Heavy Flavor Physics and Precision Tests of the Standard Model on the Lattice. Lect. 1

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VII International School on Theoretical Physics
Parma, Italy, 14-20 September 2014

Outline of the lectures

These are going to be a lot about the LATTICE approach to phenomenology.

- First talks:
  - Flavor physics with emphasis on HEAVY FLAVOR PHYSICS on the lattice.
  - Framework: Heavy Quark Effective Theory (HQET) on the lattice.
  - in between, a blackboard introduction to mixing and CP violation in the SM.
- Last two:
  - Precision tests of the Standard Model (SM): the hadronic contribution to $(g-2)_\mu$, using lattice, of course.

Flavor in the SM = quark masses and CKM matrix. How do they show up ?

The Higgs boson has been discovered. The EWSB in the SM seems to be really described by the simplest Higgs mechanism [Englert and Brout 1964, Higgs 1964, Weinberg 1967]

$$V(\phi) = m^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4 \, ,$$

where ϕ is an SU(2) doublet. After EWSB:

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

$$\begin{aligned} m_H^2 &= 2\lambda v^2 \\ m_W^2 &= \frac{1}{4} g^2 v^2 \quad \text{from the kinetic term} \Rightarrow g \simeq 0.7 \\ m_f &= h_{U/D}^f \frac{v}{\sqrt{2}} \quad \text{Including Yukawa interactions} \end{aligned}$$

A term $m\overline{\psi}\psi = m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L)$ breaks the chiral gauge symmetry (L doublets, R singlets). Using the ϕ field, one can add terms

$$L_Y = (\overline{Q}'\phi h'_D D' + \overline{Q}'\phi_c h'_U U') + h.c.$$

where the h'_X are $n \times n$ matrices in family space, $\phi_c = \epsilon\phi^*$ and

$$Q'^f = \begin{pmatrix} u'^f_L \\ d'^f_L \end{pmatrix},\ U'^f = u'^f_R,\ D'^f = d'^f_R\ .$$

The matrices h'_X can be diagonalized by left/right mult. with unitary matrices

$$h_X = V_L^{X\dagger} h'_X V_R^X$$

Then, by redefining the quark fields

$$\begin{aligned} u'_L &= V_L^U u_L\ , \qquad & u'_R &= V_R^U u_R \\ d'_L &= V_L^D d_L\ , \qquad & d'_R &= V_R^D d_R \end{aligned}$$

in the unitary gauge, after EWSB

$$L_Y = \frac{1}{\sqrt{2}}(v+H)\sum_{f=1}^{n}\left(h_D^f \overline{d}^f d^f + h_U^f \overline{u}^f u^f\right)$$

Other part of the Lagrangian changes. As up and down quarks are rotated differently, the charged currents

$$J^\mu = \sum_{f=1}^{n} \overline{Q}'^f \gamma_\mu \tau^+ Q'^f + h.c. \to \sum_{f,g=1}^{n} \overline{u}_L^f \gamma_\mu V_{fg} d_L^g + h.c.$$

with $V = V_L^{U^\dagger} V_L^D$ the CKM matrix. Since all transformations are unitary, the neutral current, defined through the commutator of the generators of the charged currents, remains diagonal (no FCNC).

With 3 families the CKM matrix is complex and depends on 4 parameters, these, as well as the 6 quark masses, are among the SM parameters. I won't discuss why they are what they are, ranging 5 orders of magnitude (flavor problem).

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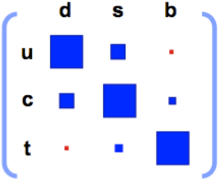
Wolfenstein parametrization $(\lambda,\ A,\ \rho,\ \eta)$. Then up to $O(\lambda^4)$

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$



$$\lambda \simeq 0.226,\ A \simeq 0.814,\ \rho \simeq 0.135,\ \eta \simeq 0.349$$

There is a hierarchy. Processes involving off-diagonal elements are "Cabibbo" suppressed.

Let's take the product of the first and third columns and normalize it by $V_{cd}V_{cb} \simeq A\lambda^3$. Unitarity gives us a triangle in the $\rho - \eta$ plane.

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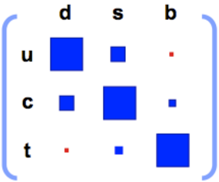

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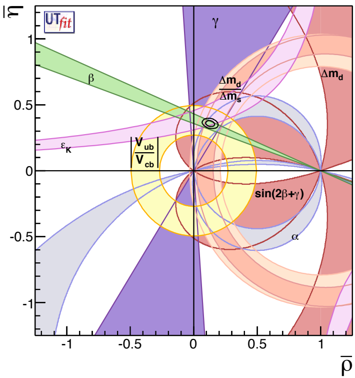
The Unitarity Triangle



Name of the game: determine the CKM matrix elements from comparison between experiments and theory. Typically constraints on the lengths or angles.

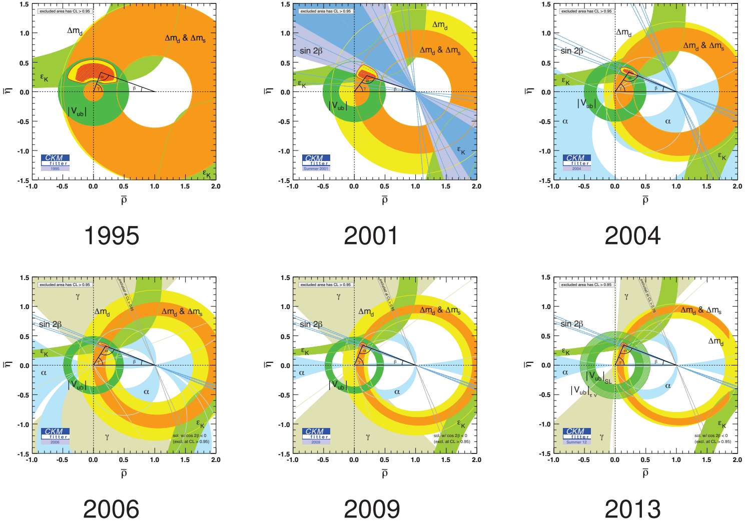
The goal is three-fold (in the quest for New Physics)

1 Try to define precisely what is “old” before claiming something is “new”.

2 Search for inconsistencies (“indirect” signals of nearby New Physics). E.g., by not using the unitarity constraint or by over-constraining the system. Possible effects of new particles in the loops (that’s “indirect”). Several two-three sigmas discrepancies in the past.

3 Put sharp constraints on any Beyond SM (BSM) model.

In this analysis hadronic matrix elements are often needed. The lattice can provide a non-perturbative first-principle determination of those [See Luigi’s lectures for an introduction to the lattice regularization].

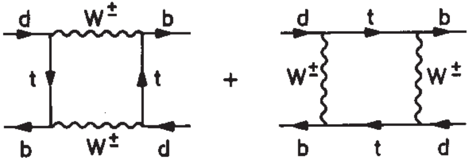


Impressive success of theory, where lattice has an extremely important role, and experiments.

Let us look in more detail at such analysis for one prototype quantity.

$B_{(s)} - \overline{B}_{(s)}$ mixing, used to fix the right side of the UT. Goal, get to the Effective Weak Hamiltonian ($\Delta F = 2$) and the bag parameter(s).

B and $\overline{B}$ are flavor eigenstates, they mix in the SM.



charm and up could also circulate, but they are GIM suppressed ...

The Effective Weak Hamiltonian is obtained by integrating out all heavy internal particles (W and t). At low-energy, box, penguins and W-exchange diagrams are replaced by point-like four-fermion local vertices (formally using OPE).

Using the Feynman rules of the SM (in the 't Hooft-Feynman gauge), the effective vertex corresponding to the box diagrams can be computed explicitly. Neglecting QCD corrections:

$$Box(\Delta B = 2) = \lambda_i \frac{G_F^2}{16\pi^2} M_w^2 S_0(x_i) \ (\bar{\mathbf{b}}\mathbf{d})_{\mathbf{V-A}}(\bar{\mathbf{b}}\mathbf{d})_{\mathbf{V-A}}$$

where $\frac{G_F}{\sqrt{2}} = \frac{8}{m_W^2}$ and $\lambda_i = V_{ib}^* V_{id}$,with $i = t, c, u$.
In the OPE picture this is fine as long as heavy particles in the loop dominate.

$S_0(x_i)$, with $x_i = \frac{m_i^2}{m_W^2}$ is the "tree-level" coupling of the four-fermion vertex (Inami-Lim function). One finds

$$S_0(x_i) \propto x_i$$

so we could neglect charm and up in the loop wrt the top term.

Beyond LO (in QCD) it is called Wilson or matching coefficient.

Remarks

- As a consequence of unitarity: $\lambda_u + \lambda_c + \lambda_t = 0$. If Wilson coeffs. were mass independent or there were horizontal mass degeneracy then there would be no FCNC processes [GIM mechanism]. The matrix element of the four-fermion operator could still be non-vanishing for $m_u = m_c = m_t$.
- The couplings renormalize and depend on the separation scale μ. If that is large enough PT can be used.
  - Several scales appear (μ, m_b, m_t, m_W). To avoid large logs, resummations and RG-improved PT has to be used.
  - Physical processes are scale independent, scale drops between matrix element of the four-fermion operator and the running of Wilson coeffs.
  - Operators mix under renormalization.
- Fermi's theory of beta decay is the pródromos of the Weak Effective Hamiltonian

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Now, the initial and final states are hadrons. We need to evaluate the matrix element non-perturbatively. Here lattice enters the game. The frequency Δm_q, $(q = d\ , s)$:

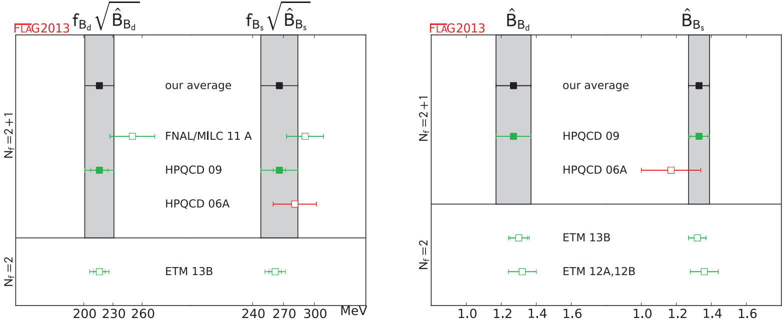
$$\Delta m_q = \frac{G_F^2 m_W^2}{6\pi^2} |V_{tq}^* V_{tb}|^2 \eta S_0(x_t) m_{B_q} F_{B_q}^2 B_{B_q}$$

$$\langle \overline{B}_q | O_{VV+AA} | B_q \rangle = \frac{8}{3} F_{B_q}^2 B_{B_q} m_{B_q}^2$$

$\langle 0 | A_\mu | P \rangle = F_P p_\mu$ describes leptonic decays of the pseudoscalar P.
η encodes QCD corrections (at NLO).

