→ goo.gl/6dtNOS (Modave '13 - lecture notes)

→ goo.gl/jxJ4Nn (Bad Honnef '15 - review talk)

## INTRODUCTION

QUANTUM FIELD THEORY ⟶ PATH INTEGRAL ($\infty$-dim'l)

Usually, perturbative expansion:

- weak coupling
- asymptotic series

⤳ Borel resummation, transseries, resurgence.

(window into nonpert. physics & strong coupling)

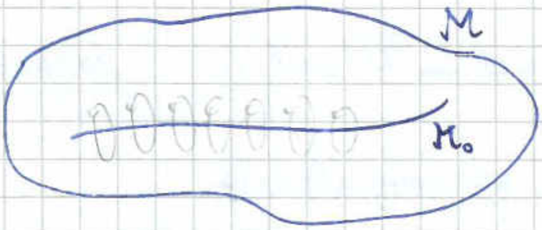
• Can we compute the path integral exactly and directly?

- Free theories
- Topological / cohomological field theories '80s - '90s
- (Rigid) SUSY THEORIES ON CURVED MANIFOLDS '00s - '10s

[Nekrasov '02
Pestun '07, ...]

KEY: LOCALIZATION (fixed point theorems)

Use (super)symmetry to reduce the dimensionality of the integration domain. Contributions only from fixed pt. locus of the symmetry
$M_0$
"LOCALIZATION LOCUS"




$\mathrm{QFT}_d$
↓
$\mathrm{QFT}_{d'<d}$
↓
$\mathrm{QFT}_0$
↓
Discrete sum

THE SEMICLASSICAL / STATIONARY PHASE APPROXIMATION IS EXACT.
(BUT FOR A MODIFIED ACTION with $\hbar_{aux}$!)

Exact results can be used to extract physical/mathematical information on the QFT (e.g. analytic properties), test/infer dualities, test general ideas in QFT.

PLAN:

- LOCALIZATION OF FINITE-DIMENSIONAL INTEGRALS
- LOCALIZATION ARGUMENT FOR SUSY QFT

Today

- LIGHTNING REVIEW OF SUSY FIELD THEORIES ON CURVED SPACE
- LOCALIZATION OF SUSY QFT's ON CURVED SPACE. (3d $\mathcal{N}=2$ on $S^3$)

Tomorrow

## REMINDER: STATIONARY PHASE APPROXIMATION

$$\boxed{Z_f(t) \equiv \int_M d^{2\ell}x \sqrt{g(x)}\, e^{it f(x)}}$$

$(M, g)$ smooth Riemannian compact $2\ell$-dim'l manifold

$f(x)$ real smooth fn

$t \to \infty$ : destructive interference unless phase $f$ is stationary.

If $f$ has isolated stationary points (Morse fn):

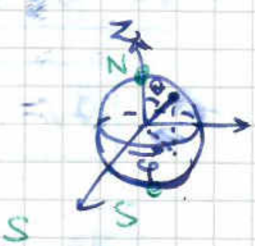
$$\boxed{Z_f(t) = \left(\frac{2\pi i}{t}\right)^{\ell} \sum_{x_k : df(x_k)=0} \underbrace{e^{itf(x_k)}}_{\text{"CLASSICAL"}} \underbrace{\frac{(-i)^{\lambda(x_k)}}{\sqrt{|\det(g^{-1}H_f(x_k))|}}}_{\text{"1-LOOP"}} + O(t^{-\ell-1})}$$

$\lambda(x_k)$ : # -ve eigenvalues of $g^{-1}H_f(x_k)$ $\quad$ (Morse index of $f$ at $x_k$)

$\hookrightarrow g^{\mu\nu}\nabla_\nu \partial_\rho f(x_k)$

* EXAMPLE : $M = S^2$ , $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

$f(\theta, \varphi) = \cos\theta = z$ "height function"




$$Z_f(t) = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, e^{it\cos\theta} \underset{t\to\infty}{=} \frac{2\pi i}{t}\Big[\underset{N}{-e^{it}\left(1+O\left(\tfrac{1}{t}\right)\right)} + \underset{S}{e^{it}\left(1+O\left(\tfrac{1}{t}\right)\right)}\Big]$$

stationary phase approx.

Evaluate $Z_f(t)$ exactly:

$$Z_f(t) = 2\pi \int_{-1}^{1} dz\, e^{itz} = \frac{2\pi}{it}\left(e^{it} - e^{-it}\right) = 4\pi \frac{\sin t}{t}$$

REMARKS : 1) sum of 2 terms, from 2 stationary pts of $f$ : STATIONARY PHASE APPROX. IS EXACT!

2) $U(1)$ symmetry $\varphi \to \varphi + c$


FIXED POINTS

The idea of LOCALIZATION was born with DUISTERMAAT - HECKMAN '82: phase space integrals where stationary phase approx. is exact.

KEY: SYMMETRY group with FIXED POINTS.

More general EQUIVARIANT LOCALIZATION THM'S. [Atiyah-Bott, Berline-Vergne, Witten '82-'84]

## ABELIAN EQUIVARIANT COHOMOLOGY AND LOCALIZATION (Cartan model)

- $M$ $2\ell$-dim'l manifold w/o boundary: $\partial M = 0$
- $V = V^\mu \partial_\mu$ vector field generating $U(1)$ "circle" symmetry ($V$ Killing vector for $(M,g)$: $\mathcal{L}_V g = \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$)
- POLYFORMS $\alpha \in \Lambda M = \{ \alpha = \sum_{n=0}^{2\ell} \alpha_n \mid \alpha_n \in \Lambda^n M \}$ ← n-form $\alpha_n = \frac{1}{n!} \alpha_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$
- V- (or U(1)-) EQUIVARIANT DIFFERENTIAL: $d_V : \Lambda M \longrightarrow \Lambda M$

$$\boxed{d_V := d - i_V}$$

$d$: exterior differential $\Lambda^n M \to \Lambda^{n+1} M$
$i_V$: contraction with $V$ $\Lambda^n M \to \Lambda^{n-1} M$

$$d_V^2 = \underbrace{d^2}_{0} + \underbrace{i_V i_V}_{0 \text{ on } \Lambda M} - (d i_V + i_V d) = -\mathcal{L}_V$$

On V-EQUIVARIANT POLYFORMS $\Lambda_V M = \{ \alpha \in \Lambda M \mid \mathcal{L}_V \alpha = 0 \}$, $d_V^2 = 0$. → differential

- $\alpha$ EQUIV. CLOSED: $d_V \alpha = 0$
- $\beta$ EQUIV. EXACT: $\beta = d_V \gamma$

- EQUIVARIANT COHOMOLOGY $\boxed{H_V^*(M) = \frac{\mathrm{Ker}\, d_V}{\mathrm{Im}\, d_V}} \ni [\alpha] = [\alpha + d_V \gamma]$, $d_V \alpha = 0$

Note: $(d_V \alpha)_n = d\alpha_{n-1} - i_V \alpha_{n+1}$ ! Recursion for equiv. closed $\alpha$: STEP=2.
relates $\alpha_{2\ell}$ to $\alpha_0$. $\alpha_{even}$, $\alpha_{odd}$ don't talk to each other.

- INTEGRAL $\boxed{\int_M \alpha := \int_M \alpha_{2l}}$ $d_V\alpha = 0$.

$$\int_M d_V\gamma \underset{\text{def}\int}{=} \int_M (d_V\gamma)_{2l} \underset{\text{def } d_V}{=} \int_M d\gamma_{2l-1} \underset{\text{Stokes}}{=} \int_{\partial M}\gamma_{2l-1} \underset{\partial M=\phi}{=} 0 .$$

(equivariant) STOKES

$$\Rightarrow \boxed{\int_M (\alpha + d_V\gamma) = \int_M \alpha}$$

$\int_M \alpha$ only depends on $[\alpha] \in H_V^*(M)$.

