QCD Thermodynamics

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- Lecture I: QCD at finite temperature and density, continuum and lattice
- Lecture II: Applications of lattice thermodynamics
- Lecture III: Towards the QCD phase diagram at finite temperature and density

Literature

- O.P., “Lattice QCD at non-zero temperature and density”,
  Les Houches lecture notes 2009, arxiv:1009.4089
- O.P., “The QCD equation of state from the lattice”
  Prog. Part. Nucl. Phys. 70 (2013) 55, arxiv:1207.5999

  Proper references to covered material in those articles

Textbooks:

- Gale, Kapusta, “Finite temperature field theory: principles and applications”
- Montvay, Münster, “Quantum fields on a lattice”
- Gattringer, Lang, “Quantum chromodynamics on the lattice”

Units for these lectures

- Natural units: $\hbar = 1, c = 1$
- Einstein sum convention: $a_i b_i \equiv \sum_i a_i b_i$

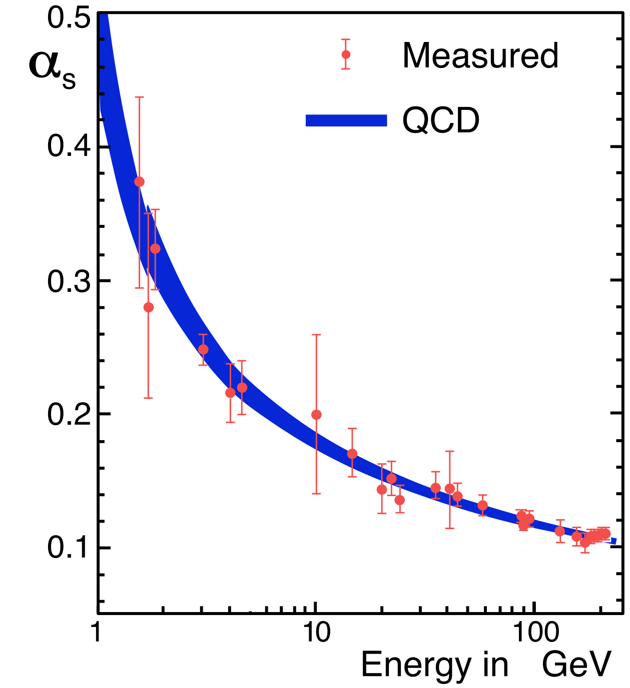
Hopefully only in a few instances (not intentional):

- Small circle approximation: $\pi = 1$
- Supra-natural approximations: $-1 = 1, i = 1$

Lecture I: QCD at finite temperature and density

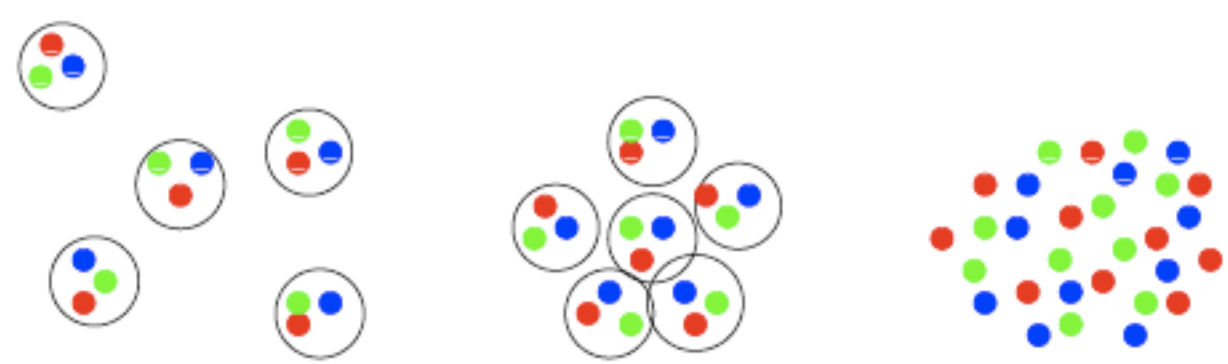
- Motivation: Why thermal QCD?
- The continuum formulation
- Differences and limitations of perturbation theory compared to T=0
- The lattice formulation

Why thermal QCD?



asymptotic freedom $\alpha_s(p \to \infty) \to 0$

$$\xrightarrow{\quad T, \mu_B \quad}$$



Hadrongas | Quark-Gluon-Plasma

Chiral symmetry: broken | (nearly) restored

Order parameters: $\langle \bar{\psi}\psi \rangle, \langle \psi\psi \rangle$

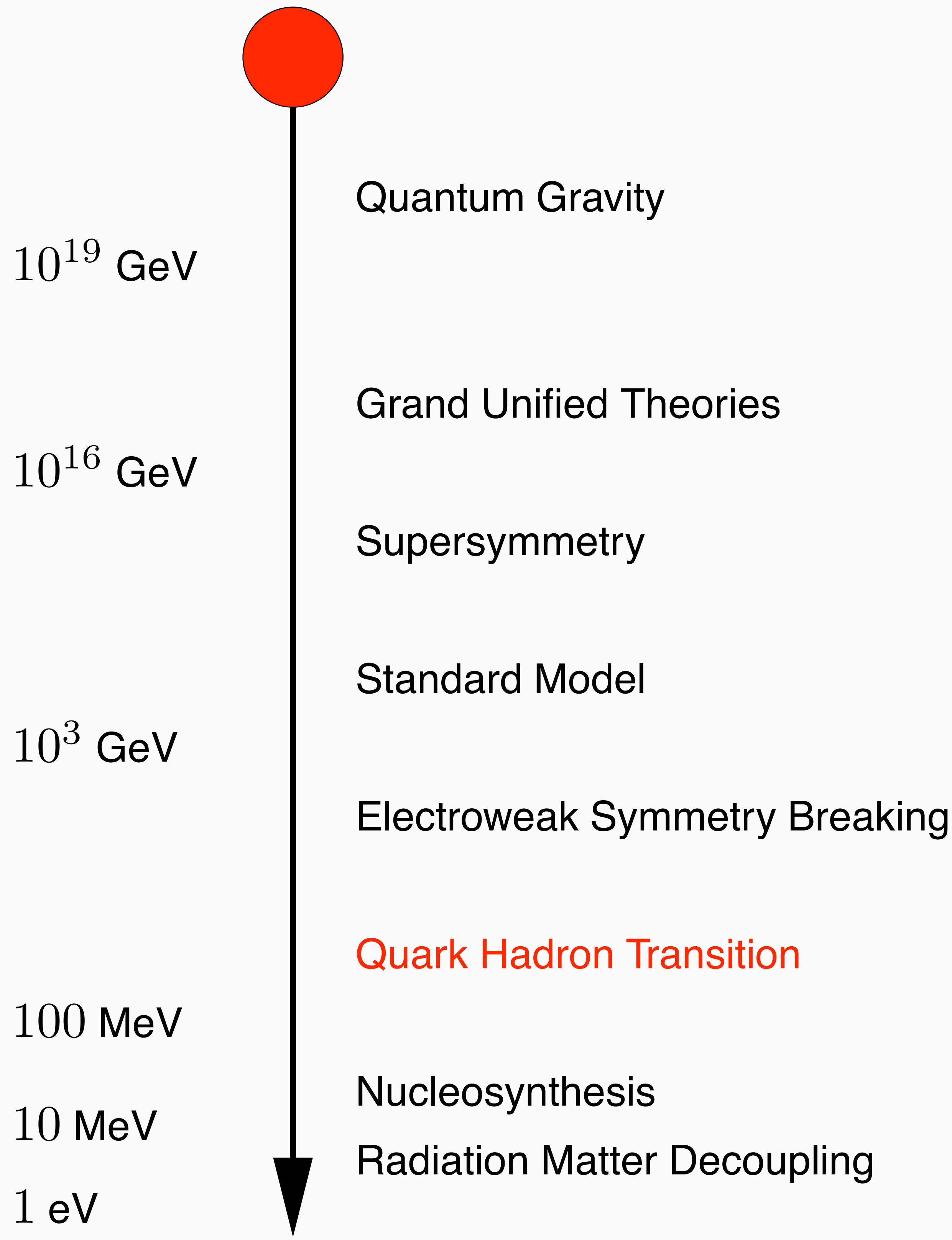
chiral condensate , Cooper pairs

Thermal QCD in nature

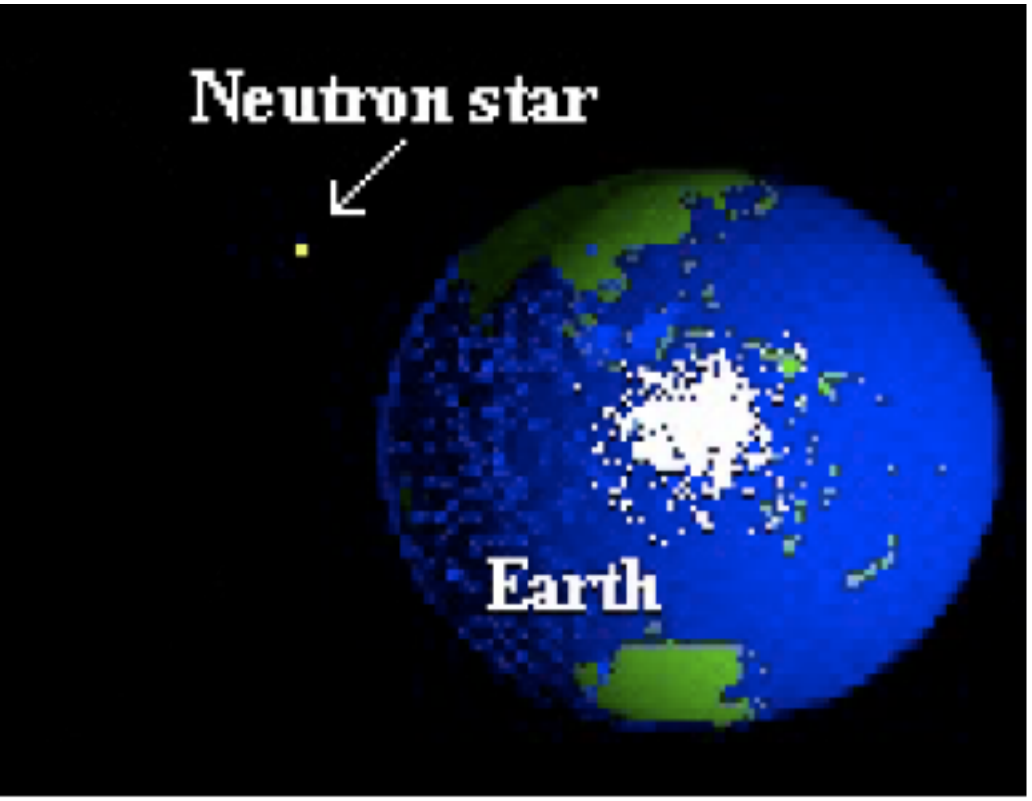
Physics of early universe:

non-abelian plasma physics
($\mu_B \approx 0$)

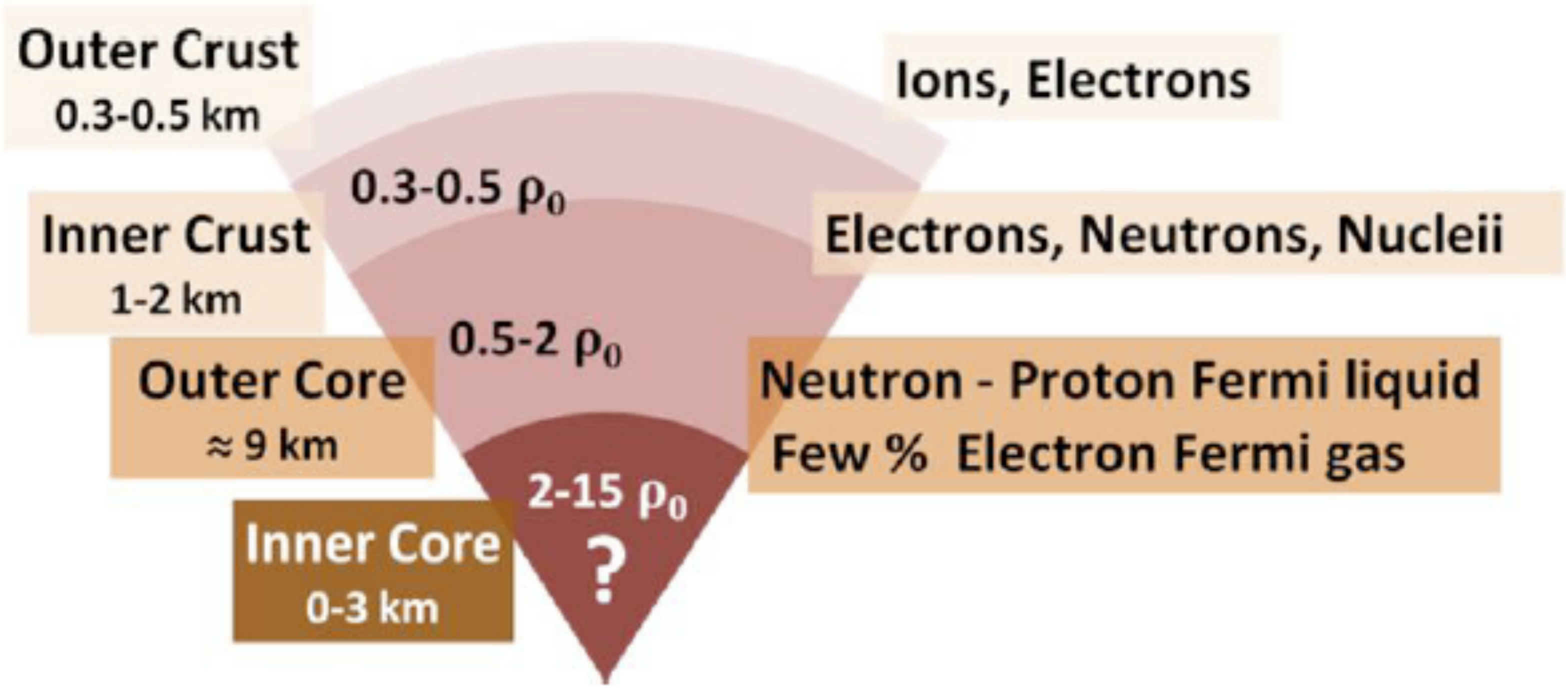
➡ QCD is prototype



What are compact stars made of?

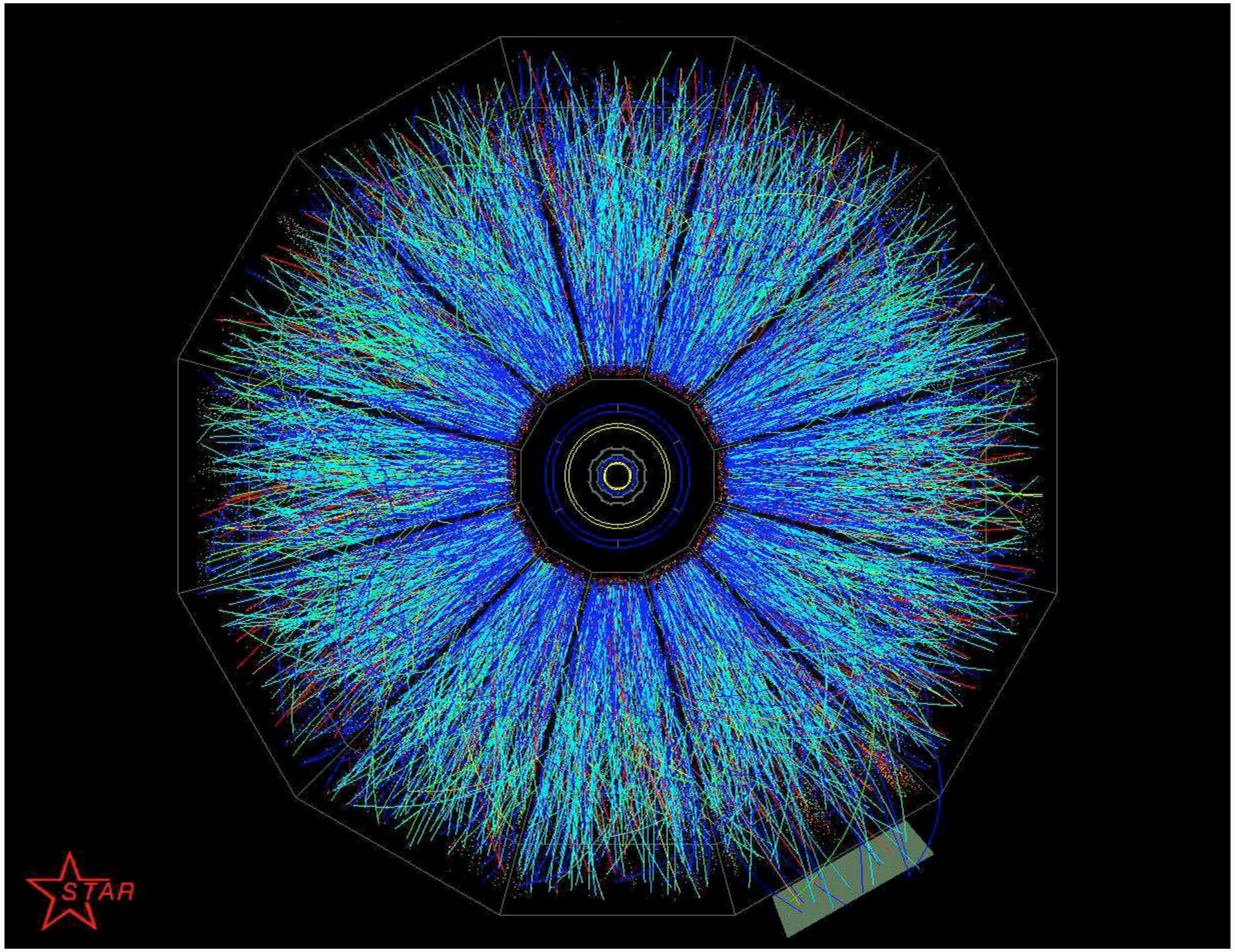


Radius ~ 10-12 km
Mass ~ 1.2-2.2 x Solar Mass



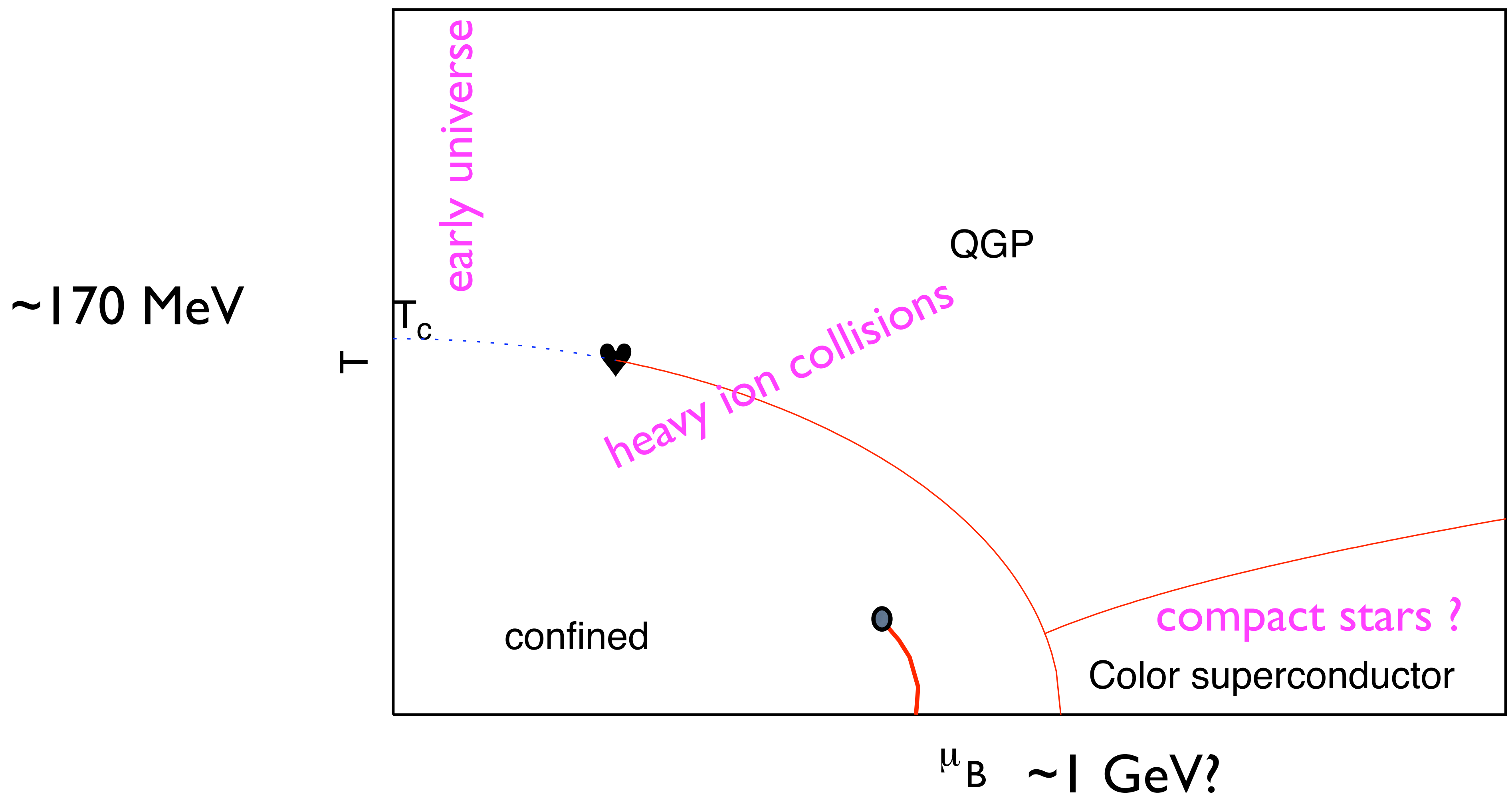
ρ_0 : nuclear density

Thermal QCD in experiment



heavy ion collision experiments at RHIC, LHC, GSI....

QCD phase diagram: theorist's view (science fiction)

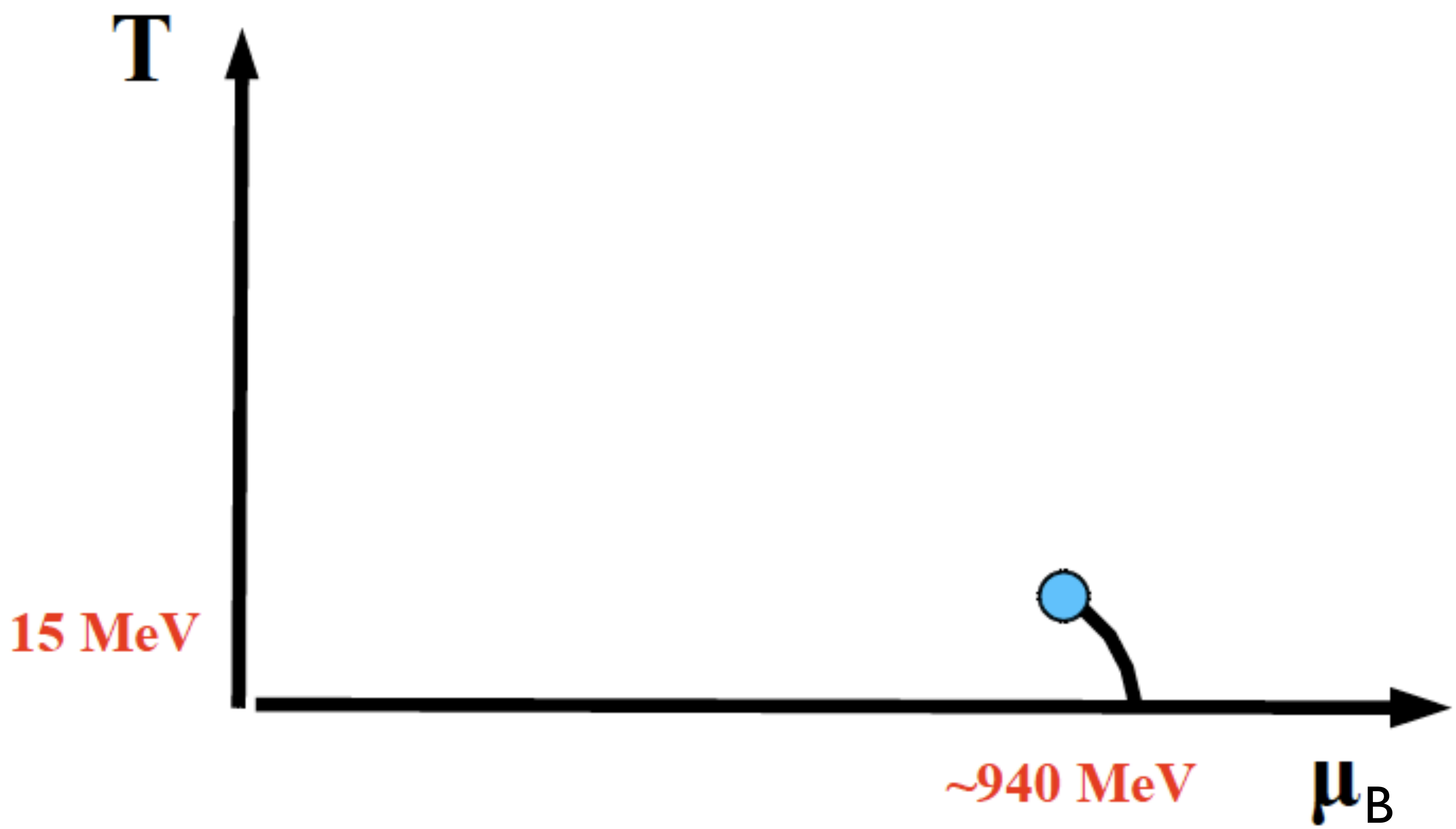


Until 2001: no finite density lattice calculations, sign problem!

Expectation based on simplifying models (NJL, linear sigma model, random matrix models, ...)

Check this from first principles QCD!

The QCD phase diagram established by experiment:



Nuclear liquid gas transition with critical end point

Statistical mechanics reminder

System of particles in volume V with conserved number operators, $N_i, i = 1, 2, \ldots$
in thermal contact with heatbath at temperature T

Canonical ensemble: exchange of energy with bath, particle number fixed

Grand canonical ensemble: exchange of energy and particles with the bath

Density matrix,
Partition function:

$$\rho = \mathrm{e}^{-\frac{1}{T}(H - \mu_i N_i)} \,, \qquad Z = \hat{\mathrm{Tr}}\rho \,, \quad \hat{\mathrm{Tr}}(\ldots) = \sum_n \langle n|(\ldots)|n\rangle$$

Thermodynamics:

$$F = -T \ln Z \,,$$
$$p = \frac{\partial(T \ln Z)}{\partial V} \,,$$
$$S = \frac{\partial(T \ln Z)}{\partial T} \,,$$

$$\bar{N}_i = \frac{\partial(T \ln Z)}{\partial \mu_i} \,,$$
$$E = -pV + TS + \mu_i \bar{N}_i$$

Densities:

$$f = \frac{F}{V}, \quad p = -f, \quad s = \frac{S}{V}, \quad n_i = \frac{\bar{N}_i}{V}, \quad \epsilon = \frac{E}{V}$$

QCD at finite temperature and density

Grand canonical partition function

$$Z(V,T,\mu;g,N_f,m_f) = \mathrm{Tr}(\mathrm{e}^{-(\mathrm{H}-\mu\mathrm{Q})/\mathrm{T}}) = \int \mathrm{DA}\,\mathrm{D}\bar{\psi}\,\mathrm{D}\psi\;\mathrm{e}^{-\mathrm{S_g}[\mathrm{A}_\mu]}\mathrm{e}^{-\mathrm{S_f}[\bar{\psi},\psi.\mathrm{A}_\mu]}$$

Action

$$S_g[A_\mu] = \int_0^{1/T} d\tau \int_V d^3x\,\frac{1}{2}\mathrm{Tr}\,F_{\mu\nu}(x)F_{\mu\nu}(x),$$

$$S_f[\bar{\psi},\psi,A_\mu] = \int_0^{1/T} d\tau \int_V d^3x \sum_{f=1}^{N_f} \bar{\psi}_f(x)\,(\gamma_\mu D_\mu + m_f - \mu_f\gamma_0)\,\psi_f(x)$$

$$A_\mu(\tau,\mathbf{x}) = A_\mu(\tau+\frac{1}{T},\mathbf{x}), \qquad \psi_f(\tau,\mathbf{x}) = -\psi_f(\tau+\frac{1}{T},\mathbf{x})$$

quark number $N_q^f = \bar{\psi}_f\gamma_0\psi_f$

Parameters

$$g^2, m_u \sim 3\mathrm{MeV}, m_d \sim 6\mathrm{MeV}, m_s \sim 120\mathrm{MeV}, V, T, \mu = \mu_B/3$$

$N_f = 2+1$ sufficient up to T~300-400 MeV

Difference to T=0: compact, periodic time direction!

Fourier expansion of the fields: discrete Matsubara frequencies

$$A_\mu(\tau,\mathbf{x}) = \frac{1}{\sqrt{VT}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} \mathrm{e}^{i(\omega_n \tau + \mathbf{p}\cdot\mathbf{x})} A_{\mu,n}(p)\ , \quad \omega_n = 2n\pi T\ , \qquad p_i = (2\pi n_i)/L$$

$$\psi(\tau,\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} \mathrm{e}^{i(\omega_n \tau + \mathbf{p}\cdot\mathbf{x})} \psi_n(p)\ , \quad \omega_n = (2n+1)\pi T$$

Thermodynamic limit: $$\frac{1}{V} \sum_{n_1,n_2,n_3} \xrightarrow{V\to\infty} \int \frac{d^3p}{(2\pi)^3}$$

Modified Feynman rules:

Inverse (bosonic) free propagator:

$$\Delta^{-1} = p^2 + m^2 = \omega_n^2 + \mathbf{p}^2 + m^2 = (2n\pi T)^2 + \mathbf{p}^2 + m^2$$

Loop integration:

$$\sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}$$

Perturbation theory at finite T

Split action into free (Gaussian) and interacting part, expand in interactions

$$Z = N\int D\phi\, \mathrm{e}^{-(S_0+S_i)} = N\int D\phi\, \mathrm{e}^{-S_0}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}S_i^l$$

$$\ln Z = \ln Z_0 + \ln Z_i = \ln\left(N\int D\phi\, \mathrm{e}^{-S_0}\right) + \ln\left(1+\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\frac{\int D\phi\, \mathrm{e}^{-S_0}S_i^l}{\int D\phi\, \mathrm{e}^{-S_0}}\right)$$

Renormalisation: Whatever renormalisation is necessary and sufficient at T=0 is also necessary and sufficient at finite temperature and density

UV behaviour: microscopic physics, depends on details of interactions

T, μ : macroscopic parameters, affect IR behaviour of the theory

Ideal gases from the Gaussian path integral

Important (sometimes unrealistic) model systems to (mis-)guide intuition

Real scalar field:

$$S_0 = \int_0^{\frac{1}{T}} d\tau \int d^3x \, \frac{1}{2}\phi(x)(-\partial_\mu\partial_\mu + m^2)\phi(x)$$

Fourier space:

$$S_0 = \frac{1}{2T^2}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{p}}(\omega_n^2+\omega^2)\phi_n(p)\phi_n^*(p)$$

$$\omega = \sqrt{\mathbf{p}^2+m^2} \qquad \phi_n^*(p) = \phi_{-n}(-p)$$

$$\begin{aligned}
Z_0 &= N\prod_{\mathbf{p}}\int d\phi_0 \exp\left[-\frac{1}{2T^2}(\omega_0^2+\omega^2)\phi_0^2(p)\right] \\
&\quad \times \prod_{n>0}\int d\phi_n\, d\phi_n^* \;\exp\left[-\frac{1}{2T^2}(\omega_n^2+\omega^2)\phi_n(p)\phi_n^*(p)\right] \\
&= N\prod_{\mathbf{p}}(2\pi)^{1/2}\left(\frac{\omega_0^2+\omega^2}{T^2}\right)^{-\frac{1}{2}}\prod_{n>0}\int d|\phi_n|\,|\phi_n|\exp\left[-\frac{1}{2T^2}(\omega_n^2+\omega^2)|\phi_n|^2\right] \\
&= N\prod_{\mathbf{p}}(2\pi)^{1/2}\left(\frac{\omega_0^2+\omega^2}{T^2}\right)^{-\frac{1}{2}}\prod_{n>0}\left(\frac{\omega_n^2+\omega^2}{T^2}\right)^{-1} = N'\prod_{n=-\infty}^{\infty}\prod_{\mathbf{p}}\left(\frac{\omega_n^2+\omega^2}{T^2}\right)^{-\frac{1}{2}} = N'(\det\Delta^{-1})^{-1/2}
\end{aligned}$$

Note: T-independent constants may be dropped (no contribution to thermodynamics)

$$\ln Z_0 = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{p}} \ln \frac{\omega_n^2+\omega^2}{T^2}$$

For Matsubara sum:

$$\ln\left[(2\pi n)^2+\frac{\omega^2}{T^2}\right] = \int_1^{\omega^2/T^2} \frac{d\theta^2}{\theta^2+(2\pi n)^2} + \ln(1+(2\pi n)^2)$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+(\frac{\theta}{2\pi})^2} = \frac{2\pi^2}{\theta}\left(1+\frac{2}{e^{\theta}-1}\right)$$

$$\begin{aligned}\ln Z_0 &= -\sum_{\mathbf{p}}\int_1^{\omega/T} d\theta\,\left(\frac{1}{2}+\frac{1}{e^{\theta}-1}\right) + \text{T-indep.}\\ &\overset{V\to\infty}{\longrightarrow} V\int\frac{d^3p}{(2\pi)^3}\left[\frac{-\omega}{2T} - \ln\left(1-e^{-\frac{\omega}{T}}\right)\right]\,.\end{aligned}$$

Vacuum energy, pressure: $E_0 = -\partial_{\frac{1}{T}}\ln Z_0 = \frac{V}{2}\int\frac{d^3p}{(2\pi)^3}\omega$ $\qquad p_0 = T\partial_V \ln Z_0 = -\frac{E_0}{V}$

divergent, zero point energy!

Renormalisation: $$p_{\text{phys}}(T) = p(T) - p(T=0)$$

Final result: $$\ln Z_0 = -V \int \frac{d^3p}{(2\pi)^3} \ln\left(1 - \mathrm{e}^{-\frac{\omega}{T}}\right)$$

m=0: $$p = \frac{\pi^2}{90} T^4$$

Fermion fields (Grassmann!): $$\ln Z_0 = 2V \int \frac{d^3p}{(2\pi)^3} \left[\ln\left(1 + \mathrm{e}^{-\frac{\omega-\mu}{T}}\right) + \ln\left(1 + \mathrm{e}^{-\frac{\omega+\mu}{T}}\right)\right]$$

two spin components

quarks and anti-quarks

m=0: $$p = \frac{7}{8} \frac{\pi^2}{90} T^4$$

General one-particle (field) expression: $$\ln Z_i^1(V,T) = \eta V \nu_i \int \frac{d^3p}{(2\pi)^3} \ln(1 + \eta\, \mathrm{e}^{-(\omega_i - \mu_i)/T})$$

$\eta = -1$ for bosons $\quad \nu_i$: spin and internal d.o.f

$\eta = 1$ for fermions

Ideal gases in QCD

Free gas of quarks and gluons: valid at infinite temperature, weak coupling limit

$$\frac{p}{T^4} = \left(2(N^2-1) + 4NN_f\frac{7}{8}\right)\frac{\pi^2}{90}$$

quark colours

2 gluon spin states   gluon colours   quark, anti-quark, each two spin states

Hadron resonance gas: at this point a model; later: strong coupling limit of full QCD

Quark-interactions "hidden" in hadrons; hadrons interact weakly

$$\ln Z(V,T) \approx \sum_i \ln Z_i^1(V,T) \qquad i = \pi, \rho, K, p, n, \ldots$$

IR-structure: divergences and mass scales

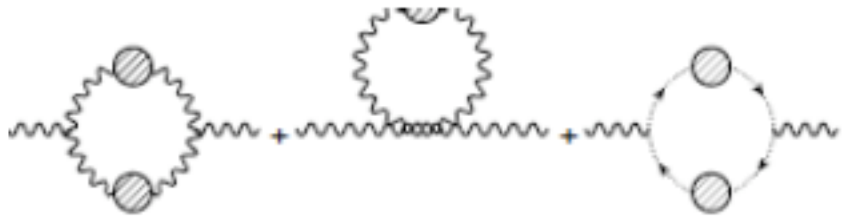
Inverse (bosonic) free propagator:

$$p^2 + m^2 = \omega_n^2 + \mathbf{p}^2 + m^2 = (2n\pi T)^2 + \mathbf{p}^2 + m^2$$

effective thermal mass ~T

n=0 mode: propagator of a 3d theory, divergent for m=0!