Experiments: $\Delta m_d = 0.507(3)(3) ps^{-1}$ [PDG]
$\Delta m_s = 17.719(36)(23) ps^{-1}$ [CDF, D0, LHCb]

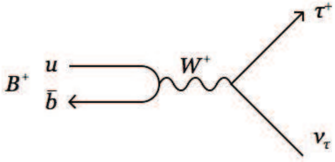
Exp. errors here are at the (sub)percent level !



Precise lattice results (e.g. including continuum limit) on these bag parameters are starting to appear now.

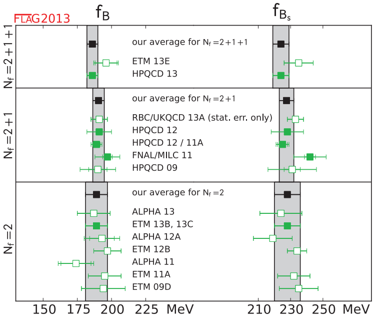
(Charged) Decay constants

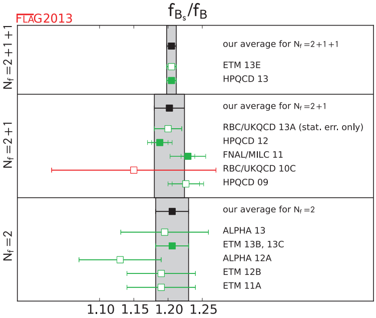
$\langle 0|\overline{u}_f \gamma_\mu \gamma_5 d_{f'}|P(p)\rangle = F_P p_\mu$ are the hadornic parameters entering leptonic decays of pseudoscalar mesons



Helicity suppression: in the rest frame of the meson, the two leptons are back to back. The $\overline{\nu}$ is right-handed, so must be the lepton (initial state is spin 0). Charged vertices are $\bar{L}L$, so ℓ must flip helicity (mass insertion).

$$\Gamma(B \to \ell\overline{\nu}_\ell) = \frac{G_F^2}{8\pi}|V_{ub}|^2 F_B^2 \left(\frac{m_\ell}{m_B}\right)^2 m_B^3 \left(1 - \frac{m_\ell^2}{m_B^2}\right)$$

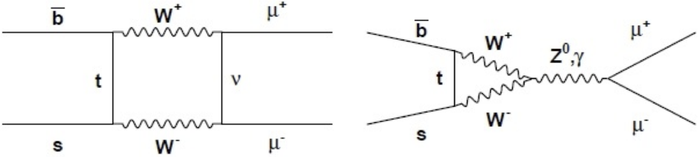




This is very well studied on the lattice, using different approaches, all giving consistent results.

The neutral case. $B_s \to \mu^+\mu^-$

In the SM through penguins and boxes

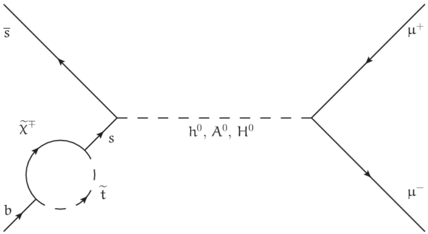


The branching ratio can be parameterized

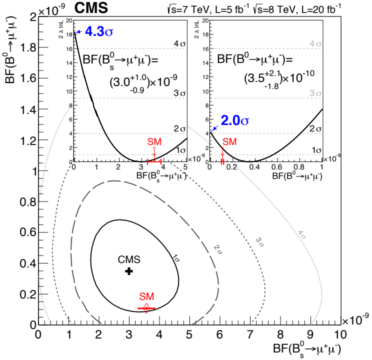
$$BR(B_s \to \mu^+\mu^-) = 3.5\cdot 10^{-9}\left[\frac{\tau(B_s)}{1.6\mathrm{ps}}\right]\left[\frac{F_{B_s}}{210\ \mathrm{MeV}}\right]^2\left[\frac{|V_{ts}|}{0.040}\right]^2\left[\frac{\overline{m_t}(m_t)}{170\ \mathrm{GeV}}\right]^{3.12}$$

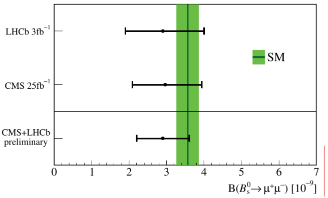
Used to be best bet for New Physics (in flavor physics)

In models with two Higgs doublets (like MSSM), one introduces $\tan(\beta) = \frac{<H_u>}{<H_d>}$. At large $\tan(\beta)$ the process can be enhanced, in particular one possible $\tan(\beta)^6$ term from the diagram [Babu and Kolda, 1999]



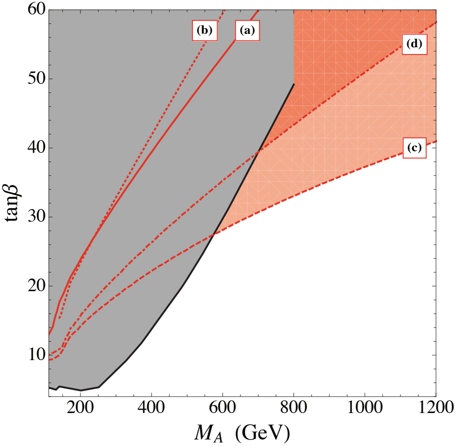
$B \to X_s \gamma$ could have be enhanced by a similar mechanism.





[CMS, arXiv:1307.5025, LHCb, arXiv:1307.5024] and [Altmannshofer et al., 2012] for constraints on m_A and $\tan(\beta)$

[arXiv:1211.1976]



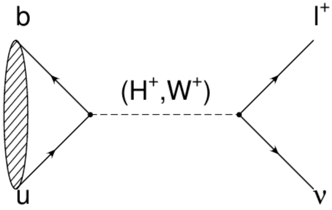
Recent measurements by Belle and BaBar observed a 3.4σ global deviation from the SM in

$$\frac{\mathcal{B}(B \to D\tau\nu)}{\mathcal{B}(B \to Dl\nu)}\,, \qquad \frac{\mathcal{B}(B \to D^*\tau\nu)}{\mathcal{B}(B \to D^*l\nu)}$$

and an *evaporating* enhancement in $B \to \tau\nu$.

(Semi)leptonic processes involving heavy quarks and τ leptons could unveil effects of particles with large coupling to heavy fermions.

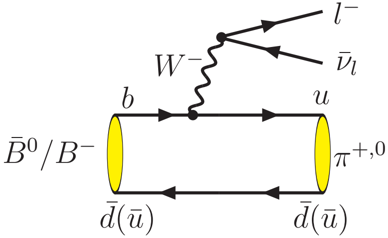
Charged scalars ? [Nierste, Trine, Westhoff '08; Kamenik, Mescia '08; Fajfer, Kamenik, Nisandzic '12, Soffer '14]



It is crucial to have precise and reliable estimate of the relevant form factors in the SM, and beyond (S, PS, T). The present theoretical knowledge is rather poor.

Form factors

Parameterizing semileptonic decay. Simplest: $B \to \pi \ell \nu$



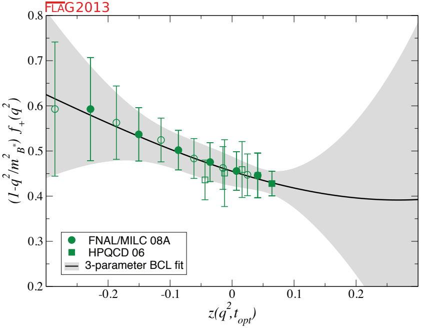
Ignoring the lepton mass:

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2}{24\pi^3} p_\pi^3 |V_{ub}|^2 |f_+(q^2)|^2$$

The hadronic matrix element is from a quark bilinear

$$\langle \pi(p_\pi)|V^\mu|B(p_B)\rangle = f_+(q^2)(p_\pi + p_B - q\Delta_{m^2})^\mu + f_0(q^2)q^\mu$$

with $\Delta_{m^2} = (m_B^2 - m_\pi^2)/q^2$



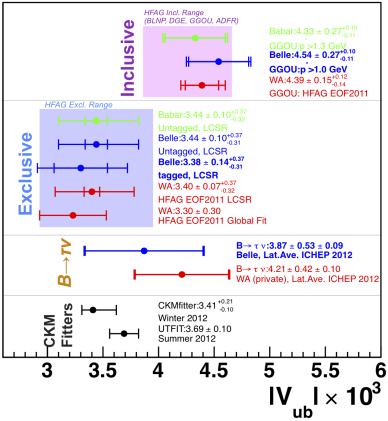
Very few lattice results, covering a part of the q^2 region only ($17\ \mathrm{GeV}^2 \leq q^2 \leq q^2_{max} = 26\ \mathrm{GeV}^2$). Parameterizations needed, BCL= [Bourelly, Caprini and Lellouch, 2009] building on BGL [Boyd, Grinstein and Lebed, '95]

- Experiments measure in the small q^2 region ($d\Gamma \propto p_\pi^3$), lattice can access the large q^2 one (a eff.).
- The kinematical factor in front of f_+ vanishes at $q_{max} = (m_B - m_\pi, \vec{0})$.

Tension between inclusive and exclusive determinations of V_{ub}.

Inclusive from $B \to X_u \ell \nu$. less clean exp. and theory wise (cuts to remove $B \to X_c$ background).