**AIM**: evaluate integrals of equivariantly closed polyforms over $M$

$$\int_M \alpha , \qquad [\alpha] \in H_V^*(M) .$$

$*$ $S^2$: $V = \frac{\partial}{\partial\varphi}$

$\alpha = e^{it\cos\theta}(d\varphi \wedge d\cos\theta + \frac{1}{it})$

ESSENCE OF EQUIVARIANT LOCALIZATION THM'S: [Witten '82, Berline, Vergne '83, Atiyah, Bott '84]

INTEGRALS OVER $M$ OF $d_V$-CLOSED POLYFORMS ONLY RECEIVE CONTRIBUTIONS FROM ("LOCALIZE TO") an infinitesimal V-invariant tubular neighbourhood of THE FIXED POINT LOCUS (ZERO LOCUS) of $V$: ("LOCALIZATION LOCUS")

$$M_V = \{x \in M \mid V|_x = 0\} .$$

WHY?

$$[\alpha] = [\alpha_{t_{aux}} \equiv \alpha\, e^{t_{aux} d_V\beta}] , \quad t_{aux} \in \mathbb{R} , \quad \mathcal{L}_V\beta = 0 .$$

defined by Taylor series

different representative of same cohomology class:

$$d_V\alpha_{t_{aux}} = 0 , \qquad (\alpha_{t_{aux}} - \underset{\alpha_0}{\underset{\parallel}{\alpha}}) = d_V\left(\alpha \sum_{k\geq 1} \frac{t_{aux}^k}{k!} \beta (d_V\beta)^{k-1}\right) .$$

$$\Rightarrow \int_M \alpha = \int_M \alpha\, e^{t_{aux} d_V\beta}$$

independent of $t_{aux}$!

Take $t_{aux} \to +\infty$, if lim exists ($(-d_V\beta)_0$ +ve definite):

integral dominated by minima of $-(d_V\beta)_0$.
(zeros)

- Canonical choice: $\beta = g(V,\cdot)$ 1-form dual to Killing vector $V = V^\mu \partial_\mu$

$\beta = V_\mu dx^\mu$

$$d_V\beta = d\beta - i_V\beta = \underbrace{-\|V\|^2}_{\text{0-form}} + \underbrace{dg(V,\cdot)}_{\text{2-form}}$$

$$\Rightarrow \int_M \alpha = \lim_{t\to+\infty} \int_M \alpha\, \underbrace{e^{t_{aux}\, dg(V,\cdot)}}_{\substack{\text{polynomial} \\ \text{in } t_{aux}}} \cdot \underbrace{e^{-t_{aux}\|V\|^2}}_{\substack{\text{exp} \\ \text{in } t_{aux}}} :$$

Gaussian factor increasingly peaked at $M_V \to \delta(V)$

$$\boxed{\text{INTEGRAL OVER } M \text{ LOCALIZES TO } M_V = \{x \in M \mid V|_x = 0\}}$$

(FIXED POINTS OF $U(1)$ GENERATED BY $V$)

Showing that $\int_M \alpha$ localizes to $\int_{M_V} \ldots$ was simple.

Understanding the precise localization formula requires more work, but we can do it in the case where $M_V$ consists of isolated fixed points.

Before that,

REMARK:

There is some freedom in the localization procedure:

1) Any Killing vector $V$ ($\bar{\bar{U}}(1)$ symmetry) under which $\alpha$ is equiv. closed can be used to localize to $M_V$.

2) Any $V$-equivariant ($\bar{\bar{U}}(1)$ symmetric) polyform $\beta$ with $-(d_V\beta)_0$ +ve definite can be used to localize to the minima (zeros) of $-(d_V\beta)_0$.

CHOICE OF $\beta$ $\leftrightarrow$ CHOICE OF "LOCALIZATION SCHEME".

- LOCALIZATION FORMULA FOR ISOLATED FIXED POINTS: $M_V = \{x_k\}$.

$$\int_M \alpha = \lim_{t_{aux}\to\infty} \int_M \alpha\, e^{t_{aux} dg(V,\cdot)}\, e^{-t_{aux}\|V\|^2}$$

Zoom near a fixed point P of the circle action (zero of V).
Adapted set of local coordinates $(x_i, y_i)_{i=1}^{\ell}$ centred at P:

$$ds^2 \simeq \sum_{i=1}^{\ell}\left(dx_i^2 + dy_i^2\right) = \sum_{i=1}^{\ell}\left(dr_i^2 + r_i^2 d\varphi_i^2\right)$$

$$V \simeq \sum_{i=1}^{\ell} \omega_{P,i}\left(-y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i}\right) = \sum_i \omega_{P,i}\frac{\partial}{\partial \varphi_i}$$


$i^{th}$ 2-plane in $T_P M$
($i^{th}$ eigenspace of $L_V(P)$)

generating "rotation"

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \longmapsto R_i(\tau)\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos(\omega_{P,i}\tau) & -\sin(\omega_{P,i}\tau) \\ \sin(\omega_{P,i}\tau) & \cos(\omega_{P,i}\tau) \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

Infinitesimal action $L_V(P)$: $\quad \underbrace{L_V(P)\big|_{i^{th}}}_{\text{eigenspace}} = R_i(\tau)^{-1}\frac{d}{d\tau}R_i(\tau) = \omega_{P,i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\omega_{P,i} = Pf\left(-L_V(P)\big|_{i^{th}}\right)$$

$$\beta = dg(V,\cdot) \simeq \sum_i \omega_{P,i}\left(-y_i dx_i + x_i dy_i\right) = \sum_i \omega_{P,i} r_i^2 d\varphi_i$$

$$d_V\beta = d\beta - \|V\|^2 \simeq 2\sum_i \omega_{P,i}\, dx_i\wedge dy_i - \sum_i \omega_{P,i}^2\left(x_i^2+y_i^2\right) = \sum_i\left[\omega_{P,i}\, d(r_i^2)\wedge d\varphi_i - \omega_{P,i}^2 r_i^2\right]$$

Local contrib'n of neighbourhood of P to $\int_M \alpha$: $\quad \left(e^{2t\omega\, dx\wedge dy} = 1 + 2t\omega\, dx\wedge dy\right)$

$$\lim_{t_{aux}\to\infty} \alpha_0(P)\prod_{i=1}^{\ell}\left(2t\,\omega_{P,i}\int dx_i\wedge dy_i\; e^{-t_{aux}\omega_{P,i}^2(x_i^2+y_i^2)}\right) = \alpha_0(P)\cdot\frac{(2\pi)^{\ell}}{\prod_{i=1}^{\ell}\omega_{P,i}}$$

$$\Rightarrow \boxed{\int_M \alpha = (2\pi)^{\ell}\sum_{x_k:\,V|_{x_k}=0}\frac{\alpha_0(x_k)}{\prod_{i=1}^{\ell}\omega_{x_k,i}} \equiv (2\pi)^{\ell}\sum_{x_k:\,V|_{x_k}=0}\frac{\alpha_0(x_k)}{Pf\left(-L_V(x_k)\right)}}$$

Atiyah-Bott '84
Berline-Vergne '83
(isolated f.p.)

$\int_M \alpha_{2\ell}$ $\xrightarrow{\text{recursion}}$ $\alpha_0$

* $\underline{S^2}$: $V = \frac{\partial}{\partial\varphi}$ , $\alpha = \frac{e^{it(\cos\theta + d\varphi\wedge d\cos\theta)}}{it}$

$\beta = g(V,\cdot) = \sin^2\theta \, d\varphi$

$\to$ Derive localization formula for $\int_{S^2} \alpha = \int \sin\theta \, d\theta \, d\varphi \, e^{it\cos\theta}$

<u>NOTE</u> the different roles of $\underset{\substack{\wr \\ \frac{1}{\hbar} \\ \text{STAYS FIXED}}}{t}$ and $\underset{\substack{\wr \\ \frac{1}{\hbar_{aux}} \\ \text{GOES TO INFINITY}}}{t_{aux}}$ !