Corrections:



$$m_E^{LO} = \left(\frac{N}{3} + \frac{N_f}{6}\right)^{1/2} gT$$

electric or Debye screening mass $\langle A_0(\mathbf{x}) A_0(\mathbf{y}) \rangle$

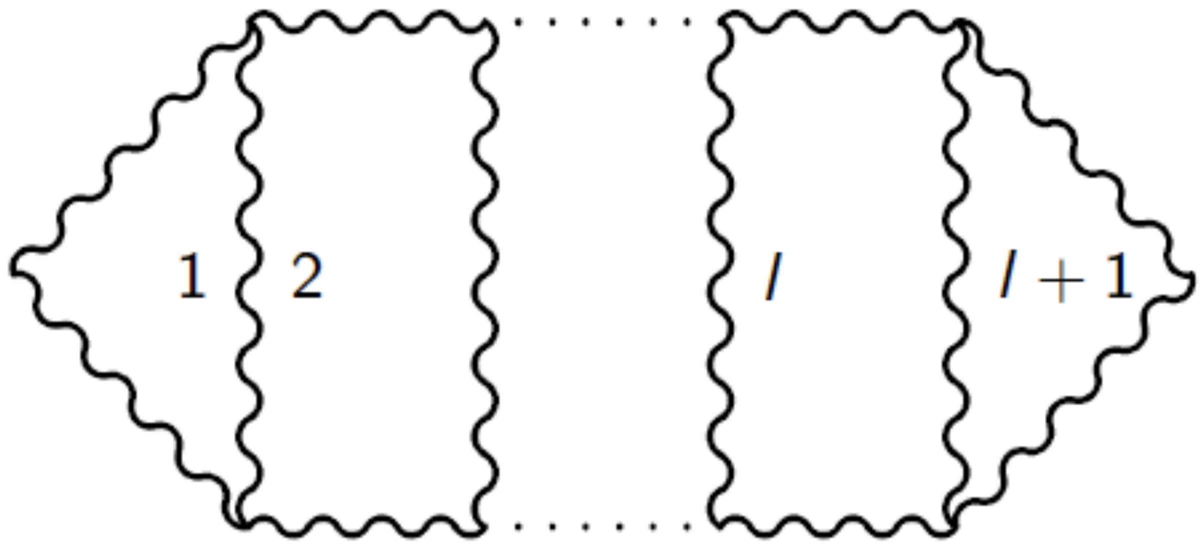
$$m_M^{LO} = 0, m_M \sim g^2 T \quad \text{from 2-loop}$$

magnetic screening mass $\langle A_i(\mathbf{x}) A_i(\mathbf{y}) \rangle$

0-mode sector of 4d QCD at finite T contains 3d Yang-Mills theory with $g_3^2 \sim g^2 T$
Confining! Doom for perturbation theory....

The Linde problem of finite T QCD / 3d YM

- $(l+1)$-loop diagram:contribution to pressure



contribution from
Matsubara 0-mode:

$$\boxed{P \sim g^{2l} \left(T \int d^3p\right)^{l+1} p^{2l}(p^2 + m^2)^{-3l}}$$

$$\begin{array}{rll} g^{2l} & \textit{for} & l = 1, 2 \\ g^6 T^4 \ln(T/m) & \textit{for} & l = 3 \\ g^6 T^4 (g^2 T/m)^{l-3} & \textit{for} & l > 3 \end{array}$$

magnetic mass $m_{mag} \sim g^2 T \Rightarrow$ **all loops** $(l > 3)$ contribute to g^6

even for weak coupling!

Same problem for all observables!
Only the order to which it occurrs is different:

E.g. for magnetic mass already at leading order (2-loop)

Perturbation theory at finite temperature works only
up to a finite, observable-dependent order,
no matter how weak the coupling!

Salvation comes as a lattice...

The lattice formulation at zero density

Hypercubic lattice: $N_s^3 \times N_\tau$, Lattice spacing a, Wilson's YM action:

$$S_g[U] = \sum_x \sum_{1 \le \mu < \nu \le 4} \beta \left(1 - \frac{1}{N} \mathrm{Re}\mathrm{Tr} U_p\right)$$

Plaquette: $U_p = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x)$ Lattice gauge coupling: $\beta = \frac{2N}{g^2}$

Periodic boundary conditions: $U_\mu(\tau, \mathrm{x}) = U_\mu(\tau + N_\tau, \mathrm{x}), U_\mu(\tau, \mathrm{x}) = U_\mu(\tau, \mathrm{x} + N_s)$

Transfer matrix formalism

Provides connection between path integral and Hamiltonian formalism

Rewrite action as sum over time slices:

$$S_g = \sum_\tau L[U_i(\tau+1), U_0(\tau), U_i(\tau)],$$

$$L[U_i(\tau+1), U_0(\tau), U_i(\tau)] = \frac{1}{2}L_1[U_i(\tau+1)] + \frac{1}{2}L_1[U_i(\tau)] + L_2[U_i(\tau+1), U_0(\tau), U_i(\tau)]$$

$$L_1[U_i(\tau)] = -\frac{\beta}{N}\sum_{p(\tau)} \mathrm{ReTr} U_p,$$ spatial plaquettes within one time slice

$$L_2[U_i(\tau+1), U_0(\tau), U_i(\tau)] = -\frac{\beta}{N}\sum_{p(\tau,\tau+1)} \mathrm{ReTr} U_p,$$ temporal plaquettes connecting slices

Transfer matrix: operator acting on square-integrable functions $\psi[U]$

Matrix elements: $$T[U_i(\tau+1), U_i(\tau)] = \int DU_0(\tau)\, \exp - L[U_i(\tau+1), U_0(\tau), U_i(\tau)]$$

Translation of states by one time-slice: $|\psi[U_i(\tau+1,\mathbf{x})]\rangle = T\,|\psi[U_i(\tau,\mathbf{x})]\rangle$

Identify: $T = e^{-aH}$

Rewrite partition function exactly:

$$Z = \int \prod_\tau \left(DU_i(\tau,\mathrm{x})\, T[U_i(\tau+1), U_i(\tau)] \right) = \hat{\mathrm{T}}\mathrm{r}(T^{N_\tau}) = \hat{\mathrm{T}}\mathrm{r}(\mathrm{e}^{-N_\tau a H})$$

Identify: $\frac{1}{T} \equiv a N_\tau$ $H|n\rangle = E_n|n\rangle$

complete set of energy eigenstates

Thermal expectation value: $\langle O\rangle = Z^{-1}\hat{\mathrm{T}}\mathrm{r}(\mathrm{e}^{-\frac{H}{T}}O) = Z^{-1}\sum_n \langle n|T^{N_\tau}O|n\rangle = \frac{\sum_n \langle n|O|n\rangle\, \mathrm{e}^{-aN_\tau E_n}}{\sum_n \mathrm{e}^{-aN_\tau E_n}}$

Thermodynamic limit: $N_s \to \infty$ but keep T finite

Vacuum expectation value: $\langle 0|O|0\rangle = \lim_{N_\tau\to\infty} \frac{\sum_n \langle n|O|n\rangle\, \mathrm{e}^{-aN_\tau(E_n-E_0)}}{\sum_n \mathrm{e}^{-aN_\tau(E_n-E_0)}}$

The space-wise transfer matrix

- Hamiltonian translates in time;
  Spectrum: particle masses, from exp. decay of correlators in time

- May also define a Hamiltonian translating the system in space;
  Spectrum: screening masses, from exp. decay of correlators in space

$$T[U(z+1),U(z)] \equiv \mathrm{e}^{-aH_z}, \quad Z = \mathrm{Tr}(\mathrm{e}^{-aN_zH_z}) \qquad U(z)\ :\ \{U_\mu(z)|\mu \neq 3\}$$

Vacuum physics: $N_{x,y,z,\tau} \to \infty$ H, H_z spectra identical

Thermal physics: $N_{x,y,z} \to \infty$ and keep N_τ

H_z acts on states defined on $N_{x,y,\tau}$ lattice;

spectrum of theory on torus with one side squeezed

Finite T physics = finite size effect of the shortened time direction!

Adding fermions

Pick a suitable fermion action:

$$S_f = \sum_{x,y} \bar{\psi}(x) M_{xy}(m_f)\, \psi(y)$$

Full QCD partition function:

$$Z(N_s, N_\tau; \beta, m_f) = \int DU \, \prod_f \det M(m_f)\, \mathrm{e}^{-S_g[U]} \, ,$$

$$U_\mu(\tau, \mathbf{x}) = U_\mu(\tau + N_\tau, \mathbf{x}),$$

$$\psi(\tau, \mathbf{x}) = -\psi(\tau + N_\tau, \mathbf{x}) \, .$$

Wilson fermions:

$$S_f^W = \frac{1}{2a} \sum_{x,\mu,f} a^4 \, \bar{\psi}_f(x)[(\gamma_\mu - r) U_\mu(x) \psi_f(x+\hat{\mu}) - (\gamma_\mu + r) U_\mu^\dagger(x-\hat{\mu}) \psi_f(x-\hat{\mu})]$$

$$+ (m + 4\frac{r}{a}) \sum_{x,f} a^4 \, \bar{\psi}_f(x) \psi_f(x)$$

pick your poison

- Wilson fermions
  add irrelevant ops. (going away in CL) to make doublers very massive
  breaks chiral symmetry for non-zero a

- staggered (Kogut-Susskind) fermions
  distribute spinor components on different sites, reduces to 4 flavours
  take 4th root of determinant to get to one flavour, keeps reduced chiral symm.
  non-local operation, have to take CL before chiral limit, mixing of spin, flavour

- domain wall fermions
  introduce 5th dimension, fermions massive in that dim. and chiral in the other
  expensive

- overlap fermions
  non-local formulation with modified chiral symmetry even for finite a
  order of magnitude more expensive than Wilson

Continuum limit

$$\frac{1}{T} \equiv a N_\tau$$

Fixed scale approach:

- For a given lattice spacing, N_τ controls temperature;
- Allows only discrete temperatures, too large for many applications;
- Continuum limit requires series of lattice spacings

Fixed N_τ approach:

- For a given N_τ, vary the lattice spacing via $\beta(a)$;
- Allows continuous temperatures, but each T value has different cut-off!
- Continuum limit requires series of N_τ

Lines of constant physics and setting the scale

- Compute observable for series of a, N_τ , $\langle O\rangle(\beta, m_f)$
- Tune bare parameters such that for each lattice spacing renormalised parameters are constant $m_f^R(am_{u,d}(\beta), am_s(\beta), \beta) = \text{const}$.
- More practical: keep physical quantities constant $O_i^{\text{phys}}(am_{u,d}(\beta), am_s(\beta), \beta) \;=\; \text{const}$
- Non-trivial because of cut-off effects: Different for different quantities and actions $O^{\text{phys}}(a) = O^{\text{phys}} + c_1 a + c_2 a^2 + \ldots$

Perturbative relation for $\beta(a)$: only good very close to continuum limit

Λ_{QCD} on lattice:

$$a\Lambda_L = \left(\frac{6b_0}{\beta}\right)^{-b_1/2b_0^2} e^{-\frac{\beta}{12b_0}},$$

$$b_0 = \frac{1}{16\pi^2}\left(11 - \frac{2}{3}N_f\right), \quad b_1 = \left(\frac{1}{16\pi^2}\right)^2\left[102 - \left(10 + \frac{2}{3}\right)N_f\right]$$

Non-perturbatively: Express computed quantity in units of another known quantity

E.g. for the critical temperature of a phase transition:

$$\frac{T_c}{m_H} = \frac{1}{a_c m_H N_\tau} = \frac{1}{a(\beta_c) m_H N_\tau}$$

Compute hadron mass at the critical lattice spacing: $a^{-1} = \frac{m_H[\mathrm{MeV}]}{(am_H)(\beta_c)}$

N.B.: Only possible when operating at physical quark masses!

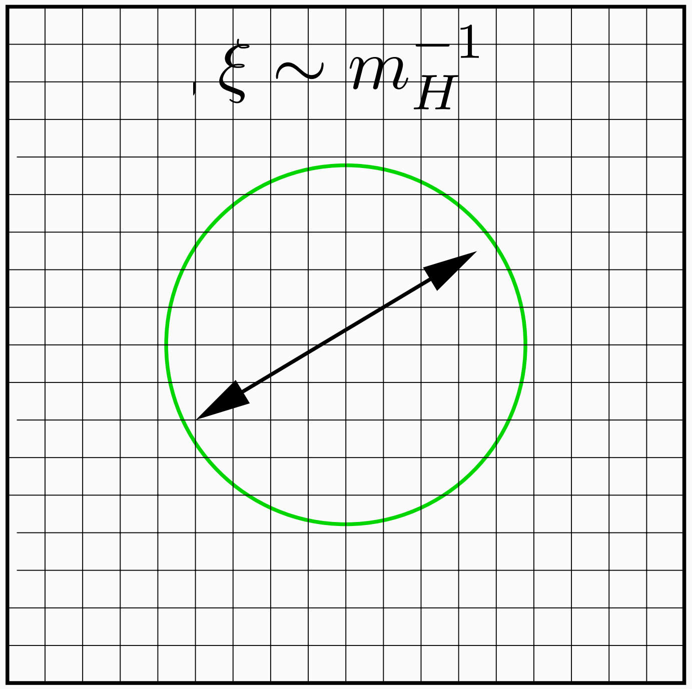
For unphysical quark masses:
(out of computational limitations or interest in certain limits, mass dependence etc.)

Take quantity that depends only weakly on quark masses:
String tension, Sommer scale

$$\frac{T}{\sqrt{\sigma}} = \frac{1}{a\sqrt{\sigma}N_\tau}, \quad \sigma \approx 425\ \mathrm{MeV}; \quad Tr_0 = \frac{r_0}{aN_\tau}, \quad r^2\frac{dV(r)}{dr} = 1.65$$

Requirements for and constraints from the lattice:

correlation length ξ : lightest gauge invariant (hadronic?) mass scale



$a \ll \xi \ll aL$!

scale of interest: $T_c \sim 200\mathrm{MeV} \sim (1\mathrm{fm})^{-1}$

feasible lattices: $32^3 \times 4, 16^3 \times 8$

$T = \frac{1}{aN_t}$ ➡ $N_\tau = 4, 8, 12$ implies $a \approx 0.25, 0.125, 0.083$ fm

➡ $aL \sim 1.5 - 3\mathrm{fm}$

low T (confined) phase: $m_\pi \gtrsim 250\mathrm{MeV}$ lighter just beginning...

high T (deconfined) phase: $m_\pi \sim T, \xi \sim 1/T$ ➡ $\frac{1}{N_t} \ll 1 \ll \frac{L}{N_t}$

➡ $T \lesssim 5T_c$

The ideal gas on the lattice

Starting point: propagator of a free scalar field

$$\begin{aligned}
\ln Z_0 &= -\frac{1}{2}\ln\det\Delta = \frac{1}{2}\mathrm{Tr}\ln\Delta^{-1} \\
&= V\sum_{n=-N_\tau/2}^{N_\tau/2-1}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^3p}{(2\pi)^3}\ln(\hat{p}^2+(am)^2) \\
&= V\sum_{n=-N_\tau/2}^{N_\tau/2-1}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^3p}{(2\pi)^3}\ln\left(4\sin^2(\frac{a\omega_n}{2})+4\hat{\omega}^2\right)
\end{aligned}$$

Lattice momenta:

$$\hat{p}^2 = 4\sin^2(\frac{a\omega_n}{2}) + 4\sum_{j=1}^{3}\sin^2(\frac{ap_j}{2}) \qquad 4\hat{\omega}^2 = 4\sum_{j=1}^{3}\sin^2(\frac{ap_j}{2}) + (am)^2$$

Matsubara sum by analytic continuation, use: $$\frac{1}{N_\tau}\sum_{n=-N_\tau/2}^{N_\tau/2-1} g(\mathrm{e}^{i\omega_n}) = -\sum_{z_i}\frac{\mathrm{Res}(\frac{g(z_i)}{z_i})}{z_i^{N_\tau}-1}$$

Substitution: $\hat{\omega} = \sinh(aE/2)$

$$\ln Z_0 = -V \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^3p}{(2\pi)^3} \ln(1 - \mathrm{e}^{-N_\tau aE})$$

Expand in small lattice spacing about continuum limit:

$$I(a) = \int_{\frac{\pi}{a}}^{\infty} \frac{d^3p}{(2\pi)^3} \ln(1 - \mathrm{e}^{-N_\tau aE}) = I(0) + \frac{dI}{da}\, a$$

$$I(0) \; = \; 0 \qquad I'(a) \; \propto \; \exp - N_\tau \pi \stackrel{a \to 0}{\longrightarrow} 0$$

So we may put a=0 in the integration limits! Now expand the dispersion relation

$$\sinh^2(\frac{aE}{2}) = \sum_{j=1}^{3} \sin^2(\frac{ap_j}{2}) + \frac{(am)^2}{4}$$

$$E(\mathbf{p}) = E^{(0)}(\mathbf{p}) + aE^{(1)}(\mathbf{p}) + a^2 E^{(2)}(\mathbf{p}) + \ldots, \qquad E^{(0)}(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$$

$$\frac{(aE)^2}{4} + \frac{(aE)^4}{48} \;\; = \;\; \sum_{j=1}^{3} \left(\frac{(ap_j)^2}{4} - \frac{(ap_j)^4}{48} \right) + \frac{(am)^2}{4} + O(a^6)$$

$$E^{(2)}(\mathbf{p}) = -\frac{1}{24E^{(0)}(\mathbf{p})}\left(\sum_{j=1}^{3} p_j^4 + E^{(0)4}(\mathbf{p})\right)$$

breaks rotation invariance

The bosonic dispersion relation has leading $O(a^2)$ cut-off effects

Improvement: subtracting these, the dispersion relation is "p4-improved"

Expansion of the pressure now simple: expand down e^{-aEN_τ} , then expand log

Use dimensionless variables: $x = p/T, \varepsilon = E/T$

$$\frac{p}{T^4} = \left(\frac{p}{T^4}\right)_{\text{cont}} - a^2 \int \frac{d^3x}{(2\pi)^3} \frac{\varepsilon^{(2)}(x)}{\mathrm{e}^{\varepsilon^{(0)}(x)} - 1} + \dots$$

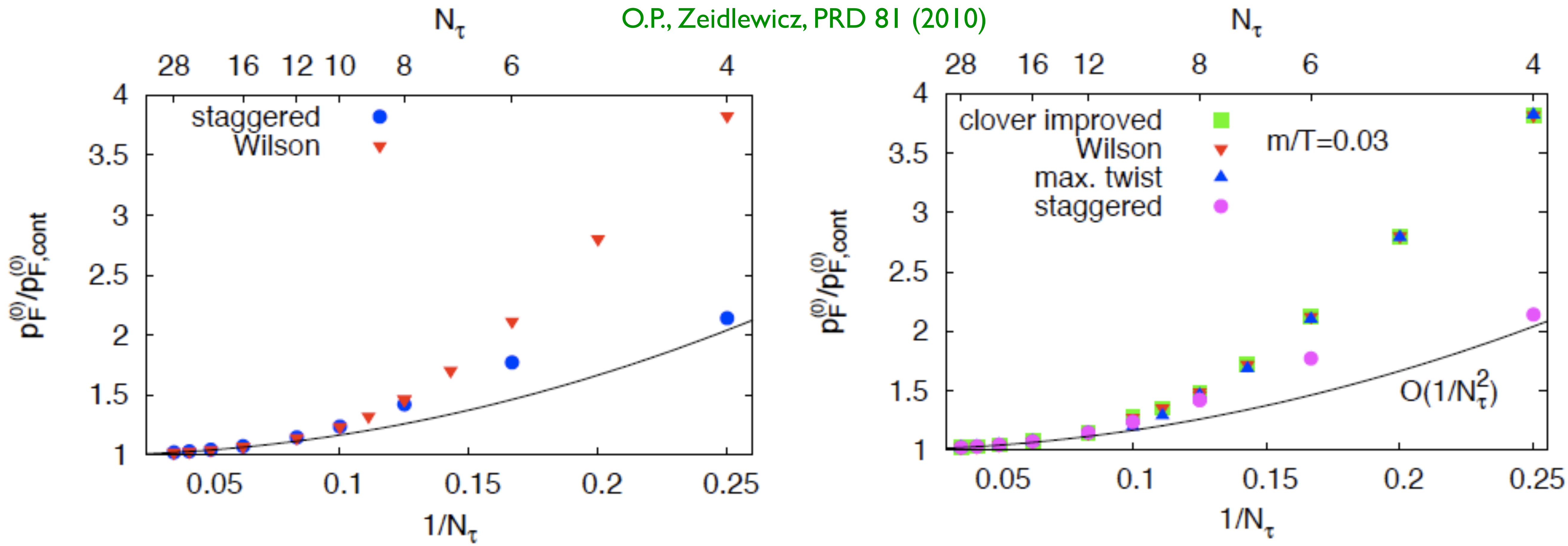
$$\frac{p}{p_{\text{cont}}} = 1 + \frac{8\pi^2}{21}\frac{1}{N_\tau^2} + O\left(\frac{1}{N_\tau^4}\right)$$

Free boson gas has leading $O(a^2)$ cut-off effects!

Free fermion gas on the lattice

Analogous calculation, massless case starts also at $O(a^2)$

O.P., Zeidlewicz, PRD 81 (2010)



For Wilson fermions with finite mass, the leading correction is $O(a)$, staggered $O(a^2)$

Note: $N_\tau \geq 10$ required for leading cut-off effects to dominate!

Summary Lecture I

- Perturbation theory of finite T QCD in continuum has infrared problems
- Long wavelength modes of finite T QCD are always confining, even at high T
- Finite T on the lattice is a finite size effect
- For simulations with fixed Nt discretisation errors are T-dependent
- Perturbation theory allows assessment of cut-off effects, but only at high T

Lecture II:

Owe Philipsen

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

- QCD in the static and chiral limit
- The equation of state
- Screening masses
- Free energy of static quarks
- Phase transitions

Quenched limit of QCD and Z(N) symmetry

Infinite quark masses (omitting flavour index) $m \rightarrow \infty$

Static quark propagator: $\langle \psi^a_\alpha(\tau, \mathbf{x}) \bar{\psi}^b_\beta(0, \mathbf{x}) \rangle = \delta_{\alpha\beta}\, e^{-m\tau} \left(T e^{i \int_0^\tau d\tau A_0(\tau, \mathbf{x})} \right)_{ab}$

On the finite T lattice: Polyakov loop $L(\mathbf{x}) = \prod_{x_0}^{N_\tau} U_0(x)$

Static QCD:
(one flavour)

$$S_{\text{static}}[U] = S_g[U] + \sum_{\mathbf{x}} \left(e^{-mN_\tau} \text{Tr} L(\mathbf{x}) + e^{-mN_\tau} \text{Tr} L^\dagger(\mathbf{x}) \right)$$

$$\xrightarrow{m \rightarrow \infty} S_g[U]$$

Gauge transformations: $U^g_\mu(x) = g(x) U_\mu(x) g^{-1}(x + \hat{\mu}), \quad g(x) \in SU(N)$

Periodic b.c.: $U_\mu(\tau, \mathbf{x}) = U_\mu(\tau + N_\tau, \mathbf{x}), \quad g(\tau, \mathbf{x}) = g(\tau + N_\tau, \mathbf{x})$

Action gauge invariant: $S_g[U^g] = S_g[U] \qquad L^g(\mathbf{x}) = g(x) L(\mathbf{x}) g^{-1}(x) \qquad \text{Tr} L^g = \text{Tr} L$

Topologically non-trivial gauge transformations:

Modified b.c. for trafo matrix: $g'(\tau+N_\tau,\mathbf{x}) = hg'(\tau,\mathbf{x})\ , \quad h \in SU(N)$

global "twist"

$U_\mu^{g'}(\tau+N_\tau,\mathbf{x}) = h\,U_\mu^{g'}(N_\tau,\mathbf{x})\,h^{-1}$ needs to be periodic for correct finite T physics!

$\Rightarrow$ $h = z\mathbf{1} \in Z(N), \quad z = \exp i\frac{2\pi n}{N}, \quad n \in \{0,1,2,\ldots N-1\}$ Centre of SU(N)

$S_g[U^{g'}] = S_g[U]$ invariant: centre symmetry of pure gauge action

Note: this is not a symmetry of H, but of H_z !

Requires compact time direction with periodic b.c. ; finite T!

$$L^{g'}(\mathbf{x}) = g'(1,\mathbf{x})L(\mathbf{x})g'^{-1}(1+N_\tau,\mathbf{x}) = g'(1,\mathbf{x})L(\mathbf{x})g'^{-1}(1,\mathbf{x})h^{-1}$$

$\mathrm{Tr}L^{g'} = z^* \mathrm{Tr}L$ Polyakov loop picks up a phase under centre transformations

Partition function in the presence of one static quark: $Z_Q = \int DU \; \mathrm{Tr}L(\mathbf{x}) \; \mathrm{e}^{-S_g[U]}$

$$\langle \mathrm{Tr}L \rangle = \frac{1}{Z} \int DU \; \mathrm{Tr}L \; \mathrm{e}^{-S_g} = \frac{Z_Q}{Z} = \mathrm{e}^{-(F_Q - F_0)/T}$$

gives free energy difference of thermal YM-system with and without a static quark

Small T: $F_Q = \infty$ because of confinement ➡ $\langle \mathrm{Tr}L \rangle = 0$

Large T: $\beta \to \infty$ $U_0 \to \mathbf{1}$ ➡ $\langle \mathrm{Tr}L \rangle \to \mathrm{Tr}\mathbf{1} = N$

Thus Polyakov loop is non-analytic function of T ➡ phase transition!

Deconfinement phase transition in YM: spontaneous breaking of Z(N) symmetry

Now add dynamical quarks:

$$\psi^g(x) = g(x)\psi(x), \quad \psi(\tau + N_\tau, \mathbf{x}) = -\psi(\tau, \mathbf{x}), \quad \psi^{g'}(\tau + N_\tau, \mathbf{x}) = -h\psi(\tau, \mathbf{x})$$

needs to be anti-periodic for correct finite T physics! $h = 1$ only

➡ Centre symmetry explicitly broken by dynamical quarks!

$\langle \mathrm{Tr} L \rangle \neq 0$ for all T!

➡ Confined and deconfined region analytically connected (only one phase!)
No need for a phase transition!

Massless QCD and chiral symmetry (continuum)

massless quarks:

S_f invariant under global chiral transformations $U_A(1) \times SU(N_f)_L \times SU(N_f)_R$

spontaneous symmetry breaking: $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$

$\Rightarrow$ $N_f^2 - 1$ massless Goldstone bosons, pions

order parameter: chiral condensate $\langle\bar{\psi}\psi\rangle = \frac{1}{L^3 N_t} \frac{\partial}{\partial m_q} \ln Z$

$$\langle\bar{\psi}\psi\rangle \begin{cases} > 0 \Leftrightarrow \text{symmetry broken phase,} & T < T_c \\ = 0 \Leftrightarrow \text{symmetric phase,} & T > T_c \end{cases}$$

chiral transition: spontaneous restoration of global $SU(N_f)_L \times SU(N_f)_R$ at high T

Chiral symmetry explicitly broken by dynamical quarks, no need for phase transition!

Physical QCD

.....breaks both chiral and Z(3) symmetry explicitly

.....but displays confinement and very light pions

- no order parameter $\Rightarrow$ no phase transition necessary!
- if there is a p.t.: are there two distinct transitions?
- if there is just one p.t.: is it related to chiral or Z(3) dynamics?
- if there is no phase transition: how do the properties of matter change?

Equation of state: ideal (non-interacting) gases

partition fcn. for one relativistic bosonic/fermionic d.o.f.:

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \ln\left(1 \pm \mathrm{e}^{-(E(p)-\mu)/T}\right)^{\pm 1}, \qquad E(p) = \sqrt{\mathbf{p}^2 + m^2}$$

equation of state for g d.o.f., two relevant limits: Stefan-Boltzmann

Relativistic Boson, $m \ll T$	$\times$ (Fermion)	Non-relativistic, $m \gg T$
$p_r = g\frac{\pi^2}{90}T^4$	$\left(\frac{7}{8}\right)$	$p_{nr} = gT\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}\exp(-m/T)$
$\epsilon_r = g\frac{\pi^2}{30}T^4$	$\left(\frac{7}{8}\right)$	$\epsilon_{nr} = \frac{m}{T}p_{nr} \gg p_{nr}$

$\Rightarrow$ $$p_r = \epsilon_r/3, \qquad p_{nr} \simeq 0$$

The QCD equation of state

Task: compute free energy density or pressure

$$f = -\frac{T}{V}\ln Z(T,V)$$

➡ all bulk thermodynamic properties follow:

$$p = -f, \qquad \frac{\epsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right), \qquad \frac{s}{T^3} = \frac{\epsilon + p}{T^4}, \qquad c_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\epsilon}$$

Technical problem: partition function in Monte Carlo normalized to 1.

➡ Z, p, f not directly calculable, only $\langle O \rangle = Z^{-1}\,\mathrm{Tr}(\rho O)$

➡ Integral method: $$\left.\frac{f}{T^4}\right|_{T_o}^{T} = -\frac{1}{V}\int_{T_o}^{T}\mathrm{d}x\,\frac{\partial x^{-3}\ln Z(x,V)}{\partial x}$$

modify for lattice action: Integration along line of constant physics!

$$\left.\frac{f}{T^4}\right|_{(\beta_o,m_{f0})}^{(\beta,m_f)} = -\frac{N_\tau^3}{N_s^3}\int_{\beta_o,m_{f0}}^{\beta,m_f}\left(\mathrm{d}\beta'\left[\left\langle\frac{\partial\ln Z}{\partial\beta'}\right\rangle - \left\langle\frac{\partial\ln Z}{\partial\beta'}\right\rangle_{T=0}\right] + \sum_f \mathrm{d}m_f'\left[\left\langle\frac{\partial\ln Z}{\partial m_f'}\right\rangle - \left\langle\frac{\partial\ln Z}{\partial m_f'}\right\rangle_{T=0}\right]\right)$$

N.B.: lower integration constant not rigorously defined, but exponentially suppressed

$$\frac{f}{T^4}(\beta_0) \sim \mathrm{e}^{-\mathrm{m}_{\text{Hadron}}/\mathrm{T}} \approx 0$$

cut-off effects in the high temperature, ideal gas limit: momenta $\sim T \sim \frac{1}{a}$

$$\left.\frac{p}{T^4}\right|_{N_\tau} = \left.\frac{p}{T^4}\right|_{\infty} + \frac{c}{N_\tau^2} + \mathcal{O}(N_\tau^{-4}) \qquad \text{(staggered)}$$

Quantities to be calculated:

$$\frac{1}{N_\tau N_s^3}\frac{\partial \ln Z}{\partial \beta} = \frac{1}{N_\tau N_s^3}\left\langle \sum_p U_p \right\rangle = \langle -s_g \rangle$$

$$\frac{1}{N_\tau N_s^3}\frac{\partial \ln Z}{\partial m_f} = \frac{1}{N_\tau N_s^3}\left\langle \sum_x \bar{\psi}_f(x)\psi_f(x) \right\rangle$$

For the numerical integration along lines of constant physics, need beta-functions!