[P. Urquijo for Belle, plenary at ICHEP 2012]



Summarizing

- Introduction of the Higgs sector of the SM, quark masses and CKM. [Cabibbo suppression]
- Weak Effective Hamiltonian using the example of $B - \overline{B}$ mixing. [GIM mechanism]
- (Charged) Decay constants. [Helicity suppression]
- $B_s \to \mu^+\mu^-$ in SM and BSM. [With extra neutral Higgs as well as charged Higgs].
- Form factors for semileptonic decays. [V_{ub} incl. vs excl. tension]

Something to read (mostly lectures)

- for SM: G. Ridolfi, `http://www.ge.infn.it/~ridolfi/notes/smcom.ps`
- for Weak Effective Hamiltonian: A. Buras, arXiv:hep-ph/9806471
- for Heavy Quark Effective Theory: M. Neubert, Phys.Rept. 245 (1994) 259-396 and hep-ph/9610266, Manohar and Wise "Heavy Quark Physics".
- for HQET on the lattice: R. Sommer, arXiv:1008.0710
- for a review of Flavor lattice results: FLAG, arXiv:1310.8555
- G. Hiller and collab. 2010 ... for NP constraints from $B \to K^{(*)}\ell^+\ell^-$.

and references therein !!

Heavy Flavor Physics and Precision Tests of the Standard Model on the Lattice. Lect. 2

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- HQET as a model independet approach to describe (some) processes involving heavy-light hadrons.
    - Heuristic derivation
- HQET on the lattice. Why and how.
    - Non-perturbative matching
    - Results

- Heavy–light systems are characterized by two scales
  - $\Lambda_{\rm QCD}$ associated with the dynamics of the light degrees of freedom (and the size of the hadron $\simeq 1/\Lambda_{\rm QCD}$)
  - $1/m_Q \simeq$ Compton wave length of the heavy quark

QED analogue $\rightarrow$ hydrogen atom

the electron wave-function does not depend on the nucleus mass.

- Separating the two scales may help
  - short distances effects may be treated using perturbation theory
  - Long distance physics might simplify

Symmetries realized in this particular kinematical situation

They are not symmetries of the full Lagrangian

- Resolving the quantum numbers of the heavy quark would requires a hard probe ($\mu \simeq m_Q$)
- light degrees of freedom exchange momenta of $O(\Lambda_{\rm QCD})$, they are blind to flavor, mass and spin of the heavy quark (heavy quark spin and flavor symmetry)
- in this respect heavy quarks are just color sources (color field extends over large distances because of confinement)

Hadronic states can be classified by the quantum numbers of the light degrees of freedom (flavor, spin, parity, etc). In general

$$m_H = m_Q + \bar{\Lambda} + \frac{\Delta m^2}{2m_Q} + O(1/m_Q^2)$$

$\bar{\Lambda}$ from terms in the Lagrangian independent of the heavy quark mass
$\Delta m^2 = -\lambda_1 + 2[J(J+1) - 3/2]\lambda_2$
λ_1 : kinetic energy of the heavy quark inside the meson
λ_2 : interaction of the heavy quark spin with the gluon field
$\bar{\Lambda},\ \lambda_1,\ \lambda_2$ independent of m_Q

$$\text{example :} \quad \begin{aligned} m_{B^*}^2 - m_B^2 &= 4\lambda_2 + O(1/m_b) \quad \simeq 0.49\ \mathrm{GeV}^2 \\ m_{D^*}^2 - m_D^2 &= 4\lambda_2 + O(1/m_c) \quad \simeq 0.55\ \mathrm{GeV}^2 \end{aligned}$$

$\Rightarrow \lambda_2 \simeq 0.12\mathrm{GeV}^2$

Semileptonic decays (in the limit $m_Q \to \infty$)

Consider mesons witha a given velocity v, elastic scattering $\bar{B}(v) \to \bar{B}(v')$ induced by a vector current. In the limit $m_b \to \infty$ this can only depend on the boost $\gamma = v \cdot v'$ (Isgur-Wise)

$$\frac{1}{m_B}\langle \bar{B}(v')|\bar{b}_{v'}\gamma_\mu b_v|\bar{B}(v)\rangle = \xi(v\cdot v')(v+v')^\mu$$

with $\xi(1) = 1$ for current conservation. From flavor-symmetry

$$\frac{1}{\sqrt{m_B m_D}}\langle \bar{D}(v')|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(v)\rangle = \xi(v\cdot v')(v+v')^\mu$$

Flavor changing currents are described by two form factors

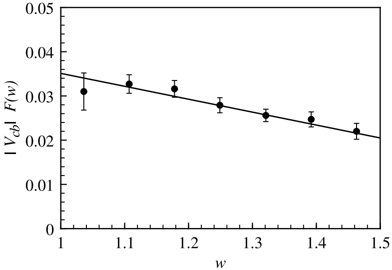
$$\langle \bar{D}(v')|\bar{c}_{v'}\gamma_\mu b_v|\bar{B}(v)\rangle = f_+(q^2)(p+p')^\mu - f_-(q^2)(p-p')^\mu$$

$$\Rightarrow f_\pm(q^2) = \frac{m_B \pm m_D}{2\sqrt{m_B m_D}}\xi(v\cdot v')$$

Using a spin-symmetry-transformation a vector can be turned into a pseudoscalar by rotating the spin of the heavy quark

$$\frac{d\Gamma(\bar{B} \to D^* l\bar{\nu})}{dw} = \frac{G_F^2}{48\pi^3}|V_{cb}|^2 \kappa(m_B, m_{D^*}, w)\xi^2(w)$$

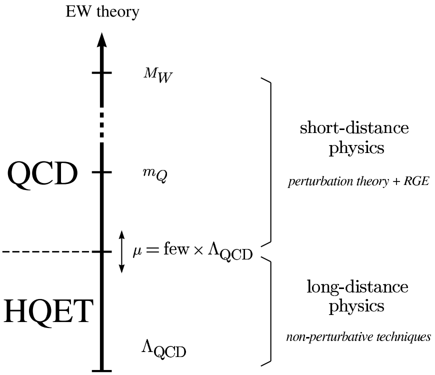
the product $|V_{cb}|\xi(w)$ can be measured at experimentally [CLEO, LEP, Belle, BABAR] and extrapolated to $w \to 1$ where HQET predictions for $\xi(1)$ can be used



$|V_{cb}|\xi(1) = (35.90 \pm 0.45) \times 10^{-3}$, $\xi(1) = (1 + O(1/m_Q^2)) \times \text{short distance effects}$

The form factor can be computed on the lattice including higher orders in $\frac{1}{m_Q}$ also for the $B \to Dl\nu$ transition [FLAG, arXiv:1310.8555, FNAL/MILC, Atoui et al., '13].

HQET [Eichten, '88, Eichten and Hill '90]



- The effective theory reproduces the full one at large distances, but short distance effects are different as high momentum modes have been removed from the theory $\rightarrow$ Wilson coeff.
- Heavy quarks are not really integrated out, only small components of the heavy quark spinor which describe fluctuations around the mass shell will be removed.
- The effective theory is still strongly interacting.

In the infinite mass limit: $p_Q^\mu = m_Q v^\mu + k^\mu$.
$k^\mu \simeq \Lambda_{\rm QCD}$. Changes in the velocity due to the residual momentum k^μ vanish (as $m_Q \to \infty$). It is useful to introduce

$$h_v(x) = e^{im_Q v\cdot x} P_+ Q(x)\ , \quad H_v(x) = e^{im_Q v\cdot x} P_- Q(x)\ , \quad P_\pm = (1 \pm \not{v})/2$$

the phase makes the fields slowly varying in space and the meaning of the projectors is clear in the rest frame $[v_\mu = (1,0,0,0)]$.
Using $\gamma_\mu v^\mu h_v = h_v$, $\gamma_\mu v^\mu H_v = -H_v$ and starting from $\bar{Q}(i\gamma_\mu D^\mu - m_Q)Q$

$$\mathcal{L}_Q = \bar{h}_v iv\cdot D h_v - \bar{H}_v(iv\cdot D + 2m_Q)H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v$$

H_v is massive $\Rightarrow$ zig-zag transitions from quarks to antiquarks are suppressed by a factor $2m_Q$. These degrees can be eliminated by making use of e.o.m.

$$\mathcal{L}_{\rm eff} = \bar{h}_v iv\cdot D h_v + \bar{h}_v i\not{D}_\perp \frac{1}{2m_Q + iv\cdot D} i\not{D}_\perp h_v$$

As $h(x)$ is slowly varying, the expression can be expanded in powers of iD/m_Q

$$\mathcal{L}_{\rm eff} = \bar{h}_v iv\cdot D h_v + \frac{1}{2m_Q}\sum_{n=0}^{\infty} \bar{h}_v i\not{D}_\perp \left(-\frac{iv\cdot D}{2m_Q}\right)^n i\not{D}_\perp h_v$$

At $O(1/m_Q)$, i.e. from $n=0$, in the rest frame, it takes the form

$$\mathcal{L}_{\rm eff} = \bar{h}_v iD_0 h_v + \frac{1}{2m_Q}\bar{h}_v(i\vec{D})^2 h_v + \frac{1}{2m_Q}\bar{h}_v \vec{S}\cdot\vec{B}_c h_v + O(1/m_Q^2)$$

EXERCISE 1: Prove it !

- the kinetic energy from the off-shell residual motion and the chromo-magnetic Pauli interaction are relativistic sub-leading effects.
- In the static action no γ-matrices appear, the interactions do not change the spin of the heavy quark ($SU(2)$ spin symmetry).
- also, $h(x) \to e^{i\eta(\vec{x})}h(x)$ is a symmetry (local flavor number conservation).

The coefficients of the $1/m_Q$ terms in the Lagrangian need perturbative and non-perturbative corrections in order to match the effective theory to the full one.
QCD (heavy-light) currents $j(\mu) = Z_j \bar{\psi}_l \Gamma \psi_h$ are expanded in HQET

$$j(\mu') = C(\mu',\mu)\tilde{j}(\mu) + \frac{1}{2m_Q}\sum_i B_i(\mu',\mu)O_i(\mu) + O(1/m_Q^2)$$

where O_i are dimension 4 operators with the proper quantum numbers.