## • DUISTERMAAT – HECKMAN THM ('82)

$(M,\omega)$ symplectic manifold

$V$ Hamiltonian vector field generating $U(1)$ action:

$\boxed{dH + i_V\omega = 0}$ $\Leftrightarrow \partial_\mu H = \omega_{\mu\nu} V^\nu$ $\Big|\Rightarrow \begin{array}{l} V^\mu \partial_\mu H = 0 \\ \mathcal{L}_V \omega = 0 \end{array}$

$\dim M = 2\ell$
$d\omega = 0$
$\omega$ non-degenerate 2-form.
Locally $\omega \simeq \sum_{i=1}^{\ell} dp_i \wedge dq^i$

—— • ——

E.g. $\omega = \sum_i dp_i \wedge dq^i$ (locally Darboux thm)

$H$ Hamiltonian for time evolution $\rightsquigarrow V = \frac{d}{dt} = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}$

$0 = dH + i_V\omega \Leftrightarrow \begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \end{cases}$ Hamilton's equs

—— • ——

$\Leftrightarrow$ $\boxed{d_V(\omega - H) = 0}$ $\omega - H$ : equiv. symplectic form.

$$\boxed{Z_H(t) = \int_M \frac{\omega^\ell}{\ell!} e^{-itH} = \left(\frac{i}{t}\right)^\ell \int_M e^{it(\omega - H)} = \left(\frac{2\pi i}{t}\right)^\ell \sum_{x_k : dH|_{x_k} = 0} \frac{e^{-itH(x_k)}}{\mathrm{Pf}\left(-L_V(x_k)\right)}}$$

EXACT STATIONARY PHASE APPROX. !

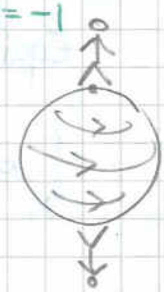
It looks like an exact stationary phase formula for $t = \frac{1}{\hbar}$, but it **really** is a stationary phase approx. for $t_{aux} = \frac{1}{\hbar_{aux}}$, that we could freely take to $\infty$.

* $\underline{S^2}$: $\omega = \sin\theta \, d\theta \wedge d\varphi = d\varphi \wedge d\cos\theta$

$$V = \frac{\partial}{\partial\varphi} \qquad H = -\cos\theta$$

$\mathrm{Pf}(-L_V|_N) = 1$

$\mathrm{Pf}(-L_V|_S) = -1$



* $\underline{\mathbb{R}^2}$: $\omega = dx \wedge dy = d\left(\frac{r^2}{2}\right) \wedge d\varphi$

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial\varphi} \qquad H = \frac{x^2+y^2}{2} = \frac{r^2}{2}$$

Compute the "EQUIVARIANT VOLUME" of $\mathbb{R}^2$.

$it \sim \varepsilon$ ($2\pi$ usually absorbed)

Nekrasov

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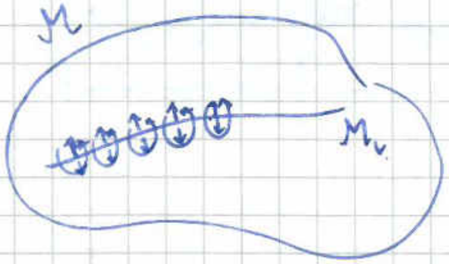
If FIXED POINTS of $V$ are NOT ISOLATED:



Potential $\sim t_{aux} \|V\|^2$

- Modes NORMAL to $M_V$: "MASSIVE"

integrate out in saddle pt / 1-loop approx as $t_{aux} \to \infty$

$\to$ "1-loop" determinant / Pfaffian

- Modes ALONG $M_V$ "MASSLESS"

treat non-linearly $\rightsquigarrow \int_{M_V} \cdots$

BERLINE-VERGNE THM

$$\boxed{\int_M \alpha = \int_{M_V} \frac{i^*\alpha}{\chi_V^{NM_V}} = \int_{M_V} \frac{i^*\alpha}{\mathrm{Pf}\left(\frac{\Omega_V - L_V^{NM_V}}{2\pi}\right)}}$$

"V-equiv. Euler class" of normal bundle to $M_V$

$i : M_V \hookrightarrow M$ embedding

$TM|_{M_V} = TM_V \oplus NM_V$

| EQUIVARIANT LOCALIZATION | SUPERSYMMETRIC LOCALIZATION |
|---|---|
| $d_v = d - i_v$ | $Q$ |
| $d_v^2 = -\mathcal{L}_v$ | $Q^2 = B$ spacetime + internal bosonic symmetry |
| Even/odd polyforms | Bosonic/fermionic observables. |
| Equiv. closed $\alpha$ : $d_v\alpha = 0$ | Supersymmetric "BPS" observable $\mathcal{O}_{BPS}$ : $Q\mathcal{O}_{BPS} = 0$ |
| $\int_M \alpha = \int_M \alpha\, e^{t d_v \beta}$ , $\mathcal{L}_v \beta = 0$ | $\int_{\mathcal{F}} [\mathcal{D}X]\, \mathcal{O}_{BPS}\, e^{-S[X]} = \int_{\mathcal{F}} [\mathcal{D}X]\, \mathcal{O}_{BPS}\, e^{-S[X] - tQ\mathcal{P}_F[X]}$ , $B\mathcal{P}_F[X]=$ |
| $M_v$ | $\mathcal{F}_Q$ (SUSY "BPS" LOCUS) |
| Equiv. Euler class of $NM_v$ | 1-loop SDet of normal fluctuations. |

# • SUPERSYMMETRIC LOCALIZATION [Witten '80s]

• QFT with Grassmann odd (fermionic) symmetry: SUPERCHARGE $Q$

$$\boxed{Q^2 = \underset{\text{bosonic}}{B}} \quad \longleftarrow \text{isometry + gauge + internal global}$$

• SUPERSYMMETRIC ("BPS") OBSERVABLE : gauge invariant and $Q$-closed $Q\mathcal{O}_{BPS} = 0$.
(local / multilocal / nonlocal)

AIM : $\boxed{\text{COMPUTE } \langle \mathcal{O}_{BPS} \rangle \text{ EXACTLY IN THE QUANTUM THEORY}}$

by localization to supersymmetric field configurations $\mathcal{F}_Q$.

We'll use again the power of cohomology – now, cohomology of $Q$.

1) $\langle Q\mathcal{O} \rangle \sim \int_{\mathcal{F}} [DX] (Q\mathcal{O}) e^{-S[X]} = \int_{\mathcal{F}} [DX]\, Q\left(\mathcal{O} e^{-S[X]}\right) = 0.$ Path integral only depends on $Q$-cohom. class

2) $\mathcal{O}_{BPS}$ : $Q\mathcal{O}_{BPS} = 0$

$$\boxed{\langle \mathcal{O}_{BPS} \rangle = \int_{\mathcal{F}} [DX]\, \mathcal{O}_{BPS}\, e^{-S[X]} = \int [DX]\, \mathcal{O}_{BPS}\, e^{-S[X] - tQ\mathcal{P}_F[X]}}$$

Q-cohomologous

$\forall t \in \mathbb{R}$, $\forall$ fermionic $\mathcal{P}_F[X]$ which is $Q^2 = B$-invariant and s.t. $Q\mathcal{P}_F[X]\big|_{bos}$ is +ve definite (don't change asymptotics)

• $t \to +\infty$: path integral dominated by saddles of $S_{loc}[X] \equiv Q\mathcal{P}_F[X]$ !

– Canonical choice: $\boxed{\mathcal{P}_F[X] = \sum_{\text{fermions } \psi_X} (Q\psi_X, \psi_X) \equiv \sum_{\text{fermions } \psi_X} \int (Q\psi_X)^\dagger \psi_X}$

$$\Rightarrow \boxed{S_{loc}\big|_{bosonic} = \sum_{\text{fermions } \psi_X} (Q\psi_X, Q\psi_X) \equiv \sum_{\text{fermions } \psi_X} \int |Q\psi_X|^2}$$ SUM OF SQUARES!

SADDLES OF $S_{loc}$ : "BPS"/ SUPERSYMMETRIC FIELD CONFIGURATIONS $\psi_X = 0$, $Q\psi_X = 0$ $\leftarrow \mathcal{F}_Q$

$$\mathcal{F}_Q = \{ [X_0] \in \mathcal{F} \mid \psi_X \ , \ Q\psi_X|_{X_0} = 0 \}$$

SUPERSYMMETRIC LOCUS ("LOCALIZATION LOCUS")

EXPAND $X = X_0 + \frac{1}{\sqrt{t}}\delta X$ and TAKE $t \to \infty$.