Directly accessible before integration: trace anomaly

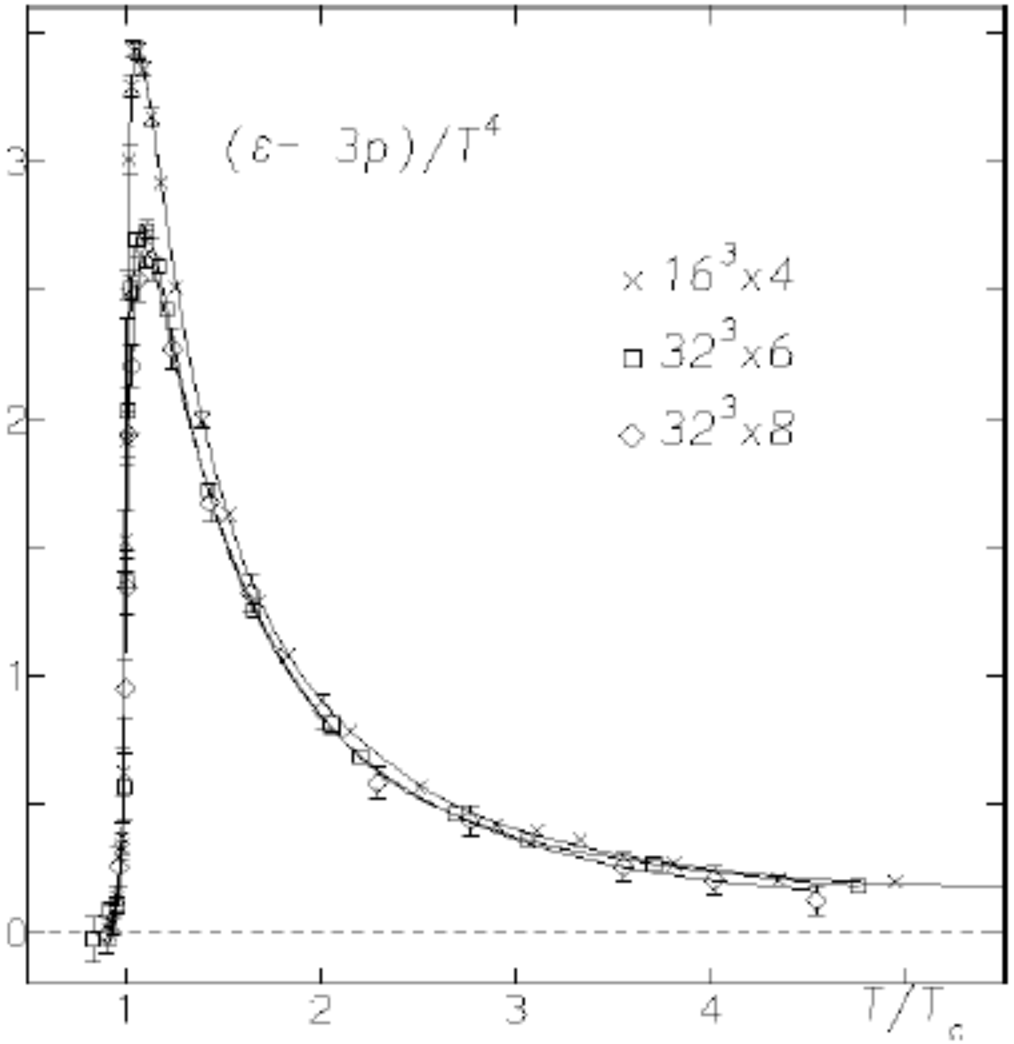
$$I(T) \equiv T^{\mu\mu}(T) = T^5 \frac{\partial}{\partial T}\frac{p(T)}{T^4} = \epsilon - 3p$$

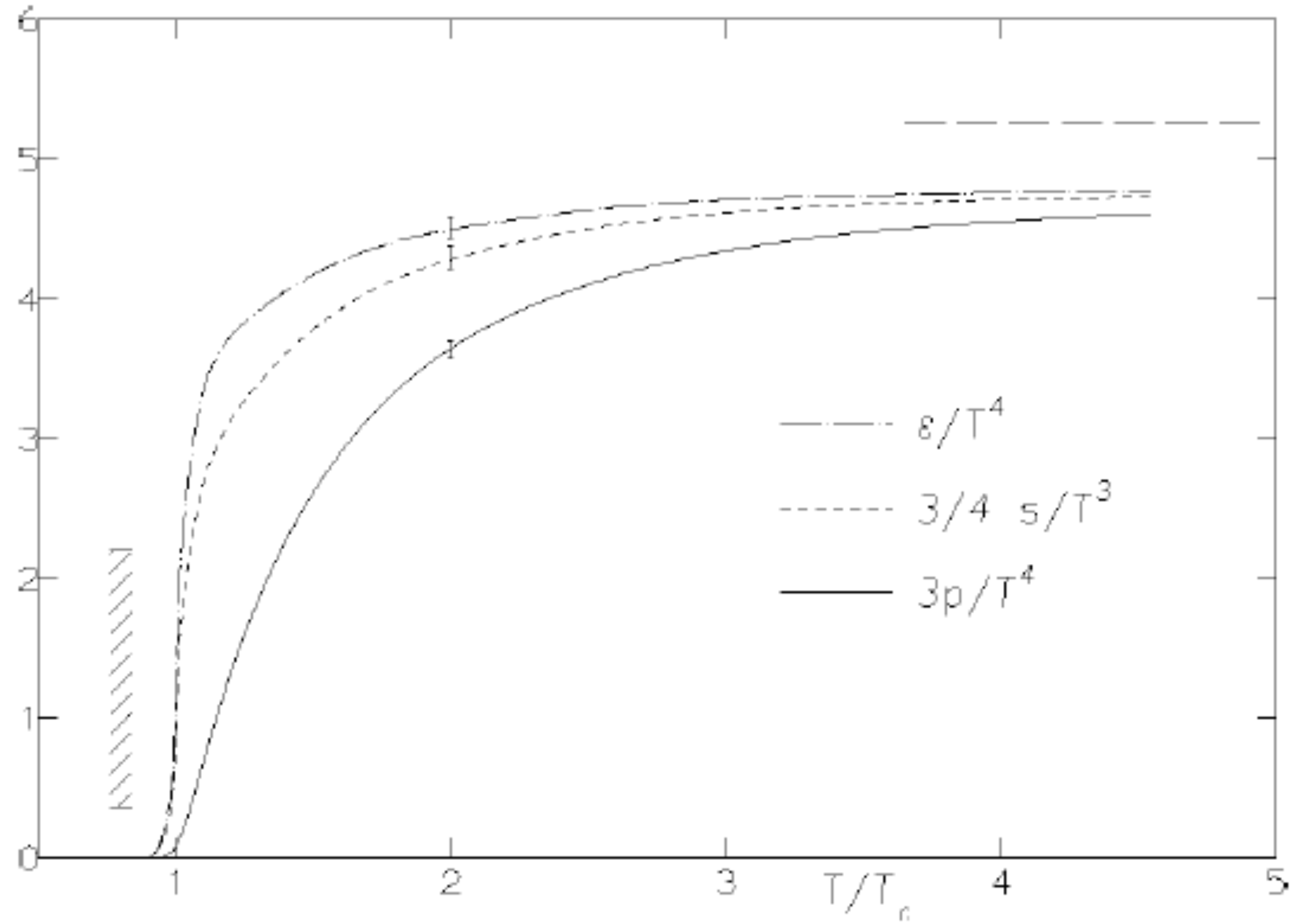
$$\frac{I(T)}{T^4}\frac{dT}{T} = N_\tau^4 \left(d\beta \langle -s_g \rangle^{\mathrm{sub}} + \sum_f dm_f \langle \bar{\psi}_f \psi_f \rangle^{\mathrm{sub}} \right) ,$$

$$\frac{I(T)}{T^4} = -N_\tau^4 \left(a\frac{d\beta}{da} \langle -s_g \rangle^{\mathrm{sub}} + \sum_f a\frac{dm_f}{da} \langle \bar{\psi}_f \psi_f \rangle^{\mathrm{sub}} \right)$$

Numerical results, pure gauge

Boyd et al., NPB 469 (1996)





Ideal gas behaviour at high and low T

Continuum extrapolation using Nt=6,8

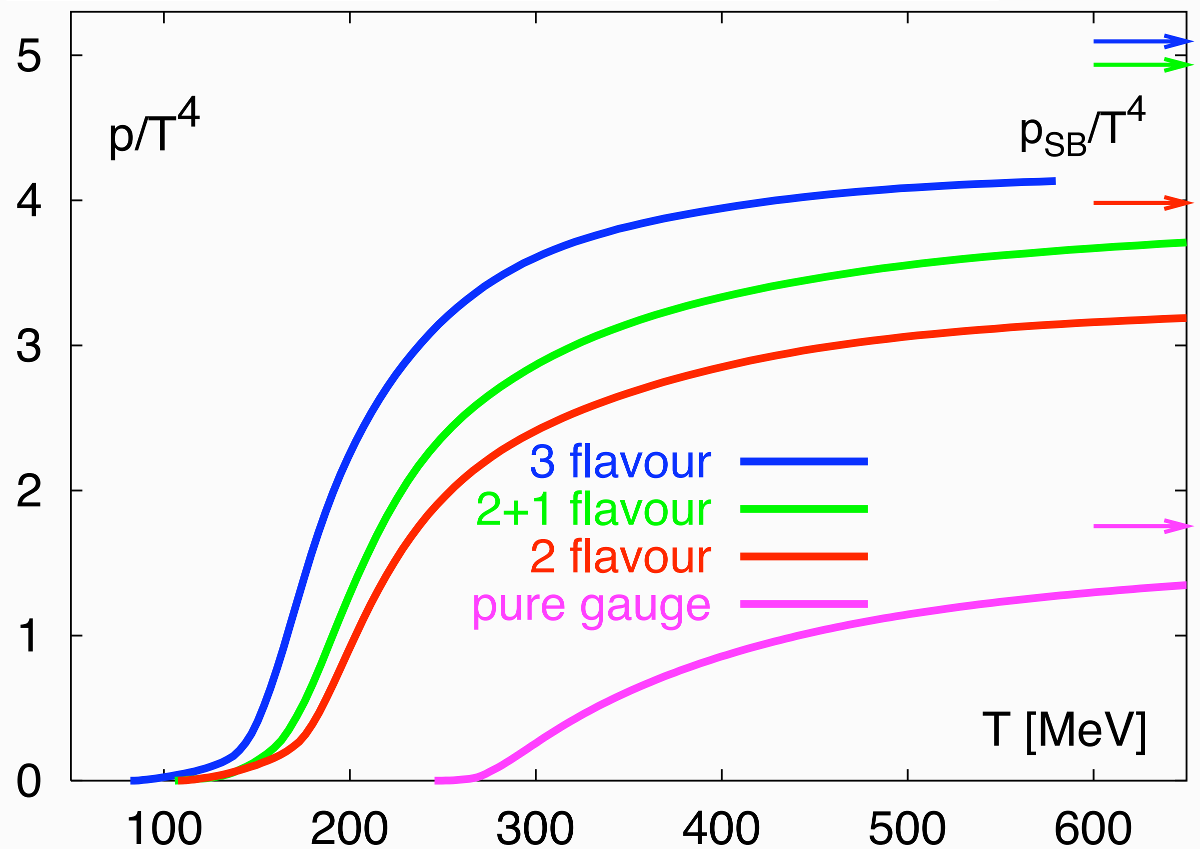
$$\left(\frac{p}{T^4}\right)_a = \left(\frac{p}{T^4}\right)_0 + \frac{c(T)}{N_\tau^2}$$

Flavour dependence of the equation of state

staggered p4-improved, $N_\tau = 4$

Bielefeld

Karsch et al., PLB 478 (2000)



compare with ideal gas:

Pions

$$\frac{\epsilon_{SB}}{T^4} = \frac{3p_{SB}}{T^4} = \begin{cases} 3\frac{\pi^2}{30} & , T < T_c \\ (16 + \frac{21}{2}N_f)\frac{\pi^2}{30} & , T > T_c \end{cases}$$

Gluons and Quarks

$T > T_c$: more degrees of freedom, but significant interaction!

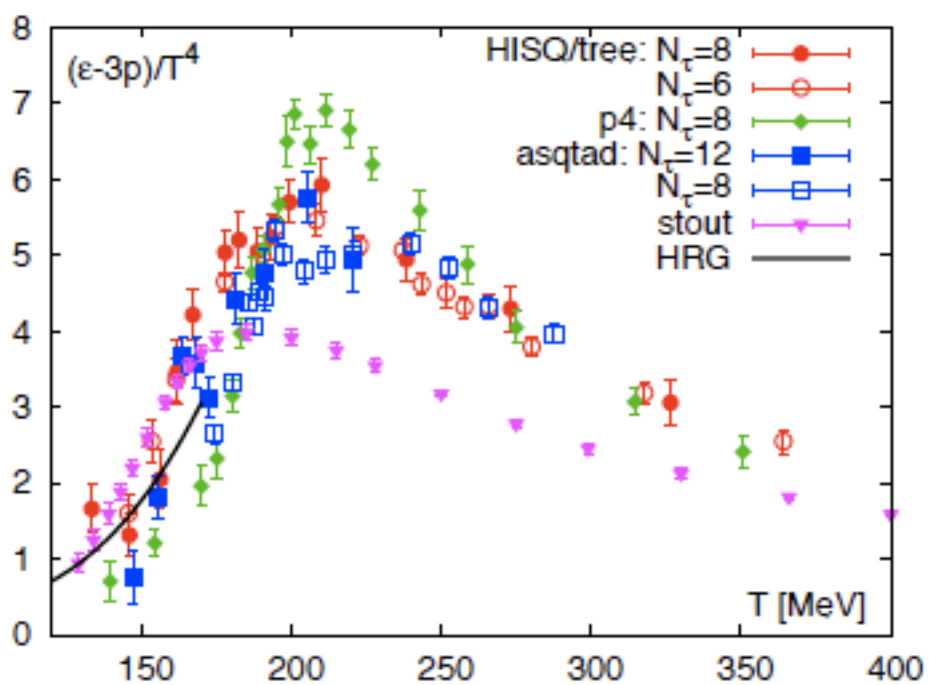
➡ sQGP or `almost ideal' gas....?

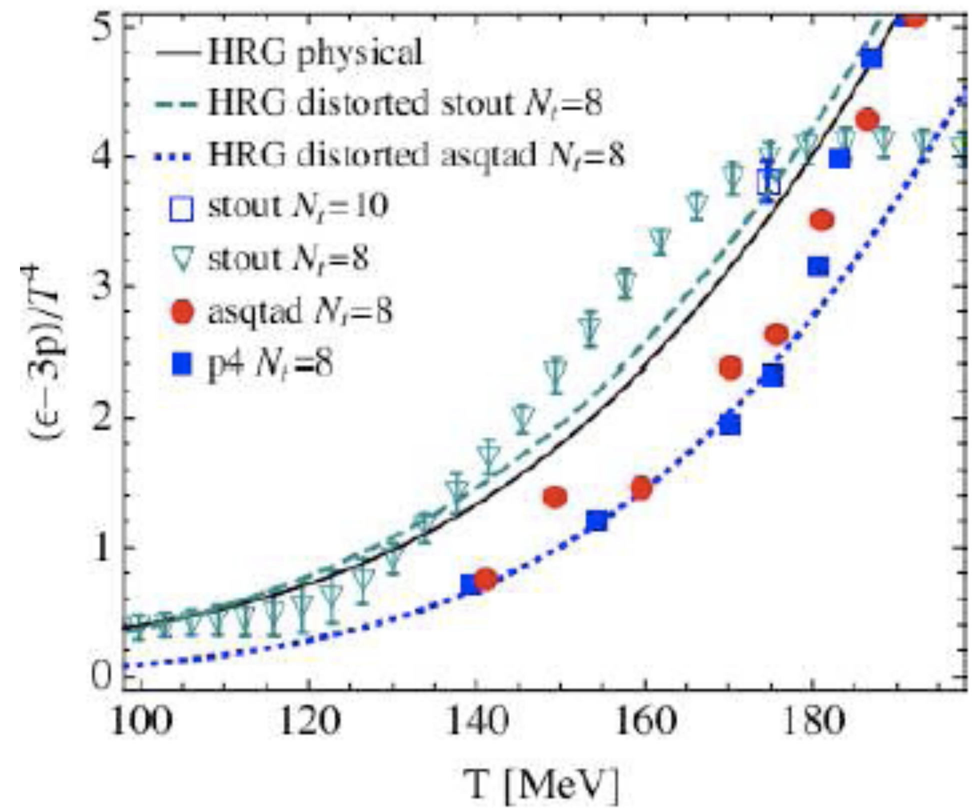
Deconfinement:

Free the Quarks!!!

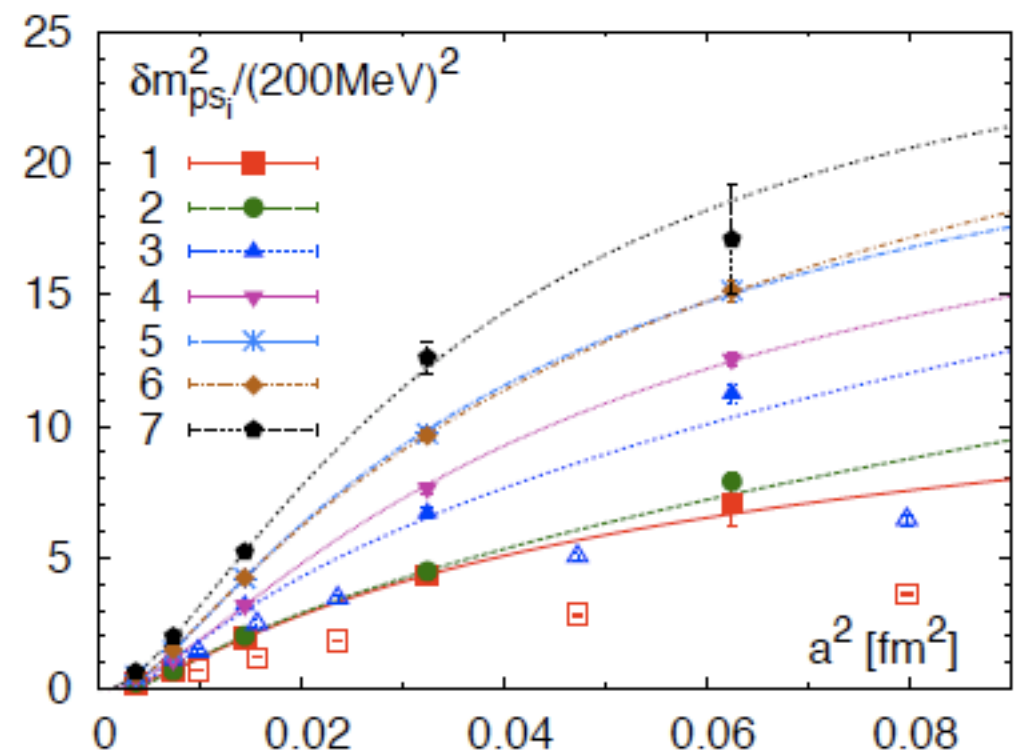
Beware of cut-off effects!

Different versions of improved staggered actions:





Taste splittings of staggered actions give different contributions to pressure



Equation of state for physical quark masses, continuum

Hadron resonance gas model

Karsch et al., PLB 478 (2000)

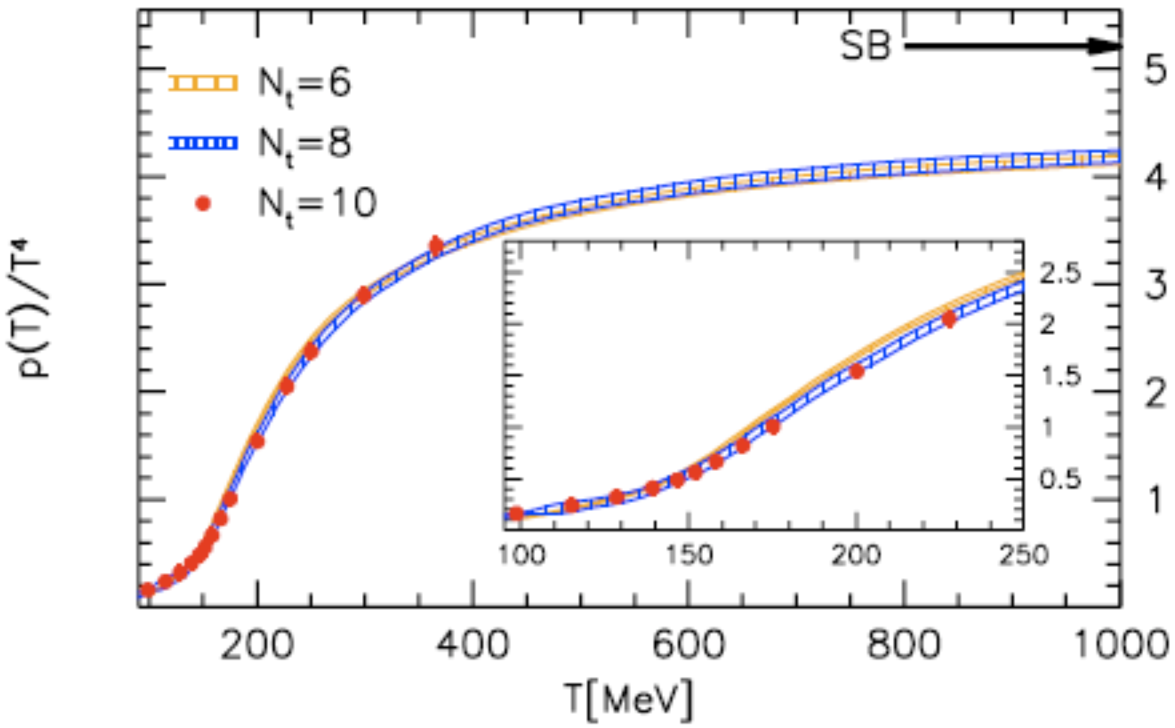


Figure 10: The pressure normalized by T^4 as a function of the temperature on $N_t = 6, 8$ and 1(lattices. The Stefan-Boltzmann limit $p_{SB}(T) \approx 5.209 \cdot T^4$ is indicated by an arrow. For our highest temperature $T = 1000$ MeV the pressure is almost 20% below this limit.

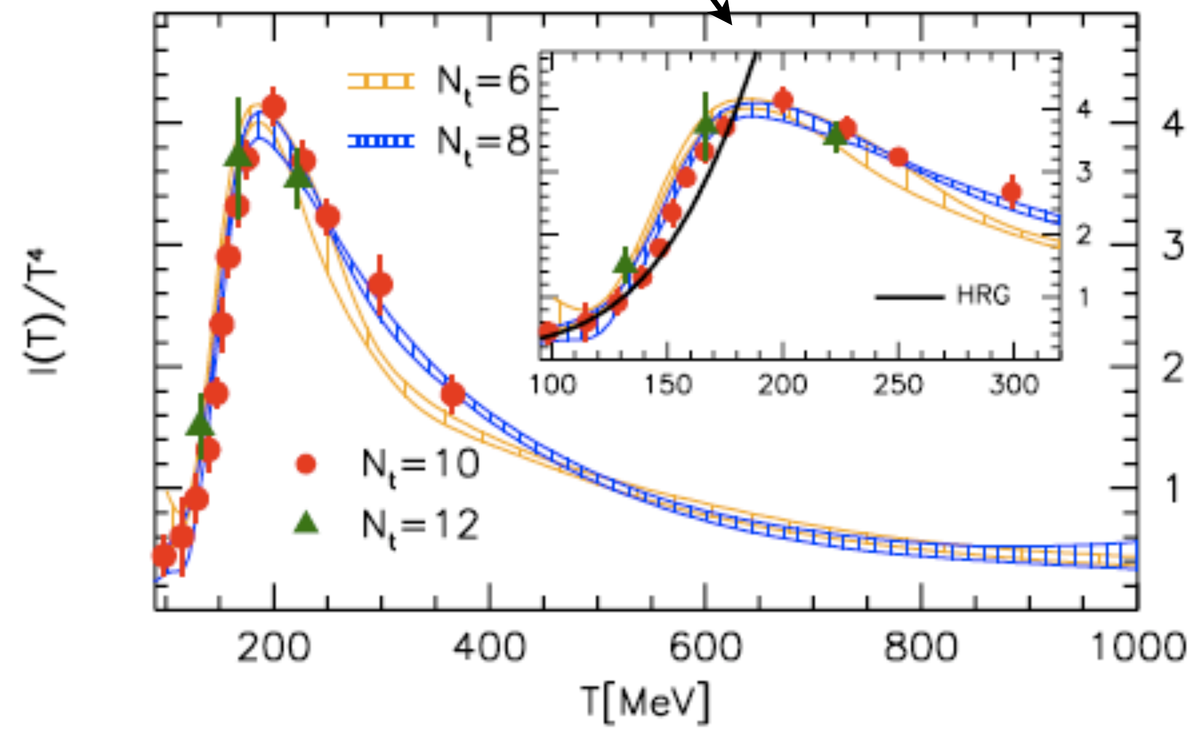
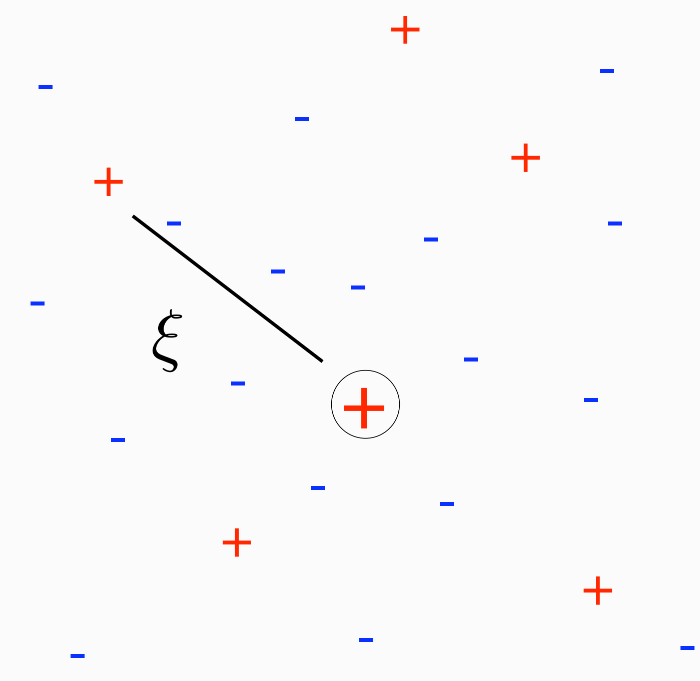


Figure 9: The trace anomaly $I = \epsilon - 3p$ normalized by T^4 as a function of the temperature $N_t = 6, 8, 10$ and 12 lattices.

Budapest-Marseille-Wuppertal

Symanzik-improved gauge action, staggered quarks with stout links

Screening masses: QED



$m_D = \xi^{-1}$ inverse screening length

determined by equal time field correlator:

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \langle E_i(\mathbf{x},t), E_j(\mathbf{x}',t)\rangle \sim \mathrm{e}^{-\mathrm{m_D}|\mathbf{x}-\mathbf{x}'|}$$

equivalently: Fourier transform of A_0 propagator
= LO potential of a static charge

$$V(r) = Q\int \frac{d^3k}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{ikr}}}{k^2+\Pi_{00}(0,\mathbf{k})} = Q\frac{\mathrm{e}^{-\mathrm{m_D r}}}{4\pi r}$$

effective photon mass $\sim T$

magn. fields unscreened

Screening masses: QCD

- analogy: screening of colour sources ➡ deconfinement
- proposed plasma signal: J/ψ suppression Matsui, Satz

BUT:

- field correlator not gauge-invariant!
- perturbatively: fix gauge, look for pole mass in A_0 propagator (gauge-inv.)

$$m_D = m_D^0 + \frac{3}{4\pi} g^2 T \ln \frac{m_D^0}{g^2 T} + c_3 g^2 T + \mathcal{O}(g^3 T)$$

Kobes et al.

Rebhan

$$m_D^0 = \left(\frac{N}{3} + \frac{N_f}{6}\right)^{1/2} gT$$

non-analytic in g^2 , c_3 receives contributions from all orders (Linde problem)

(magn. mass scale)

Generalized definition of screening

➡ spatial correlators of equilibrated, gauge inv. sources A

$$\bar{A}(\mathbf{x}) = \frac{1}{\beta}\int_0^\beta d\tau\, A(\mathbf{x}, -\mathrm{i}\tau) \qquad\qquad C(|\mathbf{x}|) = \langle \bar{A}(\mathbf{x})\bar{A}(0)\rangle_c \sim \mathrm{e}^{-M|\mathbf{x}|}$$

Technically: spectrum of spatial Hamiltonian with one compactified dimension, characterized by:

$$J_R^{PC}$$

SO(2) rotations

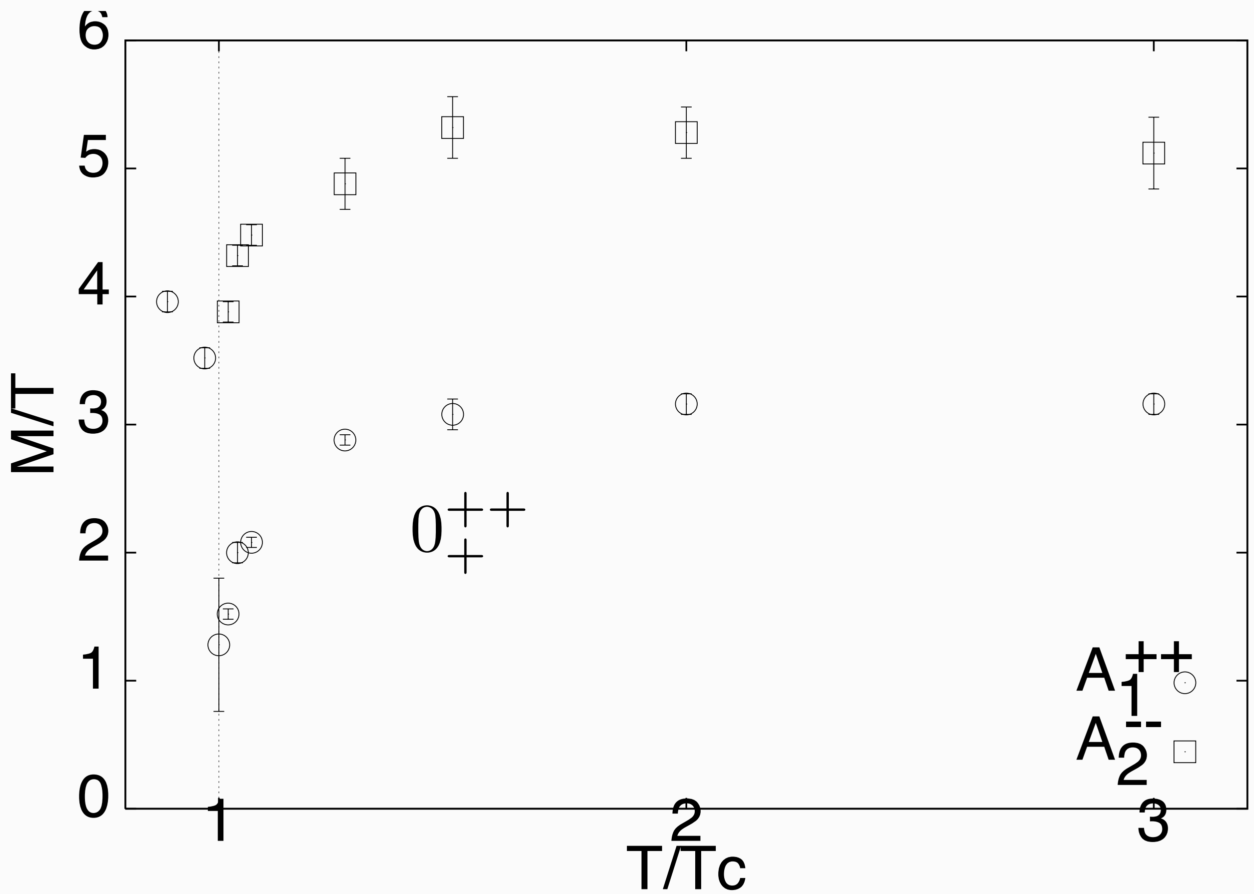
T-reflection, Z(2)

in principle all equilibrium properties encoded in screening spectrum!

Screening masses from numerical simulations

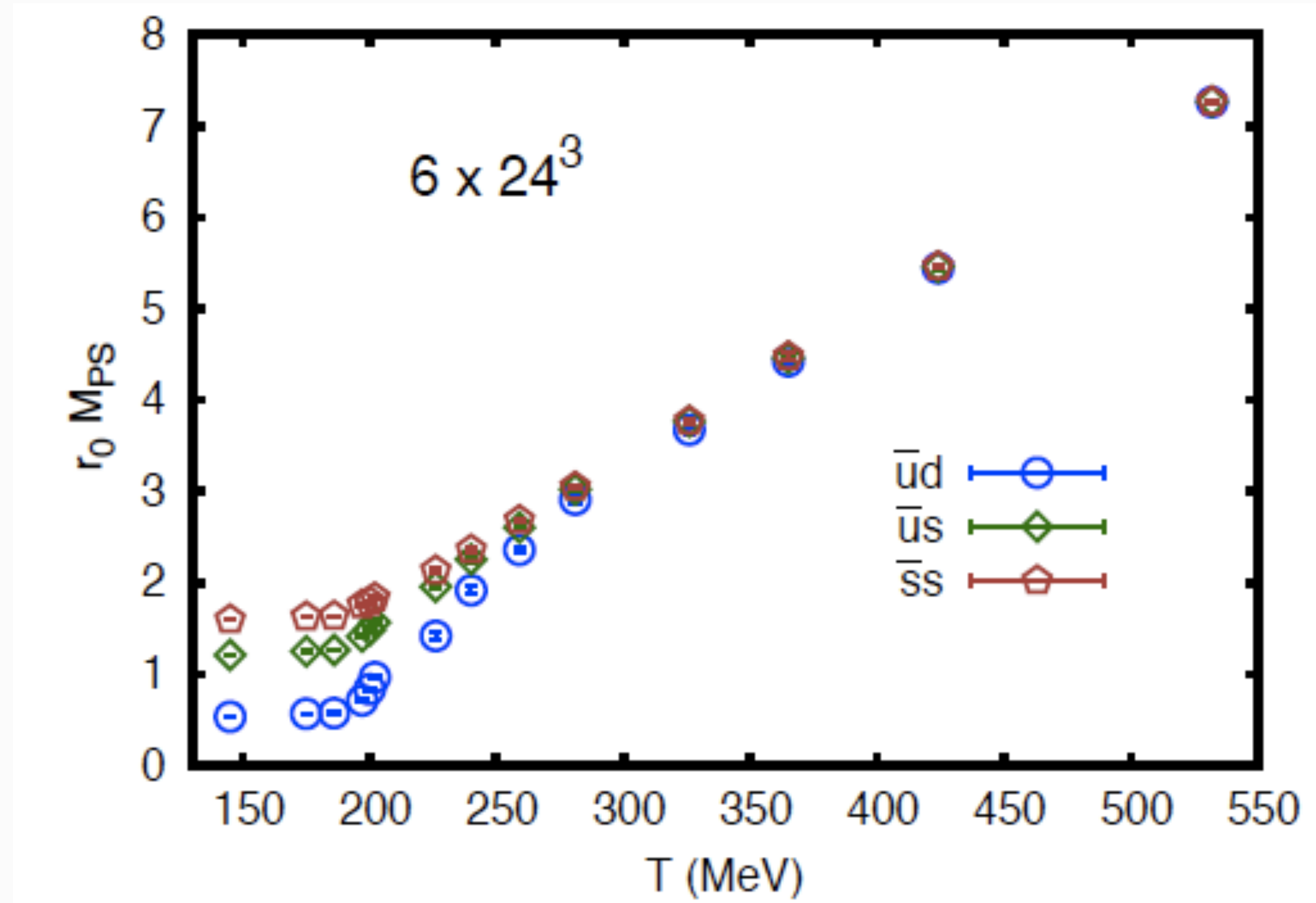
pure gauge, `glueballs`

Datta, Gupta, PRD 67 (2003)



`mesons` dynamical, staggered

RBC-Bielefeld, EPJC 71 (2011)



$T < T_c:$ screening masses stable and close to the $T = 0$ masses

$T > T_c:$ pion mode massive, degeneracies V, AV ➡ chiral symmetry restoration

$T = T_c:$ dip in screening mass=peak in suscept. $\chi = \int_0^{1/T} d\tau \int d^3x\, C(\tau, \mathbf{x})$

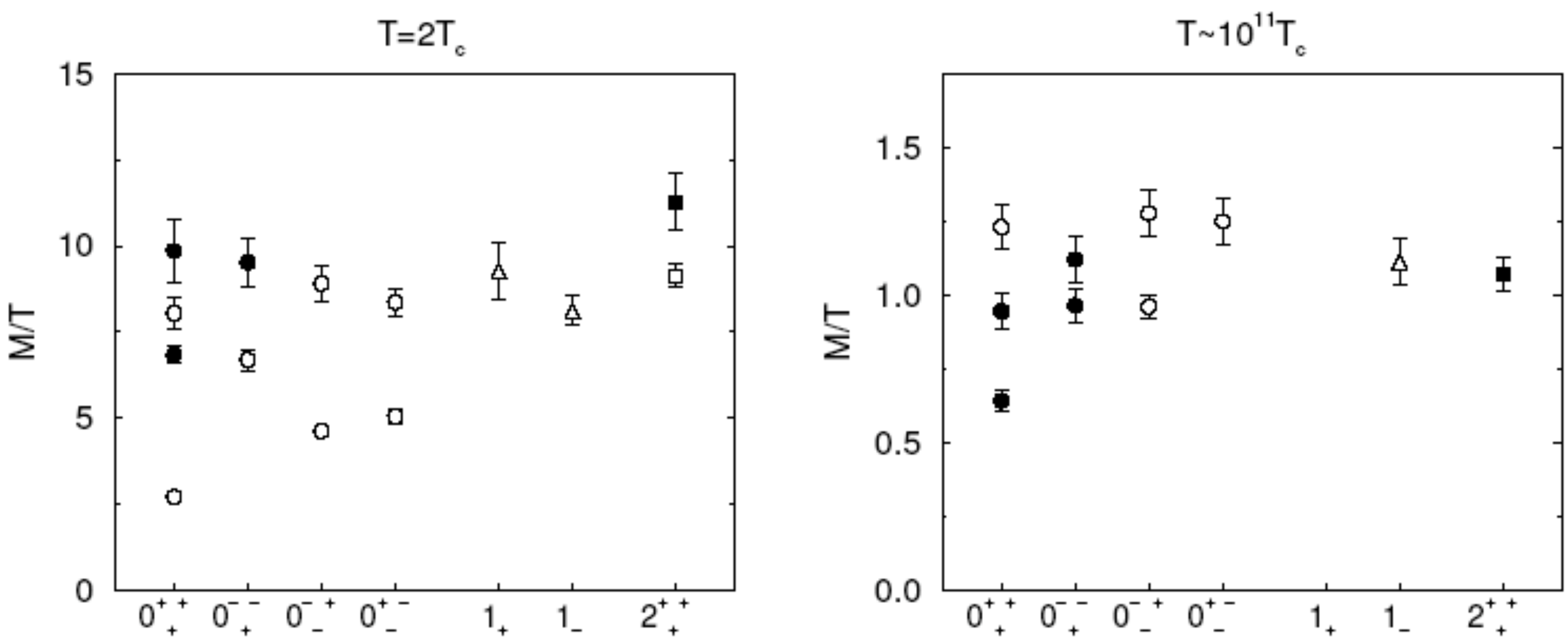
Identify screening d.o.f. from mixing analyses

●: $\mathrm{Tr}\,F_{12}^2, \mathrm{Tr}\,F_{12}^3, \ldots$ glueballs as in 3d YM

○: $\mathrm{Tr}\,A_0^2, \mathrm{Tr}\,A_0^3, \mathrm{Tr}\,A_0 F_{12}, \ldots$ scalar bound states

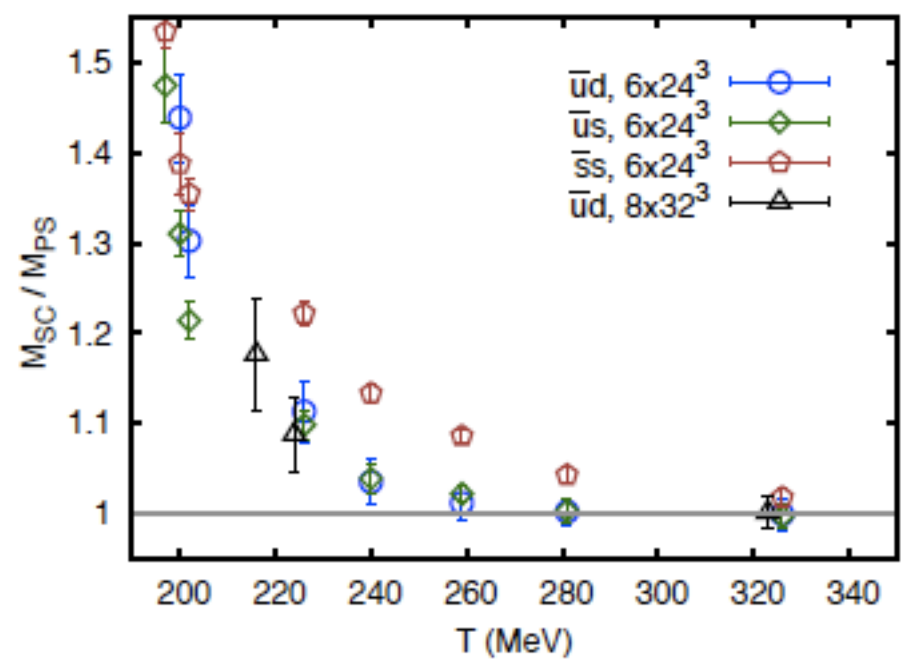
$\Rightarrow$ Practically no mixing between ● and ○