The C coefficient *corrects* for short distance effects not included in the effective theory and *can* be perturbatively estimated

$$C(\mu',\mu) = C(m_Q,m_Q)\exp\left[\int_{\alpha_s^{(n_f)}(m_Q)}^{\alpha_s^{(n_f)}(\mu')} \frac{d\alpha_s}{\alpha_s}\frac{\gamma_j(\alpha_s)}{2\beta^{(n_f)}(\alpha_s)} - \int_{\alpha_s^{(n_l)}(m_Q)}^{\alpha_s^{(n_l)}(\mu)} \frac{d\alpha_s}{\alpha_s}\frac{\tilde{\gamma}_j(\alpha_s)}{2\beta^{(n_l)}(\alpha_s)}\right]$$

having performed the matching at the scale $\mu = \mu' = m_Q$.

Let us consider the example

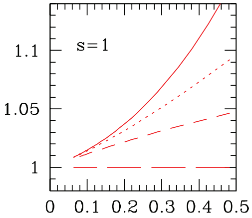
$$m^2_{B^*} - m^2_B = C_{mag}(m_b/\Lambda_{\rm QCD})\ \langle B|\bar{\psi}_h \sigma \mathbf{B} \psi_h|B^*\rangle_{RGI} \times (1 + \mathrm{O}(1/m_b))$$

$C_{mag}(m_b/\Lambda_{\rm QCD})$ has a perturbative expansion. The truncation at O(n-1)

$$\simeq \alpha(m_b)^n \simeq \left\{ \frac{1}{2b_0 \ln(m_b/\Lambda_{\rm QCD})} \right\}^n >> \frac{\Lambda_{\rm QCD}}{m_b} \quad \text{as } m_b \to \infty$$

The PT corrections to the leading term are larger than the $1/m_b$ ones !

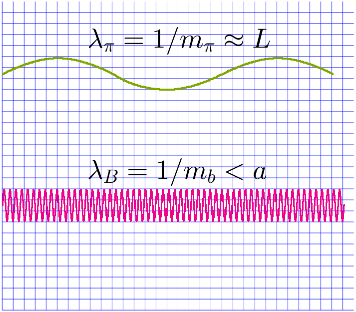
In addition the perturbative series isn't always well behaved [R. Sommer, 2010].
C_{PS}/C_V vs $1/\ln(\Lambda/m_b)$



[Chetyrkin and Grozin 2003, Broadhurst and Grozin '91, '95, Bekavac et al. 2010]

- finite volume effects are mainly triggered by the light degrees of freedom. The usual requirement is $m_{PS}L > 4$ and m_{PS} is typically around 250 MeV in actual simulations $\Rightarrow L \simeq 4\ \mathrm{fm}$.
- cutoff effects are related to the heavy quark mass. $a << 1/m_b \simeq 0.03\ \mathrm{fm}$.

$\Rightarrow L/a \simeq 100$ is needed to have those systematics under control !!
Integrating out the heavy quark mass in this case is useful !!



In addition the autocorrelation of observables grows as $1/a^n$ with $n \geq 2$ [Schäfer, Sommer and Virotta '10, Lüscher and Schäfer, '11]

Field Content (and Euclidean notation and rest-frame)

$$S_{HQET} = a^4 \sum_x \left\{ \bar{\psi}_h (D_0 + m_{\rm bare})\psi_h + \omega_{spin}\bar{\psi}_h(-\sigma\mathbf{B})\psi_h + \omega_{kin}\bar{\psi}_h\left(-\frac{1}{2}\mathbf{D}^2\right)\psi_h + \dots \right\}$$

We also consider the currents

$$\begin{aligned}
A_0^{\rm HQET}(x) &= Z_{\rm A}^{\rm HQET}[A_0^{\rm stat}(x) + \sum_{i=1}^{2} c_{\rm A}^{(i)} A_0^{(i)}(x)] \,, \\
A_0^{(1)}(x) &= \overline{\psi}_{\rm l}\tfrac{1}{2}\gamma_5\gamma_i(\nabla_i^{\rm S} - \overleftarrow{\nabla}_i^{\rm S})\psi_{\rm h}(x) \,, \\
A_0^{(2)}(x) &= -\tilde{\partial}_i A_i^{\rm stat}(x)/2 \,, \quad A_i^{\rm stat}(x) = \overline{\psi}_{\rm l}(x)\gamma_i\gamma_5\psi_{\rm h}(x) \,,
\end{aligned}$$

$$A_k^{\rm HQET}(x) = Z_{{\rm A}_k}^{\rm HQET}[A_k^{\rm stat}(x) + \sum_{i=3}^{6} c_{\rm A}^{(i)} A_k^{(i)}(x)] \,,$$

$$\begin{aligned}
A_k^{(3)}(x) &= \overline{\psi}_{\rm l}(x)\frac{1}{2}(\nabla_i^{\rm S} - \overleftarrow{\nabla}_i^{\rm S})\gamma_i\gamma_5\gamma_k\psi_{\rm h}(x) \,, \quad A_k^{(4)}(x) = \overline{\psi}_{\rm l}(x)\frac{1}{2}(\nabla_k^{\rm S} - \overleftarrow{\nabla}_k^{\rm S})\gamma_5\psi_{\rm h}(x) \,, \\
A_k^{(5)}(x) &= \widetilde{\partial}_i \left(\overline{\psi}_{\rm l}(x)\gamma_i\gamma_5\gamma_k\psi_{\rm h}(x)\right)/2 \,, \quad A_k^{(6)}(x) = \widetilde{\partial}_k A_0^{\rm stat}/2
\end{aligned}$$

and analogous expressions for the vector current, 19 coeffs in total.

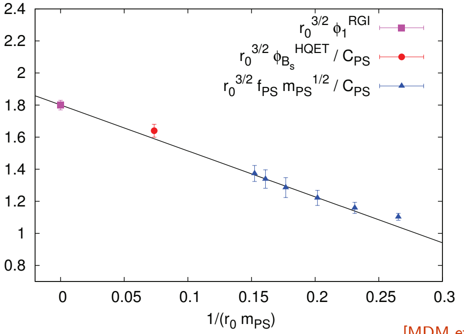
- Theoretically very sound. In particular, the continuum limit is well defined and can be reached numerically [ALPHA, '03].
  NB. The static theory is not renormalizable by power-counting (static prop. $\propto \delta(\vec{x}-\vec{y})$). Still, only dim. 4 ops. in the action.
  In [Grinstein, '90] it has been shown that QCD correlators are reproduced to all orders in α_s at LO in $1/m_h$.
- Can be treated non-perturbatively including renormalization and $O(1/m_h)$ [Heitger and Sommer, '03 and ALPHA ...].

  Next to leading order terms in the $1/m_h$ expansion are not included in the action, that would produce couplings of negative dimension. They are treated as insertions into correlation functions evaluated in the static theory

$$e^{-(S_{rel}+S_{HQET})} = e^{-(S_{rel}+S_{stat})} \times [1 - a^4 \sum_x \mathcal{L}^{(1)}(x, \omega_{spin}, \omega_{kin}) + \ldots]$$

and $S_{stat} = a^4 \sum_x \bar{\psi}_h(x) D_0^{HYP} \psi_h(x)$ to minimize the noise/signal ratio.

- It is self-consistent. The validity of the $1/m_h$ expansion can be tested down to the charm mass, as opposed to what is done within other approaches, where results are extrapolated from m_c to m_b assuming HQET.



[MDM et al., arXiv:1006.5816]

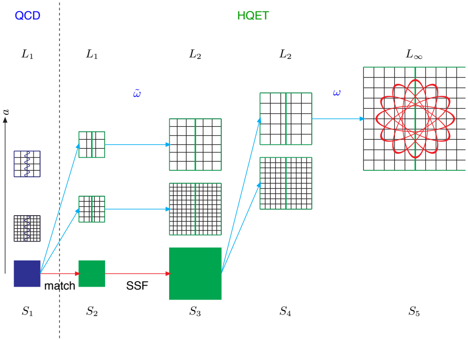
- Numerically it is as expensive as other approaches, the matching between QCD and HQET is perfomed in small volumes and it is very cheap concerning CPU-time. The costy part is the large volume, as for everybody.

[ALPHA, arXiv:1001.4783, 1004.2661, 1006.5816 and 1203.6516, 1311.5498]

- $a = f(\beta)$, $\beta = 6/g_0^2$. large β = small a.
- The parameters are renormalization factors. They depend on a but not on L.
- L/a can't be arbitrarily large.
- Eventually we want them for $a \simeq 0.1 - 0.05$ fm (large volumes for phenomenology).
- Idea: at small L and very fine a we simulate HQET and QCD with a relativistic b-quark. We get the parameters by matching 19 suitable quantities [MDM et al., arXiv:1312.1566]

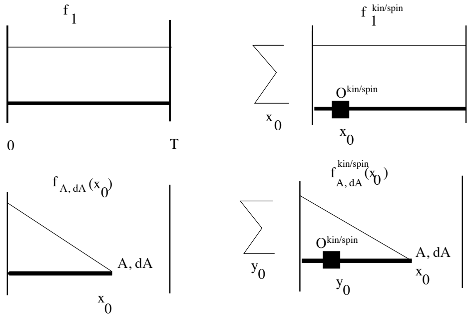
$$\Phi_i^{\rm QCD}(m_{\rm b}, 0) = \Phi_i^{\rm HQET}(\omega_{\dots}, c^{(j)}, Z_{\dots}^{\rm HQET}, a)$$

- By a sequence of evolution (in L, fixed a) and matching (continuum vs finite a, fixed L) steps in HQET, one can obtain the parameters at larger a.