Semiclassical expansion (in $\hbar_{aux} = \frac{1}{t}$!) reduces to

$$S[X_0] + \frac{1}{2}\iint \frac{\delta^2 S_{loc}[X]}{\delta X^2}\Big|_{X=X_0} (\delta X)^2 .$$

1-loop (super)det

$$\boxed{\langle O_{BPS} \rangle = \int_{\mathcal{F}_Q} [\mathcal{D}X_0]\, O_{BPS}\big|_{X=X_0}\, e^{-S[X_0]} \frac{1}{\mathrm{Sdet}\left[\frac{\delta^2 S_{loc}}{\delta X^2}\Big|_{X=X_0}\right]}}$$

Path Integration domain reduced from $\mathcal{F}$ to $\mathcal{F}_Q$:

- Integral over normal (infinitesimal) modes $\to$ 1-loop Sdet
- Integral over modes along $\mathcal{F}_Q$

The ART is to apply this procedure in such a way that $\mathcal{F}_Q$ is simple, e.g. constant field configurations. This is often possible.

- FREEDOM OF CHOOSING LOCALIZATION SCHEME:
  - Choice of $Q$
  - Choice of $P_F[X]$.

$\to$ Different representations for the same path integral.

If time allows, mention computation of 1-loop det as in slides (cohomological reorganization, index thm), otherwise perhaps explicit pairing later

# Localization of SUSY QFTs in curved space


Stefano Cremonesi

King's College London



VIII Parma International School of Theoretical Physics, 10/9/2016

# SUSY on curved space and Localization

Today we will see how the idea of localization applies to the path integral of a supersymmetric quantum field theory on a compact curved manifold $\mathcal{M}$.

- Compact space provides IR cutoff, making path integral better defined
- Supersymmetric localization reduces it to a finite-dimensional integral

$$Z_{\mathcal{M}}[J] = \left\langle e^{-\int J\mathcal{O}} \right\rangle = \int [\mathcal{D}X]\; e^{-S[X]-\int J\mathcal{O}}$$

$J$ is a supersymmetric source, coupled to a supersymmetric observable $\mathcal{O}$.

The dependence on $\mathcal{M}$ is hidden in $S[X]$ and the notion of supersymmetry.

# Plan

1. Supersymmetric localization
2. Rigid SUSY on curved space
3. Localization of 3d $\mathcal{N} = 2$ gauge theories on $S^3_b$

# Supersymmetric localization

# The path integral of a supersymmetric QFT

Consider a supersymmetric QFT with fields $X \in \mathcal{F}$:

- Supercharge $\mathcal{Q}$: $\mathcal{Q}^2 = \mathcal{H}$
- Action $S[X]$: $\mathcal{Q}S[X] = 0$
- Supersymmetric observable $\mathcal{O}$: $\mathcal{Q}\mathcal{O} = 0$

We wish to compute

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} [\mathcal{D}X] \; \mathcal{O} \; e^{-S[X]}$$

Note that [Witten 1988]

$$\langle \mathcal{Q}\mathcal{O}' \rangle = \int_{\mathcal{F}} [\mathcal{D}X] \; (\mathcal{Q}\mathcal{O}')e^{-S[X]} = \int_{\mathcal{F}} [\mathcal{D}X] \; \mathcal{Q} \left( \mathcal{O}' e^{-S[X]} \right) = 0 \, ,$$

therefore expectation values only depend on the $\mathcal{Q}$-cohomology class:

$$\langle \mathcal{O} + \mathcal{Q}\mathcal{O}' \rangle = \langle \mathcal{O} \rangle$$

# Localization argument



We can exploit the fact that the expectation value of a $\mathcal{Q}$-closed observable $\mathcal{O}$ only depends on its $\mathcal{Q}$-cohomology class $[\mathcal{O}]$. Change representative:

$$\langle \mathcal{O} \rangle = \langle \mathcal{O}\, e^{-t\mathcal{Q}\mathcal{V}[X]} \rangle = \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{O}\, e^{-S[X]-t\mathcal{Q}\mathcal{V}[X]} \qquad \forall\, t,\ \mathcal{V}[X]\ \text{ s.t. }\ \mathcal{Q}^2\mathcal{V}[X] = 0\ .$$

We will assume that $\mathrm{Re}\mathcal{Q}\mathcal{V}[X]|_{bos}$ is positive semi-definite and consider $t \geq 0$.

$$\langle \mathcal{O} \rangle = \lim_{t\to+\infty} \int_{\mathcal{F}} [\mathcal{D}X]\ \mathcal{O}\ e^{-S[X]-t\mathcal{Q}\mathcal{V}[X]}\ .$$

- $t \to +\infty$: integral dominated by the saddle points of the

**Localizing action**

$$S_{loc}[X] = \mathcal{Q}\mathcal{V}[X]\ ,$$

**Localization locus**

$$\mathcal{F}_{loc} = \{X_0 \in \mathcal{F} | \left.\frac{\delta S_{loc}[X]}{\delta X}\right|_{X_0} = 0\}\ .$$

Exact semiclassical approximation in $\hbar_{aux} = 1/t$, around saddles $X_0$ of $S_{loc}$:

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X$$

$$S[X] + tS_{loc}[X] \xrightarrow[t\to+\infty]{} S[X_0] + \frac{1}{2}\iint \frac{\delta^2 S_{loc}[X]}{\delta X^2}\Big|_{X_0} (\delta X)^2$$

Integrating out the transverse fluctuations $\delta X$, one obtains

**Localization formula**

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}_{loc}} [\mathcal{D}X_0]\ \mathcal{O}|_{X_0}\ e^{-S[X_0]} \frac{1}{\mathrm{Sdet}\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right]} .$$

*E.g.*, for the standard choice (here $\Psi = \{\text{fermions}\}$) [Pestun 2007]

$$\mathcal{V}_P = (\mathcal{Q}\Psi, \Psi) \quad \Longrightarrow \quad S_{loc}|_{bos} = (\mathcal{Q}\Psi, \mathcal{Q}\Psi) \geq 0 ,$$

the path integral over $\mathcal{F}$ localizes to $\mathcal{F}_{loc} = \mathcal{F}_{\mathcal{Q}}$, the BPS locus.

# The 1-loop determinant $Z_{1-loop}$

$$Z_{1-loop} = \mathrm{Sdet}\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right] = \left(\frac{\det K_{ferm}}{\det K_{bos}}\right)^{1/2} ,$$

where the kinetic operators for bosonic and fermionic fluctuations $K_{bos}$, $K_{ferm}$ are modified Laplace and Dirac operators.

**General observations**:

- Computing their spectrum can be difficult.
- Many cancellations because SUSY pairs bosons and fermions.
- One can localize to fixed points of $\mathcal{Q}^2$ in spacetime.

The computation is best done by organising fields cohomologically (*i.e.* in multiplets of $\mathcal{Q}$) and applying index theorems. [Pestun 2007]

1. Reorganise fields in $\mathcal{Q}$-multiplets $\{X\} = \{\phi,\ \psi' = \mathcal{Q}\phi,\ \psi,\ \phi' = \mathcal{Q}\psi\}$:

$$\mathcal{Q}\phi = \psi' , \qquad \mathcal{Q}\psi' = \mathcal{Q}^2\phi .$$
$$\mathcal{Q}\psi = \phi' , \qquad \mathcal{Q}\phi' = \mathcal{Q}^2\psi .$$

For simplicity, [Hosomichi 2015]

$$\mathcal{V}_H = (\phi, \mathcal{Q}\phi) + (\psi, \mathcal{Q}\psi)$$
$$\mathcal{Q}\mathcal{V}_H = (\psi', \psi') + (\phi, \mathcal{Q}^2\phi) + (\phi', \phi') - (\psi, \mathcal{Q}^2\psi)$$

$$\boxed{Z_{1-loop} = \left(\frac{\det_\psi \mathcal{Q}^2}{\det_\phi \mathcal{Q}^2}\right)^{1/2}}$$

2. If there is a differential operator $\mathcal{D}$ that commutes with $\mathcal{Q}^2$,

$$\begin{array}{cccc} \mathcal{D}: & \Gamma(E_0) & \to & \Gamma(E_1) \\ & \cup & & \cup \\ & \phi & & \psi \end{array} \qquad\qquad \begin{array}{cccc} \mathcal{D}^\dagger: & \Gamma(E_1) & \to & \Gamma(E_0) \\ & \cup & & \cup \\ & \psi & & \phi \end{array}$$

then

$$\boxed{Z_{1-loop} = \left(\frac{\det_{\mathrm{coker}\mathcal{D}} \mathcal{Q}^2}{\det_{\ker\mathcal{D}} \mathcal{Q}^2}\right)^{1/2} .}$$