Hart et al., NPB 586 (2000)



RBC-Bielefeld, EPJC 71 (2011)

Studies of chiral symmetry restoration:



meson	σ	$\vec{\pi}$	$\vec{\delta}$	η'
operator	$\overline{\psi}\psi$	$\overline{\psi}\gamma^5\vec{\tau}\psi$	$\overline{\psi}\vec{\tau}\psi$	$\overline{\psi}\gamma^5\psi$

$$
\begin{array}{ccc}
 & \leftarrow \quad SU(2) \times SU(2) & \rightarrow \\
\uparrow & \sigma & \vec{\pi} \\
U_A(1) & & \\
\downarrow & \eta' & \vec{\delta}
\end{array}
$$

The free energy of a static quark anti-quark pair

$$\langle \mathrm{Tr} L^\dagger(\mathbf{x}) \mathrm{Tr} L(\mathbf{0})\rangle = \frac{Z_{\bar{Q}Q}}{Z} = \exp -\frac{F_{\bar{Q}Q}(\mathbf{x},T) - F_0(T)}{T}$$

Transfer matrix in temporal gauge: $(T_0)_{\tau+1,\tau} \equiv \mathrm{e}^{-aH_0} = \exp -L[U_i(\tau+1), 1, U_i(\tau)]$

Acts on Hilbert space of states with static charges

$$\begin{aligned}\langle \mathrm{Tr} L^\dagger(\mathbf{x}) \mathrm{Tr} L(\mathbf{0})\rangle &= \frac{1}{Z}\hat{\mathrm{Tr}}\left(T_0^{N_\tau - 1}\int DU_0(N_\tau)\ U^\dagger_{0\alpha\alpha}(N_\tau,\mathbf{x}) U_{0\beta\beta}(N_\tau,\mathbf{0})\ \mathrm{e}^{-L[U_i(1),U_0(N_\tau),U_i(N_\tau)]}\right) \\ &= \frac{1}{Z}\hat{\mathrm{Tr}}(T_0^{N_\tau} P_{\alpha\alpha\beta\beta})\end{aligned}$$

Projection operator: $P_{\alpha\beta\mu\nu} = \int Dg\ g^\dagger_{\alpha\beta}(\mathbf{x}) g_{\mu\nu}(\mathbf{0}) R(g)$ $\qquad R(g)|\psi\rangle = |\psi^g\rangle$

Projects on sector with fundamental charge at x and anti-charge at 0; annihilates all other states

$$|\psi_{\beta\mu}[U^g]\rangle = g_{\beta\gamma}(\mathbf{x}) g^\dagger_{\delta\mu}(\mathbf{0})|\psi_{\gamma\delta}[U]\rangle$$

$$P_{\alpha\beta\mu\nu}|\psi_{\gamma\delta}\rangle = \frac{1}{N^2}\delta_{\beta\gamma}\delta_{\mu\delta}|\psi_{\alpha\nu}\rangle$$

$$\langle \mathrm{Tr}L^\dagger(\mathbf{x})\mathrm{Tr}L(\mathbf{0})\rangle = \frac{1}{N^2 Z}\sum_{n,\alpha,\beta}\langle n_{\alpha\beta}|n_{\beta\alpha}\rangle\, \mathrm{e}^{-\frac{E_n^{\bar{Q}Q}(|\mathbf{x}|)}{T}} = \frac{1}{Z}\sum_n \mathrm{e}^{-\frac{E_n^{\bar{Q}Q}(|\mathbf{x}|)}{T}}$$

Zero temperature limit:

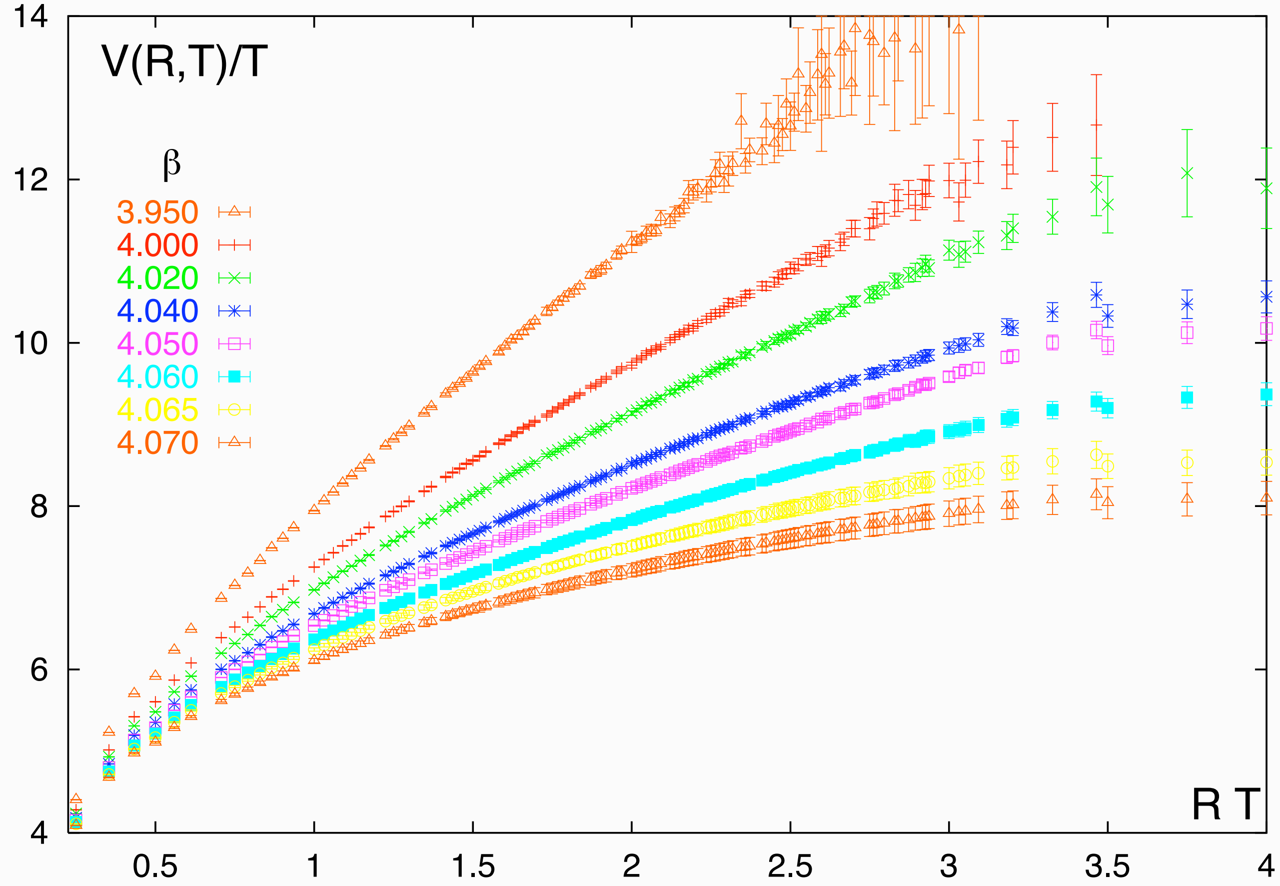
$$\lim_{T\to 0}\frac{\sum_n \mathrm{e}^{-\frac{E_n^{\bar{Q}Q}}{T}}}{\sum_n \mathrm{e}^{-\frac{E_n}{T}}} = \lim_{T\to 0}\mathrm{e}^{-(E_0^{\bar{Q}Q}-E_0)/T}\,\frac{1+\mathrm{e}^{-(E_1^{\bar{Q}Q}(|\mathbf{x}|)-E_0^{\bar{Q}Q})/T}+\dots}{1+\mathrm{e}^{-(E_1-E_0)/T}+\dots} \to \mathrm{e}^{-\frac{V(|\mathbf{x}|)}{T}}$$

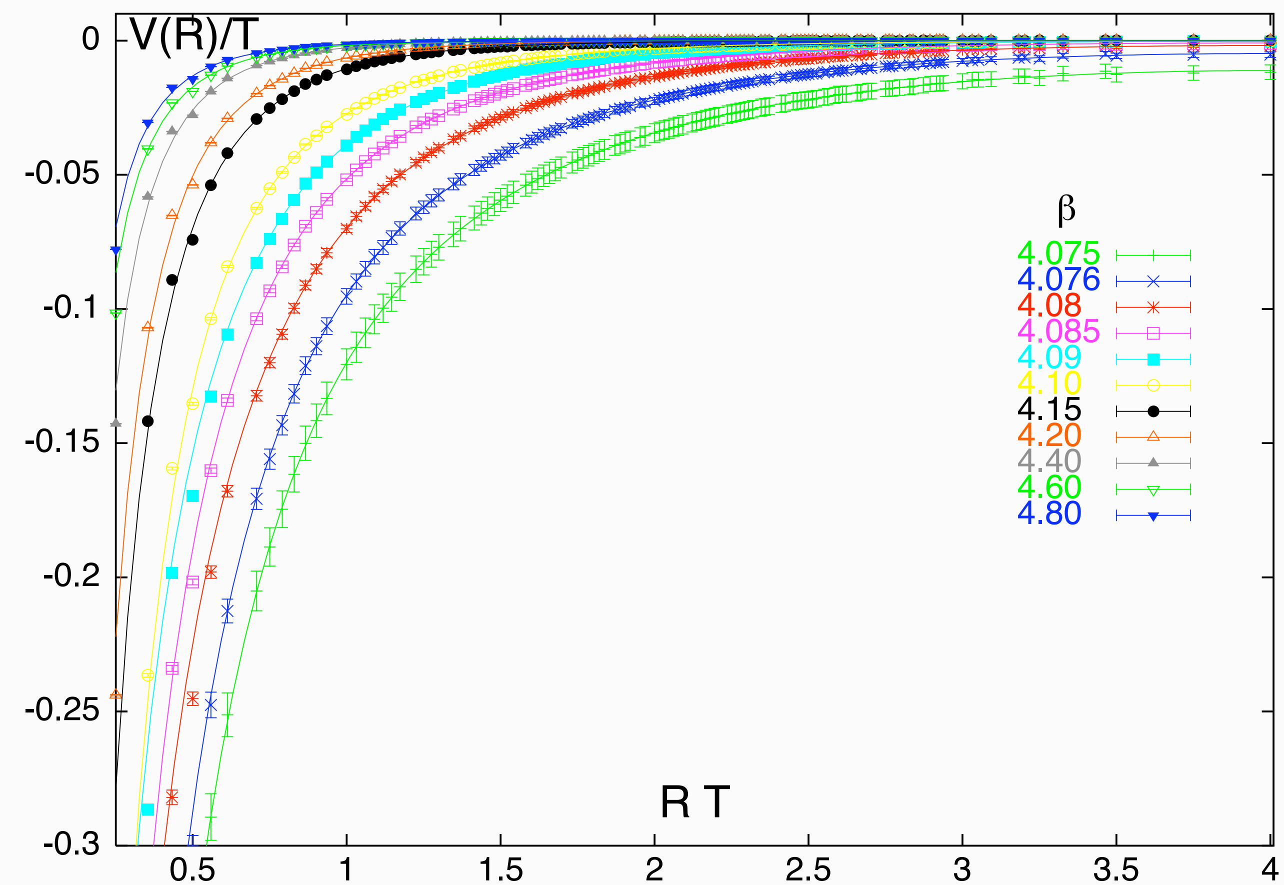
This is the usual static potential of a quark anti-quark pair at distance |x|.

The free energy is the Boltzmann-weighted sum over all excited states.

The static quark free energy in the quenched limit

Kaczmarek et al., PRD 62 (2000)





$T < T_c$

$T > T_c$

$$\frac{\sigma(T)}{\sigma(0)} = a\sqrt{1 - b\frac{T^2}{T_c^2}}$$

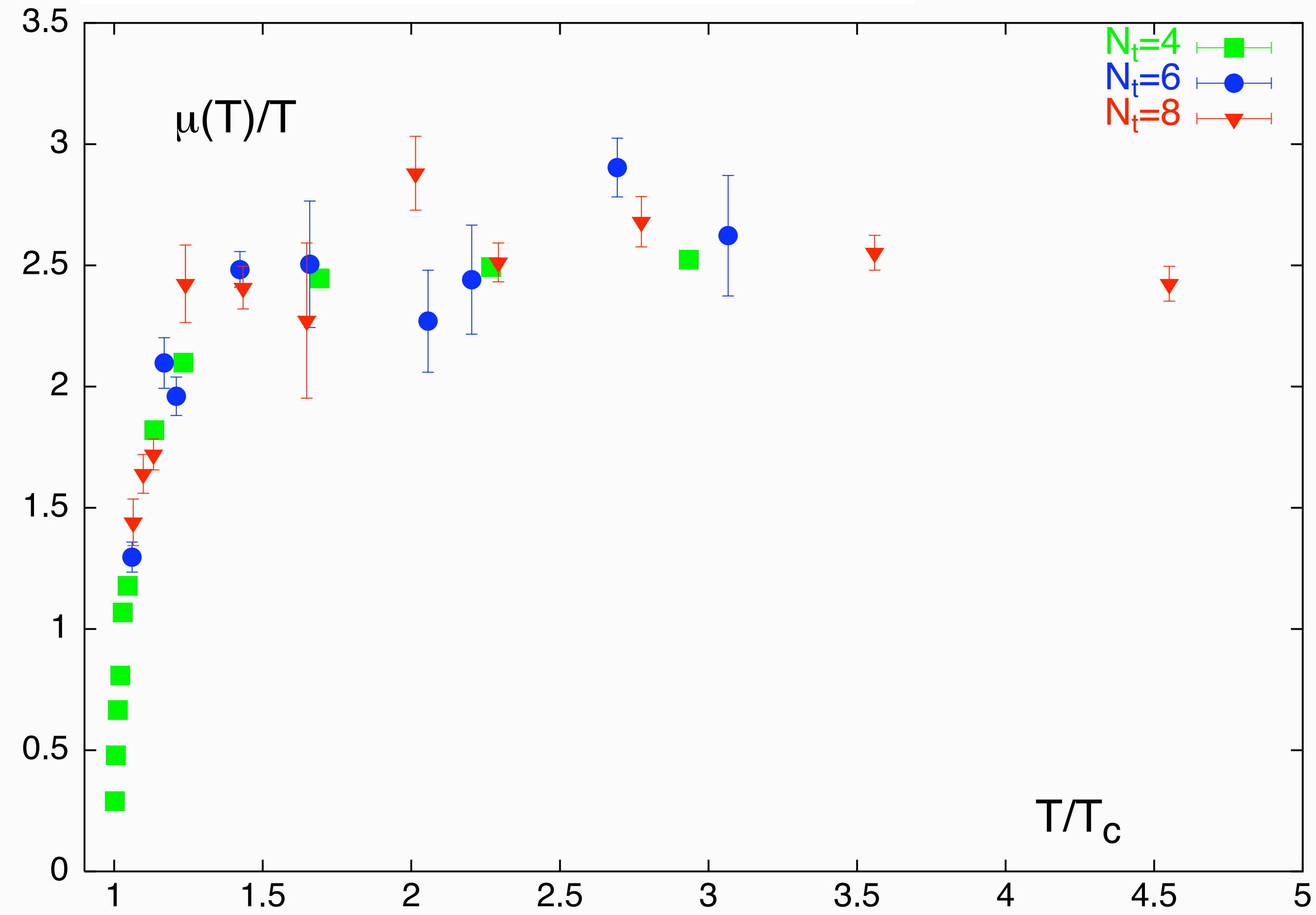
$$\frac{F_{q\bar{q}}(r,T)}{T} = -\frac{c(T)}{(rT)^d}\mathrm{e}^{-\mu(T)r}$$

Screening of the static quark free energy

...in pure gauge theory

$$\frac{F_{q\bar{q}}(r,T)}{T} = -\frac{c(T)}{(rT)^d} \mathrm{e}^{-\mu(T)r}$$

Kaczmarek et al., PRD 62 (2000)



perturbation theory: $d = 2, \mu = 2m_D^0$ exchange of two A_0

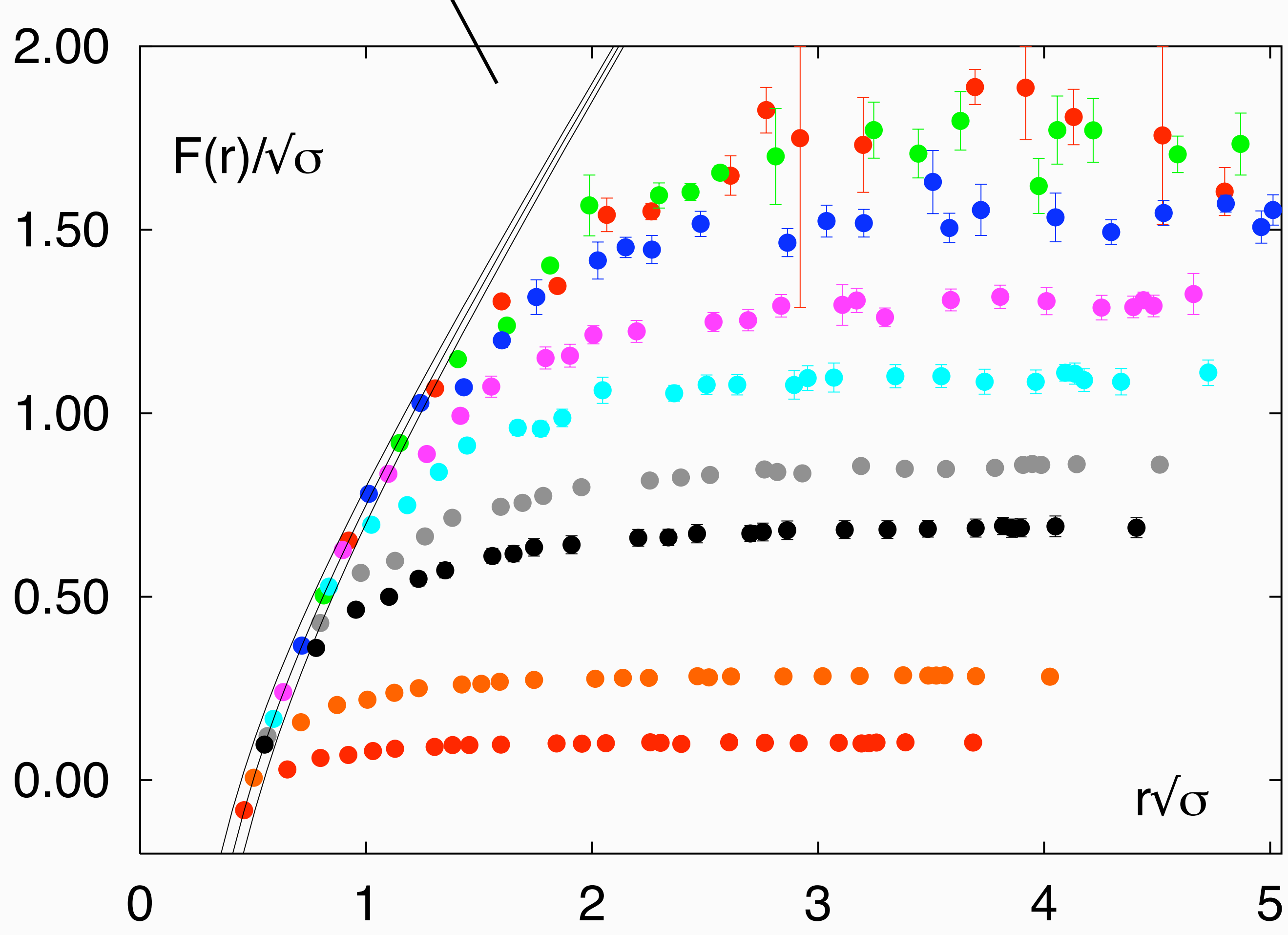
numerically: $d \approx 3/2, \mu \approx M_{0_+^{++}}$ lightest gauge inv. screening mass

The static quark free energy, dynamical

Bielefeld

with dynamical light quarks: $N_f = 3$

$$V(r) = -\alpha/r + r\sigma$$



$T = 0.58T_c$

$T = 1.15T_c$

- screening of colour force at T=0 by dynamical fermions
- with increasing T screening by the plasma

Decomposition in different colour channels

McLerran, Svetitsky PRD 81

$$
\begin{aligned}
\mathrm{e}^{-F_{\bar{q}q}(r,T)/T} &= \frac{1}{N^2}\langle \mathrm{Tr}\, L^\dagger(\mathbf{x})\, \mathrm{Tr}\, L(\mathbf{y})\rangle = \frac{1}{N^2}\mathrm{e}^{-F_1(r,T)/T} + \frac{N^2-1}{N^2}\mathrm{e}^{-F_8(r,T)/T} \\
\mathrm{e}^{-F_1(r,T)/T} &= \frac{1}{N}\langle \mathrm{Tr}\, L^\dagger(\mathbf{x}) L(\mathbf{y})\rangle, \\
\mathrm{e}^{-F_8(r,T)/T} &= \frac{1}{N^2-1}\langle \mathrm{Tr}\, L^\dagger(\mathbf{x})\, \mathrm{Tr}\, L(\mathbf{y})\rangle - \frac{1}{N(N^2-1)}\langle \mathrm{Tr}\, L^\dagger(\mathbf{x}) L(\mathbf{y})\rangle.
\end{aligned}
$$

- correlators in 'singlet' and 'octet' channels gauge dependent, non-pert. meaning?

- $$F_1(r,T) \sim \frac{\mathrm{e}^{-m_D(T)r}}{4\pi r}$$ Nadkarni PRD 86

Spectral analysis of Polyakov loop correlators

Jahn, O.P., PRD 05

$\hat{T}_0 = \mathrm{e}^{-a\hat{H}_0}$ with Kogut-Susskind Hamiltonian in temporal gauge

$$\mathrm{e}^{-F_{\bar{q}q}/T} = \frac{1}{ZN^4}\hat{\mathrm{Tr}}\,[\hat{T}_0^{N_t}\;\hat{P}^{\mathrm{F}\otimes\bar{\mathrm{F}}}] = \frac{1}{ZN^2}\sum_n \langle n_{\alpha\beta}|n_{\beta\alpha}\rangle\;\mathrm{e}^{-E_n/T}$$

$$\mathrm{e}^{-F_1/T} = \frac{1}{ZN^2}\sum_n \langle n_{\delta\gamma}|\hat{U}_{\gamma\delta}(\mathbf{x},\mathbf{y})\hat{U}^\dagger_{\alpha\beta}(\mathbf{x},\mathbf{y})|n_{\beta\alpha}\rangle\;\mathrm{e}^{-E_n/T}$$

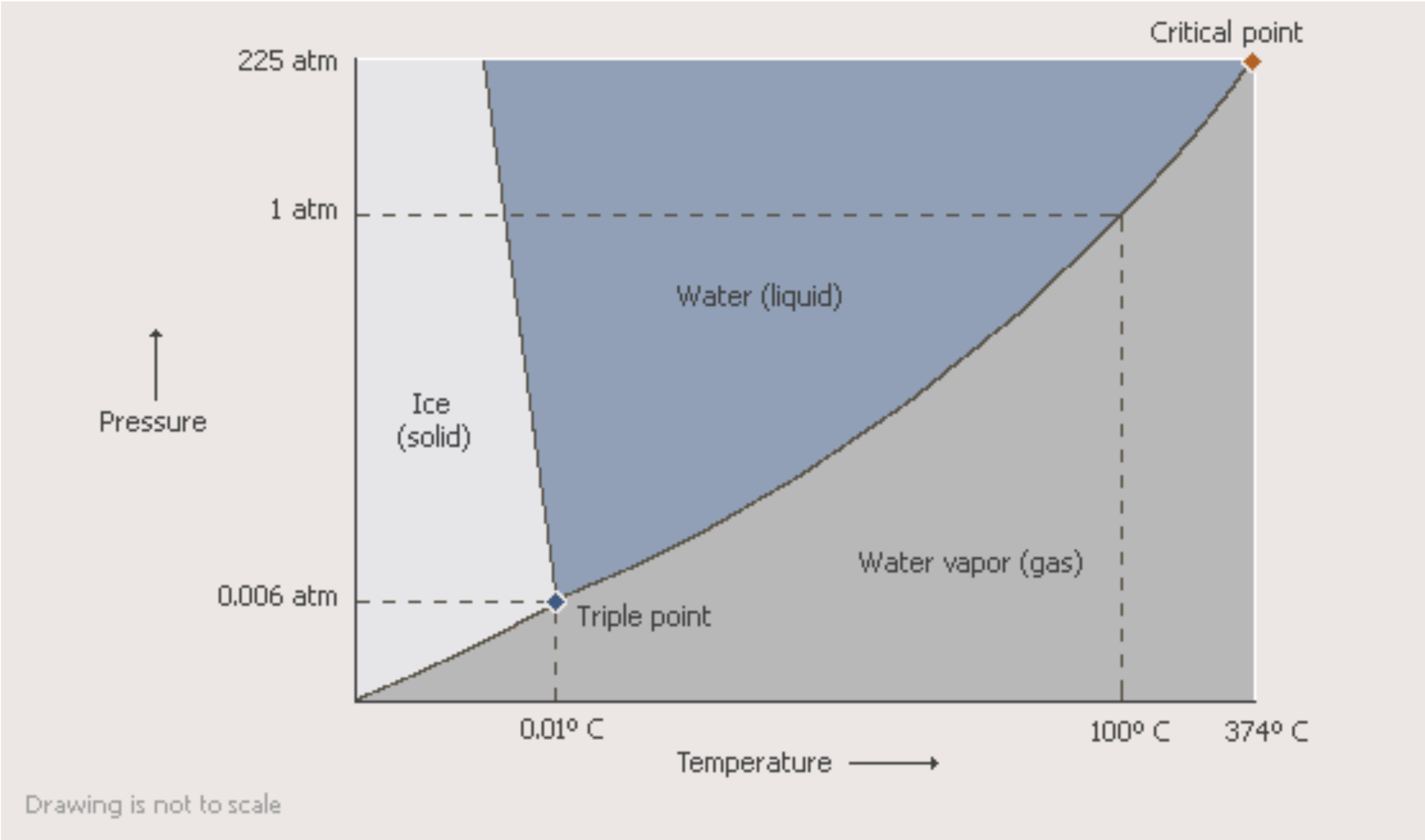
$$\mathrm{e}^{-F_8/T} = \frac{1}{ZN^2}\sum_n \langle n_{\delta\gamma}|\hat{U}^a_{\gamma\delta}(\mathbf{x},\mathbf{y})\hat{U}^{\dagger a}_{\alpha\beta}(\mathbf{x},\mathbf{y})|n_{\beta\alpha}\rangle\;\mathrm{e}^{-E_n/T}$$

- energies: usual (T=0) colour singlet potential + excit. in all three channels
- non-vanishing matrix elements in singlet and octet channel
- matrix elements path/gauge dependent but contribute

Phase transitions and phase diagrams

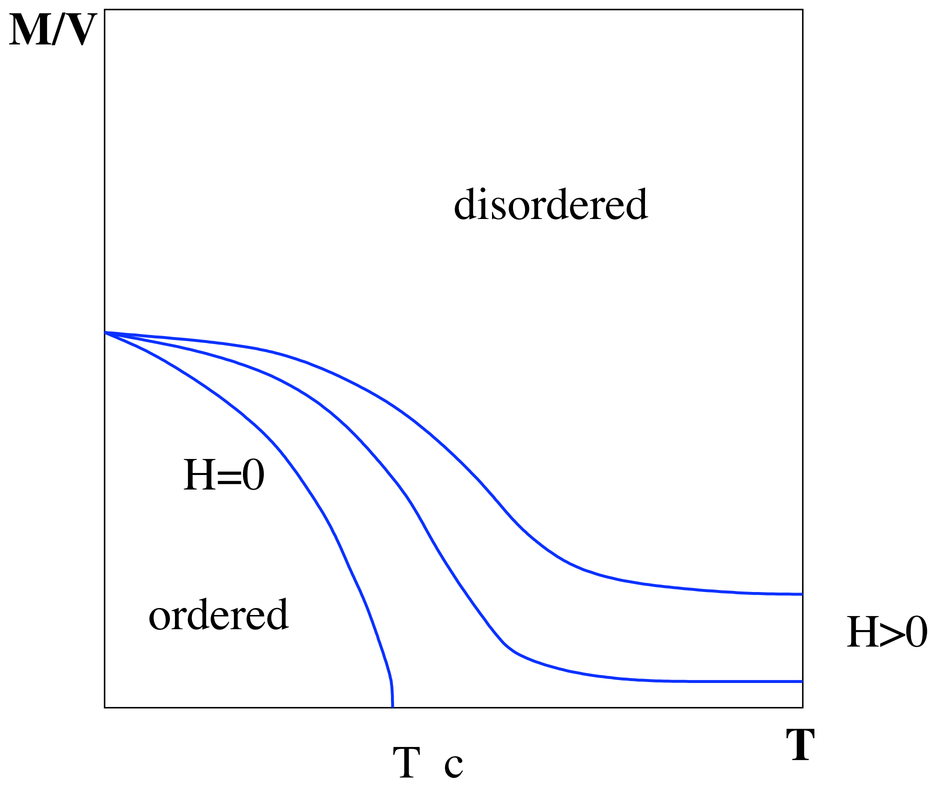
- phase transitions: singularities in free energy $F \Rightarrow$zeroes in partition function Z
  only in thermodynamic limit! (Lee, Yang)
- first order: jump in order parameter, latent heat, phase coexistence
- second order: diverging correlation length
- crossover smooth, analytic transition

Example 1: water



order parameter: density ρ

Example 2: ferromagnetism



Ising model, Z(2) symmetry

spins with nearest neighbour interaction

$$E = -\sum_{ij} \epsilon_{i,j} s_i s_j - H \sum_i s_i$$

$$t = (T - T_c)/T_c$$

Universality of 2.o. phase transitions, critical exponents:

Correlation length diverges: microscopic dynamics unimportant, only global symmetries

specific heat $C \sim |t|^{-\alpha}$, magnetization $M \sim |t|^{\beta}$, $\chi \sim |t|^{-\gamma}$ and $\xi \sim |t|^{-\nu}$

exponents the same for all systems within one universality class!

Critical endpoint of water shows 3d Ising universality, Z(2)!

Finding a phase transition: fluctuations

Fluctuations visible in any observable, but largest in "order parameter":

$$O \in \{\mathrm{Tr}L, \bar{\psi}\psi, \mathrm{Tr}U_p, \ldots\}$$

Generalised susceptibilities:

$$\chi_O = \int d^3x \left(\langle O(\mathbf{x})O(0)\rangle - \langle O(\mathbf{x})\rangle\langle O(0)\rangle \right)$$

(Note: can be generalised to 4d, but the QCD equilibrium system is 3d!)

Volume averages (intensive variables):

$$\bar{O} = \frac{1}{V}\int d^3x\, O(\mathbf{x})$$

$$\chi_{\bar{O}} = N_s^3(\langle \bar{O}^2\rangle - \langle \bar{O}\rangle^2) = N_s^3\langle (\delta\bar{O})^2\rangle$$

fluctuation: $\delta\bar{O} = \bar{O} - \langle\bar{O}\rangle$

Pseudo-critical couplings (finite V!):

$$\chi(\beta_c, m_f) = \chi_{\max} \Rightarrow \beta_c(m_f)$$

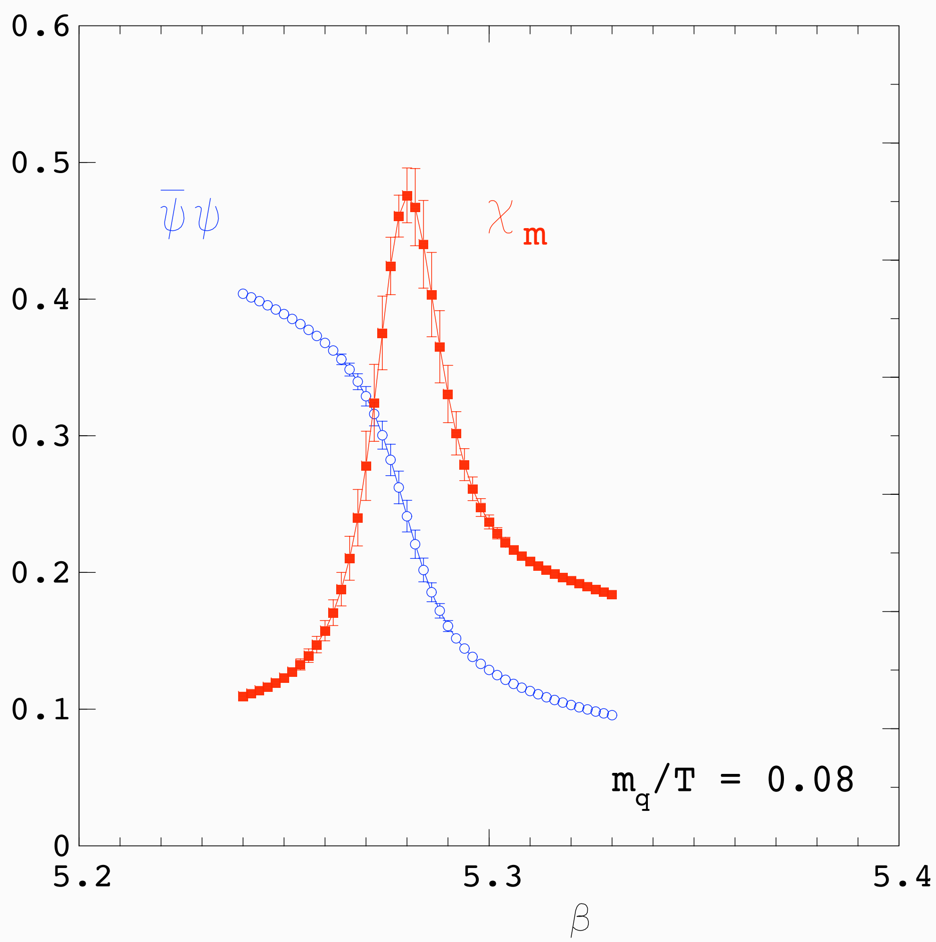
- fluctuations maximal but finite!
- pseudo-critical parameters not unique!