Let's take the easy one, ω_{spin} as an example. In QCD in a finte volume (Schrödinger Functional) we define VV and AA boundary to boundary correlators and their corresponding HQET expansion.

$$
\begin{aligned}
f_1 &= -\frac{a^{12}}{2L^6} \sum_{\vec{u},\vec{v},\vec{y},\vec{z}} \left\langle \overline{\zeta}'_\ell(\vec{u})\gamma_5\zeta'_{\rm b}(\vec{v})\,\overline{\zeta}_{\rm b}(\vec{y})\gamma_5\zeta_\ell(\vec{z}) \right\rangle \,, \\
k_1 &= -\frac{a^{12}}{6L^6} \sum_{k} \left\langle \overline{\zeta}'_\ell(\vec{u})\gamma_k\zeta'_{\rm b}(\vec{v})\,\overline{\zeta}_{\rm b}(\vec{y})\gamma_k\zeta_\ell(\vec{z}) \right\rangle \,,
\end{aligned}
$$



The expansions read

$$[f_1]_R^{HQET} = Z_{\zeta_h}^2 Z_\zeta^2 e^{-m_{\rm bare} T} \left\{ f_1^{stat} + \omega_{kin} f_1^{kin} + \omega_{spin} f_1^{spin} \right\} ,$$

$$[k_1]_R^{HQET} = Z_{\zeta_h}^2 Z_\zeta^2 e^{-m_{\rm bare} T} \left\{ f_1^{stat} + \omega_{kin} f_1^{kin} - \tfrac{1}{3}\omega_{spin} f_1^{spin} \right\} ,$$

- The matching equation (in a size L_1, usually around 0.5 fm)

$$\Phi_{spin}(L_1, m_b, a) = \frac{3}{4} \ln\left(\frac{f_1}{k_1}\right)(L_1, m_b, a) = \omega_{spin}(m_b, a) \frac{f_1^{spin}}{f_1^{stat}}(L_1, a) + \ldots$$

can be solved for ω_{spin} at a and m_b where the matching is performed. This a is very fine, not suitable for computing the spectrum or decay constants or ...

- Evolution (SSF) an re-matching (from now on in HQET only).

At the same a (and m_b), we consider $L_2 = 2L_1$, simply by doubling the number of points. Using the same ω_{spin} we compute $\Phi_{spin}(L_2, m_b, a)$, with RHS above.

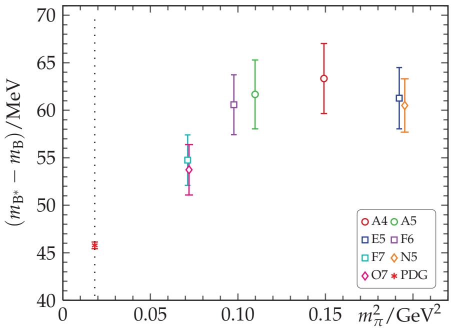
Then we change $a \to 2a$ and solve for $\omega_{spin}(m_b, 2a)$ the equation

$$\Phi_{spin}(L_2, m_b, a) = \Phi_{spin}(L_2, m_b, 2a)$$

so, we set to 0 cutoff effects on Φ_{spin}. One or two steps are usually enough.

Remark. In the LHS of matching equations the $\lim a \to 0$ is usually taken.

Finally, in large volume $\frac{4}{3}\omega_{spin}\langle B|O_{spin}|B\rangle$ give the V-PS splitting



[MDM and ALPHA '12]

Similarly, the parameters entering the b-quark mass and the B-meson decay constant have all been determined non-perturbatively.
Matching-quantites have been defined and studied in perturbation theory far all the 19 parameters in the action and vector and axial currents at $O(1/m_h)$ [MDM, Dooling, Hesse, Heitger and Simma, '13].

More results from HQET on the lattice at O($1/m_h$)

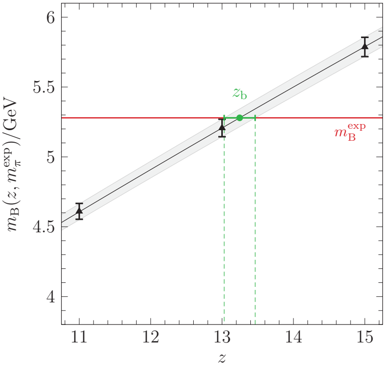
We generate $N_f = 2$ dynamical configurations, with NP O(a) improved Wilson fermions and plaquette gauge action.

β	a[fm]	L/a	m_π[MeV]	$m_\pi L$	#cfgs	$\frac{\#\text{cfgs}}{\tau_{\text{exp}}}$	id
5.2	0.075	32	380	4.7	1012	122	A4
		32	330	4.0	1001	164	A5
		48	280	5.2	636	52	B6
5.3	0.065	32	440	4.7	1000	120	E5
		48	310	5.0	500	30	F6
		48	270	4.3	602	36	F7
		64	190	4.1	410	17	G8
5.5	0.048	48	440	5.2	477	4.2	N5
		48	340	4.0	950	38	N6
		64	270	4.2	980	20	O7

The b-quark mass is determined by computing, as a function of the heavy quark mass m_h used in the matching, the large-volume quantity

$$m_B(m_h) = m_{bare}(m_h) + E^{stat} + \omega_{spin}(m_h)E^{spin} + \omega_{kin}(m_h)E^{kin}$$

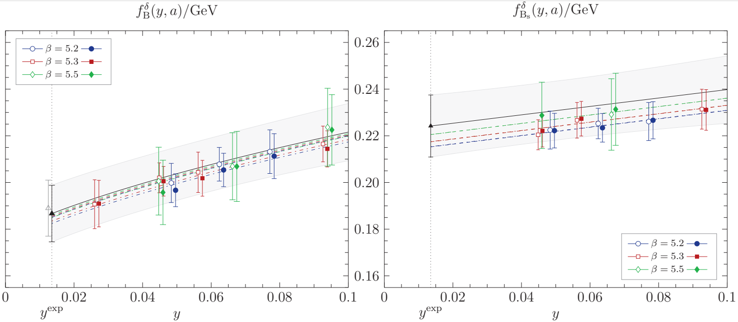
and then solving $m_B(m_h) = m_B^{exp}$, with m_h as unknown.



$z = L_1 m_h$,   [MDM and ALPHA, arXiv:1311.5498]

N_f	Ref.	M	$\overline{m}_{\overline{MS}}(\overline{m}_{\overline{MS}})$	$\overline{m}_{\overline{MS}}(4\,\mathrm{GeV})$	$\overline{m}_{\overline{MS}}(2\,\mathrm{GeV})$
0	[36]	6.76(9)	4.35(5)	4.39(6)	4.87(8)
2	this work	6.58(17)	4.21(11)	4.25(12)	4.88(15)
5	PDG13 [1]	7.50(8)	4.18(3)	4.22(4)	4.91(5)

Convergence at lower scales may be due to the common low-energy input (m_B).



Continuum-Chiral extrapolations, using

$$
\begin{aligned}
f_{B_s}(m^2_{\mathrm{PS}}, a^2) &= b + c m^2_{\mathrm{PS}} + d a^2 \\
f_B(m^2_{\mathrm{PS}}, a^2) &= b' \left[1 - \frac{3}{4}\frac{1+3\hat{g}^2}{(4\pi f_\pi)^2} m^2_{\mathrm{PS}} \ln(m^2_{\mathrm{PS}})\right] + c' m^2_{\mathrm{PS}} + d' a^2
\end{aligned}
$$

with f_π from exp. and $\hat{g} = 0.51(2)$ [Bulava et al. PoS LAT10]

give $f_B = 187(12)(7)_\chi$ MeV, $f_{B_s} = 224(13)$ MeV and $\frac{f_{B_s}}{f_B} = 1.195(61)(20)_\chi$

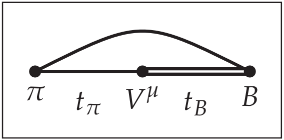
Differential decay rate in $B \to \pi l \nu$

$$\frac{d\Gamma}{dq^2} = \frac{G_{\rm F}^2}{24\pi^3} p_\pi^3 |V_{\rm ub}|^2 |f_+(q^2)|^2 \ ,$$

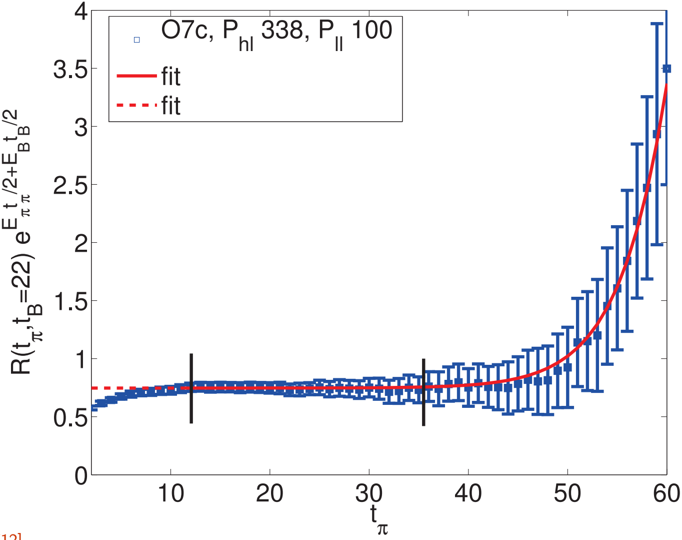
where q is the lepton pair momentum. The form factor $f_+(q^2)$ can be extracted from the matrix element of the vector current

$$\langle \pi(p_\pi)|V^\mu|B(p_{\rm B})\rangle = f_+(q^2)(p_\pi + p_{\rm B} + q\Delta_{m^2})^\mu + f_0(q^2)q^\mu\Delta_{m^2} \ ,$$

Setting $\vec{p}_{\rm B} = \vec{0}$, for each $\vec{p}_\pi$, one has to study a ratio of 3 over 2 -point functions on the lattice looking for a plateau in the insertion time of the current.



$\vec{p}_\pi = 1, 0, 0 \times \frac{2\pi}{L}$



[ALPHA, LAT12]

Conclusions

- Heavy-flavor physics is a rich area of research, with a lot of interplay between theory and experiments.
- Powerful framework to determine EW-parameters, test the SM and constrain NP.
- HQET is a modern, model independent framework to perform quantitative computations in heavy-flavor physics.
- The physics is mostly non-perturbative. Lattice plays a key rôle and HQET is a viable and solid approach to obtain precise predictions.

Heavy Flavor Physics and Precision Tests of the Standard Model on the Lattice. Lect. 3

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VII International School on Theoretical Physics
Parma, Italy, 14-20 September 2014

- Introduction and definitions.
- Motivations: probe of Possible New Physics.
  Three σ between experiment and "theory".
- Challenges of the lattice approach.
- Field theoretical description of connected and disconnected quark diagrams. PqQCD and PqχPT.
- Improved momentum resolution of the connected contribution by using twisted boundary conditions.
- Numerical results (Wilson clover, $N_{\rm f} = 2$).
- Review of existing results for $a_\mu^{\rm HLO}$.