← unpaired $\psi$

← unpaired $\phi$

The 1-loop determinant can be deduced from the $\mathcal{Q}^2$-equivariant index of $\mathcal{D}$

$$\mathrm{Ind}(\mathcal{D}; e^{\mathcal{Q}^2}) := \mathrm{tr}_{\ker \mathcal{D}}(e^{\mathcal{Q}^2}) - \mathrm{tr}_{\mathrm{coker}\mathcal{D}}(e^{\mathcal{Q}^2}) = \sum_j d_j \, e^{h_j}$$

as

$$Z_{1-loop} = \left( \frac{\det_{\mathrm{coker}\mathcal{D}} \mathcal{Q}^2}{\det_{\ker \mathcal{D}} \mathcal{Q}^2} \right)^{1/2} = \prod_j h_j^{-d_j/2} \, .$$

If $\mathcal{D}$ is transversally elliptic, which ensures that $d_j$ are finite, the equivariant index can be computed by the Atiyah-Bott fixed point formula *e.g.* [Atiyah 1974]

$$\mathrm{Ind}(\mathcal{D}; e^{\mathcal{Q}^2}) = \sum_{p | e^{\mathcal{Q}^2} \cdot p = p} \frac{\mathrm{tr}_{E_0(p)} \, e^{\mathcal{Q}^2} - \mathrm{tr}_{E_1(p)} \, e^{\mathcal{Q}^2}}{\det_{T\mathcal{M}(p)}(1 - e^{\mathcal{Q}^2})} \, .$$

This reduces the computation of $Z_{1-loop}$ to determining the local action of $\mathcal{Q}^2$ around fixed points in field space $\mathcal{F}$ and in spacetime $\mathcal{M}$.

# Rigid SUSY on curved space

# The problem of defining rigid SUSY on curved space

**Supersymmetric QFT on flat space** $(\mathbb{R}^d, g^{(0)}_{\mu\nu} = \eta_{\mu\nu})$**:**

- Flat space SUSY algebra → SUSY transformations $\delta^{(0)}X$
- $\mathcal{L}^{(0)}$ SUSY Lagrangian → $\delta^{(0)}\mathcal{L}^{(0)} = \partial_\mu(\dots)^\mu$

$$\downarrow$$

**Supersymmetric QFT on curved space** $(\mathcal{M}_d, g_{\mu\nu})$**:**

- Curved space SUSY algebra → SUSY transformations $\delta X$
- $\mathcal{L}$ SUSY Lagrangian → $\delta\mathcal{L} = \nabla_\mu(\dots)^\mu$

?

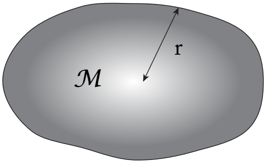
We would like to know:

1. For which flat space SUSY algebras and $(\mathcal{M}_d, g_{\mu\nu})$ this is possible
2. What are $\delta X$ and $\mathcal{L}(X, \partial_\mu X)$

# Approach 1: Trial and error

$$\delta = \delta^{(0)}\Big|_{\substack{\eta\to g\\ \partial\to\nabla}} + \sum_{n\geq 1}\frac{1}{r^n}\delta^{(n)}$$

$$\mathcal{L} = \mathcal{L}^{(0)}\Big|_{\substack{\eta\to g\\ \partial\to\nabla}} + \sum_{n\geq 1}\frac{1}{r^n}\mathcal{L}^{(n)}$$




until SUSY algebra closes and $\delta\mathcal{L} = \nabla_\mu(\dots)^\mu$.

Drawbacks:

- No guarantee it will work
- Case by case
- When it succeeds, expansions stop at $n = 1$ and $n = 2$ resp. Why?

# Approach 2: Background supergravity [Festuccia, Seiberg 2011]

see also [Karlhede, Roček 1988; Johansen 1995; Adams, Jockers, Kumar, Lapan 2011]

- Nonlinearly couple supersymmetric FT to an *off-shell* supersymmetric background for supergravity multiplet $(g_{\mu\nu}, \psi_{\mu\alpha}, \mathrm{aux})$.
- Rigid limit of supergravity: gravity multiplet becomes non-dynamical.
- Only require that the background is supersymmetric:

**Generalised Killing spinor equations**

$$\psi_{\mu\alpha} = 0 \,, \qquad \delta_\zeta \psi_{\mu\alpha} = 0$$

Advantages:

- Model independent: only input is flat space SUSY algebra.
- $\delta_{SuGra}|_{\mathrm{bg}} X_{SFT} = \delta X_{SFT} \,, \quad \mathcal{L}_{SFT+SuGra}|_{\mathrm{bg}} = \mathcal{L}_{SFT} \quad .$
- $1/r$ expansion above due to auxiliary fields.
- Supersymmetric backgrounds $\mathrm{bg} = (\mathcal{M}_d, g_{\mu\nu}, \mathrm{aux}, \zeta)$ can be classified.

# 3d $\mathcal{N}=2$ SUSY with $U(1)_R$ symmetry

**SUSY algebra on $\mathbb{R}^3$:**

$$\{Q_\alpha, \widetilde{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z$$

$$\{Q_\alpha, Q_\beta\} = 0 \qquad \{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\} = 0$$

$$[R, Q_\alpha] = -Q_\alpha \qquad [R, \widetilde{Q}_\alpha] = +\widetilde{Q}_\alpha$$

$$[Z, Q_\alpha] = [Z, \widetilde{Q}_\alpha] = [Z, R] = 0$$

[Dumitrescu, Seiberg 2011]

| | | | | | | |
|---|---|---|---|---|---|---|
| **Supercurrent $\mathcal{R}$-multiplet:** | $T^{\mu\nu}$ | $S^{\mu\alpha}$ | $\widetilde{S}^{\mu\alpha}$ | $j^\mu_{(R)}$ | $j^\mu_{(Z)}$ | $i\epsilon^{\mu\nu\rho}\partial_\rho J_{(Z)}$ |
| **New min'l SUGRA multiplet:** | $h_{\mu\nu}$ | $\psi_{\mu\alpha}$ | $\widetilde{\psi}_{\mu\alpha}$ | $A_\mu$ | $C_\mu$ | $B_{\mu\nu}$ |

3d version of [Sohnius, West 1981/82]

$$C_\mu \longleftrightarrow V^\mu = -i\epsilon^{\mu\nu\rho}\partial_\mu C_\rho \qquad B_{\mu\nu} \longleftrightarrow H = \frac{i}{2}\epsilon^{\mu\nu\rho}\partial_\mu B_{\nu\rho}$$

$$\delta\mathcal{L}^{lin}_{min} = -T^{\mu\nu}h_{\mu\nu} - \frac{1}{2}S^\mu\psi_\mu + \frac{1}{2}\widetilde{S}^\mu\widetilde{\psi}_\mu + j^\mu_{(R)}(A_\mu - \frac{3}{2}V_\mu) + j^\mu_{(Z)}C_\mu + J_{(Z)}H$$

# 3d $\mathcal{N} = 2$ SUSY with $U(1)_R$ symmetry on $\mathcal{M}_3$

[Closset, Dumitrescu, Festuccia, Komargodski 2012]

$\delta_\zeta \psi_{\mu\alpha}$, $\delta_{\widetilde{\zeta}} \widetilde{\psi}_{\mu\alpha}$ in the rigid limit can be inferred from linear theory, diffeo + local R invariance and dimensional analysis, without knowing the full SuGra.

$$\delta_\zeta \psi_\mu = 2(\nabla_\mu - iA_\mu)\zeta + H\gamma_\mu\zeta + 2iV_\mu\zeta + \epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta + (\dots)$$
$$\delta_{\widetilde{\zeta}} \widetilde{\psi}_\mu = 2(\nabla_\mu + iA_\mu)\widetilde{\zeta} + H\gamma_\mu\widetilde{\zeta} - 2iV_\mu\widetilde{\zeta} - \epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta} + (\dots) \,,$$

**(Generalised) Killing spinor equations**

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{H}{2}\gamma_\mu\zeta - iV_\mu\zeta - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta$$
$$(\nabla_\mu + iA_\mu)\widetilde{\zeta} = -\frac{H}{2}\gamma_\mu\widetilde{\zeta} + iV_\mu\widetilde{\zeta} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta} \,.$$

Supersymmetric background:

$(\mathcal{M}_3, g_{\mu\nu}, A_\mu, V_\mu, H)$ allowing solutions $(\zeta, \widetilde{\zeta}) \neq 0$ of GKSE.