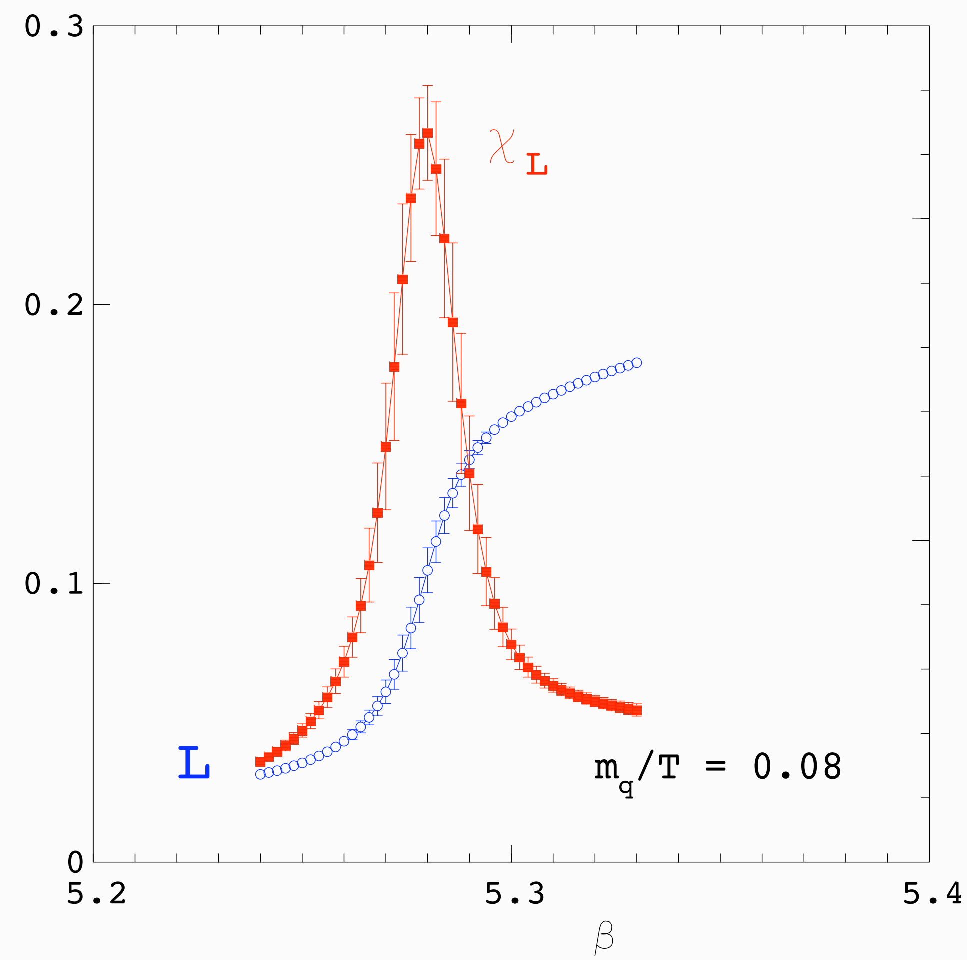
Finding the phase transition: the critical temperature

Measuring the `order parameter' as function of lattice coupling (viz. T)

$$\beta = \frac{2N_c}{g^2(a)} \qquad T = \frac{1}{aN_t}$$

here: $N_f = 2$





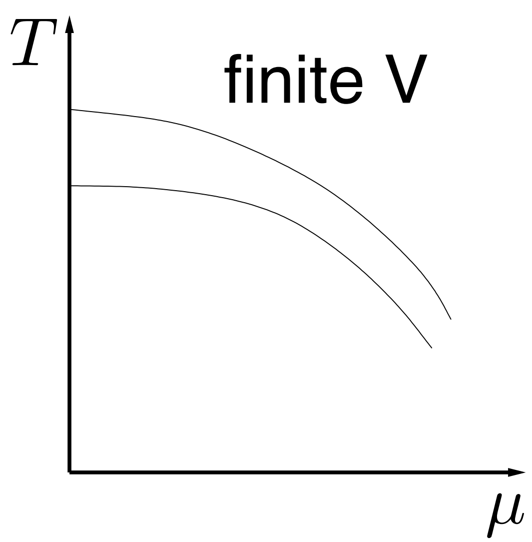
Susceptibilities: $\chi = VN_t(\langle\bar{\mathcal{O}}^2\rangle - \langle\bar{\mathcal{O}}\rangle^2) \quad \Rightarrow \chi_{max} = \chi(\beta_0) \quad \Rightarrow T_0$

$$T_{deconf} \approx T_{chiral}$$

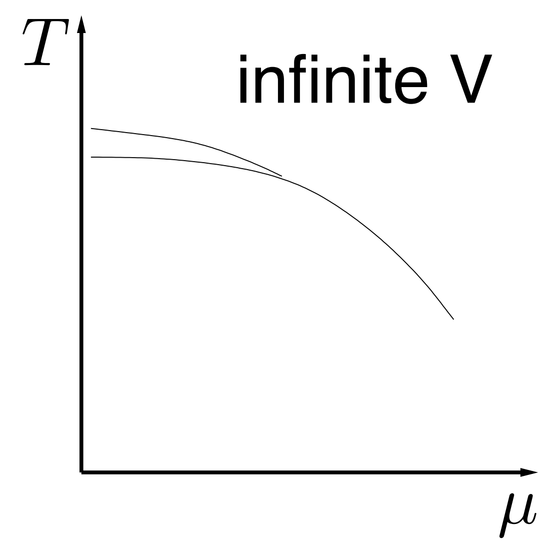
Approaching the thermodynamic limit

different definitions (e.g. scanning in different directions, different observables etc.)

$\beta_0(\mu)$ not unique



$\beta_c(\mu)$ unique for p.t., not for crossover



Critical line unique in thermodynamic limit!

Order of transition: finite volume scaling $$(\beta_0(V) - \beta_0(\infty)) \sim V^{-\sigma}$$

$\sigma = 1$	1st order
$\sigma < 1$	2nd order
$\sigma = 0$	crossover

Scaling analyses employing universality

Effective Hamiltonian analogous to Ising model: $$\frac{H_{eff}}{T} = \tau E + hM$$

Extensive operators: E energy-like M magnetisation-like

Parameters: τ temperature-like h magnetic field-like

At a critical point, the singular part of the free energy has the scaling form:

$$f_s(\tau, h) = b^{-d} f_s(b^{D_\tau}\tau, b^{D_h} h)$$

$b = LT = N_s/N_\tau$ dim.less scale factor

Relation between scaling dimensions and critical exponents:

$$D_\tau = \frac{1}{\nu}, \quad \gamma = \frac{2D_h - d}{D_\tau}, \quad \alpha = 2 - \frac{d}{D_\tau}$$

$$\chi_E = V^{-1}\langle(\delta E)^2\rangle = -\frac{1}{T}\frac{\partial^2 f}{\partial \tau^2} \sim b^{\alpha/\nu}$$

$$\chi_M = V^{-1}\langle(\delta M)^2\rangle = -\frac{1}{T}\frac{\partial^2 f}{\partial h^2} \sim b^{\gamma/\nu}$$

How to map parameters and fields of QCD to those of the Ising model?

For many applications not necessary...

$$E(S_p, \bar{\psi}\psi, ...), M(S_p, \bar{\psi}\psi, ...), \tau(\beta, m_f, \mu_f), h(\beta, m_f, \mu_f)$$

$\Rightarrow$ $\chi_{\bar{\psi}\psi}(E, M)$ mix of energy and magnetic susceptibilities, in thermodynamic limit the more divergent one dominates!

Symmetry groups relevant for QCD: Z(2), O(4), O(2)

$\Rightarrow$ γ/ν

First order scaling: $\chi_{\bar{O}} \sim V$

Analytic crossover: no divergence, susceptibilities have finite thermodynamic limit

Summary Lecture II

- In the strong coupling limit QCD reduces to hadron resonance gas
- Equation of state accessible at physical masses in the continuum limit
- Screening masses give information about relevant scales, symmetries
- Static quark free energy gives information about deconfinement;
  But not to be used in potential models
- Phase transitions are non-analyticities in the thermodynamic functions;
  Only visible in infinite volume: finite size scaling necessary!

Lecture III:

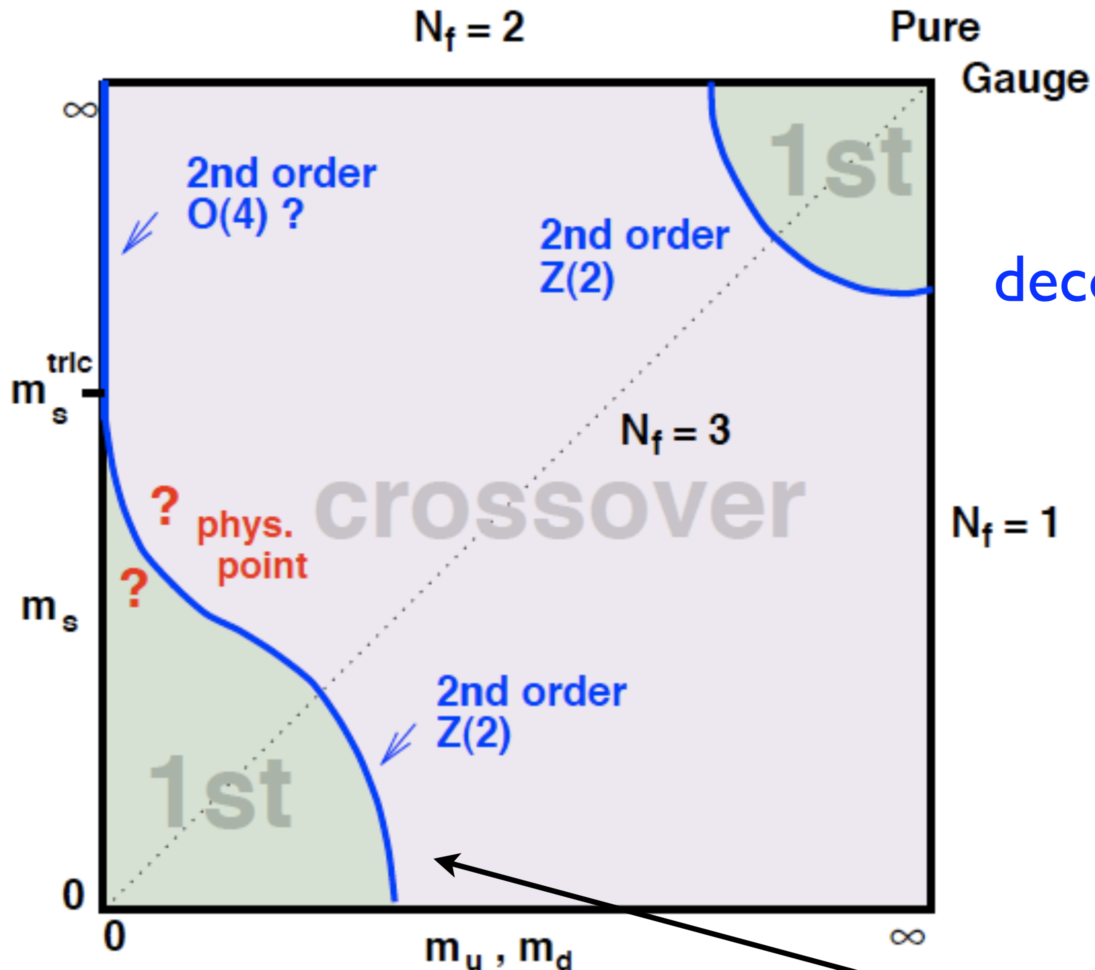
Owe Philipsen

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

- The QCD phase transition at zero density
- Lattice QCD at finite temperature and density
- Towards the QCD phase diagram

The order of the QCD thermal transition, $\mu = 0$

deconfinement p.t.:
breaking of global $Z(3)$



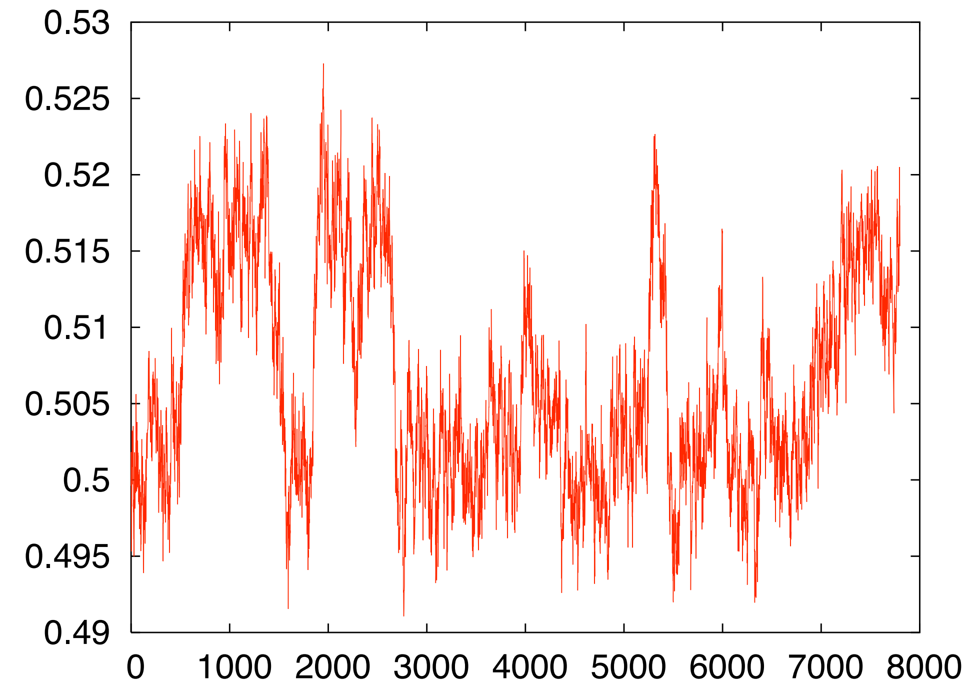
deconfinement critical line

chiral critical line

chiral p.t.
restoration of global
$$SU(2)_L \times SU(2)_R \times U(1)_A$$
anomalous

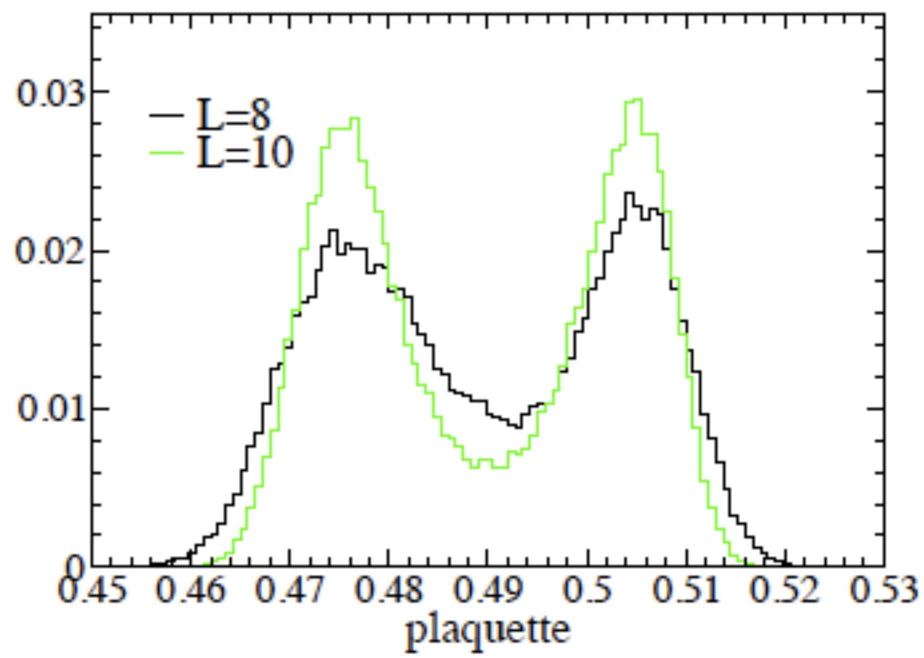
Very difficult!

Monte Carlo history,
plaquette near phase boundary

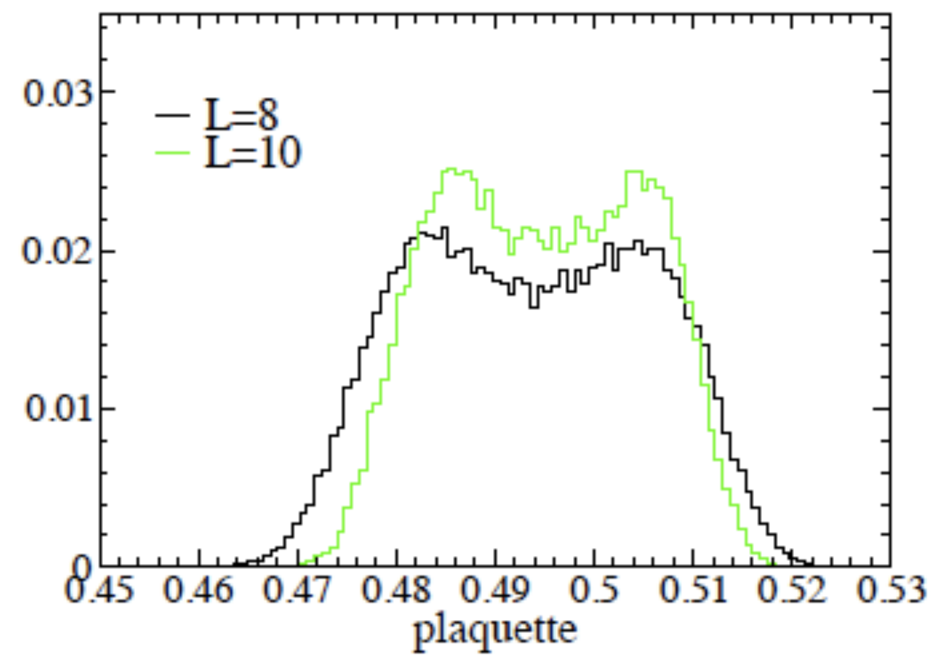


Distribution:

first-order



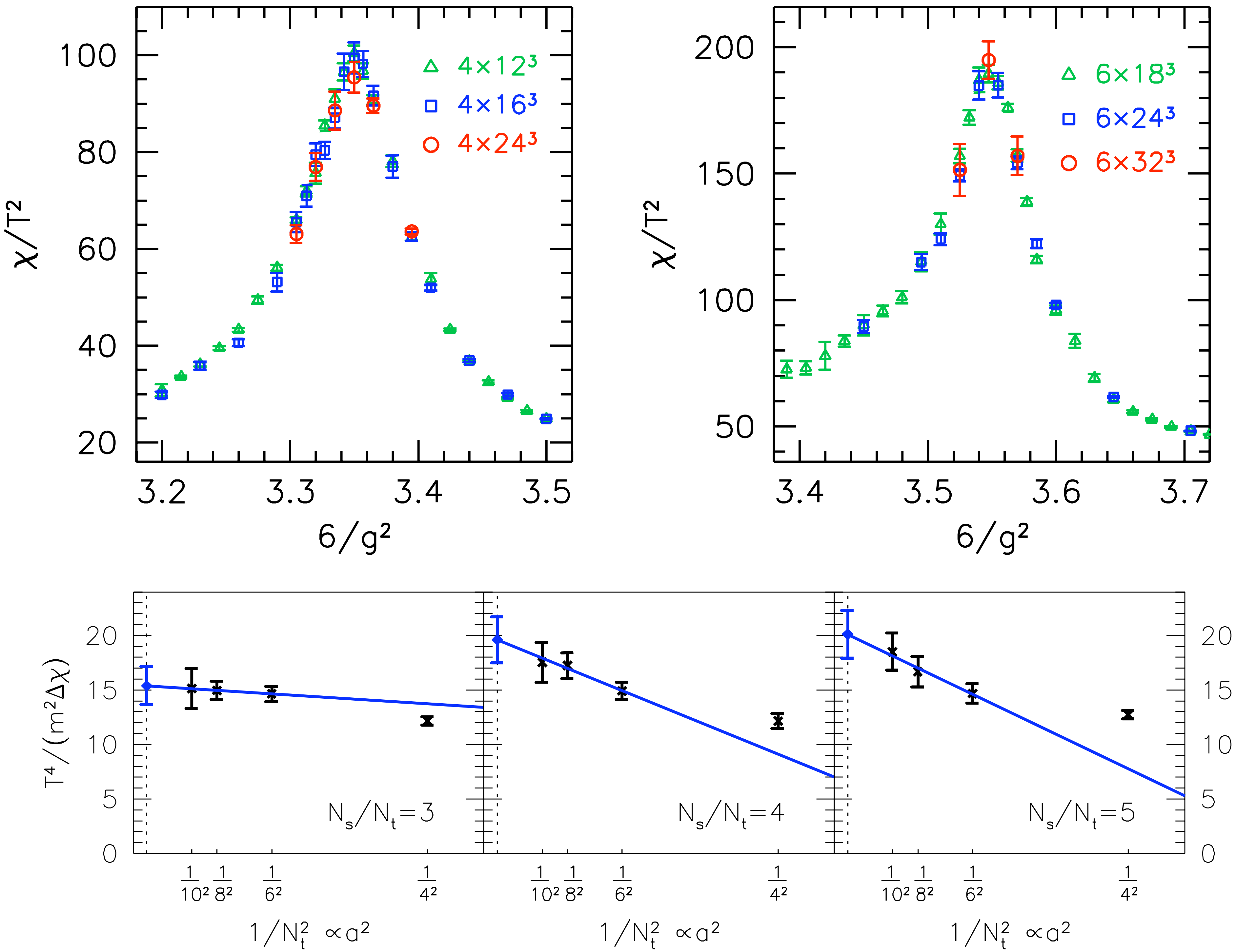
crossover



The nature of the transition for phys. masses

Aoki et al. 06

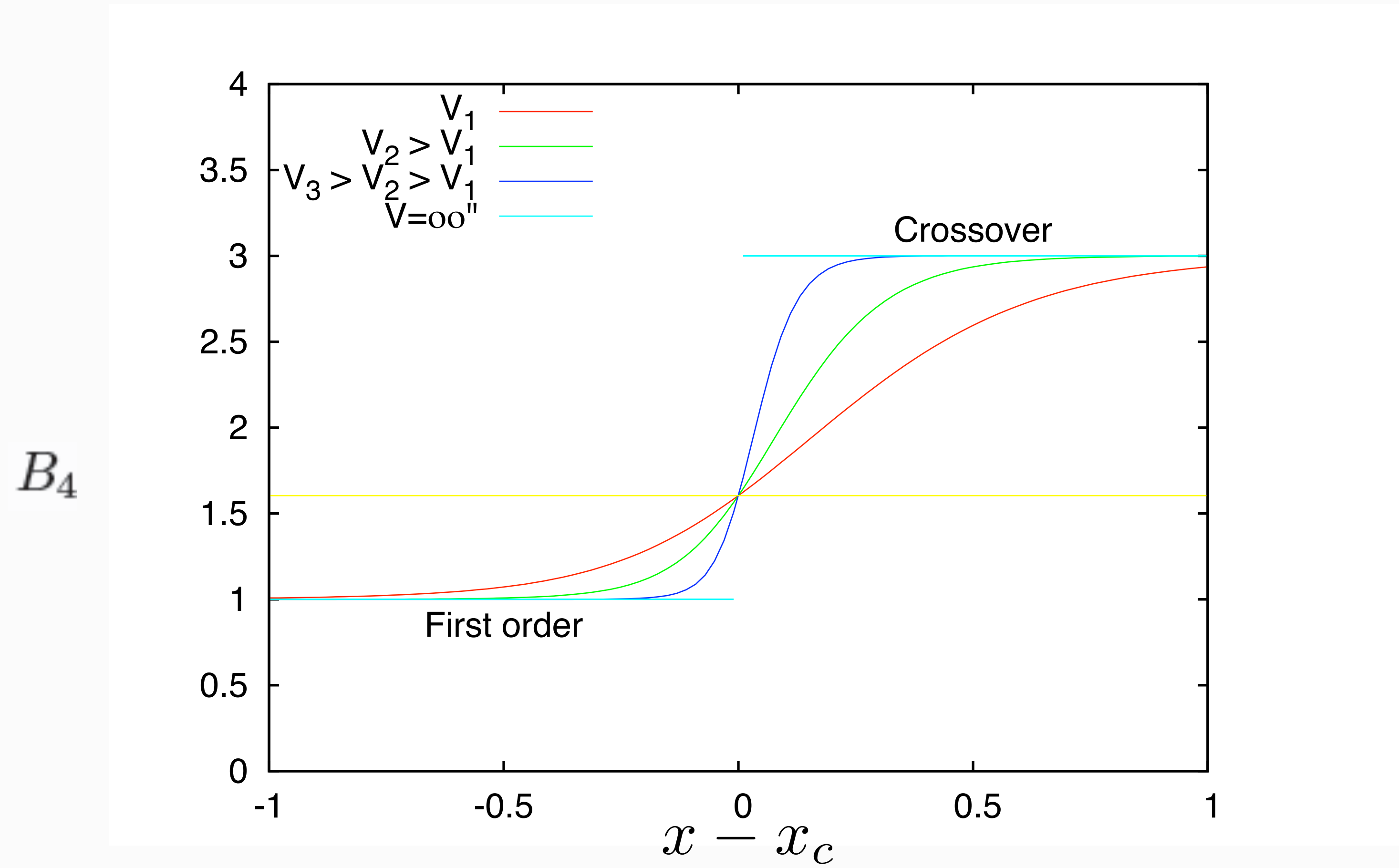
...in the staggered approximation...in the continuum...is a crossover!

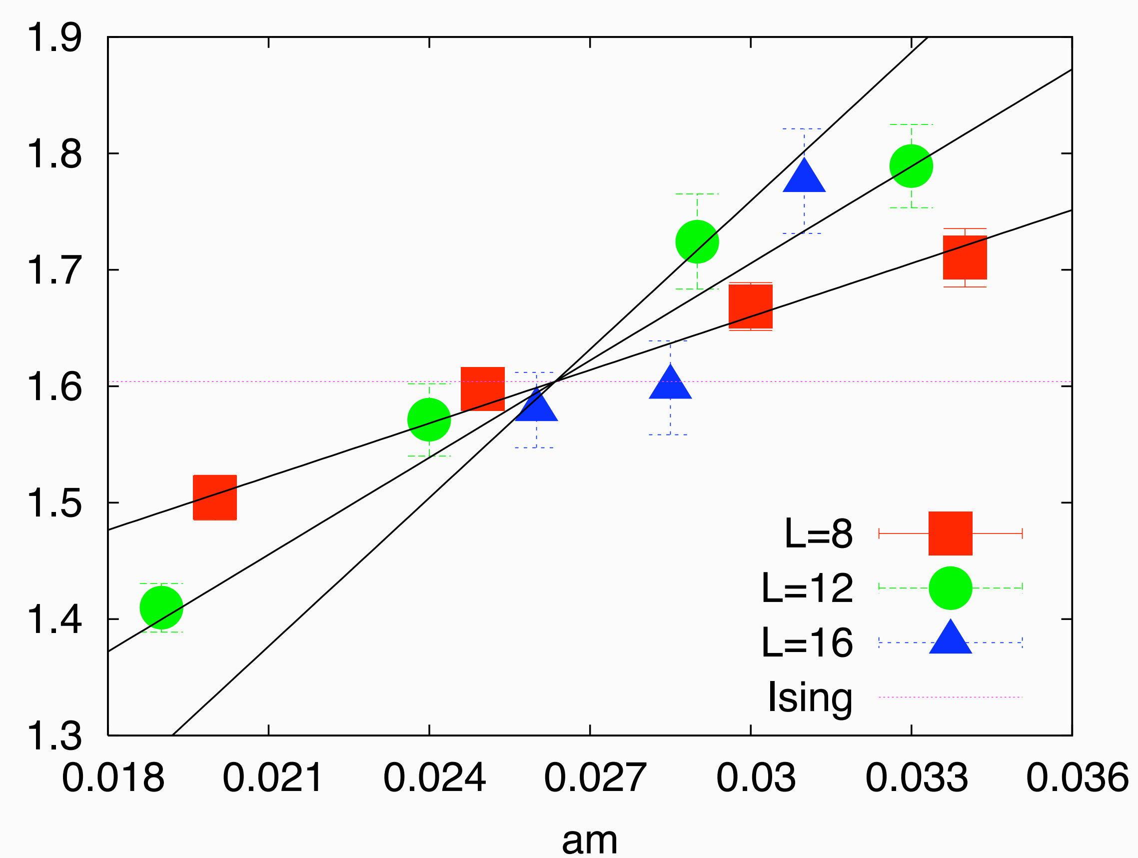


How to identify the order of the phase transition

$$B_4(\bar\psi\psi) \equiv \frac{\langle(\delta\bar\psi\psi)^4\rangle}{\langle(\delta\bar\psi\psi)^2\rangle^2} \stackrel{V\to\infty}{\longrightarrow} \begin{cases} 1.604 & \text{3d Ising} \\ 1 & \text{first}-\text{order} \\ 3 & \text{crossover} \end{cases}$$

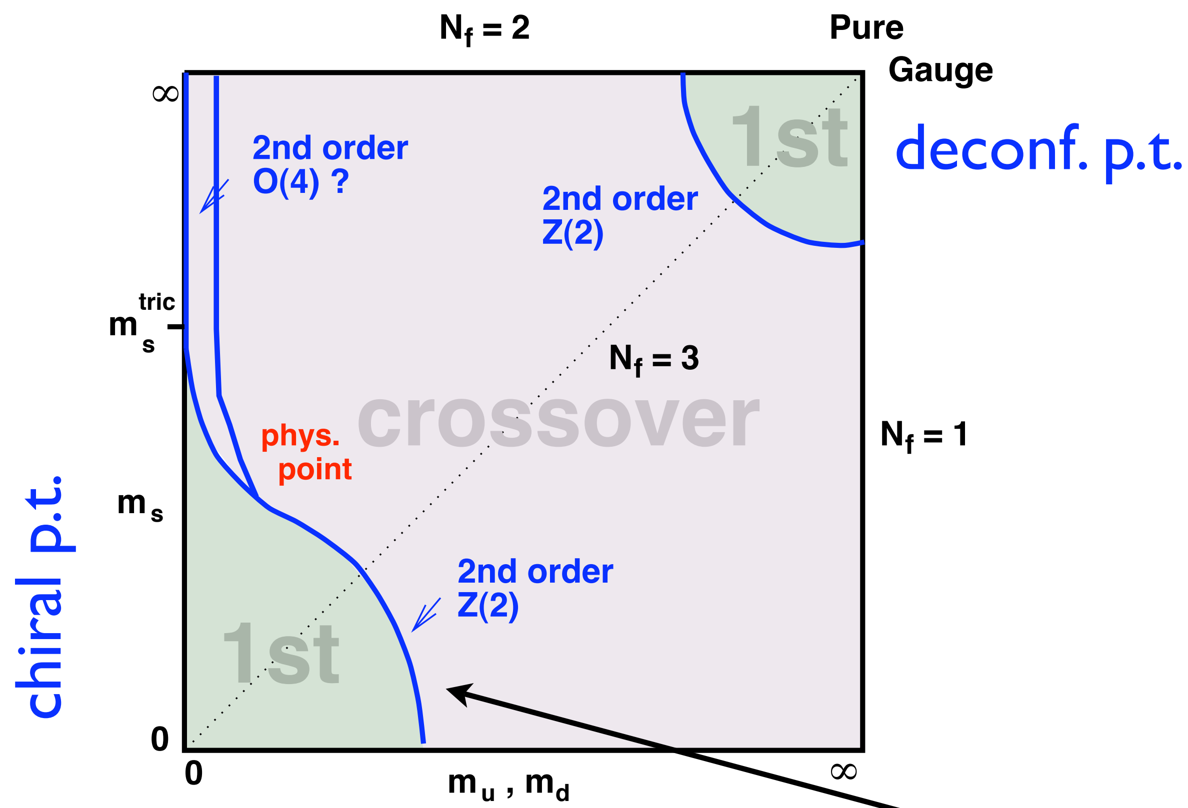
$\mu = 0:$ $$B_4(m, L) = 1.604 + bL^{1/\nu}(m - m_0^c), \quad \nu = 0.63$$

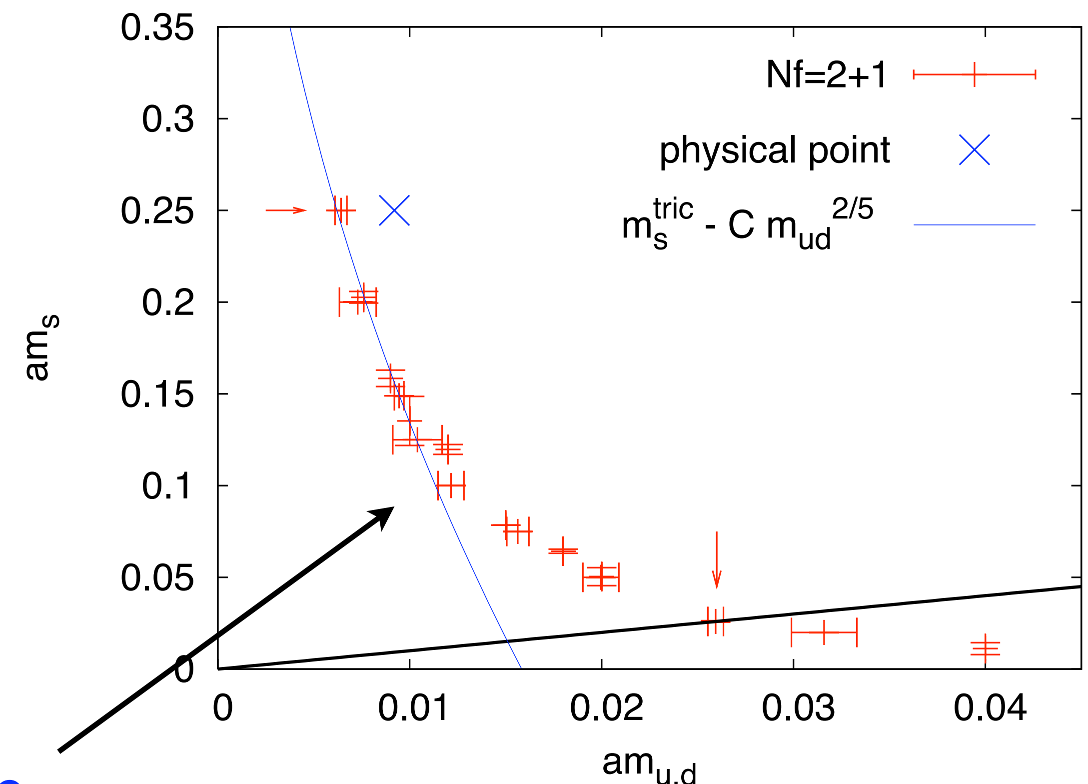




parameter along phase boundary, $T = T_c(x)$

Order of p.t., arbitrary quark masses $\mu = 0$

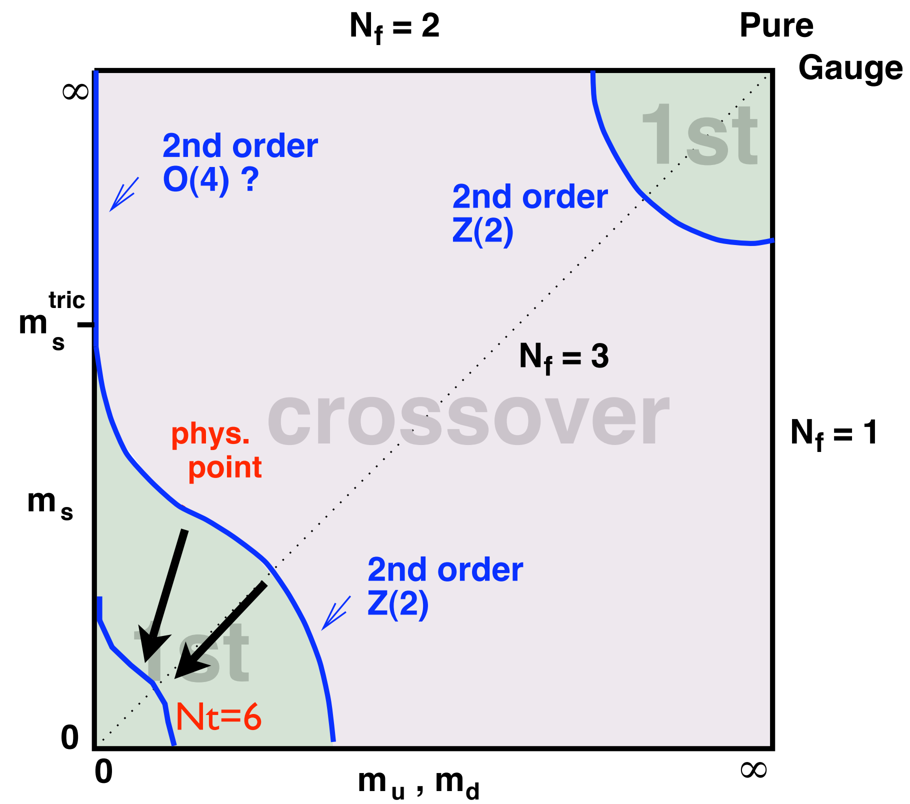




chiral critical line

- physical point: crossover in the continuum — Aoki et al 06
- chiral critical line on $N_t = 4, a \sim 0.3$ fm — de Forcrand, O.P. 07
- consistent with tri-critical point at $m_{u,d} = 0, m_s^{\text{tric}} \sim 2.8T$
- But: $N_f = 2$ chiral O(4) vs. 1st still open
  $U_A(1)$ anomaly! — Di Giacomo et al 05, Kogut, Sinclair 07; Chandrasekharan, Mehta 07; Cossu et al. 12, Aoki et al. 12

Towards the continuum: $N_t = 6, a \sim 0.2$ fm



$$\frac{m_\pi^c(N_t=4)}{m_\pi^c(N_t=6)} \approx 1.77 \qquad N_f = 3$$

de Forcrand, Kim, O.P. 07
Endrödi et al 07

First order region shrinks drastically, continuum limit not yet known...

N.B.: for fixed masses in physical units the order of the p.t. depends on the cut-off!

Lattice QCD at finite baryon density

$$Z = \hat{\mathrm{Tr}}\, \mathrm{e}^{-(H-\mu Q)}, \quad Q = \int d^3x\, \bar{\psi}(x)\gamma_0\psi(x) = \int d^3x\, \psi^\dagger(x)\psi(x)$$

Quark number and chemical potential: $Q = B/3, \mu = \mu_B/3$

Necessary for real world applications: heavy ion collisions, nuclear matter, compact stars,...