Introduction and definitions [Jegerlehner and Nyffler, Phys.Rept. 477 (2009) 1-110]

An orbiting particle of mass m and electric charge e has a magnetic dipole moment

$$\vec{\mu} = \frac{e}{2m}\vec{L} = \mu_B \vec{L} \quad \text{(if the particle is an electron)}$$

and interacts with a magnetic field through a term

$$H = -\vec{\mu} \cdot \vec{B}$$

The spin produces an intrinsic magnetic moment

$$\vec{\mu} = g\frac{e}{2m}\vec{S}$$

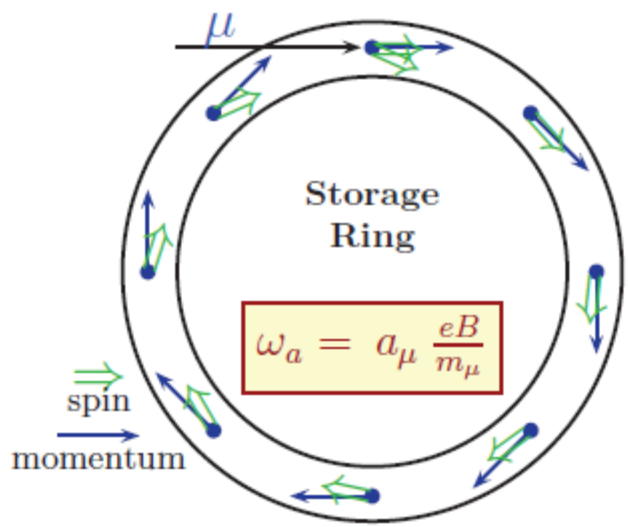
g is the gyromagnetic ratio and the deviation from the tree level value of 2

$$a_l = \frac{g_l - 2}{2}, \quad l = \text{lepton index}$$

is the anomalous magnetic moment.

$\Rightarrow$ (anomalous) Zeeman effect. Level splitting of atomic spectra in magnetic field (eg to measure a_e).

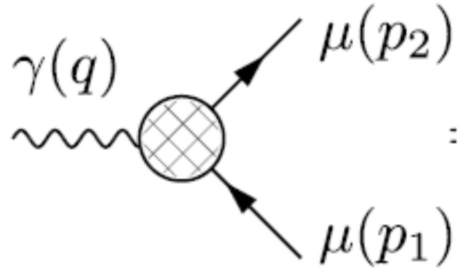
To measure a_μ the motion of the spin is studied for polarized muons (as produced in $\pi \to \mu + \bar{\nu}_\mu$) in homogeneous magnetic fields [BNL and CERN from '60, Fermilab E989 summer 2014].



actual precession × 2

The μ decays into polarized e $(+\nu_\mu \bar{\nu}_e)$. The elctron momentum provides (or strongly correlates to) the muon spin. Experimentally (and theoretically) a_e is known with below 1 ppb accuracy, a_μ below 1 ppm.

In a QFT the interaction of the lepton with the electromagnetic field is described by the vertex



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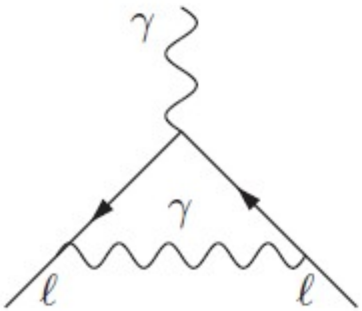
which indeed can be parameterized in terms of electric and magnetic form factors. a_l is basically the $q^2 \to 0$ limit of the magentic FF.

It can be shown [Berestetskii, '56] (argument similar to helicity suppression) that the sensitivity of a_l to new physics states of mass M is proportional to

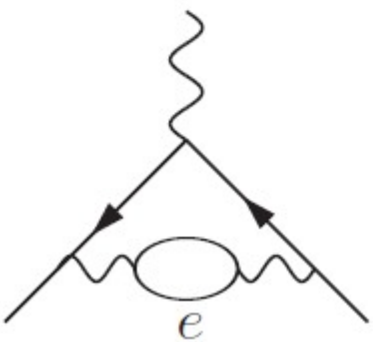
$$\delta a_l \propto \frac{m_l^2}{M^2}$$

:-) Good sensitivity to nearby New Physics, for a_μ (a_τ can't be measured).
:-(Hadronic effects ($M \simeq \Lambda_{QCD}$) maybe large and are notoriously difficult to estimate.

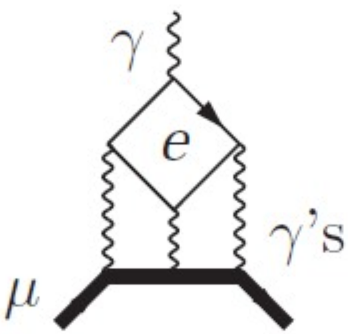
More Dictionary



Schwinger, 1948 $a_l = \int dq^2 f(q^2) = \frac{\alpha}{\pi}\frac{1}{2}$



Vacuum polarization



Light-by-Light scattering

Up to partial 5 loops [Remiddi, Laporta, Kinoshita ...]

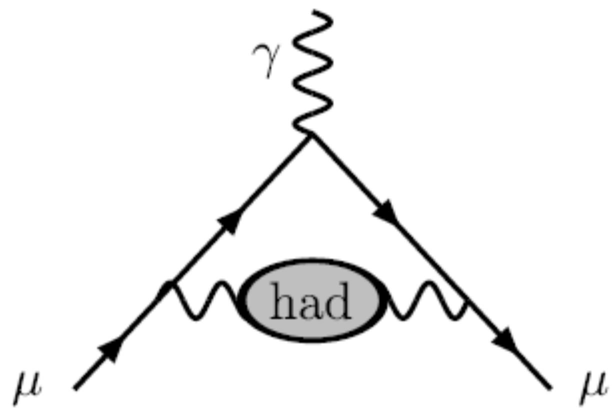
Motivations to study a_μ on the lattice

⋆ 3 sigmas discrepancy between exp and theo [Jegerlehner and Nyffeler, 2009] :

$$a_\mu^{exp} = 1.16592080(63) \times 10^{-3}$$

$$a_\mu^{the} = 1.16591790(65) \times 10^{-3}$$

Contribution	Value	Error
QED incl. 4-loops+LO 5-loops	116 584 718.1	0.2
Leading hadronic vacuum polarization	6 903.0	52.6
Subleading hadronic vacuum polarization	-100.3	1.1
Hadronic light–by–light	116.0	39.0
Weak incl. 2-loops	153.2	1.8
Theory	116 591 790.0	64.6



That is used to discuss and constrain possible New Physics models, eg in the MSSM

$$a_\mu^{\mathrm{SUSY}} = \pm 130 \times 10^{-11} \cdot \left(\frac{100 \text{ GeV}}{m_{\mathrm{SUSY}}}\right)^2 \tan(\beta)$$

$+$ other speculations:

[Hamaguchi and collab., arXiv:1206.0161: "Muon g-2 anomaly and 125 GeV Higgs : Extra vector-like quark and LHC prospects"]

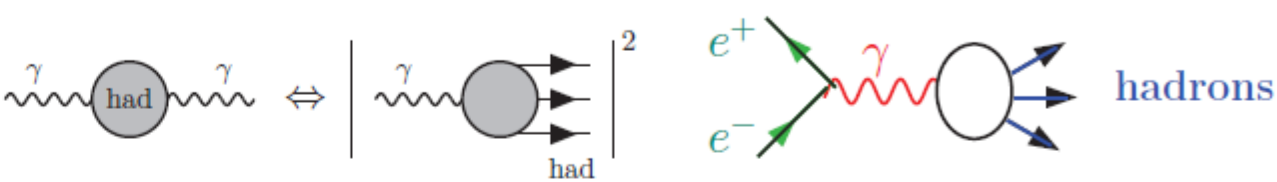
however ...

The *theory* number is obtained by estimating the hadronic contribution to the photon propagator from

- The experimentally measured hadronic e^+e^- annihilation cross-section:

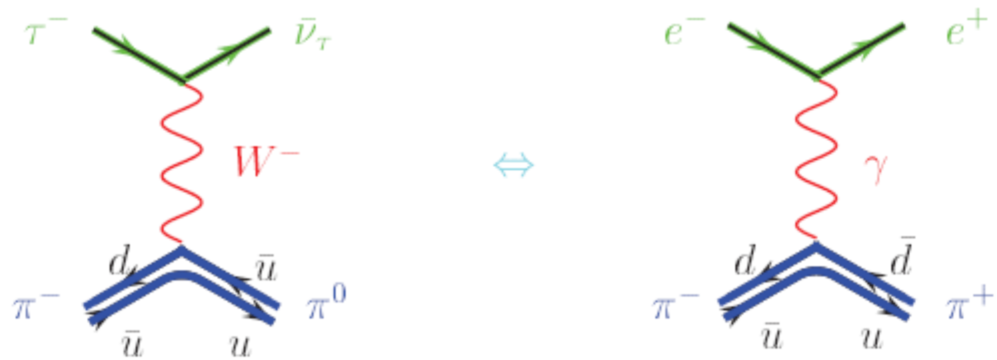
$$\mathrm{DR}: \quad \Pi(k^2) - \Pi(0) = \frac{k^2}{\pi} \int_0^\infty ds \frac{\mathrm{Im}\Pi(s)}{s(s-k^2-i\epsilon)}$$

$$+ \text{ optical theorem} \quad \mathrm{Im}\Pi(s) \propto s\sigma_{\mathrm{tot}}(e^+e^- \to \text{anything})$$



EXERCISE 2: Prove the optical theorem starting from the unitarity of $S = \mathcal{I} + iT$

- Experimentally measured vector spectral functions from hadronic τ decays. The hadronic matrix elements are related to those in e^+e^- annihilation by isospin (broken symmetry).



