all perturbed quantities in terms of Ω, say ($X = X^0 + \Omega X^1 + ...,\quad \delta\psi = \delta\psi^0 + \Omega\delta\psi^1 + ...,\quad etc.$). Let us consider eq. (116) at lower order in Ω. Since ϖ is of order Ω, we shall substitute to $(\delta\psi, \delta\nu, \delta\mu_2, \delta\mu_3, \xi_2, \xi_3)$, their zero order terms in Ω, i.e. $(\delta\psi^0, \delta\nu^0, \delta\mu_2^0, \delta\mu_3^0, \xi_2^0, \xi_3^0)$. Consequently, the axial perturbations X on the left-hand side of eq. (116) will be of order one in Ω (X^1):

$$
\begin{aligned}
&(e^{-3\psi+\nu-\mu_2+\mu_3} X^1_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3} X^1_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3} X^1 \qquad (117)\\
&= \varpi_{,r}(3\delta\psi^0 - \delta\nu^0 - \delta\mu_2^0 + \delta\mu_3^0)_{,\theta} - 4[(\epsilon+p)e^{\nu+\mu_2}\xi_2^0\varpi]_{,\theta} + 4[(\epsilon+p)e^{\nu+\mu_3}\xi_3^0\varpi]_{,r}.
\end{aligned}
$$

In a similar manner, the zero-order (with respect to Ω) axial perturbations X^0 will be the source for the first order polar modes, $(\delta\psi^1, \delta\nu^1, \delta\mu_2^1, \delta\mu_3^1, \xi_2^1, \xi_3^1)$, of a slowly rotating star, a case that we are not going to treat in the present paper.

Since the left-hand side of eq. (117) is the same as eq. (16), we can expand X^1 in terms of Gegenbauer polynomials (see eq. (20)). It should be stressed that $(\delta\psi^0, \delta\nu^0, \delta\mu_2^0, \delta\mu_3^0, \xi_2^0, \xi_3^0)$ are the solution of the polar equations to order zero in Ω, namely the solution appropriate to a non-rotating star that we have discussed in section 5. Therefore, the 'source term' on the right-hand side can be separated in terms of Legendre polynomials as indicated in eqs. (25)-(26). By introducing the variable r_* defined in eq. (19), and the function $Z^1 = X^1/r$, we find that eq. (117) reduces to

$$
\begin{aligned}
&\sum_{l=2}^{\infty}\left\{\frac{d^2 Z_l^1}{dr_*^2} + \sigma^2 Z_l^1 - \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon-p) - 6m(r)]Z_l^1\right\} C_{l+2}^{-\frac{3}{2}}(\mu) \qquad (118)\\
&\qquad = 6\frac{e^{2\nu}}{r^3} J(1-\mu^2)^2 \sum_{l=2}^{\infty} S_l^0(r,\mu),
\end{aligned}
$$

where

$$S_l^0 = \varpi_{,r}[(2W_l^0 + N_l^0 + 5L_l^0 + 2nV_l^0 P_{l,\mu} + 2\mu V_l^0 P_{l,\mu,\mu}] + 2\varpi W_l^0 (Q-1)\nu_{,r} P_{l,\mu}, \qquad (119)$$

and Q has been defined in eq. (35). Eq. (118) is valid from the center of the star up to radial infinity, remembering that outside the star, ϵ, p and W are zero. In order to eliminate the angular dependence in eqs. (118), we multiply by $C_{m+2}^{-\frac{3}{2}}$ and integrate over the range $\mu = \cos\theta = (-1, 1)$. Since $C_{m+2}^{-\frac{3}{2}}$, $P_{l,\mu}$ and $\mu P_{l,\mu,\mu}$ are of opposite parities, it follows that the polar modes belonging to *even* l can couple only with the axial modes belonging to *odd* l, and conversely, and it must be

$$l = m + 1, \qquad \text{or} \qquad l = m - 1. \qquad (120)$$

Moreover, a *propensity rule* is true. Due to the behaviour of the source term S_l^0 near the origin (for details, see Paper III, eqs. (61)-(63)), the transition $l \to l+1$ is strongly favoured over the transition $l \to l-1$. It is interesting to note that these 'coupling rules' are known in atomic theory: the first is the Laporte rule, while the propensity rule has been formulated (Fano,1985) in the context of light absorption. Once again, we are dealing with a phenomenon in general relativity that has a counterpart in the theory of quantum mechanics.