## Curved space supersymmetry algebra

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\phi_{(r,z)} = -2i\left(\mathcal{L}'_K + \zeta\widetilde{\zeta}(z - rH)\right)\phi_{(r,z)}$$

$$\{\delta_\zeta, \delta_\eta\}\phi_{(r,z)} = 0 \qquad \{\delta_{\widetilde{\zeta}}, \delta_{\widetilde{\eta}}\}\phi_{(r,z)} = 0$$

where $\mathcal{L}'_K$ is a fully covariant Lie derivative along the Killing vector $K^\mu = \zeta\gamma^\mu\widetilde{\zeta}$,

$$\mathcal{L}'_K\varphi_{(r,z)} = \left(K^\mu D_\mu + \tfrac{i}{2}(D_\mu K_\nu)S^{\mu\nu}\right)\varphi_{(r,z)} \; ,$$

$$D_\mu\varphi_{(r,z)} = \left(\nabla_\mu - ir(A_\mu - \tfrac{1}{2}V_\mu) - izC_\mu\right)\varphi_{(r,z)}$$

the totally covariant derivative of a field $\varphi_{(r,z)}$ of $R$-charge $r$ and $Z$-charge $z$.

The representation of this SUSY algebra on a general multiplet is known.

We will be mostly interested in vector and chiral multiplets.

# Vector multiplet $V$

## SUSY transformations:

$$
\begin{aligned}
\delta a_\mu &= -i(\zeta\gamma_\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma_\mu\lambda) \\
\delta\sigma &= -\zeta\widetilde{\lambda} + \widetilde{\zeta}\lambda \\
\delta\lambda &= +\zeta\,(D + iH\sigma) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\zeta f_{\mu\nu} - \gamma^\mu\zeta\,(iD_\mu\sigma - V_\mu\sigma) \\
\delta\widetilde{\lambda} &= -\widetilde{\zeta}\,(D + iH\sigma) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\widetilde{\zeta} f_{\mu\nu} + \gamma^\mu\widetilde{\zeta}\,(iD_\mu\sigma + V_\mu\sigma) \\
\delta D &= D_\mu\big(\zeta\gamma^\mu\widetilde{\lambda} - \widetilde{\zeta}\gamma^\mu\lambda\big) - iV_\mu\big(\zeta\gamma^\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma^\mu\lambda\big) - H\big(\zeta\widetilde{\lambda} - \widetilde{\zeta}\lambda\big)
\end{aligned}
$$

## SUSY Lagrangians:

$$
\begin{aligned}
\mathcal{L}_{YM} &= \frac{1}{g_{YM}^2}\,\mathrm{Tr}\Big(\frac{1}{2}f_{\mu\nu}f^{\mu\nu} + D_\mu\sigma D^\mu\sigma + (D + iH\sigma)^2 + i\sigma\varepsilon^{\mu\nu\rho}V_\mu f_{\nu\rho} - V^\mu V_\mu\sigma^2 \\
&\qquad - 2i\widetilde{\lambda}\gamma^\mu(D_\mu + \tfrac{i}{2}V_\mu)\lambda - 2i\widetilde{\lambda}[\sigma,\lambda] + iH\widetilde{\lambda}\lambda\Big) \\
\mathcal{L}_{CS} &= i\frac{k}{4\pi}\,\mathrm{Tr}\left(\varepsilon^{\mu\nu\rho}(a_\mu\partial_\nu a_\rho + i\frac{2}{3}a_\mu a_\nu a_\rho) + 2D\sigma + 2\widetilde{\lambda}\lambda\right) \\
\mathcal{L}_{FI} &= -i\frac{\xi}{2\pi}\,\mathrm{Tr}\,(D - iH\sigma - iV^\mu a_\mu)
\end{aligned}
$$

# Chiral multiplet $\Phi_{(r,z)}$

**SUSY transformations**:

$$\begin{aligned}
\delta\phi &= \sqrt{2}\zeta\psi \\
\delta\psi &= \sqrt{2}\zeta F - \sqrt{2}i\,(z-\sigma-rH)\,\widetilde{\zeta}\phi - \sqrt{2}i\gamma^\mu\widetilde{\zeta}D_\mu\phi \\
\delta F &= \sqrt{2}i\,(z-\sigma-(r-2)H)\,\widetilde{\zeta}\psi + 2i\widetilde{\zeta}\widetilde{\lambda}\phi
\end{aligned}$$

**SUSY Lagrangians**:

$$\begin{aligned}
\mathcal{L}_{mat} &= D^\mu\widetilde{\phi}D_\mu\phi - i\widetilde{\psi}\gamma^\mu D_\mu\psi - \widetilde{F}F - i\widetilde{\phi}V^\mu D_\mu\phi \\
&+ \widetilde{\phi}\Big(-i(D+iH\sigma) + (z-\sigma-rH)^2 + 2H(z-\sigma) + \frac{r}{2}(\frac{1}{2}R + V^\mu V_\mu - H^2)\Big)\phi \\
&+ i\widetilde{\psi}\Big(z-\sigma-(r-\frac{1}{2})H\Big)\psi - \frac{1}{2}\widetilde{\psi}\gamma^\mu V_\mu\psi + \sqrt{2}i\big(\widetilde{\phi}\lambda\psi + \phi\widetilde{\lambda}\widetilde{\psi}\big) \\
\mathcal{L}_W &= F_{W(\Phi)} + \widetilde{F}_{\widetilde{W}(\widetilde{\Phi})} = \left(F\frac{\partial W}{\partial\phi} + \psi\psi\frac{\partial^2 W}{\partial\phi^2}\right) + \left(\widetilde{F}\frac{\partial\widetilde{W}}{\partial\widetilde{\phi}} + \widetilde{\psi}\widetilde{\psi}\frac{\partial^2\widetilde{W}}{\partial\widetilde{\phi}^2}\right)
\end{aligned}$$

# Compact supersymmetric backgrounds

**1 supercharge $\zeta$: $\mathcal{M}_3$ has a transversely holomorphic foliation**

Coordinates $(\tau, z, \bar{z})$: $\tau' = \tau + t(z,\bar{z})$ , $z' = f(z)$ .

Metric: $ds^2 = (d\tau + h(\tau,z,\bar{z})dz + \bar{h}(\tau,z,\bar{z})d\bar{z})^2 + c(\tau,z,\bar{z})^2 dz d\bar{z}$

$\zeta$ determines all background fields, up to invariance of GKSE.