Behaviour under charge conjugation: $C = \gamma_0\gamma_2$ $\quad \gamma_\mu = \gamma_\mu^\dagger, \{\gamma_5, \gamma_\mu\} = 0$

$$A_\mu^C = -A_\mu^*, \quad \psi^C = \gamma_0\gamma_2\bar{\psi}^T, \quad \bar{\psi}^C\gamma_0\psi^C = -\bar{\psi}\gamma_0\psi$$

sign flip in Q!

$$\begin{aligned} \mu > 0 \quad &: \quad \text{net baryon number} \\ \mu < 0 \quad &: \quad \text{net anti-baryon number} \end{aligned}$$

Exact symmetry of the continuum grand canonical partition function:

$$Z(\mu) = \int DA^C \, D\bar{\psi}^C D\psi^C \; \exp - \left[S_g^C + S_f^C(\mu = 0) - \mu \int_0^{1/T} dx_0 \, Q^C \right]$$

$$= \int DA \, D\bar{\psi} D\psi \; \exp - \left[S_g + S_f(\mu = 0) + \mu \int_0^{1/T} dx_0 \, Q \right] \; = Z(-\mu)$$

Lattice implementation, naive: $S_f[M(\mu)] = S_f[M(0)] + a\mu \sum_x \psi(x)\gamma_0\psi(x)$

Introduces divergence, which is absent at zero density: failure! $\epsilon = \frac{1}{V}\frac{\partial}{\partial(\frac{1}{T})} \ln Z \stackrel{a \to 0}{\longrightarrow} \infty$

Another symmetry broken by the discretisation!

Continuum fermion number like current coupling to (imaginary) gauge field:

$$j^0 = \bar{\psi}\gamma^0\psi \qquad \mu Q = -ig \int d^3x \, A_0 j_0 \quad \text{with} \quad A_0 = i\frac{\mu}{g}$$

Effectively part of covariant derivative, "gauged" U(1), protects against renormalisation

Lattice implementation: lattice covariant derivative with external gauge field

$$U_{0,\mathrm{ext}} = \mathrm{e}^{iagA_0} = \mathrm{e}^{-a\mu}$$

Wilson fermions:

$$S_f^W = a^3 \sum_x \Big(\bar\psi(x)\psi(x) - \kappa \left[\mathrm{e}^{a\mu}\bar\psi(x)(r-\gamma_0)U_0(x)\psi(x-\hat 0) + \mathrm{e}^{-a\mu}\bar\psi(x+\hat 0)(r+\gamma_0)U_0^\dagger(x)\psi(x) \right] \\ -\kappa \sum_{j=1}^{3} \left[\bar\psi(x)(r-\gamma_j)U_j(x)\psi(x+\hat j) + \bar\psi(x+\hat j)(r+\gamma_j)U_j^\dagger(x)\psi(x) \right] \Big)$$

(Discretisation not unique, only continuum limit)

Now use $\det(\not{D}(U^\dagger)+m+\gamma_0\mu) = \det(\not{D}(U)+m-\gamma_0\mu)$ $\quad S_g[U^\dagger] = S_g[U]$

$$\Rightarrow \quad Z(\mu) = Z(-\mu)$$

The sign problem

Dirac operators satisfy
(continuum, Wilson, staggered,...)

$$(\not{D}+m)^\dagger = \gamma_5(\not{D}+m)\gamma_5$$

With complex chemical potential:

$$\gamma_5(\not{D}+m-\gamma_0\mu)\gamma_5 = (-\not{D}+m+\gamma_0\mu) = (\not{D}+m+\gamma_0\mu^*)^\dagger$$

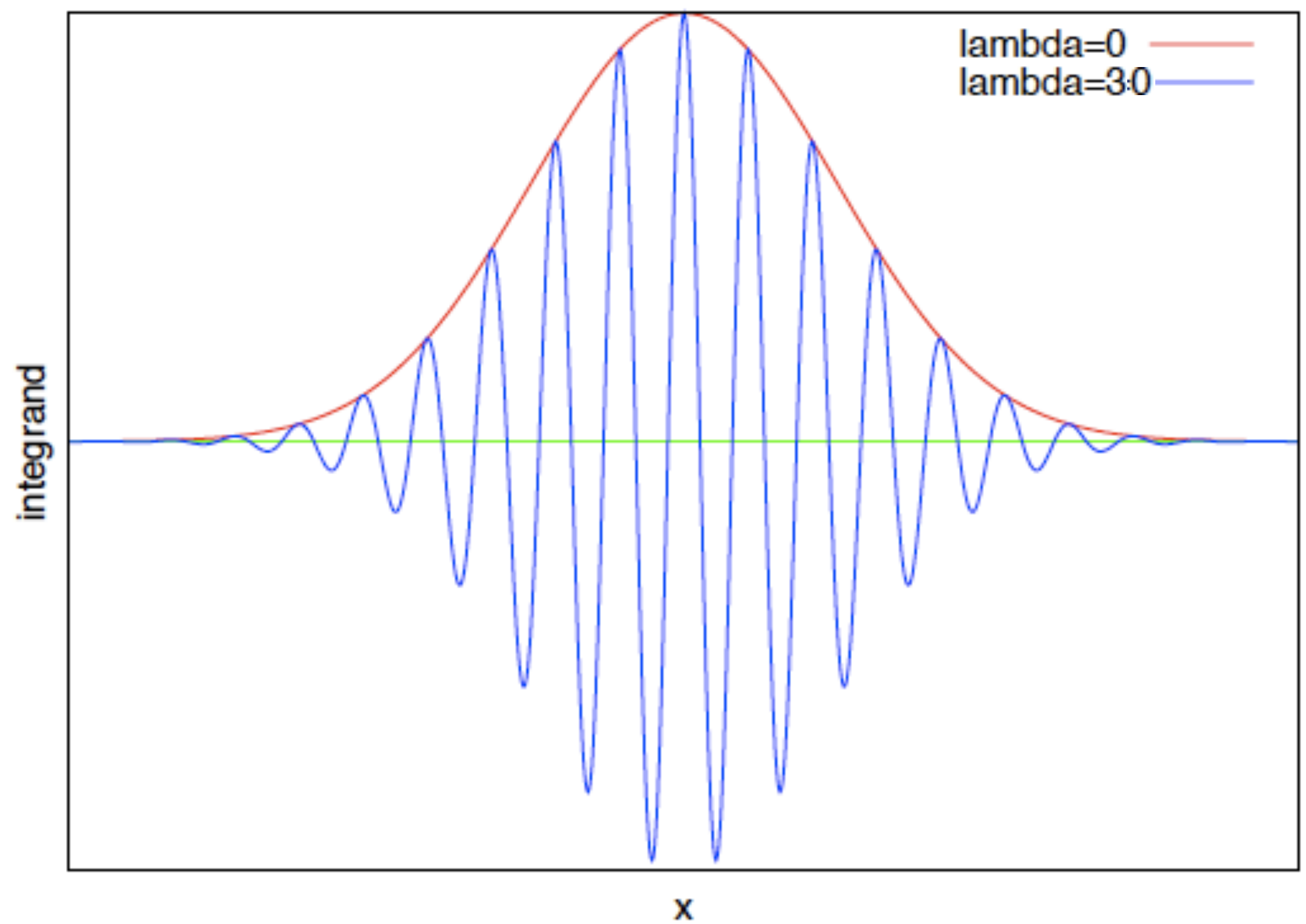
$$\det(\not{D}+m-\gamma_0\mu) = \det{}^*(\not{D}+m+\gamma_0\mu^*)$$

"Sign problem" of QCD

- Complex measure cannot be used for MC importance sampling
- After integration over gauge fields the partition function is real!
- Generic for systems with anti-particles, necessary for physics!

1 dim. illustration

- Example: $Z(\lambda) = \int dx \exp(-x^2 + \mathbf{i}\lambda\mathbf{x})$



- $Z(\lambda)/Z(0) = \exp(-\lambda^2/4)$: exponential cancellations

Example: Polyakov loop

$$\langle \ldots \rangle_g = \int DU \ldots \exp -S_g[U]$$

$$\langle \mathrm{Tr} L \rangle = \mathrm{e}^{-\frac{F_Q}{T}} = \langle \mathrm{Re}\mathrm{Tr} L \ \mathrm{Re} \det M - \mathrm{Im}\mathrm{Tr} L \ \mathrm{Im} \det M \rangle_g$$
$$\langle (\mathrm{Tr} L)^* \rangle = \mathrm{e}^{-\frac{F_{\bar{Q}}}{T}} = \langle \mathrm{Re}\mathrm{Tr} L \ \mathrm{Re} \det M + \mathrm{Im}\mathrm{Tr} L \ \mathrm{Im} \det M \rangle_g$$

Static quarks and anti-quarks must have different free energy at finite density!

Sign problem expresses property under C-conjugation!

$$\det(\not{D} + m - \gamma_0 \mu) \xrightarrow{C} \det(\not{D} + m + \gamma_0 \mu)$$

Fixes:

- Cluster algorithms find configs. with conjugate determinant works for particular Hamiltonians, but not QCD
- Simulation with Langevin algorithms (no importance sampling) Only proven to work for real actions, but work for some ranges of coupling constants

Special cases without sign problem

Imaginary chemical potential:

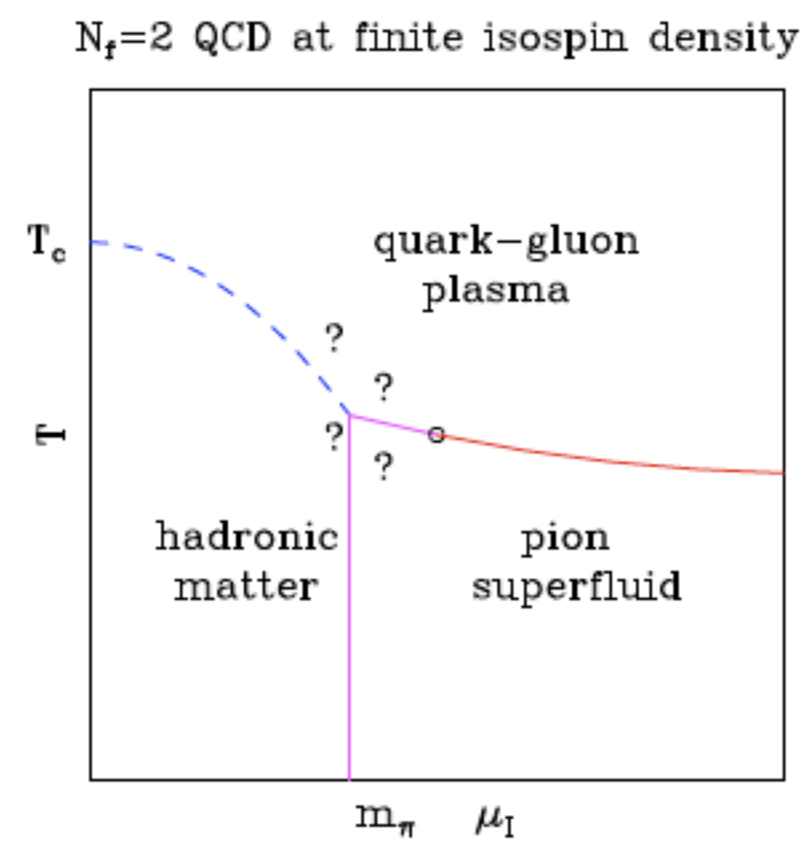
$$\det(\not{D} + m - \gamma_0\mu) = \det{}^*(\not{D} + m + \gamma_0\mu^*)$$ real for $\mu = i\mu_i, \mu_i \in \mathbb{R}$

Two flavours, finite isospin chemical potential:

$$\mu_u = -\mu_d \equiv \mu_I$$

$$\det(\not{D} + m - \gamma_0\mu_I)\det(\not{D} + m + \gamma_0\mu_I)$$

$$= |\det(\not{D} + m - \gamma_0\mu_I)|^2 \geq 0$$



Two colours, SU(2) QCD:

$$S[\not{D} + m - \gamma_0\mu]S^{-1} = [\not{D} + m - \gamma_0\mu^*]^* \qquad S = C\gamma_5\sigma^2 \qquad ST^aS^{-1} = -T^{a*}$$

real reps.

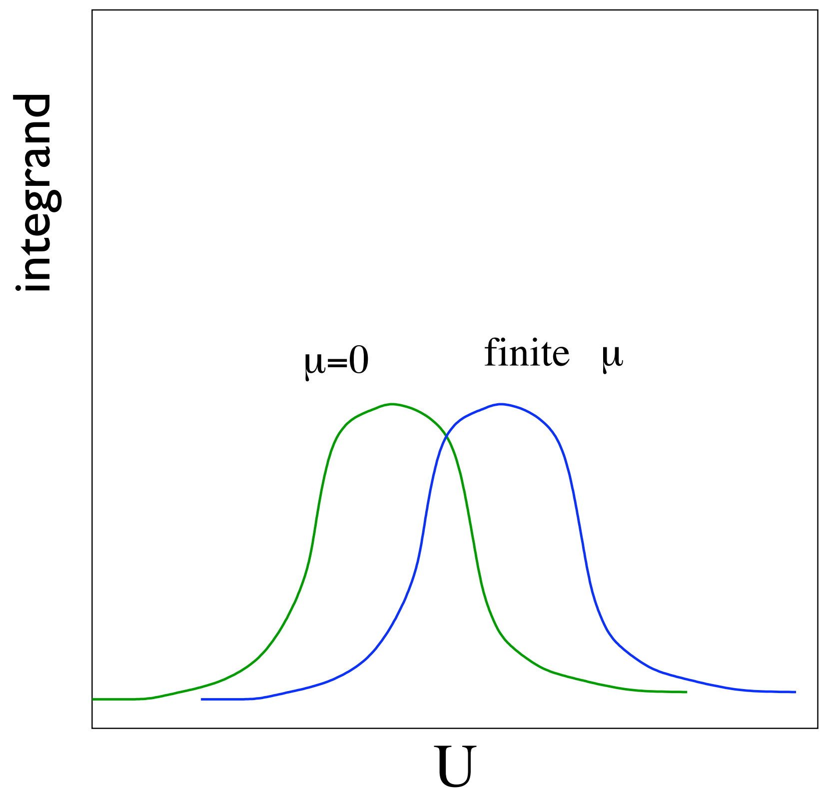
Approximate methods to evade the sign problem: Reweighting

Based on exact relation:

$$Z(\mu) = \int DU \ \det M(\mu)\, \mathrm{e}^{-S_g[U]} = \int DU \ \det M(0)\, \frac{\det M(\mu)}{\det M(0)}\, \mathrm{e}^{-S_g[U]}$$
$$= Z(0) \left\langle \frac{\det M(\mu)}{\det M(0)} \right\rangle_{\mu=0} .$$

I. Numerically difficult, signal exponentially suppressed with volume

$$\frac{Z(\mu)}{Z(0)} = \exp -\frac{F(\mu) - F(0)}{T} = \exp -\frac{V}{T}\left(f(\mu) - f(0)\right)$$



II. Overlap problem, because of importance sampling

With increasing difference the most frequent configs. are increasingly unimportant

Finite density by Taylor expansion

Taylor expansion of the pressure around zero density:

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n} \equiv \Omega(T,\mu)$$

$$c_0(T) = \frac{p}{T^4}(T,\mu=0), \quad c_{2n}(T) = \frac{1}{(2n)!} \left.\frac{\partial^{2n}\Omega}{\partial(\frac{\mu}{T})^{2n}}\right|_{\mu=0}$$

The coefficients can be computed at zero density!

Other physical quantities follow:

$$\frac{n}{T} = \frac{\partial\Omega}{\partial(\frac{\mu}{T})} = 2c_2\frac{\mu}{T} + 4c_4\left(\frac{\mu}{T}\right)^3 + \ldots,$$

$$\frac{\chi_q}{T^2} = \frac{\partial^2\Omega}{\partial(\frac{\mu}{T})^2} = 2c_2 + 12c_4\left(\frac{\mu}{T}\right)^2 + 30c_6\left(\frac{\mu}{T}\right)^4 + \ldots$$

No sign problem, but need small μ/T

Higher coeffs. increasingly difficult:

$$\frac{\partial\langle O\rangle}{\partial\mu} = \left\langle\frac{\partial O}{\partial\mu}\right\rangle + N_f\left(\left\langle O\frac{\partial\ln\det M}{\partial\mu}\right\rangle - \langle O\rangle\left\langle\frac{\partial\ln\det M}{\partial\mu}\right\rangle\right)$$

QCD at imaginary chemical potential

No sign problem; general idea:

Observables have definite symmetry,
even or odd in chemical potential

$$\langle O\rangle(\mu_i) = \sum_{k=1}^{N} c_k \left(\frac{\mu_i}{T}\right)^{2k}$$

- Simulate left side without further systematic error
- Check if fit to low order polynomial is possible $\mu/T < 1$
- Analytic continuation trivial (in the absence of singularities) $\mu_i \to -i\mu_i$

General considerations:

Partition function is periodic $Z = \hat{\mathrm{Tr}}\, e^{-\frac{(H - i\mu_i Q)}{T}}$

Is this a healthy theory?

Yes! Recall $\mu Q = -ig \int d^3x\, A_0 j_0 \quad \text{with} \quad A_0 = i\frac{\mu}{g}$

Equivalent to theory in real external field!

Periodicity non-trivial:

Chemical potential can be absorbed by boundary conditions

$$Z^{(1)}(i\mu_i) = \int DU \det M(0) \mathrm{e}^{-S_g}, \quad \text{b.c.:} \quad \psi(\tau + N_\tau, \mathbf{x}) = -\mathrm{e}^{i\frac{\mu_i}{T}} \psi(\tau, \mathbf{x})$$

Consider the topological gauge trafo $\quad g'(\tau + N_\tau, x) = \mathrm{e}^{-i\frac{2\pi n}{N}} g'(\tau, \mathbf{x})$

Measure and action are invariant, hence

$$Z^{(2)}(i\mu_i) = \int DU \det M(0) \mathrm{e}^{-S_g}, \quad \text{b.c.:} \quad \psi(\tau + N_\tau, \mathbf{x}) = -\mathrm{e}^{-i\frac{2\pi n}{N}} \mathrm{e}^{i\frac{\mu_i}{T}} \psi(\tau, \mathbf{x})$$

$$Z^{(2)}\left(i\frac{\mu_i}{T} + i\frac{2\pi n}{N}\right) = Z^{(1)}\left(i\frac{\mu_i}{T}\right)$$

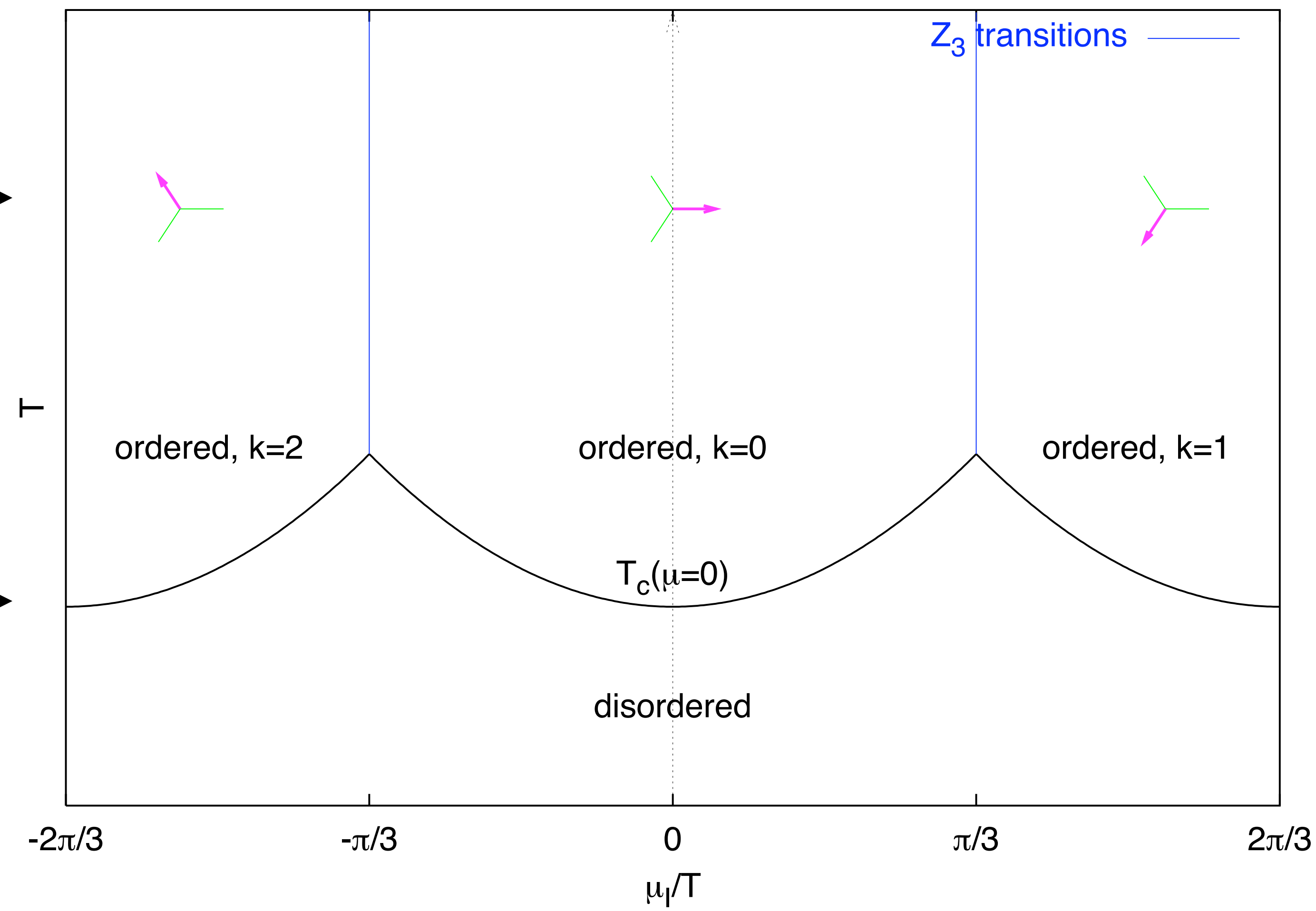
Both partition fcns. related by gauge trafo, identical!

Roberge-Weiss symmetry: $\quad Z\left(i\frac{\mu_i}{T} + i\frac{2\pi n}{N}\right) = Z\left(i\frac{\mu_i}{T}\right)$

The phase diagram at imaginary chemical potential

Phase of Polyakov loop

Analytic continuation of chiral/deconfinement transition, depends on Nf, quark masses



Roberge-Weiss: Z(3) transitions are first order for large T (perturbation theory)
crossover for small T (strong coupling limit)

analytic continuation within:

$$|\mu|/T \leq \pi/3 \Rightarrow \mu_B \lesssim 550\mathrm{MeV}$$

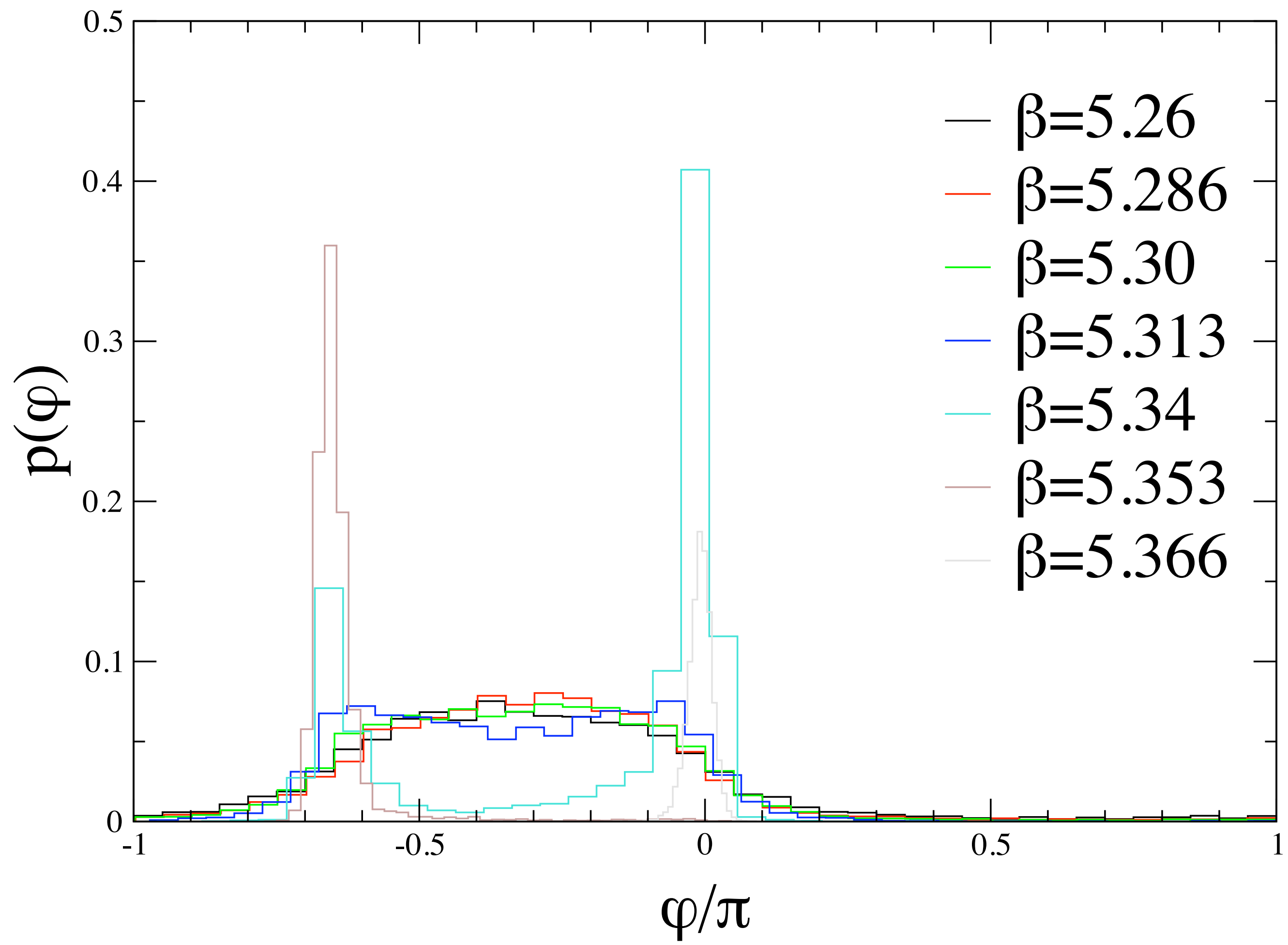
Limited by singularity (phase transition) closest to $\mu = 0$

The Z(3) transition numerically

Nf=2: de Forcrand, O.P. 02

Nf=4: D'Elia, Lombardo 03

Sectors characterised by phase of Polyakov loop: $\langle L(x) \rangle = |\langle L(x) \rangle| \, \mathrm{e}^{i\varphi}$



Low T: crossover High T: first order p.t.

Towards the QCD phase diagram

Analyticity of the (pseudo-)critical line

Recall definition by peak of susceptibilities: $\chi_{max} = \chi(\beta_c, m_f, \mu)$

Implicit definition of pseudo-critical line $\beta_c(m_f, \mu)$

Implicit function theorem:

For analytic susceptibility, also the implicitly defined pseudo-critical coupling is analytic (always true on finite V!)

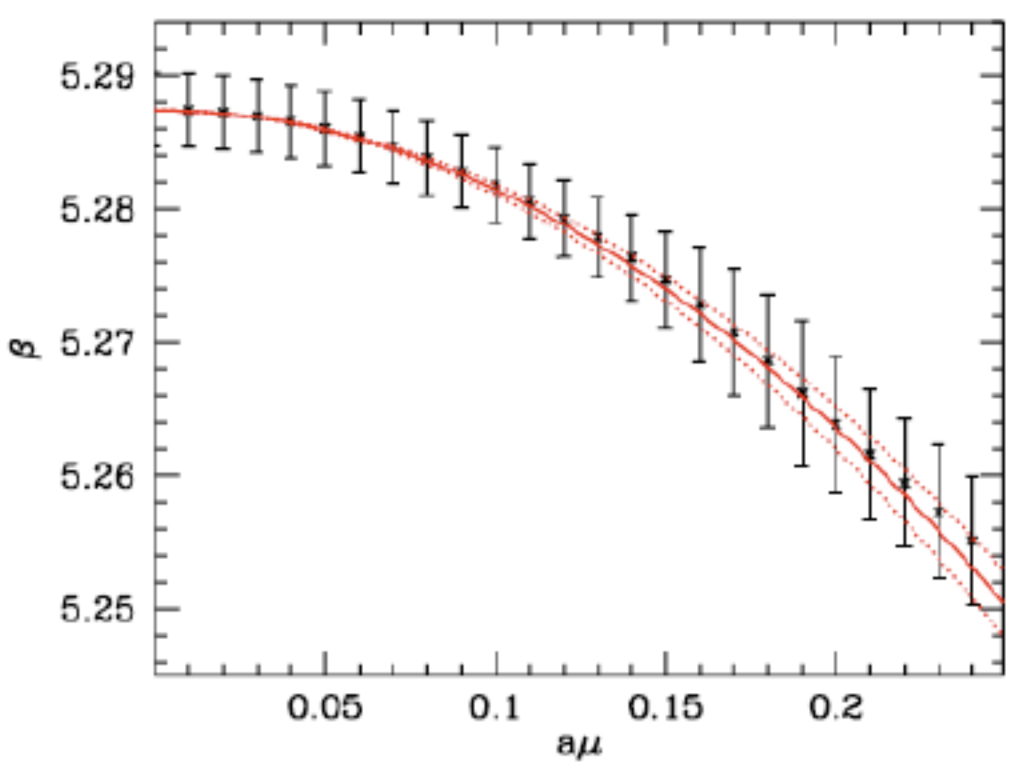
$$\beta_c(m_f, \frac{\mu}{T}) = \sum_n b_{2n}(m_f) \left(\frac{\mu}{T}\right)^{2n}$$

$$\Rightarrow \quad \frac{T_c(m_f,\mu)}{T_c(m_f,0)} = 1 + t_2(m_f)\left(\frac{\mu}{T}\right)^2 + t_4(m_f)\left(\frac{\mu}{T}\right)^4 + \ldots$$

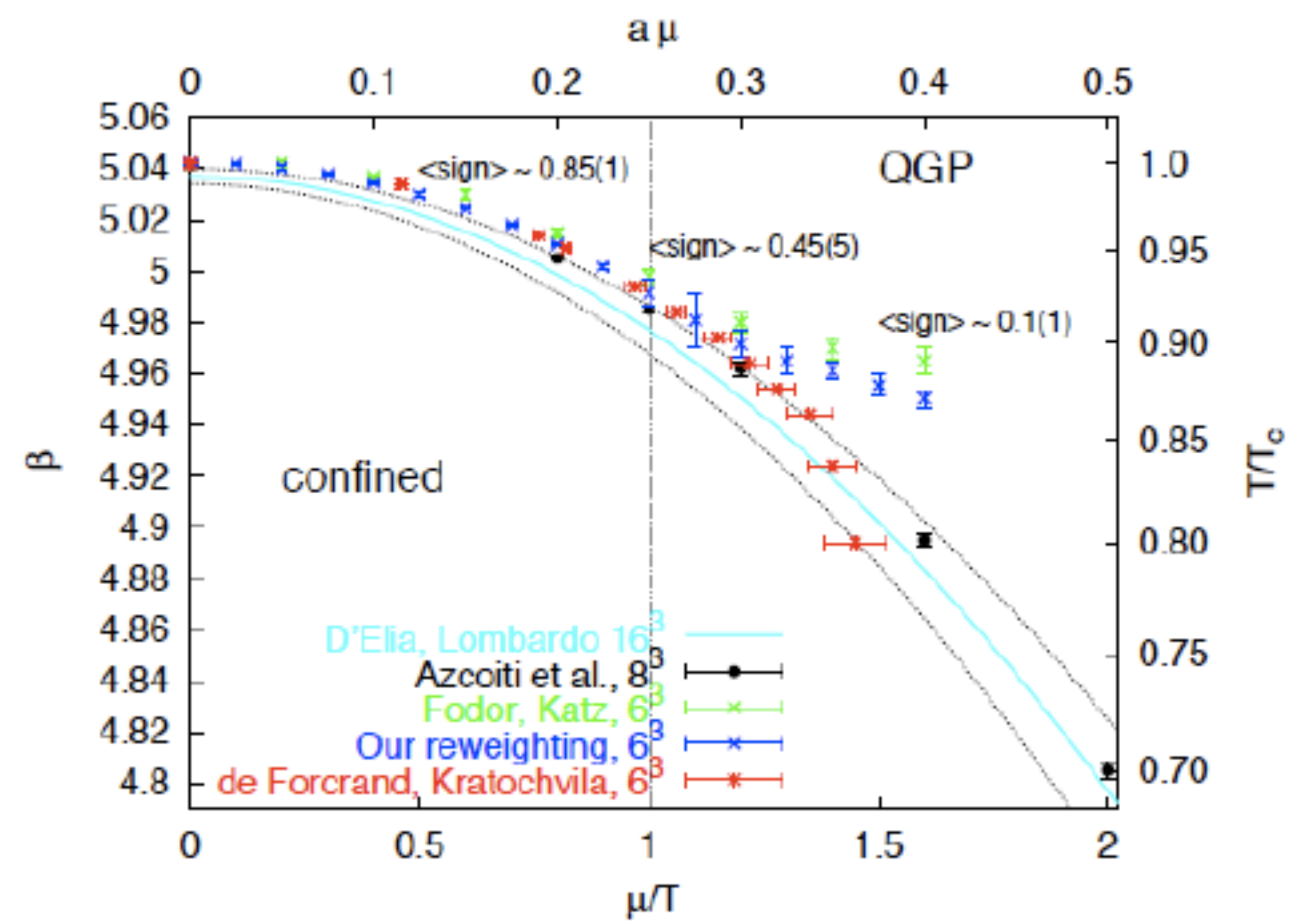
- Accessible to all methods discussed for sufficiently small chemical potential
- Crosscheck, in particular between Taylor coefficients and imaginary chem. pot.

Test of methods: comparing $T_c(\mu)$

Reweighting vs. imag. μ (FK, FP)

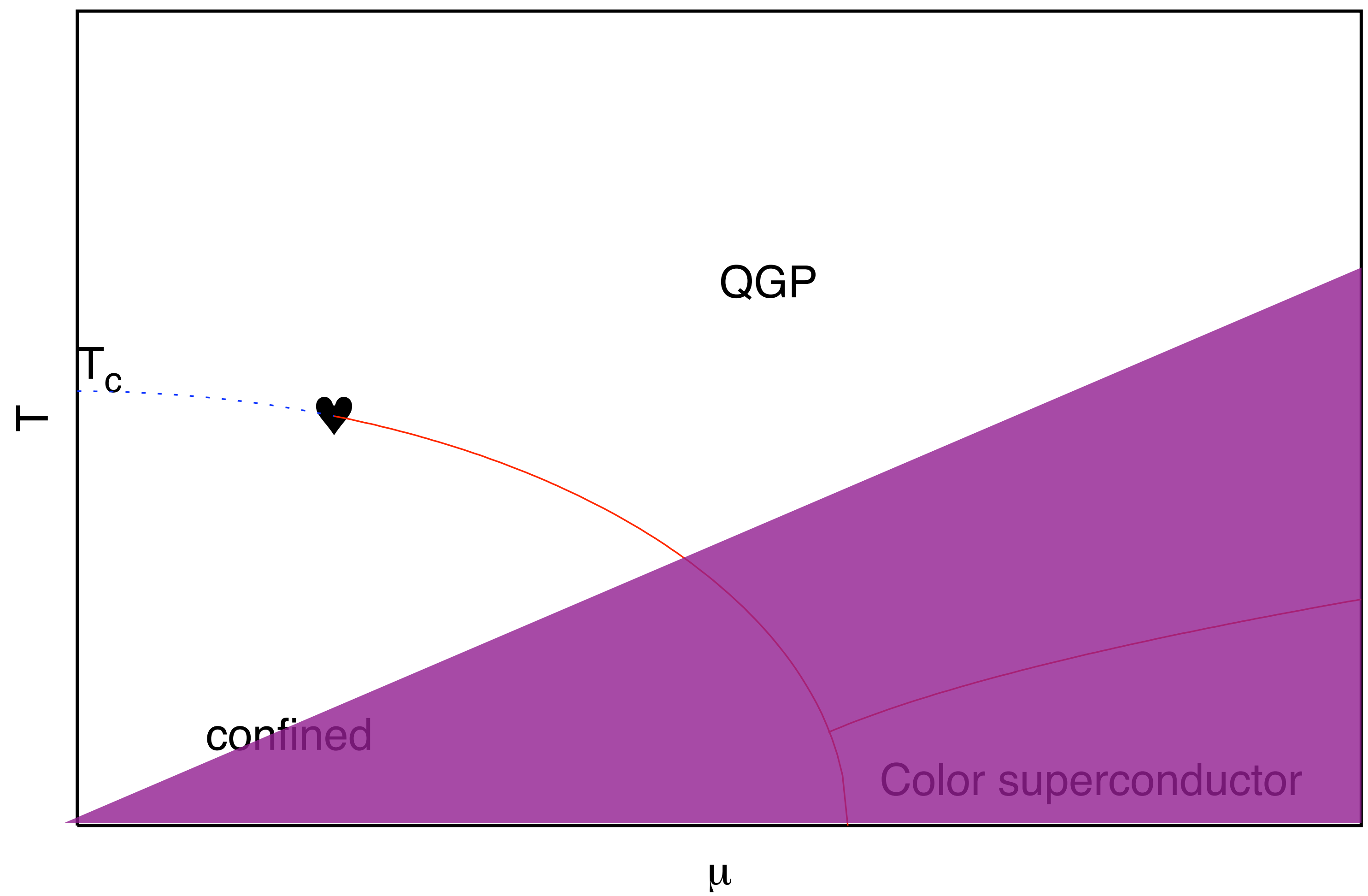


Rew., imag. μ, canonical ensemble ...