The problem which we have formulated is essentially a two-channel problem, the two channels being the axial and the polar modes, and it is clear that a whole range of problems with different initial conditions can be formulated. We have seen that in general the axial modes of a non-rotating star *are not resonant*, unless the star is extremely compact. Conversely, the polar modes are *always* resonant. In a slowly rotating star the axial and the polar modes couple in the manner that we have now described, and it is interesting to ask whether, due to this coupling, the axial modes may exhibit resonances. To answer this question we consider the following situation. Suppose that a polar gravitational wave of frequency σ excites the star in its quadrupole polar mode $l = 2$. If the star is slowly rotating, the polar perturbation of order zero in Ω, (the same as if the star were non-rotating), will act as a source for the axial perturbation with $m = 3$, according to the Laporte and the propensity rule, as shown in eqs. (118) and (119). We can solve eq. (118) and find the values of σ for which the solution at infinity reduces to a pure outgoing wave. All the methods developed in the previous sections can now be applied, since at infinity the right-hand side of eq. (118) goes to zero at least as fast as r^{-3}, and the wave equation tends to a homogeneous Schroedinger equation. As an example, in Paper III we have applied this procedure to a politropic model of star, with a politropic index $n = 1.5$, for different values of the angular velocity Ω. For this star the axial modes were not resonant in the non-rotating case. We have found that when the star does rotate *the axial modes become resonant. Their resonances are different from that of the polar modes, and in particular, the damping times are considerably longer (hundred times longer in the example we have considered).* Thus, in a slowly rotating star, the axial modes are resonant even if the star is not extremely compact, and this resonant behaviour is a consequence of the coupling between the polar and the axial modes, that is induced by the dragging of inertial frames.

10. Concluding remarks

The idea that certain types of variable stars owe their variability to periodic oscillations, originally due to Shapley (1914), received a first mathematical formulation in 1919 (Eddington 1919*a,b*). Since then, stellar pulsations have been studied both in the framework of the newtonian theory, and in general relativity, and one might think that nothing new can be said on the subject. However, if the search is focused on those phenomena that are of pure relativistic origin, some new interesting effects emerge which disclose the original content of the theory of general relativity.

A first result of this approach is a totally new interpretation of the phenomenon of non-radial oscillations of stars: we have shown that it can be studied as a problem of pure scattering of gravitational waves by the curvature of the spacetime. This interpretation is straightforward for the axial modes, since they are governed by a single Schroedinger equation with a potential barrier depending on the particular distribution of energy density and pressure inside the star. In the case of the polar modes, the scattering nature of the problem emerges as a consequence of the decoupling of the equations that govern the perturbations of the gravitational field from those that describe the perturbations of the fluid.

Moreover, we have shown that, although the axial modes do not produce a pulsating motion in the fluid, they can exhibit a resonant behaviour, either if the star is non-rotating but compact enough, or if the star is slowly rotating. In this case the resonances are induced by a coupling between the polar and the axial modes due to the dragging of inertial frames.

These effects are new. They could not have been anticipated by the newtonian theory of gravity, and they were obscured in the existing relativistic treatment of the problem.

An interesting possibility follows from these results. When a Schwarzschild black hole is perturbed, both the axial and the polar modes are resonant, and they *have exactely the same resonances.* Conversely, when a star is perturbed the spectrum of the axial and the polar modes *is different.* Thus, there is a clear signature in the spectrum of the quasi-normal modes which allows to distinguish whether the emitting source is a star or a black hole. An unanbiguous identification of black holes will therefore be possible when axial and polar gravitational waves will be detected.

But perhaps one of the most interesting consequences of our approach is that it discloses analogies and correspondences between the theory of general relativity and the theory of quantum mechanics. The fact that we can evaluate the frequencies of the quasi-normal modes, and compute the flux of gravitational radiation by generalizing the Breit-Wigner and the Regge theory, or the existence of a Laporte, selection and propensity rule which govern the coupling between the axial and polar modes of a slowly rotating star, provide an example of a close interconnection between the two theories, that has remained veiled for more than fifthy years.

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FIGURE CAPTIONS

fig. 1

The potential V for $l = 2$, computed for a model of homogeneous star and for different values of the ratio R/M. The discontinuity at $r = R$ is due to the discontinuity of ϵ. The dashed lines are the values of $(\sigma_0 M)^2$ corresponding to the quasi-stationary states.