**2 supercharges $\zeta, \widetilde{\zeta}$: $\mathcal{M}_3$ is a Seifert manifold $(S^1 \hookrightarrow M_3 \to \Sigma)$**

Metric: $ds^2 = \Omega(z,\bar{z})^2 (d\psi + h(z,\bar{z})dz + \bar{h}(z,\bar{z})d\bar{z})^2 + c(z,\bar{z})^2 dz d\bar{z}$

**4 supercharges $\zeta_1, \zeta_2, \widetilde{\zeta}_1, \widetilde{\zeta}_2$:**

- $T^3$
- Round $S^2 \times S^1$ with $H = 0$, $A = V = \pm \frac{i}{R_{S^1}} d\tau$ [Imamura, S. Yokoyama 2011]
- (Squashed) $S^3$ with $SU(2) \times U(1)$ isometry: [Imamura, D. Yokoyama 2011]

$$ds^2 = R^2\left((\mu^1)^2 + (\mu^2)^2 + h^2(\mu^3)^2\right) , \quad H = \frac{ih}{R} , \quad A = V = 2\sqrt{h^2-1}\mu^3$$

# The 3d ellipsoid $S_b^3$

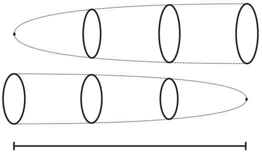
[Hama, Hosomichi, Lee 2011]

$$b^2|z_1|^2 + b^{-2}|z_2|^2 = R^2$$

$$z_1 = R b^{-1} \sin\vartheta \, e^{i\varphi_1}$$
$$z_2 = R b \cos\vartheta \, e^{i\varphi_2}$$

$S^3$ topology

$U(1)^2$ isometry



**Background**:

$$ds^2 = R^2 \left( b^2 \sin^2\vartheta \, d\varphi_1^2 + b^{-2}\cos^2\vartheta \, d\varphi_2^2 + f(\vartheta)^2 d\vartheta^2 \right)$$

$$H = -\frac{i}{Rf(\vartheta)} \,, \quad 2A = \left(1 - \frac{b}{f(\vartheta)}\right) d\varphi_1 + \left(1 - \frac{b^{-1}}{f(\vartheta)}\right) d\varphi_2$$

$$f(\vartheta) = (b^{-2}\sin^2\vartheta + b^2\cos^2\vartheta)^{1/2}$$

$$\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(\varphi_1+\varphi_2+\vartheta)} \\ e^{\frac{i}{2}(\varphi_1+\varphi_2-\vartheta)} \end{pmatrix} \,, \quad \widetilde{\zeta} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-\frac{i}{2}(\varphi_1+\varphi_2-\vartheta)} \\ e^{-\frac{i}{2}(\varphi_1+\varphi_2+\vartheta)} \end{pmatrix}$$

# Localization of $3d$ $\mathcal{N}=2$ gauge theories on $S^3_b$

[Hama, Hosomichi, Lee 2011], building on [Kapustin, Willett, Yaakov '09; Jafferis '10; HHL '10]

$$Z[\widehat{V}] = \int [\mathcal{D}V][\mathcal{D}\Phi][\mathcal{D}\widetilde{\Phi}]\, e^{-(S_{YM}[V]+S_{CS}[V]+S_{FI}[V]+S_{mat}[\Phi,\widetilde{\Phi},V,\widehat{V}]+S_W[\Phi,\widetilde{\Phi}])}$$

$\widehat{V}$: background vector multiplet (global symmetry)

- Localizing supercharge: $$\mathcal{Q} = \delta_\zeta + \delta_{\widetilde{\zeta}}$$
- Localizing action: $$S_{loc} = \mathcal{Q}\mathcal{V}_P\,, \qquad \mathcal{V}_P = \sum_{\Psi\in\{\lambda,\widetilde{\lambda},\psi,\widetilde{\psi}\}} (\mathcal{Q}\Psi, \Psi)$$
- Localization locus $\mathcal{F}_\mathcal{Q}$: $$D = -iH\sigma, \quad a_\mu = 0, \quad \sigma = \text{const.}$$ $$\phi = \widetilde{\phi} = F = \widetilde{F} = 0$$
- Classical action: $$S[X_0] = -ik\pi\,\mathrm{tr}(R\sigma)^2 + 2\pi i(\xi R)\,\mathrm{tr}(R\sigma)$$

  [$S_{YM}$, $S_{mat}$, $S_W$ are $\mathcal{Q}$-exact] $$Z_{class} = e^{ik\pi\,\mathrm{tr}(R\sigma)^2 - 2\pi i(\xi R)\,\mathrm{tr}(R\sigma)}$$

- Diagonalise $\sigma = \sigma^i H_i$: $$|J| = \prod_{\alpha\in\Delta_+} \alpha(R\sigma)^2$$

- 1-loop det of $\Phi_{(r,z)}$: $$Z^{\Phi}_{1-loop} = \prod_{m,n=0}^{\infty} \frac{(m+1)b + (n+1)b^{-1} + iRz_{\mathbb{C}}}{mb + nb^{-1} - iRz_{\mathbb{C}}} = \Gamma_h(Rz_{\mathbb{C}})$$ $$Rz_{\mathbb{C}} = Rz + i\frac{b+b^{-1}}{2}r\,, \qquad z = \rho(\sigma) + \widehat{\rho}(\widehat{\sigma})$$

- 1-loop det of $V$: $$Z^{V}_{1-loop} = \prod_{\alpha\in\Delta_+} \frac{4\sinh(\pi b\alpha(R\sigma))\sinh(\pi b^{-1}\alpha(R\sigma))}{\alpha(R\sigma)^2}$$

**Coulomb branch localization formula**

$$Z_{S^3_b}(\widehat{\sigma}; k, \xi, r) = \frac{1}{|\mathcal{W}_G|}\int \prod_{i=1}^{\mathrm{rk}(G)} d\sigma_i\, Z_{class}(\sigma;\xi,k)\, Z_{1-loop}(\sigma,\widehat{\sigma},r)\ .$$

This result generalizes to any background with $S^3$ topology: $b \in \mathbb{C}$ is the modulus of the transversely holomorphic foliation on $S^3$.

[Closset, Dumitrescu, Festuccia, Komargodski 2013; Alday, Martelli, Richmond, Sparks 2013]

# An alternative: Higgs branch localization

[Benini, SC '12] in 2d; [Fujitsuka, Honda, Yoshida '13; Benini, Peelaers '13] in 3d; ...

Localizing action:

$$S'_{loc} = \mathcal{Q}\left(\mathcal{V}_P + \mathcal{V}_{Higgs}\right)$$

$$\mathcal{V}_{Higgs} = \int d^3x\sqrt{g}\,\mathrm{tr}\left(\frac{\widetilde{\zeta}\lambda - \zeta\widetilde{\lambda}}{2i} M(\phi,\widetilde{\phi})\right)$$

$$M(\phi,\widetilde{\phi}) = \sum_\alpha \phi^\alpha \phi^\dagger_\alpha - \widehat{\xi}\,, \qquad \widehat{\xi} = \sum_{i\in\mathrm{Cartan}(g)} \widehat{\xi}^i h_i$$

When the "fake FI parameter" $\widehat{\xi} \to \infty$ (in an appropriate chamber):

- Coulomb branch saddles are suppressed
- Higgs branch saddles controlled by $M$ (plus zero size vortices) dominate.

## Higgs branch localization formula

$$Z = \sum_{\text{Higgs vacua}} Z_{class}\, Z'_{1-loop}\, Z_v^{(NP)}\, Z_{av}^{(SP)}$$

proving the factorisation of $Z$ observed in [Pasquetti '11], [Beem, Dimofte, Pasquetti '12].

# Examples and applications

# Partition function and field theory dualities

The partition function $Z_{\mathcal{M}}(\widehat{V};\lambda)$ computed exactly by localization allows detailed tests of field theory dualities. If theory $A$ is dual to theory $B$, then

$$Z^{(A)}_{\mathcal{M}}(\widehat{V}^{(A)};\lambda^{(A)}) = Z^{(B)}_{\mathcal{M}}(\widehat{V}^{(B)};\lambda^{(B)})$$

with a duality map

$$\begin{aligned} \widehat{V}^{(A)}_a &= \sum\nolimits_b c_a{}^b \widehat{V}^{(B)}_b \\ \lambda^{(A)} &= f(\lambda^{(B)}) \,. \end{aligned}$$

These tests have been performed for a variety of theories:

[Dolan, Osborn '08; Spiridonov, Vartanov '08-'12; Kapustin, Willett, Yaakov '10; Willett, Yaakov '11; Benini, Closset, SC '11; Benini, SC '12; Doroud, Gomis, Le Floch, Lee '12; . . . ]

- Identities between integrals of special functions
- Useful to determine the duality map
- Can be extended to supersymmetric operators

# 2d $\mathcal{N} = (2,2)$ with $U(1)_{R_V}$ on $S^2$, $m = \frac{1}{2\pi}\int_{S^2} dA = 0$

[Benini, SC 2012; Doroud, Gomis, Le Floch, Lee 2012]