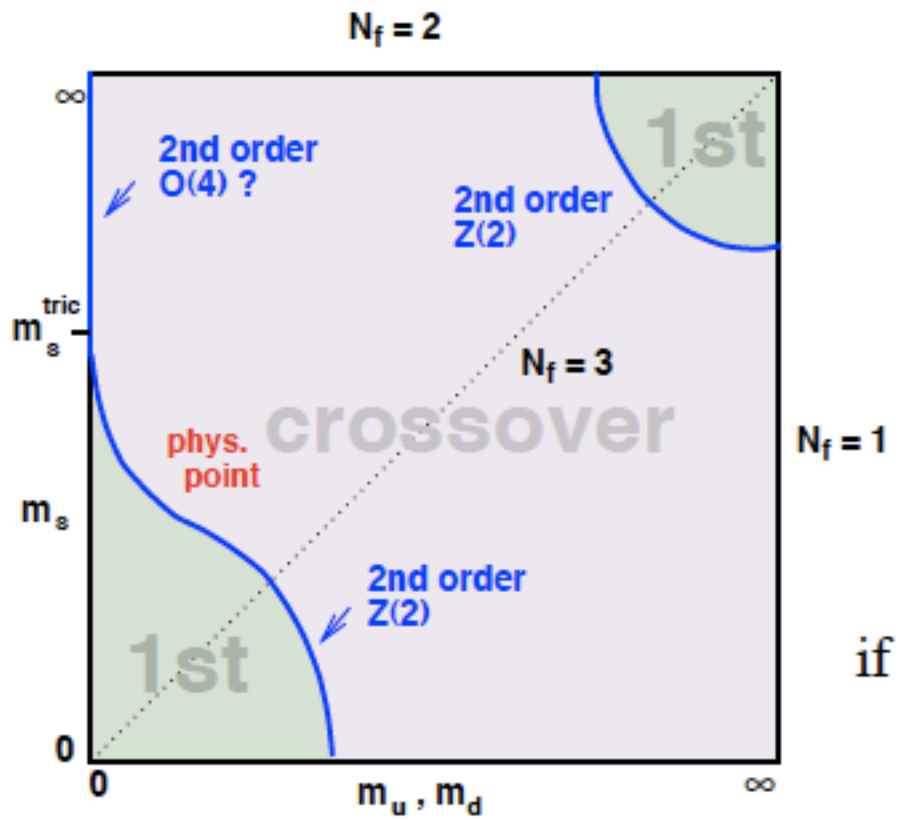
All agree on $T_0(m,\mu)$!!! $(\mu/T \lesssim 1)$

The calculable region of the phase diagram

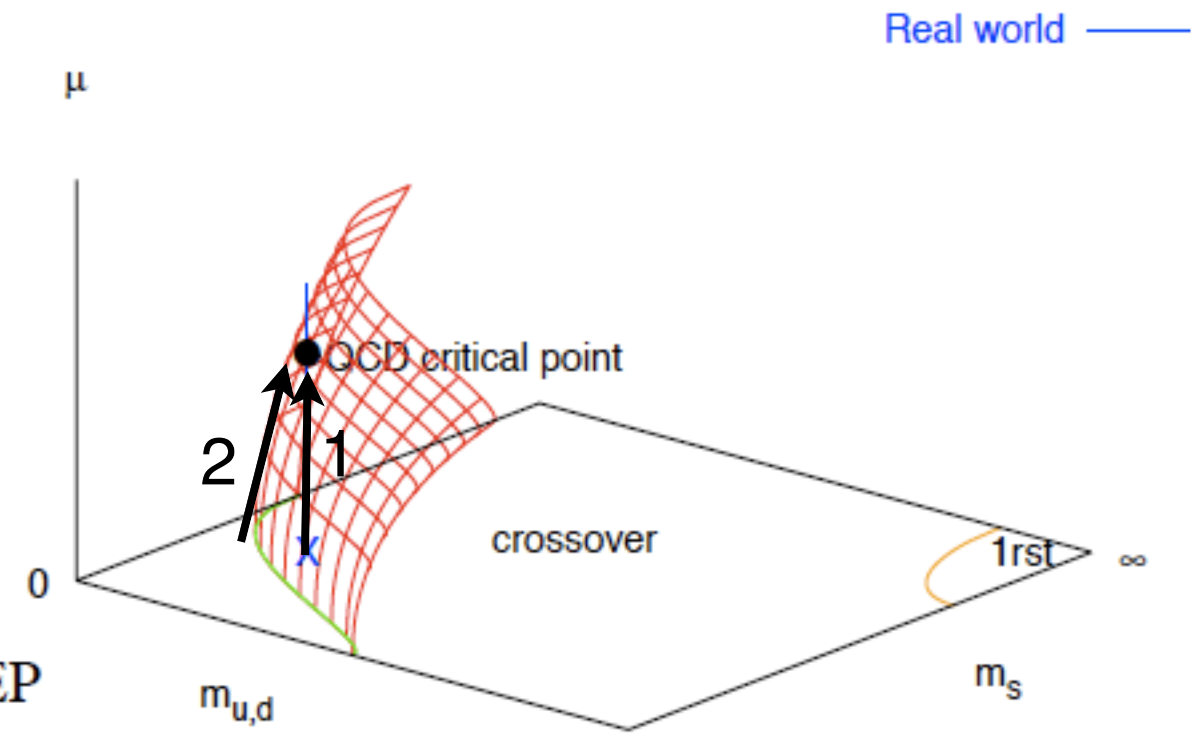


- need $\mu/T \lesssim 1 \quad (\mu = \mu_B/3)$
- Upper region: equation of state, screening masses, quark number susceptibilities etc. under control

Much harder: is there a QCD critical point?



$\underbrace{\longrightarrow}_{\text{if } \textit{chiral} \text{ CEP}}$



Two strategies:

1 follow vertical line: $m = m_{\text{phys}}$, turn on μ

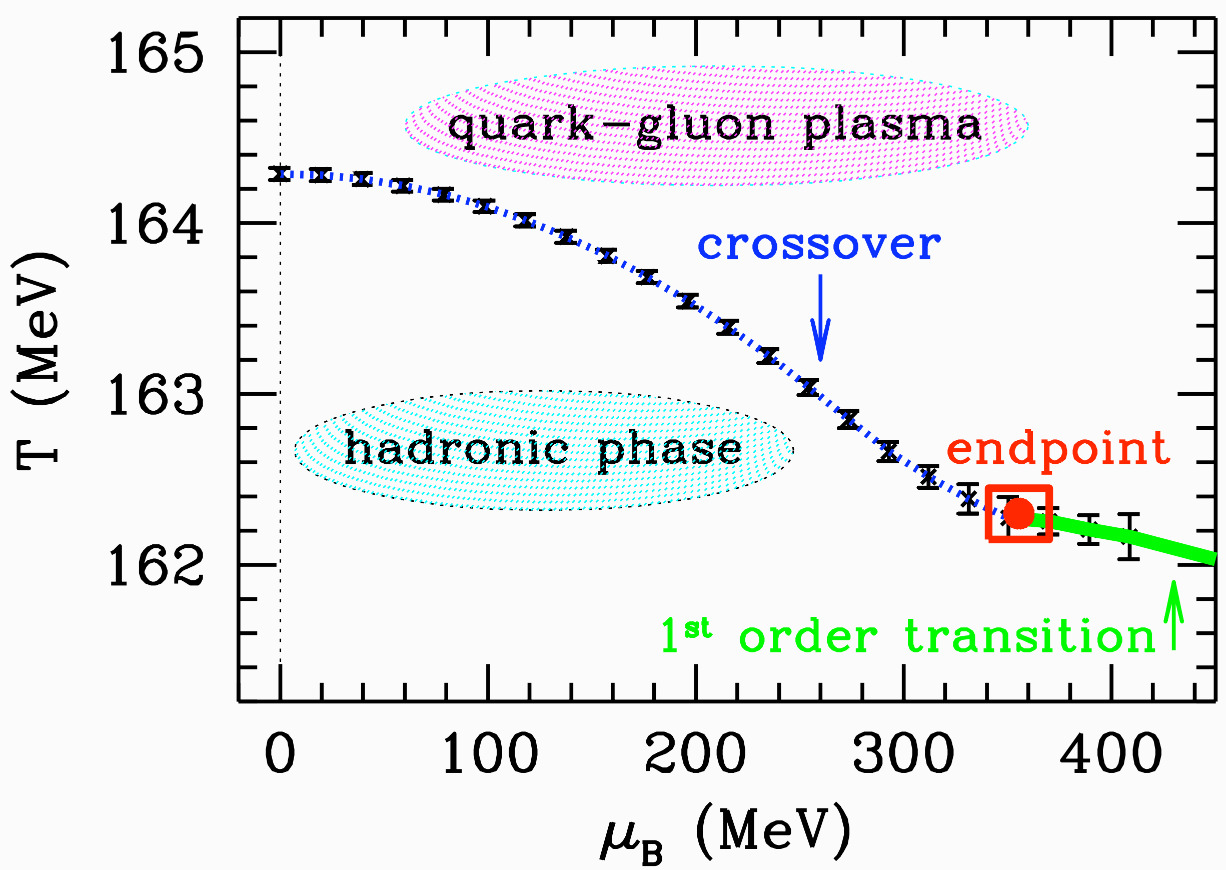
2 follow critical surface: $m = m_{\text{crit}}(\mu)$

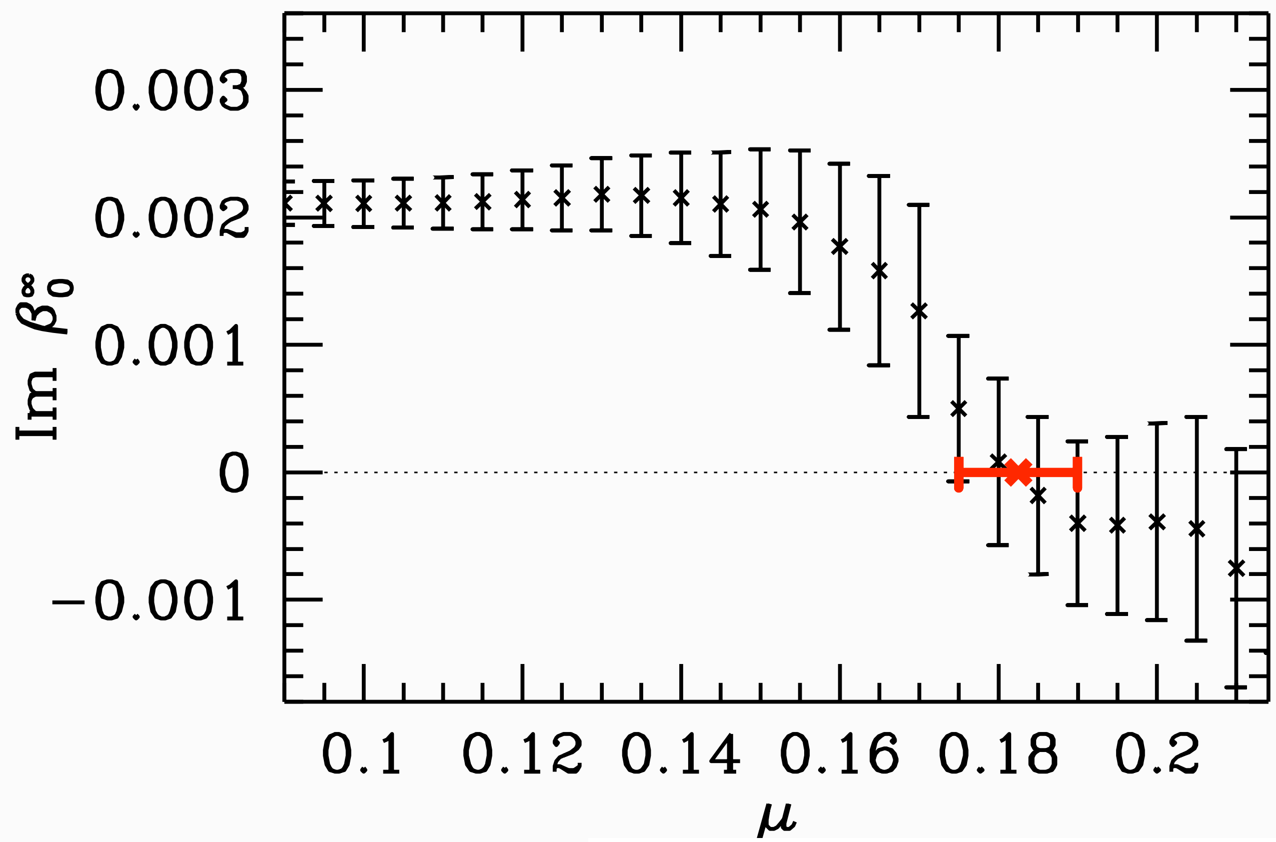
Approach 1a: CEP from reweighting

Fodor, Katz 04

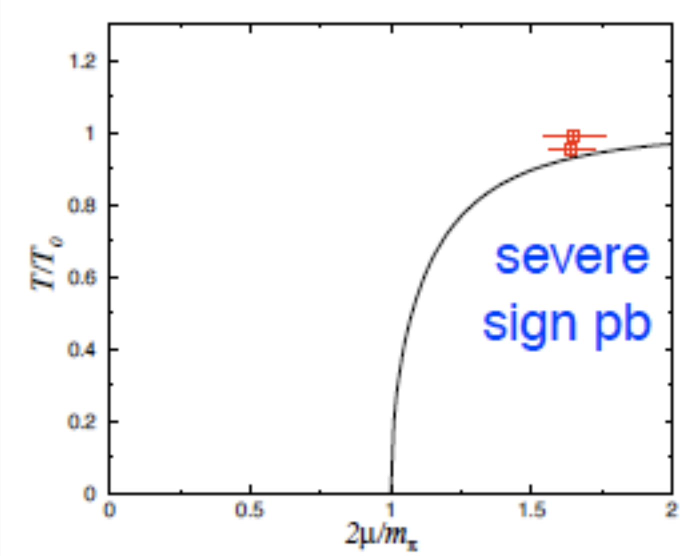
$N_t = 4, N_f = 2+1$ physical quark masses, unimproved staggered fermions

Lee-Yang zero:





$$(\mu_E^q, T_E) = (120(13), 162(2)) \text{ MeV}$$



abrupt change: caused by baryon or pion condensation?

Splittorf 05, Stephanov 08

Approach 1b: CEP from Taylor expansion

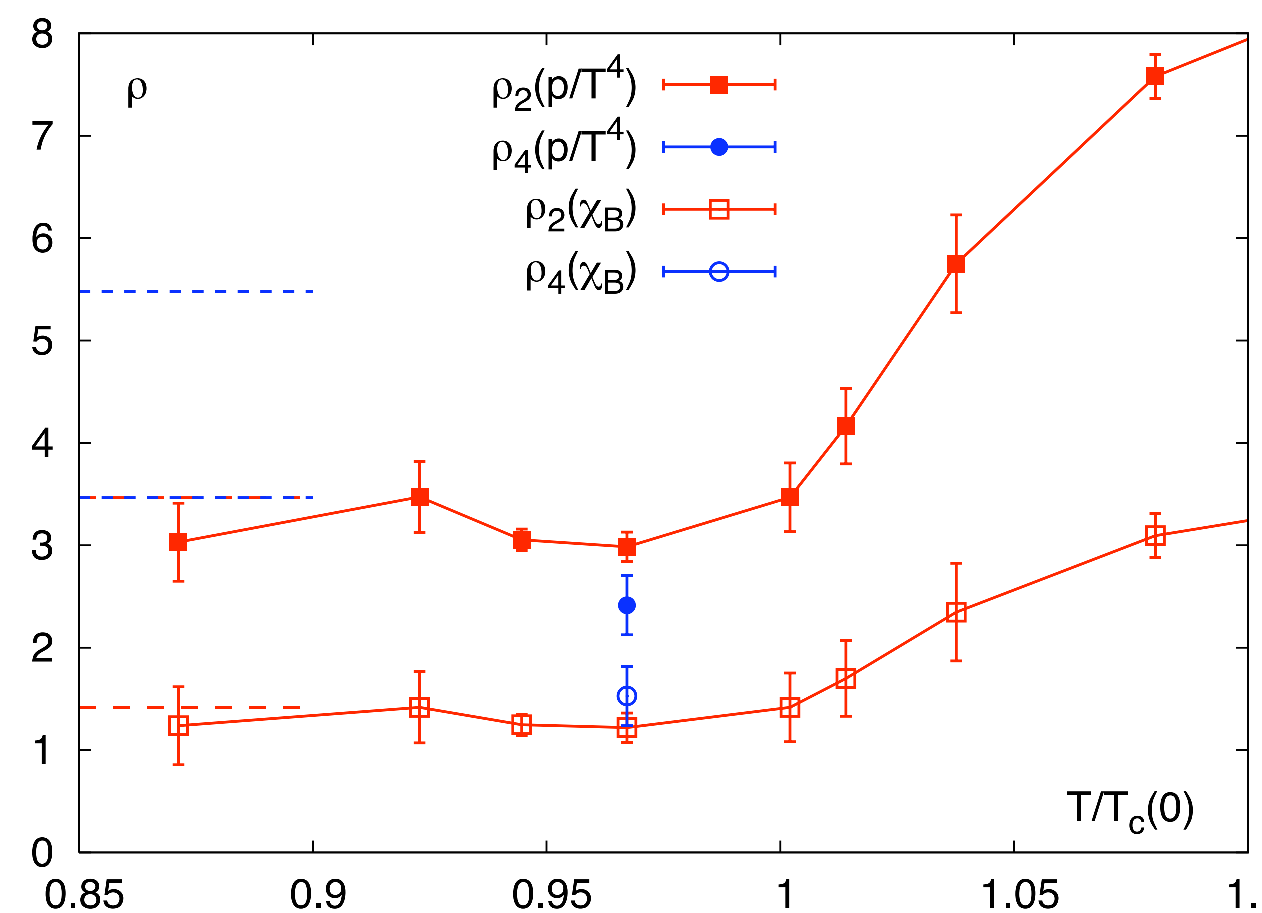
$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n}$$

Nearest singularity=radius of convergence $\quad \frac{\mu_E}{T_E} = \lim_{n\to\infty} \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}, \quad \lim_{n\to\infty} \left|\frac{c_0}{c_{2n}}\right|^{\frac{1}{2n}}$

Different definitions agree only for $n \to \infty$
not n=1,2,3,...
control of systematics?

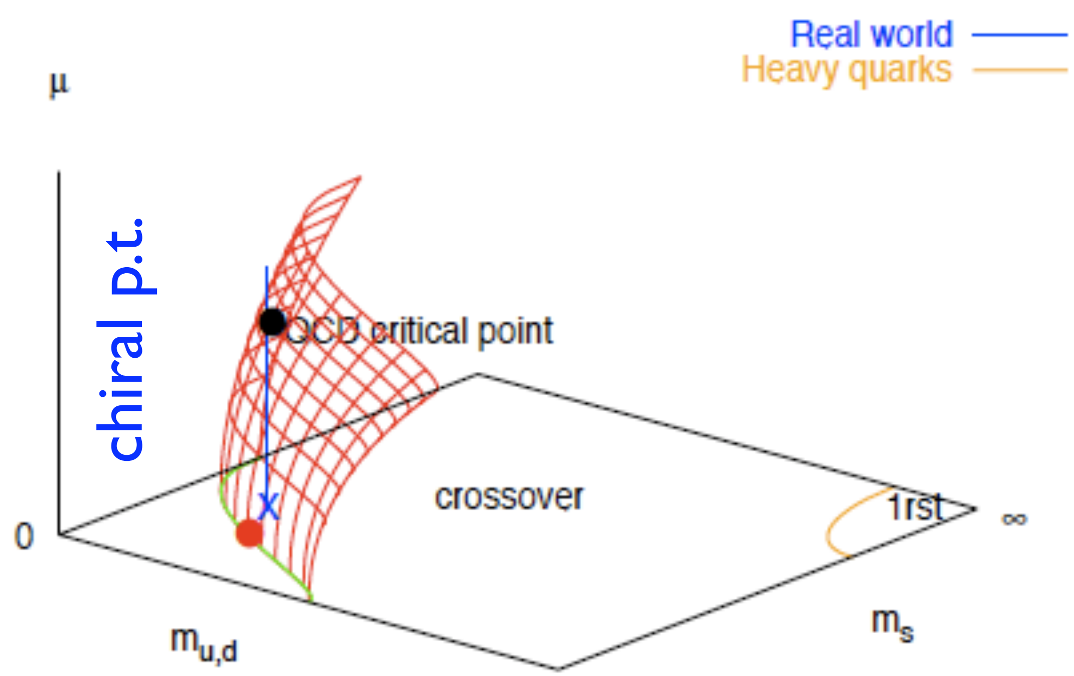
C.Schmidt, hotQCD 09

Hadron resonance gas

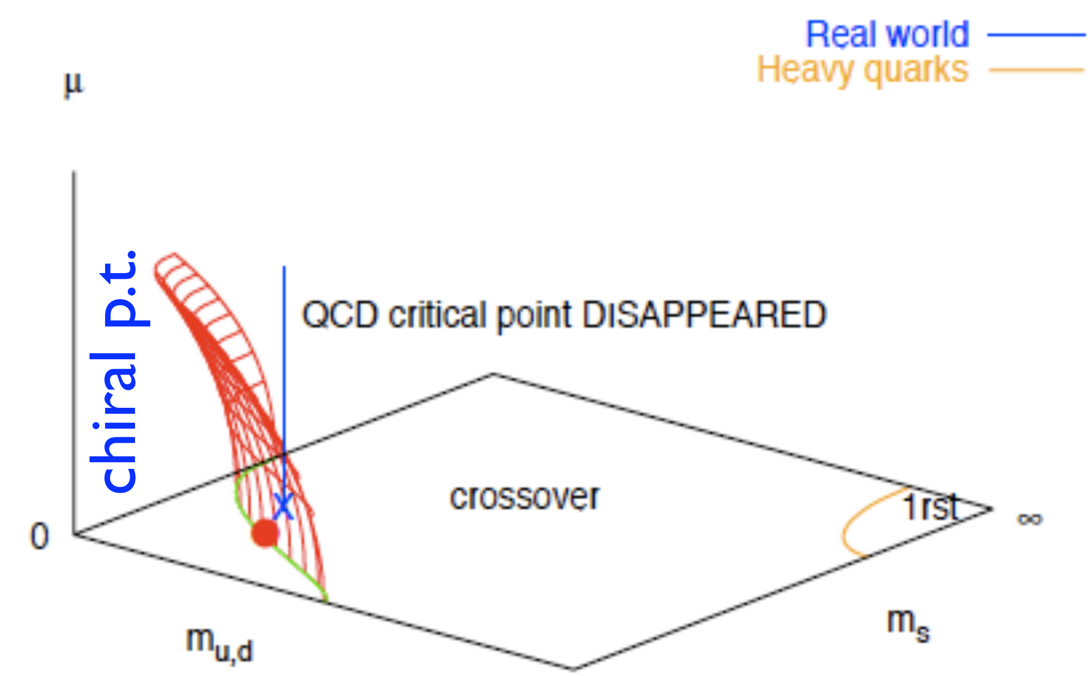


Radius of convergence necessary condition for CEP, but can it proof its existence?

Approach 2: follow chiral critical line → surface



$c_1 > 0$



$c_1 < 0$

$$\frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1} c_k \left(\frac{\mu}{\pi T}\right)^{2k}$$

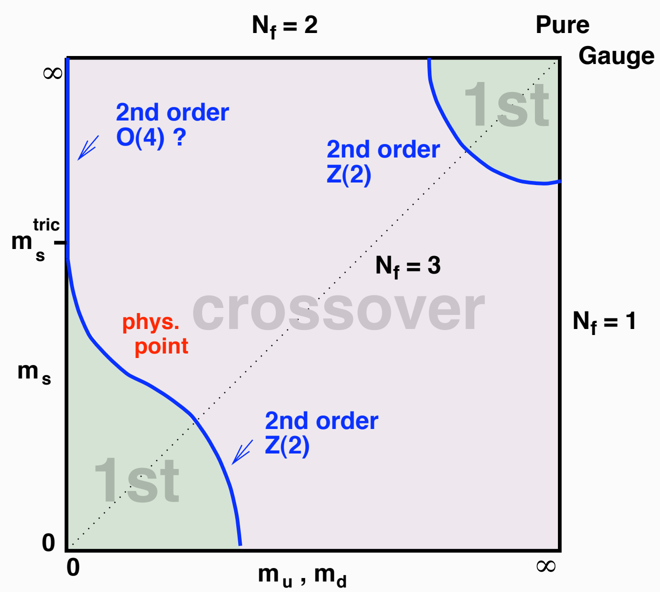
1. Tune quark mass(es) to $m_c(0)$: 2nd order transition at $\mu = 0$, $T = T_c$
   known universality class: $3d$ Ising

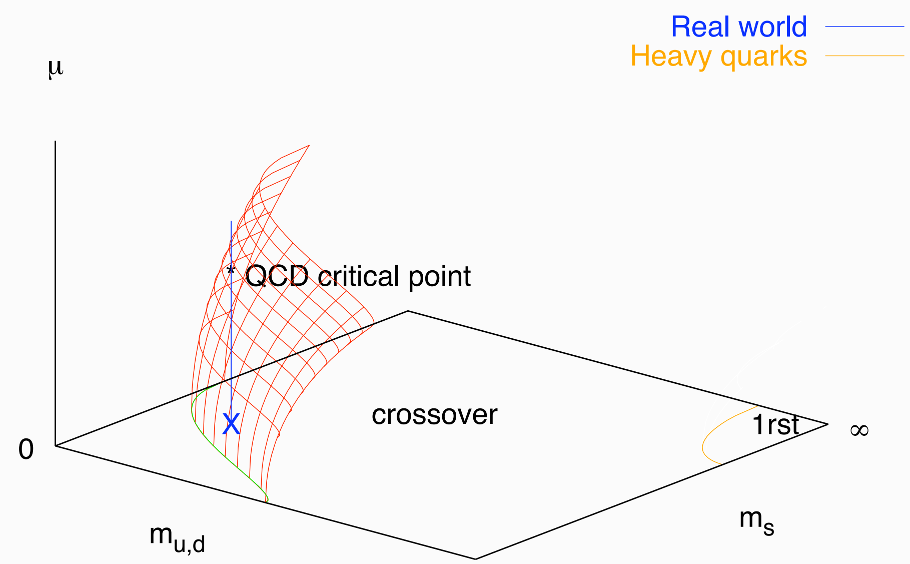
2. Measure derivatives $\frac{d^k m_c}{d\mu^{2k}}|_{\mu=0}$:

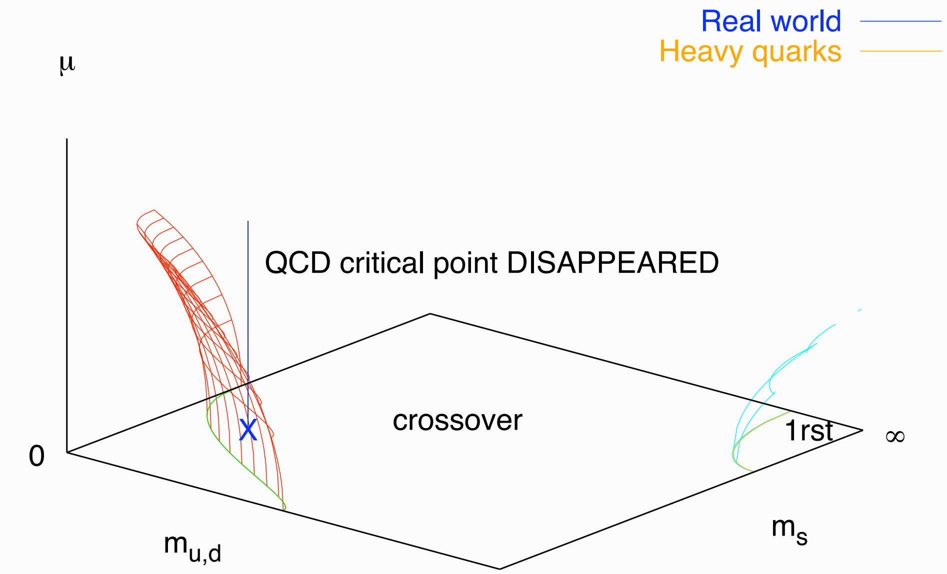
   Turn on imaginary μ and measure $\frac{m_c(\mu)}{m_c(0)}$

de Forcrand, O.P. 08,09

Finite density: chiral critical line $\longrightarrow$ critical surface





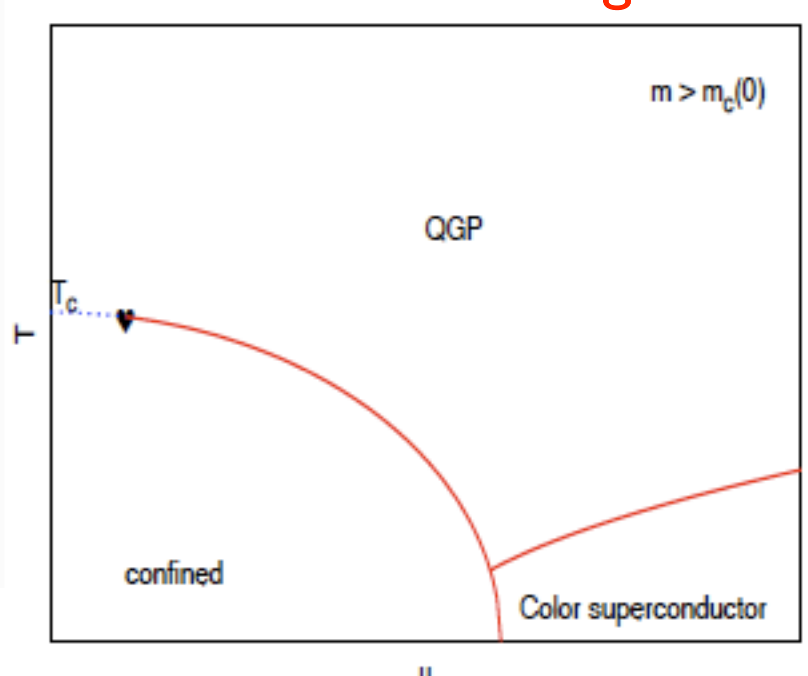


$$\frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1} \mathbf{c_k} \left(\frac{\mu}{\pi T}\right)^{2k}$$

$c_1 > 0$ $\qquad$ $c_1 < 0$

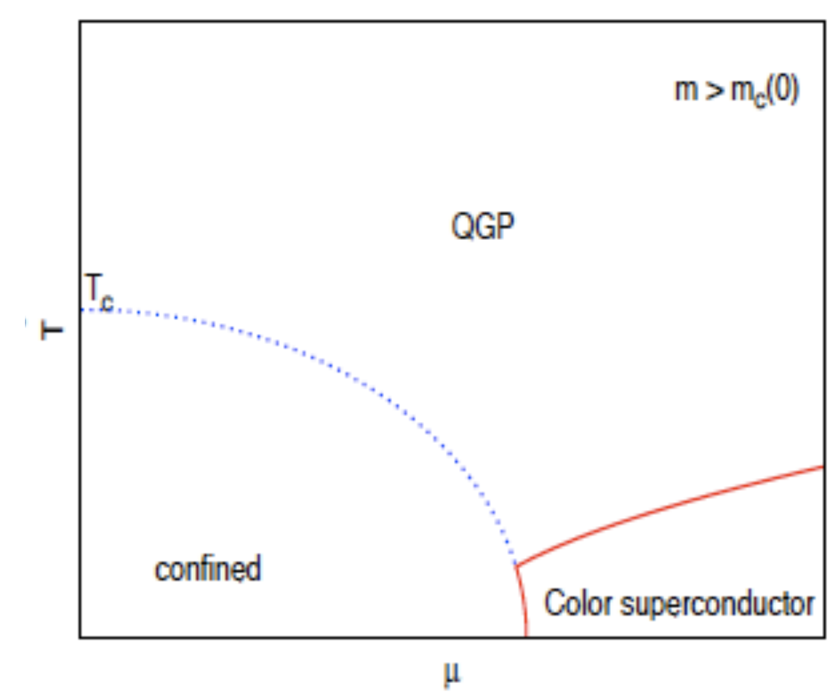
Standard scenario

transition strengthens

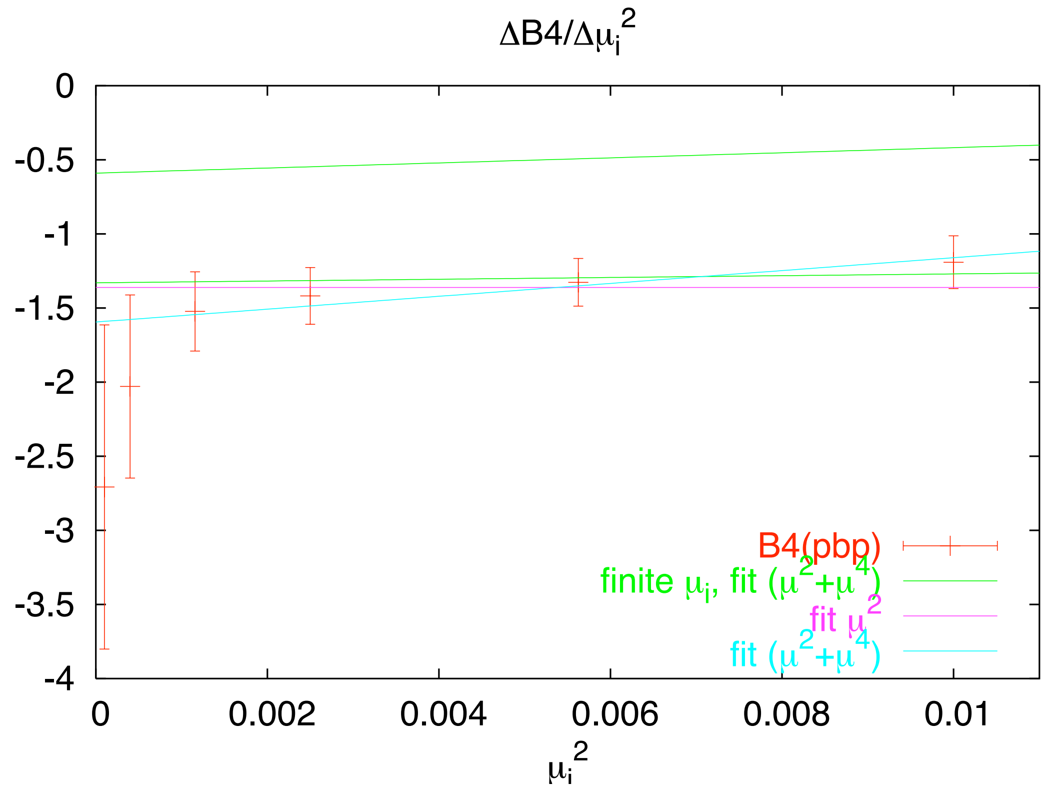


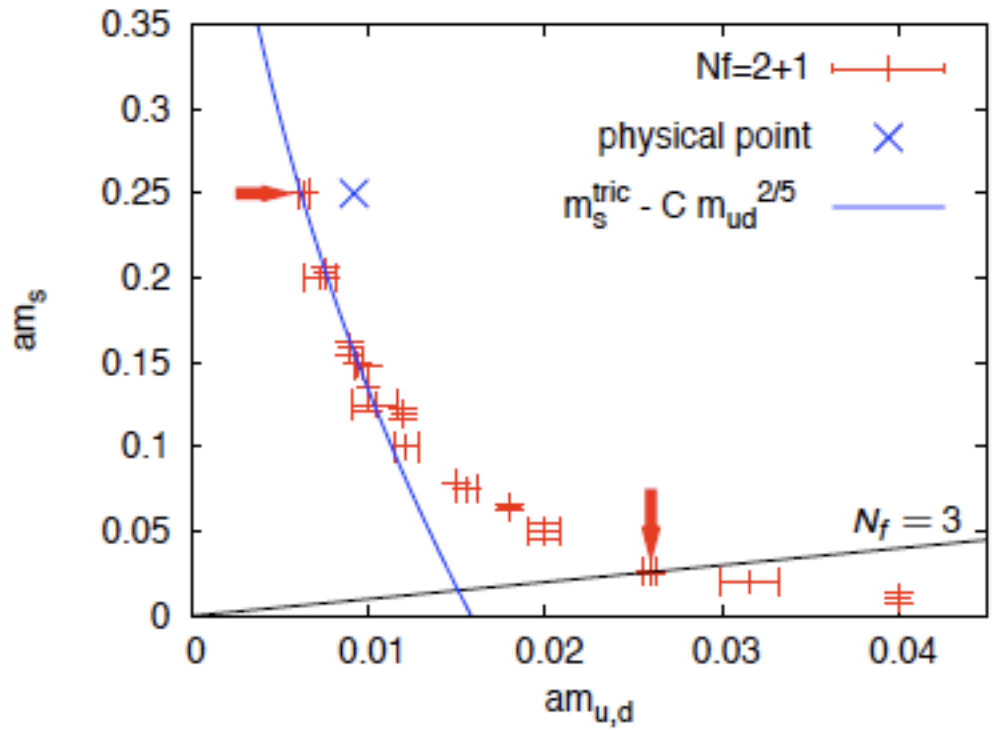
Exotic scenario

transition weakens



Curvature of the chiral critical surface





Nf=3: a) fit to imaginary chemical potential
b) calculation of coefficient by finite differences

consistent $8^3 \times 4$ and $12^3 \times 4$, $\sim 5 \times 10^6$ traj.