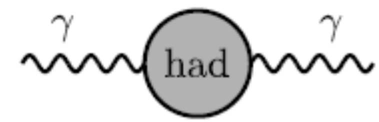
The two (semi-phenomenological) approaches provide similar numbers if isospin breaking effects and $\rho - \gamma$ mixing are properly teaken into account [Jegerlehner and Szafron, 2011].

$\Rightarrow$ Need for a purely theoretical number.

Attempts also using models [Greynat, De Rafael] or Schwinger-Dyson approach [Goecke, Fisher and Williams].

The Euclidean hadronic vacuum polarisation tensor is defined as

$$\Pi^{(N_f)}_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle J^{(N_f)}_{\mu}(x) J^{(N_f)}_{\nu}(0) \rangle$$



Euclidean invariance and current conservation imply

$$\Pi^{(N_f)}_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi^{(N_f)}(q^2)$$

The relation between $\Pi^{(N_f)}_{\mu\nu}(q^2)$ and a_μ^{HLO} is [E. De Rafael, 1994 and T. Blum, 2002]

$$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 \, f(q^2) \hat{\Pi}(q^2)$$

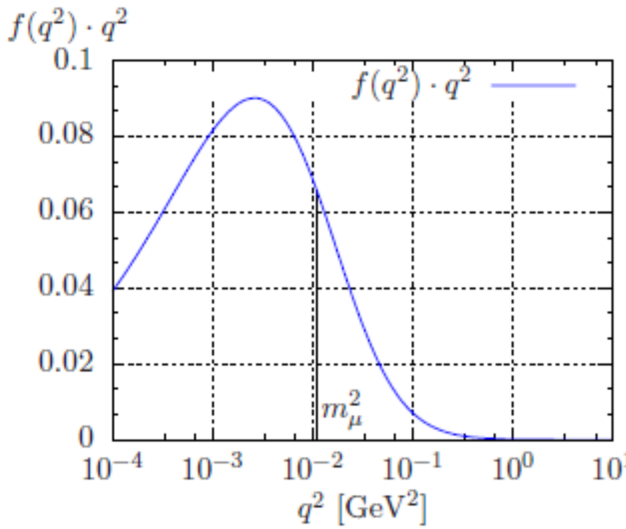
with

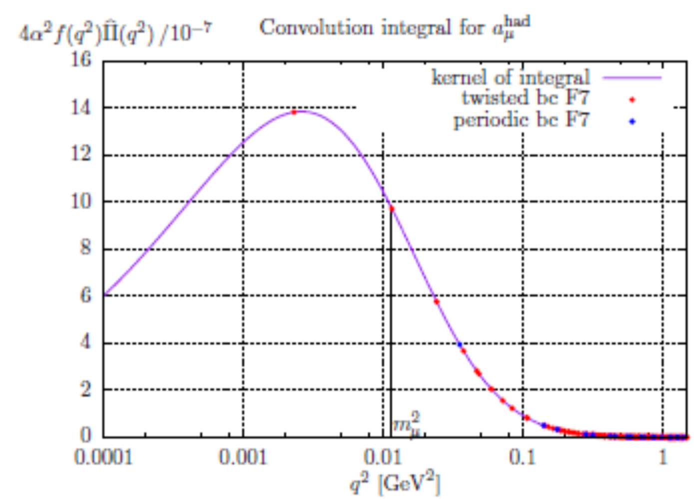
$$f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + m_\mu^2 q^2 Z^2}, \qquad Z = -\frac{q^2 - \sqrt{q^4 + 4 m_\mu^2 q^2}}{2 m_\mu^2 q^2}$$

and $\hat{\Pi}(q^2) = 4\pi^2 \left[\Pi(q^2) - \Pi(0)\right]$.

Main challenges

- The integral is dominated by the $q^2 \simeq m_\mu^2$ region





The lowest Fourier mode $2\pi/L$ is typically much larger than m_μ, and the VP has to be extrapolated to the relevant region [FSE].

- Correlator of flavor-diagonal currents $\rightarrow$ disconnected diagrams.
- The vector resonances (ρ , ω , ϕ) give a large contribution. Those have to be properly included in the lattice computation. u, d, s should be dynamical, almost physical and non-degenerate.

◇ Lattice efforts

$$a_\mu^{HLO} = 674(21)(18) \times 10^{-10}$$ [ETMC ,2013] $N_f = 2+1+1$.

$$a_\mu^{HLO} = 748(21) \times 10^{-10}$$ [Aubin and Blum, 2007] $N_f = 2+1$ ($\sqrt{\text{staggered}}$).

$$a_\mu^{HLO} = 641(33)(32) \times 10^{-10}$$ [RBC-UKQCD ,2011] $N_f = 2+1$.

$$a_\mu^{HLO} = 618(64) \times 10^{-10}$$ [MDM, Jäger, Jüttner and Wittig ,2011] $N_f = 2+1_q$.

$$a_\mu^{HLO} = 572(16) \times 10^{-10}$$ [ETMC ,2011] $N_f = 2$.

$$a_\mu^{HLO} = 446(23) \times 10^{-10}$$ [QCDSF, 2003] quenched.

Disconnected contributions aren't always thoroughly included.

From the spread of results and the errors, the uncertainty is $5-10\%$ well above the phenomenological one ($< 1\%$).

I will discuss some ideas to tackle the first two problems.

Connected and disconnected diagrams as correlators in an unphysical theory [Bernard and Golterman '92, Sharpe and Shoresh 2000, MDM and Jüttner 2011]

Let's consider the 2 flavor case and the contribution form the u quark only

$$C^{uu}_{\mu\nu,QCD}(q) = \frac{4}{9}\int d^4x e^{iqx}\langle J^{uu}_\mu(0)J^{uu}_\nu(x)\rangle$$

After Wick contractions the correlator $\langle J^{uu}_\mu(0)J^{uu}_\nu(x)\rangle$ is written in terms of connected and disconnected quark diagrams

We add a quark r degenerate with u and quenched it away (valence quark)

$$L^{(2)}_{QCD} \to L^{(2)}_{QCD} + \bar{r}(\not{D} + m_u)r + \tilde{r}^\dagger(\not{D} + m_u)\tilde{r} = L_{PqQCD}$$

$\tilde{r}$ is a commuting spin 1/2 filed, a ghost [Morel, '87].

- PqQCD is an unphysical theory, it violates the spin-statistics theorem.
- However the partition function is the same as in QCD, because on each gauge background the integrations over r and $\tilde{r}$ cancel
- In addition in this theory the (dis)connected diagrams above (actually any Wick Contraction) can be written as correlation functions.

$$
\begin{aligned}
C^{uu}_{\mu\nu,QCD}(q) &= \frac{4}{9}\int d^4x e^{iqx}\langle J^{ur}{}_{\mu}(0) J^{ru}{}_{\nu}(x)\rangle_{PqQCD} + \\
&\quad \frac{4}{9}\int d^4x e^{iqx}\langle J^{uu}_{\mu}(0) J^{rr}{}_{\nu}(x)\rangle_{PqQCD} \\
&= C^{conn}_{\mu\nu,PqQCD}(q) + C^{disc}_{\mu\nu,PqQCD}(q)
\end{aligned}
$$

– A low energy effective theory (PqχPT) exists, which should describe correlation functions in PqQCD. This theory is based on an extended (graded) flavor-symmetry group (ghosts can be rotated into fermions).
– The connected part is not flavor-diagonal anymore in PqQCD. Twisting [Sachrajda and Villadoro, 2005] can be used there to improve the momentum resolution.

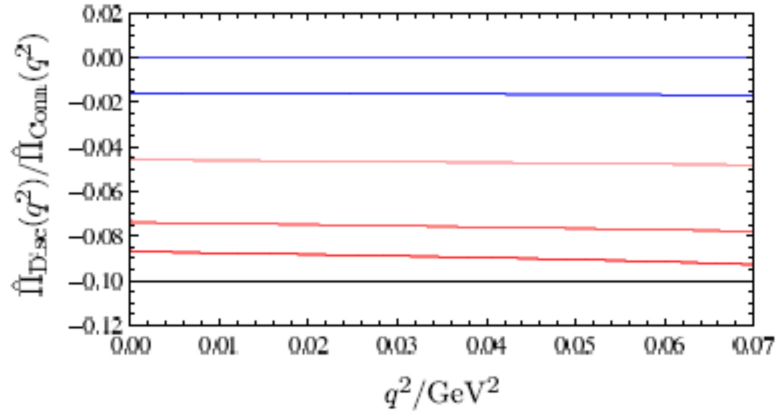
This leads to the simple result

$$\frac{\hat{\Pi}^{(2)}_{disc}(q^2)}{\hat{\Pi}^{(2)}_{conn}(q^2)} = -\frac{1}{10}$$

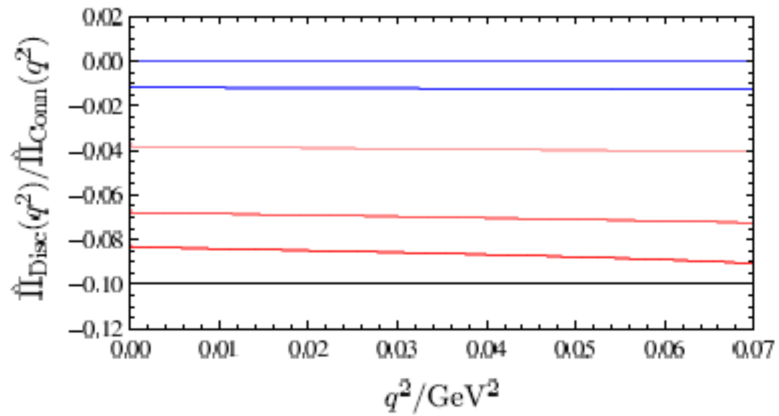
at NLO in mesonic χPT.

For the 2+1 or 2+ quenched strange cases one has to introduce one and two valence quarks at least (there's no degenerate partner of the strange quark) and therefore consider $SU(4|1)$ and $SU(4|2)$ PqχPT.