## $S^2$ partition function (Coulomb branch representation)

$$Z_{S^2,m=0}(\xi,\theta;\widehat{\sigma},\widehat{m}) = \frac{1}{|\mathcal{W}_G|}\sum_{m\in\Gamma_{G^\vee}}\int_{\mathfrak{t}_G}\left(\prod_{i=1}^{\mathrm{rk}(G)}\frac{d\sigma_i}{2\pi}\right) Z_{class}(\xi,\theta;\sigma,m)\, Z_{1-loop}(\sigma,m;\widehat{\sigma},\widehat{m})$$

$$Z_{class}(\xi,\theta;\sigma,m) = e^{-4\pi i\xi\mathrm{Tr}(\sigma)-i\theta\mathrm{Tr}(m)}$$

$$Z_{1-loop}(\sigma,m;\widehat{\sigma},\widehat{m}) = Z^V_{1-loop}(\sigma,m)\prod_i Z^{\Phi_i}_{1-loop}(\sigma,m;\widehat{\sigma},\widehat{m})$$

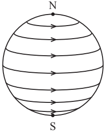
$$Z^V_{1-loop}(\sigma,m) = \prod_{\alpha\in\Delta_+}\left(\frac{\alpha(m)^2}{4}+\alpha(\sigma)^2\right)$$

$$Z^\Phi_{1-loop}(\sigma,m;\widehat{\sigma},\widehat{m}) = \prod_{\rho\in\mathcal{R}^G_\Phi}\prod_{\widehat{\rho}\in\mathcal{R}^{\widehat{G}}_\Phi}\frac{\Gamma\left(\frac{r}{2}-i(\rho(\sigma)+\widehat{\rho}(\widehat{\sigma})-\frac{\rho(m)+\widehat{\rho}(\widehat{m})}{2}\right)}{\Gamma\left(1-\frac{r}{2}+i(\rho(\sigma)+\widehat{\rho}(\widehat{\sigma}))-\frac{\rho(m)+\widehat{\rho}(\widehat{m})}{2}\right)}$$

# 2d $\mathcal{N} = (2,2)$ with $U(1)_{R_V}$ on $S^2_{\epsilon_\Omega}$, $m = \frac{1}{2\pi}\int_{S^2} dA = 1$

$\Omega$-deformation of the topological $A$-twist on $S^2$ [Witten 1988]:

$U(1)$ isometry generated by $\quad V = \partial_\varphi = i\,(z\partial_z - \bar{z}\partial_{\bar{z}})$ .




## Background

$$\{e^1 = g^{\frac{1}{4}}(|z|^2)\,dz,\; e^{\bar{1}} = g^{\frac{1}{4}}(|z|^2)\,d\bar{z}\} \qquad \Rightarrow \qquad ds^2 = e^1 e^{\bar{1}}$$

$$A_\mu = \frac{1}{2}\omega_\mu\,, \qquad C_\mu = \frac{\epsilon_\Omega}{2}V_\mu\,, \qquad \widetilde{C}_\mu = 0 \qquad (\epsilon_\Omega \in \mathbb{C})$$

$$\zeta = \begin{pmatrix}\zeta_-\\ \zeta_+\end{pmatrix} = \begin{pmatrix}\epsilon_\Omega V_1\\ 1\end{pmatrix}, \qquad \widetilde{\zeta} = \begin{pmatrix}\widetilde{\zeta}_-\\ \widetilde{\zeta}_+\end{pmatrix} = \begin{pmatrix}1\\ -\epsilon_\Omega V_{\bar{1}}\end{pmatrix} \quad \Rightarrow \quad K = \epsilon_\Omega V\,.$$

[Closset, SC 2014]

## Supersymmetry algebra ($z, \widetilde{z}$: central charges; $s$: spin)

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\varphi_{(r,z,\widetilde{z},s)} = -2i\big(z + \epsilon_\Omega \mathcal{L}_V|_{s\to s+\frac{r}{2}}\big)\varphi_{(r,z,\widetilde{z},s)}\,,$$

$$\delta_\zeta^2\,\varphi_{(r,z,\widetilde{z},s)} = 0\,, \qquad\qquad \delta_{\widetilde{\zeta}}^2\,\varphi_{(r,z,\widetilde{z},s)} = 0\,.$$

## $S^2_{\epsilon_\Omega}$ correlators (Coulomb branch representation) [Closset, SC, D. Park 2015]

$$\langle \mathcal{O}(\sigma)|_{N,S} \rangle = \frac{1}{|\mathcal{W}_G|} \sum_{k \in \Gamma_{G^\vee}} \oint_{\mathcal{C}_k^+} \prod_{a=1}^{\mathrm{rk}(G)} \left[ \frac{d\sigma_a}{2\pi i} q_a^{k_a} \right] Z_{1-loop,k}(\sigma, \widehat{\sigma})\ \mathcal{O}(\sigma \mp \tfrac{1}{2} k \epsilon_\Omega)$$

- $\mathcal{C}_k^+$ picks poles of "positively charged" fields (Jeffrey-Kirwan residue).
- $k = \{k_a\}$ are magnetic fluxes: only some chambers contribute residues.
- $q_a = e^{2\pi i \tau_a}$ formally associated to Cartan $\mathfrak{t}_G$ to ensure convergence.
- The 1-loop det of a chiral $\Phi$ is ($\widehat{\sigma}$, $\widehat{k}$ can be included)

$$Z^{\Phi}_{1-loop,k}(\sigma) = \prod_{\rho \in \mathcal{R}^G_\Phi} \frac{\Gamma\left(\frac{r}{2} + \rho(\frac{\sigma}{\epsilon_\Omega} - \frac{k}{2})\right)}{\Gamma\left(1 - \frac{r}{2} + \rho(\frac{\sigma}{\epsilon_\Omega} + \frac{k}{2})\right)} .$$

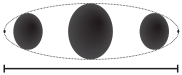
- The 1-loop det of a vector $V$ is as for a chiral of $r = 2$ and $\rho = \alpha$.
- Twisted chiral operators $\mathcal{O}$ are inserted at the poles.

Compact formula encompassing and generalising [Morrison, Plesser 1994].

**4d ellipsoid $S_b^4$:**

[Pestun 2007; Hama, Hosomichi 2012]

$$x^2 + b^2|z_1|^2 + b^{-2}|z_2|^2 = R^2$$



$$Z_{S_b^4} = \frac{1}{|\mathcal{W}_G|} \int \left[\frac{da}{2\pi}\right] |q|^{\mathrm{tr}(a^2)} Z_{1-loop}(a, \widehat{a}) \left| Z_{inst}(q; a, \widehat{a}, \epsilon_1 = b, \epsilon_2 = b^{-1}) \right|^2 ,$$

$$Z_{1-loop} = \frac{\prod_{\alpha\in\Delta} \Upsilon_b(i\alpha(a))}{\prod_{\rho\in\mathcal{R}_H^G} \prod_{\widehat{\rho}\in\mathcal{R}_H^{\widehat{G}}} \Upsilon_b(1 + i\rho(a) + i\widehat{\rho}(\widehat{a}))}$$

$$\Upsilon_b(x) = \prod_{m,n\geq 0} (mb + nb^{-1} + x)((m+1)b + (n+1)b^{-1} - x) .$$

# 5d $\mathcal{N} = 1$ theories on $S^5$

$S^1 \hookrightarrow S^5 \to \mathbb{CP}^2$:

$$Z_{S^5} = \frac{1}{|\mathcal{W}_G|} \int [da]\; e^{-\frac{8\pi^3 R}{g_{YM}^2}\operatorname{tr}(a^2) - \frac{\pi k}{3}\operatorname{tr}(a^3)}\; Z_{1-loop}(a,\widehat{a})\; Z_{inst}^{(1)} Z_{inst}^{(2)} Z_{inst}^{(3)}$$

[Kallen, Qiu, Zabzine '12; Kim$^3$ '12], based on [Hosomichi, Seong, Terashima '12]

$$Z_{1-loop}^{V} = \prod_{\alpha\in\Delta} \prod_{t\neq 0} (t - i\alpha(a))^{1+\frac{3}{2}t+\frac{1}{2}t^2}$$

$$Z_{1-loop}^{H} = \prod_{\rho\in\mathcal{R}_H^G} \prod_{\widehat{\rho}\in\mathcal{R}_H^{\widehat{G}}} \prod_{t} (t - i\rho(a) - i\widehat{\rho}(\widehat{a}))^{-(1+\frac{3}{2}t+\frac{1}{2}t^2)}$$