$$\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3)\left(\frac{\mu}{\pi T}\right)^2 \underbrace{- 47(20)\left(\frac{\mu}{\pi T}\right)^4}_{\text{8th derivative of P}} - \ldots$$

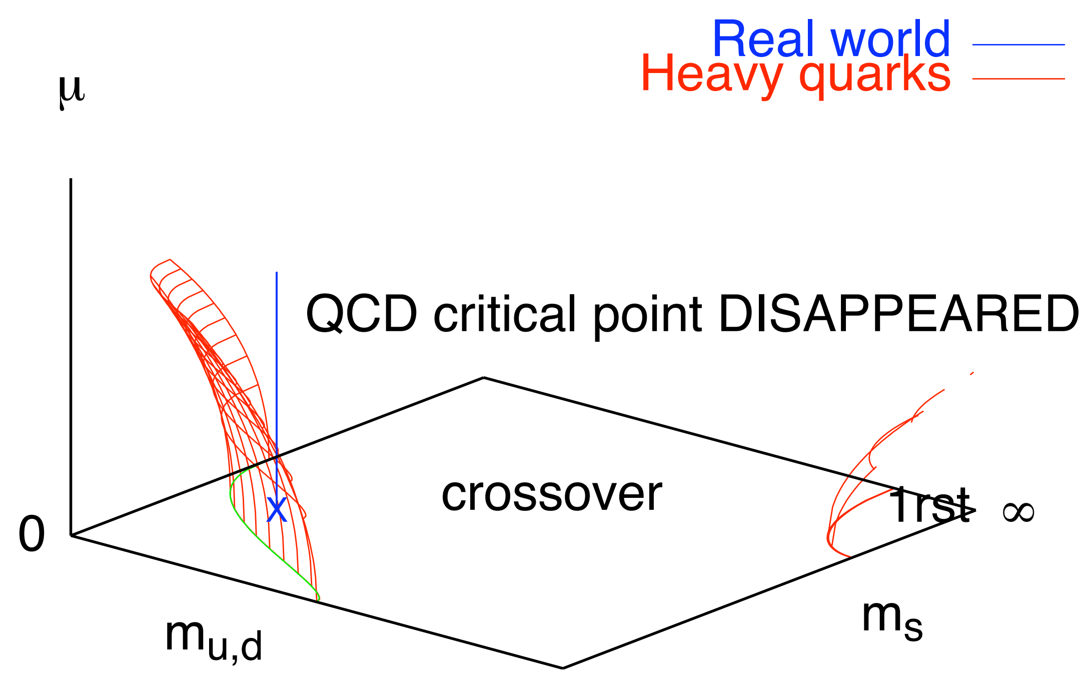
$16^3 \times 4$, Grid computing, $\sim 10^6$ traj.

$$\frac{m_c^{u,d}(\mu)}{m_c^{u,d}(0)} = 1 - 39(8)\left(\frac{\mu}{\pi T}\right)^2 - \ldots$$

Importance of higher order terms ?

de Forcrand, O.P. 08,09

On coarse lattice exotic scenario: no chiral critical point at small density



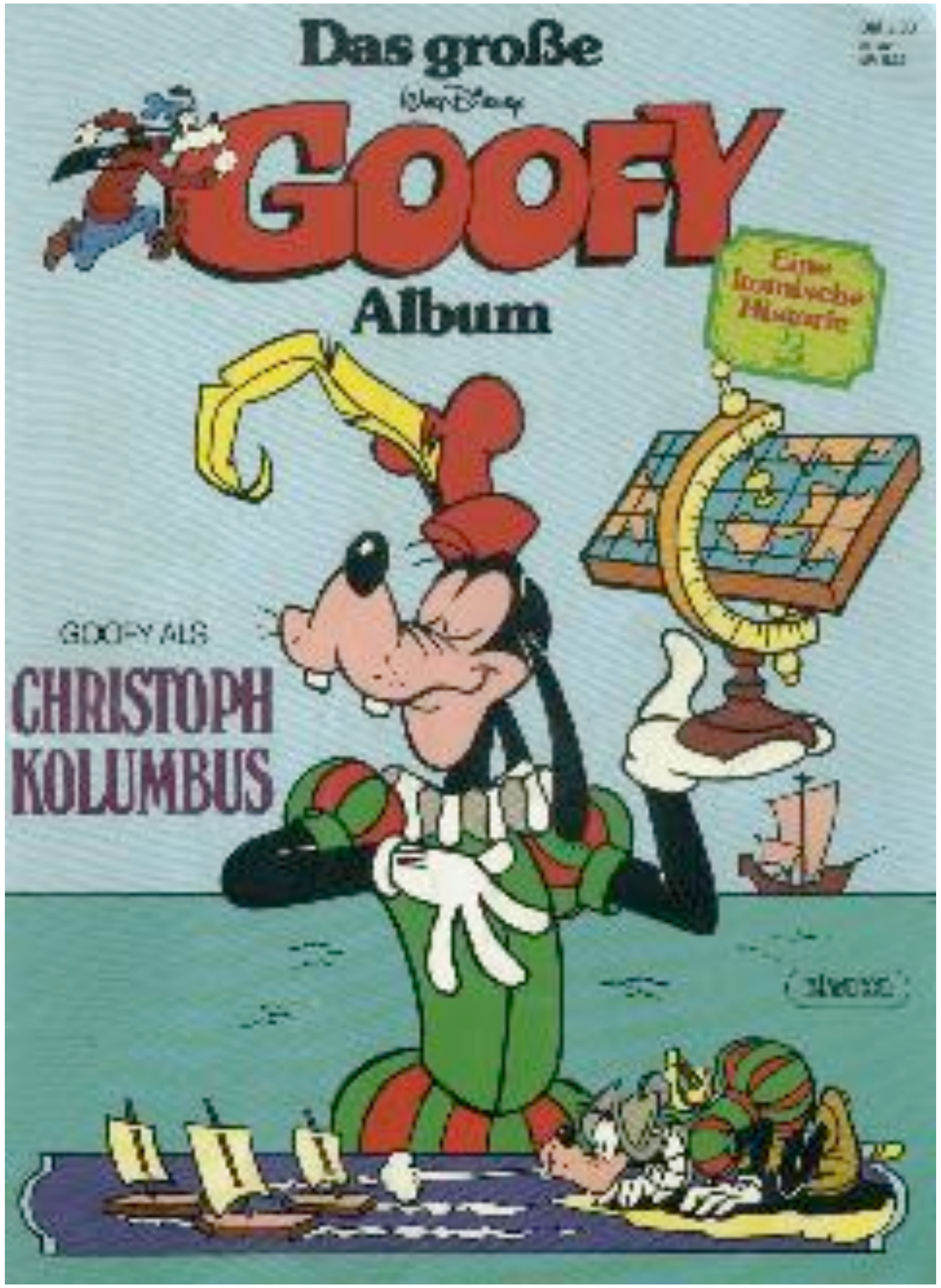
Weakening of p.t. with chemical potential also for:

-Heavy quarks — de Forcrand, Kim, Takaishi 05

-Light quarks with finite isospin density — Kogut, Sinclair 07

-Electroweak phase transition with finite lepton density — Gynther 03

Un-discovering a critical point feels like...

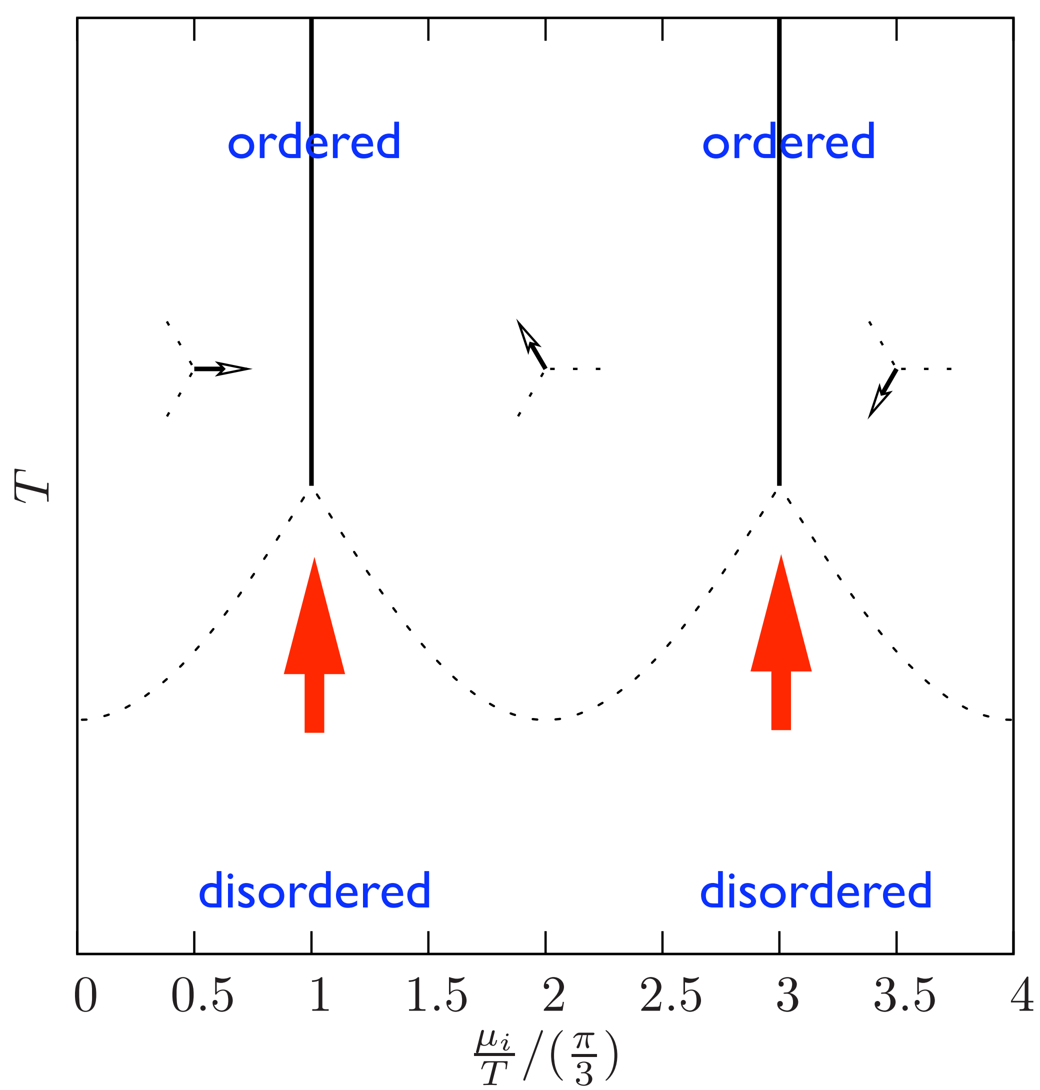


Understanding the curvature from imaginary μ

Nf=4: D'Elia, Di Renzo, Lombardo 07   Nf=2: D'Elia, Sanfilippo 09   Nf=3: de Forcrand, O.P. 10

Strategy: fix $\frac{\mu_i}{T} = \frac{\pi}{3}, \pi$, measure Im(L), order parameter at $\frac{\mu_i}{T} = \pi$

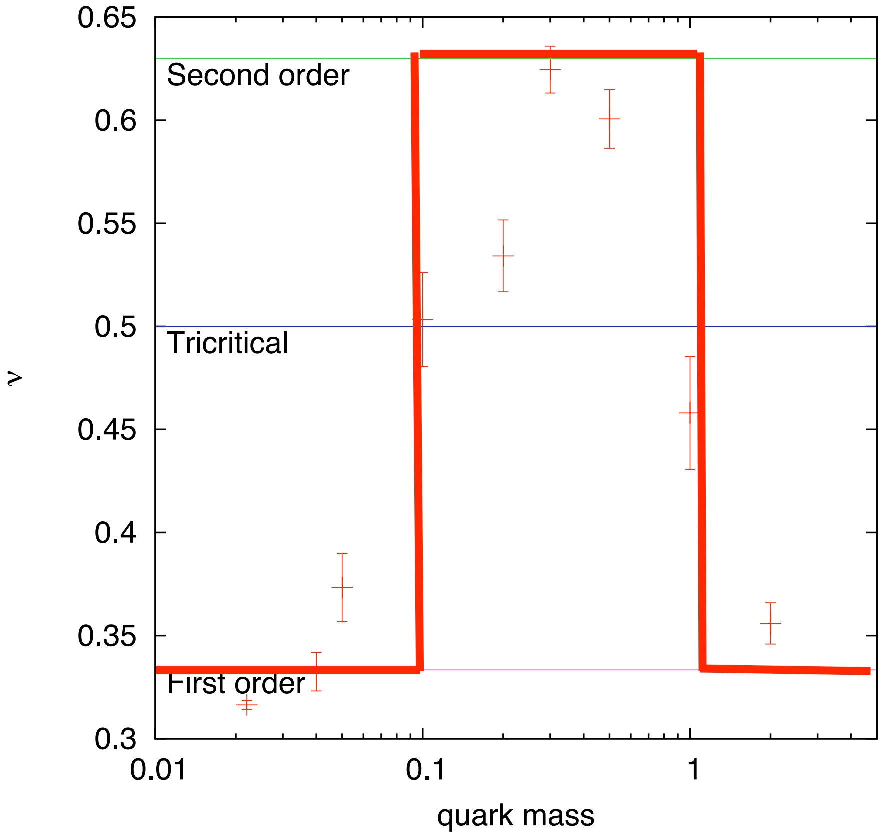
determine order of Z(3) branch/end point as function of m

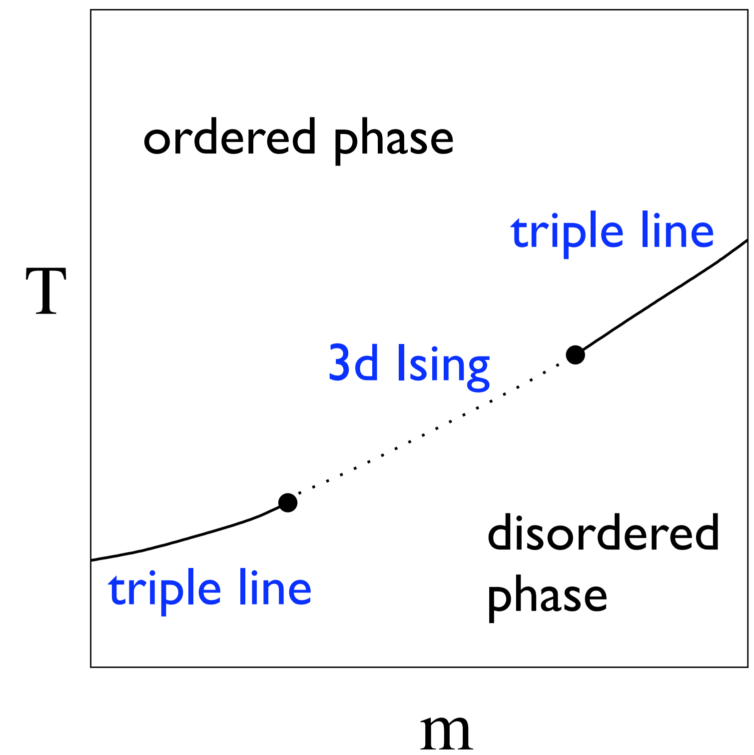


Scaling of Binder cumulant: $\nu = 0.33, 0.5, 0.63$

for 1st order, tri-critical, 3d Ising

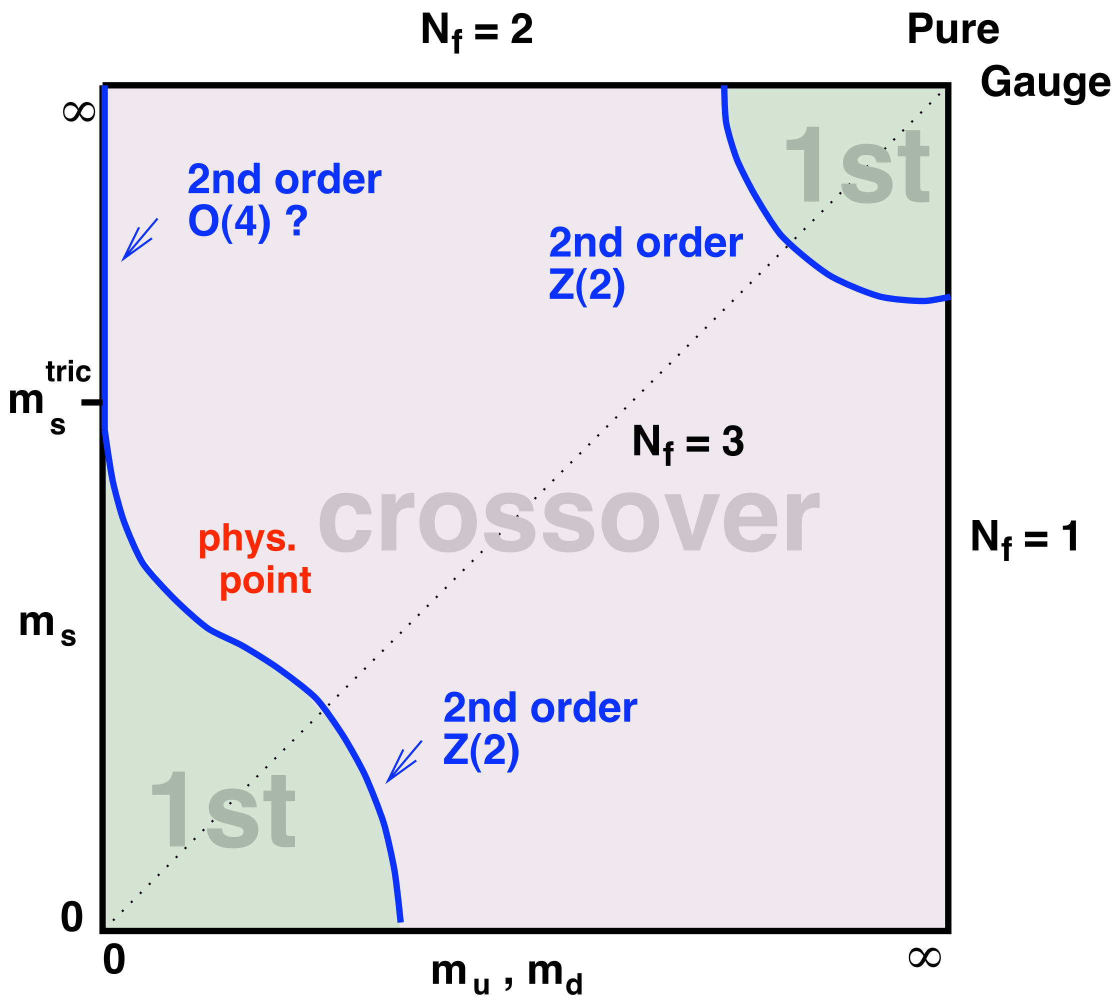
Phase diagram at fixed $\frac{\mu_i}{T} = \frac{\pi}{3}, \pi$

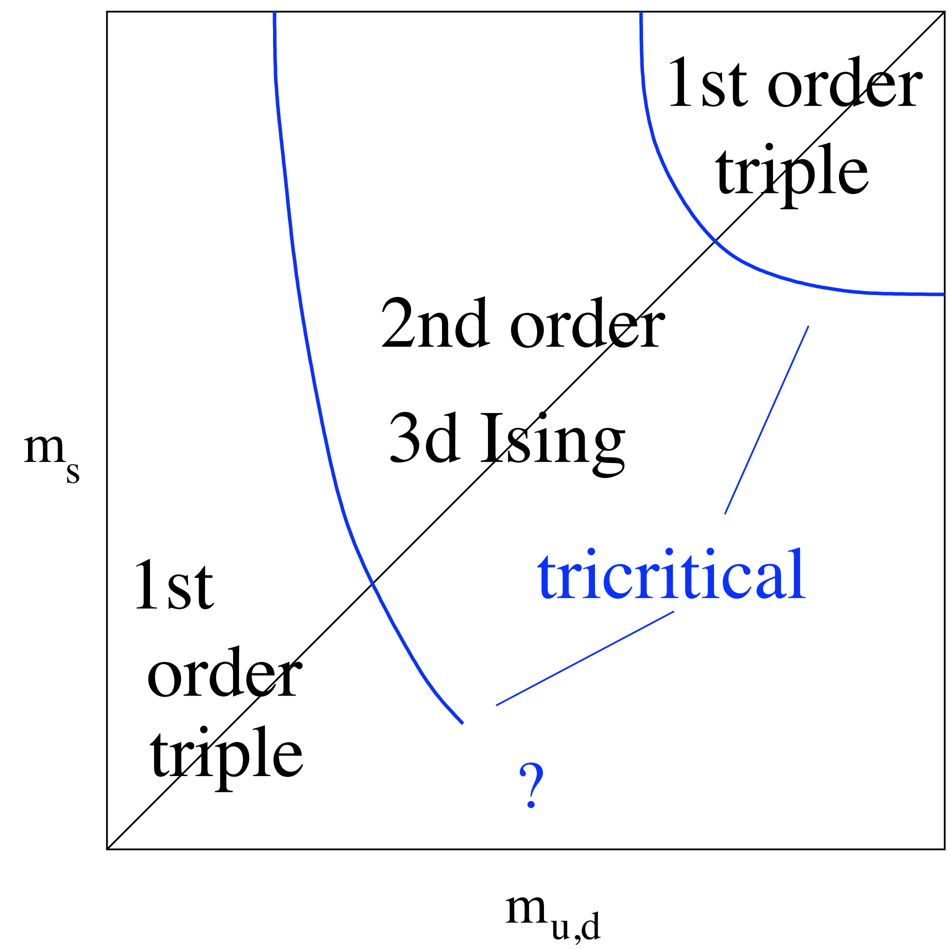




On infinite volume, this becomes a step function, smoothness due to finite L

Critical lines at imaginary μ





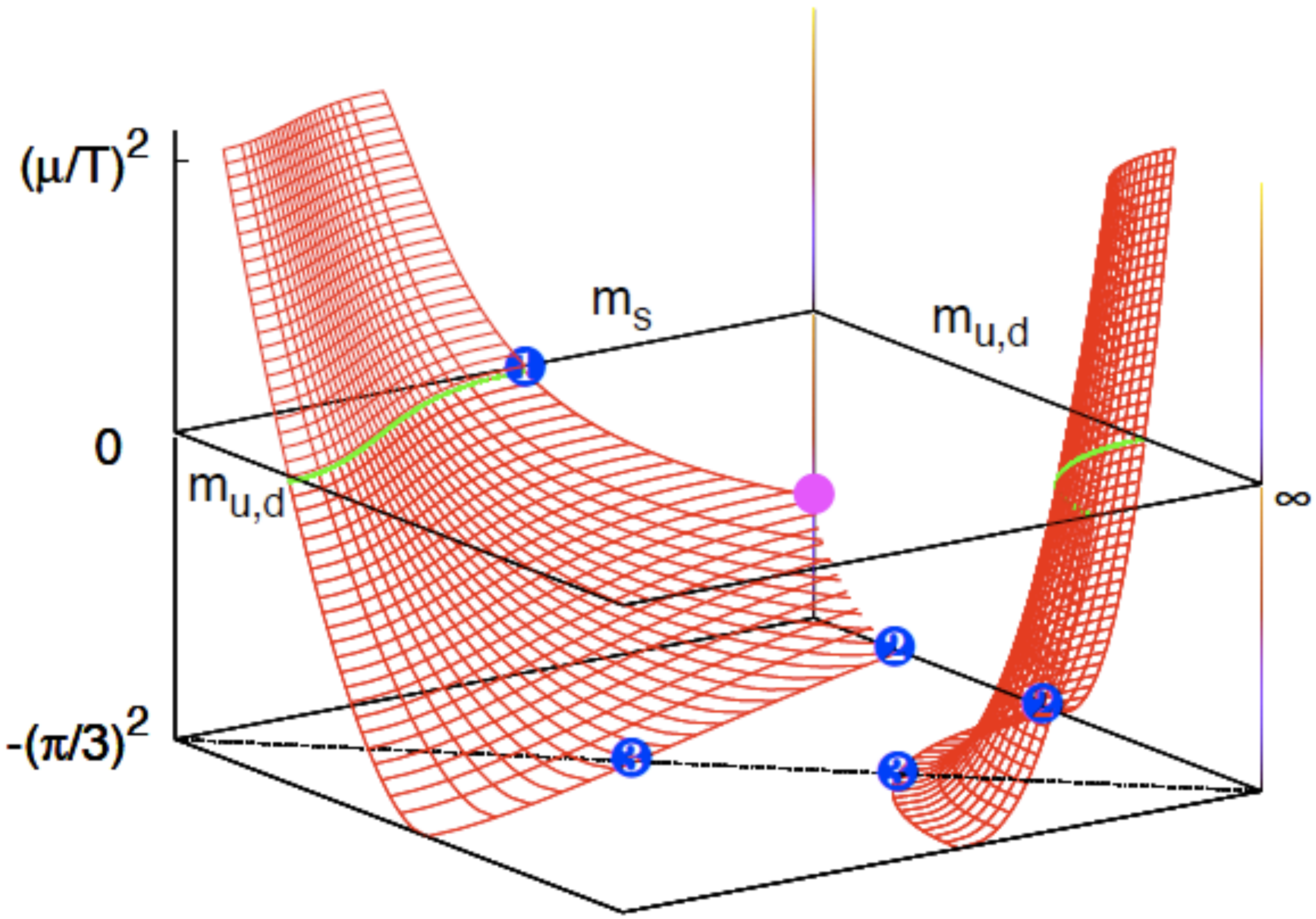
$\mu = 0$

$\mu = i\frac{\pi T}{3}$

-Connection computable with standard Monte Carlo!

-Here: heavy quarks in eff. theory

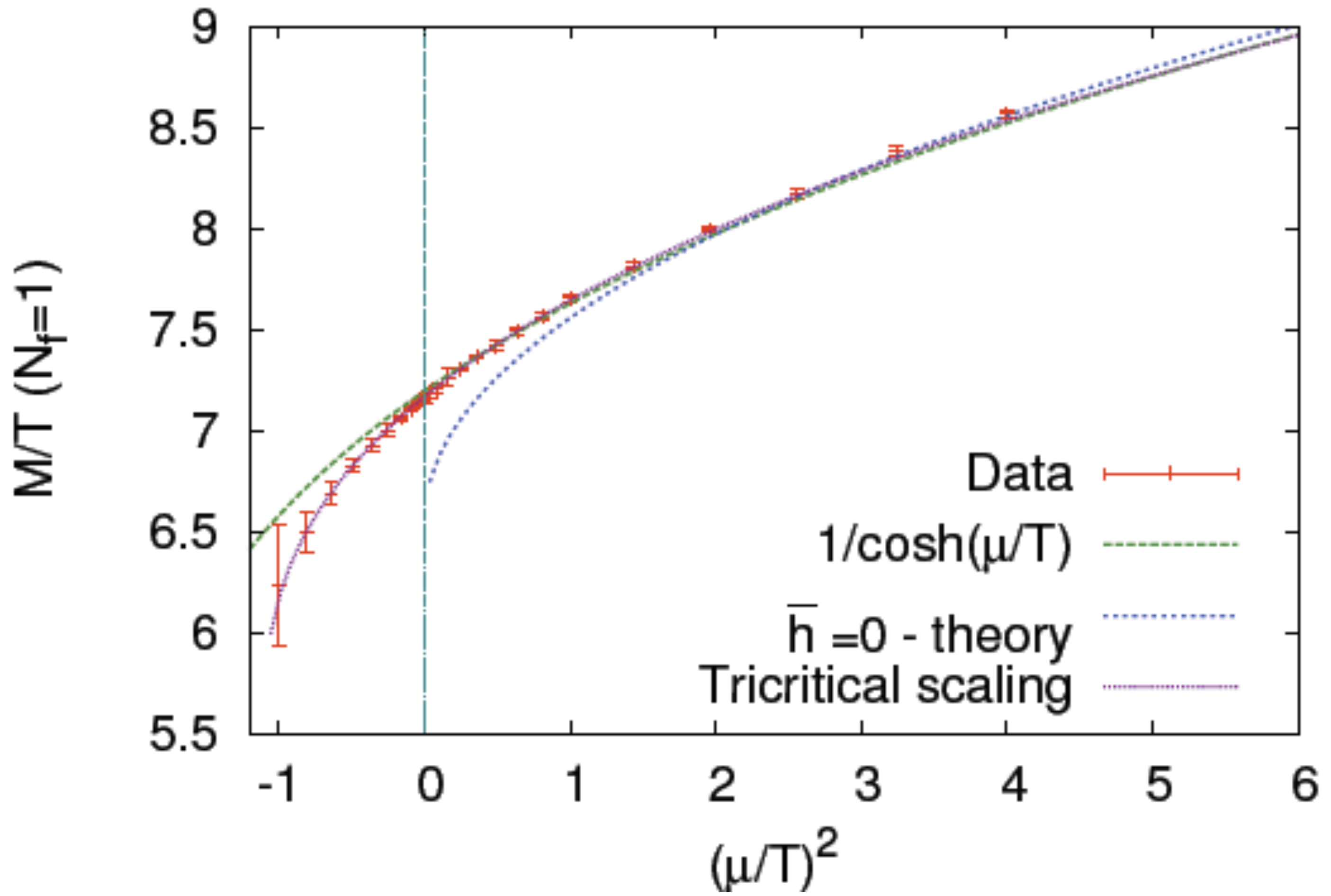
3d, imaginary chemical potential included:



Heavy quarks

Deconfinement critical line

Fromm, Langelage, Lottini, O.P. 11



tri-critical scaling: $$\frac{m_c}{T}(\mu^2) = \frac{m_{tric}}{T} + K\left[\left(\frac{\pi}{3}\right)^2 + \left(\frac{\mu}{T}\right)^2\right]^{2/5}$$ exponent universal

Summary Lecture IV

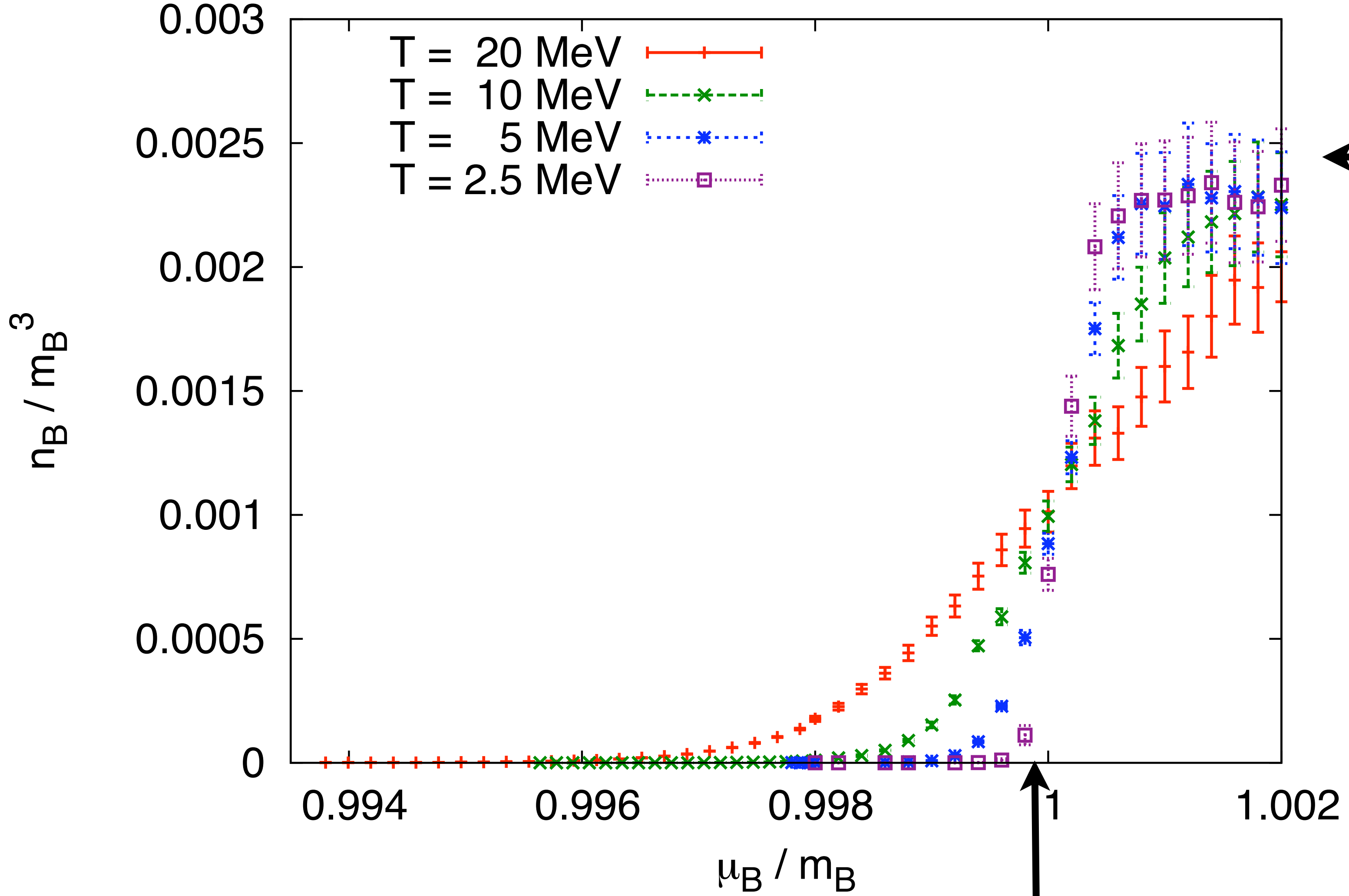
- Thermal transition at zero density is a crossover
- The sign problem is related to C-symmetry
- Direct MC methods to circumvent only at small chemical potential
- In the controlled region there is no evidence for a chiral critical point!
- Langevin algorithms?

New horizon: onset of cold nuclear matter

Based on 3d effective action by strong coupling and hopping exp.

... with very heavy quarks $m_\pi = 20 \text{ GeV}$

continuum limit with 5-7 lattice spacings per point



$\frac{\mu}{T} \sim 4000$

consistent with physical nuclear density

$\sim 0.16 \text{ fm}^{-3} \approx 0.15 \cdot 10^{-2}\, m_{\text{proton}}^3$

Frankfurt group, PRL 13

Complex Langevin: no sign problem
convergence criteria satisfied
cf. Seiler, Stamatescu; Aarts, James