For our purposes $Pq\chi PT$ is obtained by $Tr \to Str$ in the χPT Lagrangian



(a)



(b)

(a) $N_f = 2$ with a quenched strange quark,
(b) $N_f = 2 + 1$. From top to bottom: $M_\pi = M_K$, then fixed $M_K = 495$MeV and $M_\pi = 400, 300, 200, 139$ MeV. The bottom most-line at -1/10 is the result for $N_f = 2$. [MDM, Jüttner, 2011]

(Partially) Twisted boundary conditions

We exploit the possibility of changing the spatial periodicity of the fermions [ALPHA, '96, in the SF], which can be interpreted as injecting momentum (flavour twisted boundary conditions, [Sachrajda and Villadoro, 2005])

$$\psi(\vec{x} + \hat{k}L) = e^{i\theta_k}\psi(\vec{x})$$

at tree level (on the torus) then

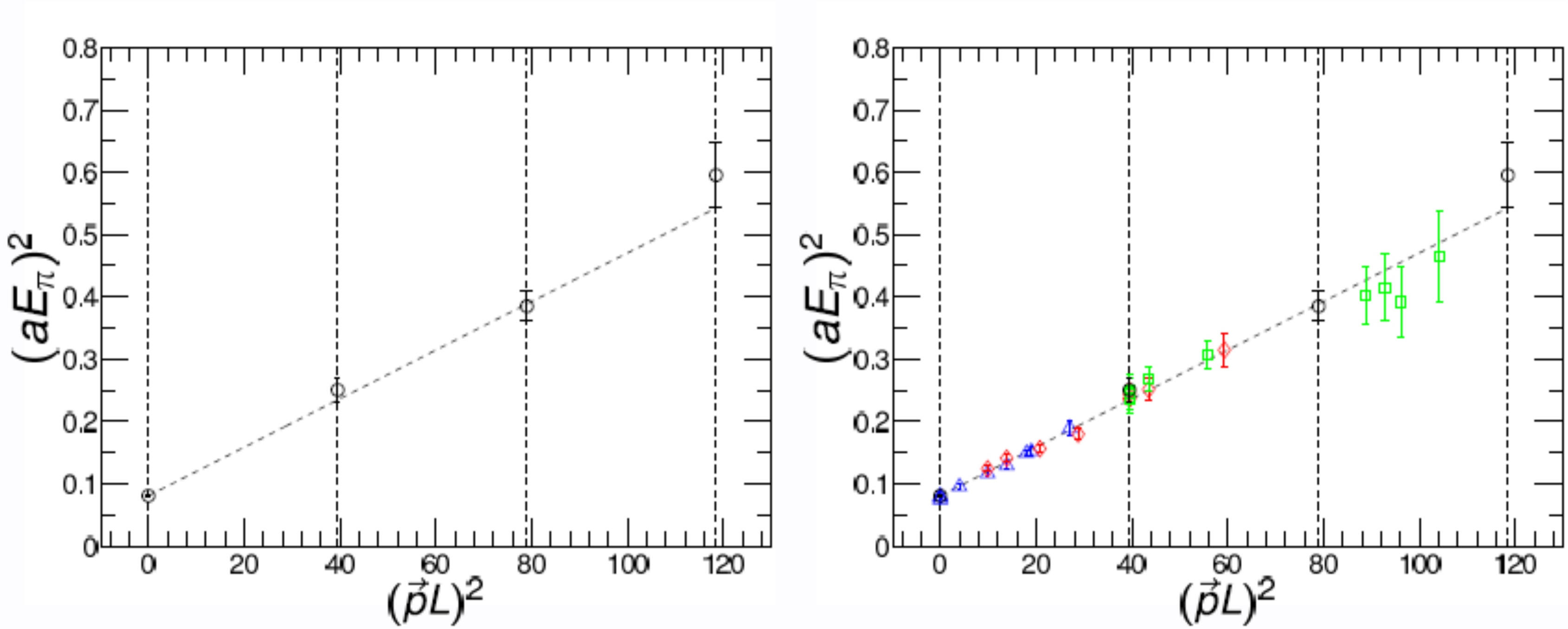
$$S(x) = \langle \psi(x)\bar{\psi}(0)\rangle = \int \frac{dk_4}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} \frac{e^{i(\vec{k}+\vec{\theta}/L)\cdot\vec{x}}}{\not{k}+\frac{\not{\theta}}{L}+m} \; ,$$

The fermions effectively have a momentum which differs from multiples of $2\pi/L$. Anti-fermions have the opposite momentum.

Valence quarks periodic in space up to a phase $\rightarrow$ spatial momentum to non-flavor singlet mesons up to exponentially small finite volume effects

[Sachrajda and Villadoro, 2005]

$$E_\pi = \sqrt{m_\pi^2 + (\vec{n}\tfrac{2\pi}{L})^2} \qquad E_\pi = \sqrt{m_\pi^2 + (\vec{n}\tfrac{2\pi}{L} + \tfrac{\vec{\theta}_u - \vec{\theta}_d}{L})^2}$$



[Flynn, Jüttner and Sachrajda, 2006]

For our application: The PqQCD expression of the connected contribution: $\langle J_\mu^{ur}(0) J_\nu^{ru}(x)\rangle$ now involves non-singlet currents and we can use twisting to improve the momentum resolution and get closer to m_μ^2.

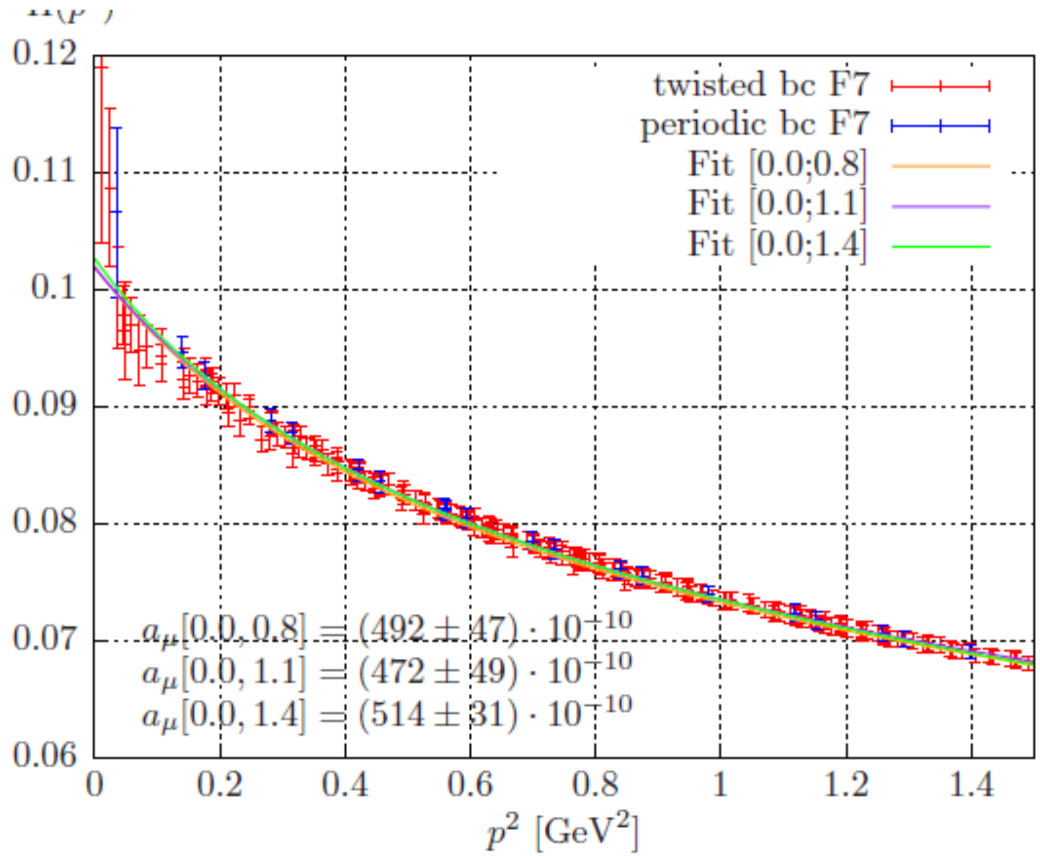
Numerical results [JHEP 1203 (2012) 055]

Ensembles of gauge configurations for $N_f = 2$ NP O(a) improved Wilson fermions generated within the CLS community effort by using the DD-HMC [Lüscher, 2005] algorithm.

	$T \cdot L^3$	β	m_π [MeV]	a [fm]	θ	κ_s	# meas.
A2	$64 \cdot 32^3$	5.2	580	0.082	0.8; 1.8; 2.6	-	500
A3	$64 \cdot 32^3$	5.2	465	0.082	0.8; 1.8; 2.6	-	532
A4	$64 \cdot 32^3$	5.2	355	0.082	0.8; 1.8; 2.6	-	800
A5	$64 \cdot 32^3$	5.2	305	0.082	0.8; 1.8; 2.6	-	432
D5	$48 \cdot 24^3$	5.3	415	0.069	0.9; 1.7; 2.1; 2.5	0.13574	1183
E4	$64 \cdot 32^3$	5.3	555	0.069	0.8; 1.8; 2.6	0.13605	648
E5	$64 \cdot 32^3$	5.3	415	0.069	0.8; 1.8; 2.6	0.13574	672
F6	$96 \cdot 48^3$	5.3	300	0.069	0.4; 1.9; 2.3	0.13575	804
F7	$96 \cdot 48^3$	5.3	250	0.069	0.4; 1.9; 2.3	0.13570	820
N4	$96 \cdot 48^3$	5.5	505	0.053	0.8; 1.9; 2.6	0.13639	532
N5	$96 \cdot 48^3$	5.5	405	0.053	0.8; 1.9; 2.6	0.13629	644

$0.05 \leq a \leq 0.08$ fm.

We twist the d quark in $C^{conn}_{\mu\nu}(q^2) \propto \int d^4x e^{iqx} \langle J^{ud}_\mu(0) J^{du}_\nu(x) \rangle$.
We use the conserved 1-point split vector current.



We now have to

- Parameterize the function and extrapolate to $p^2 = 0$ to subtract $\Pi(0)$.
- Insert the parameterized expression in the convolution integral and propagate the errors to get a_μ^{HLO}.

We use 3 different fit ansätze in the region $0 < p^2 < p^2_{\rm max} \simeq 1.5 - 2{\rm GeV}^2$

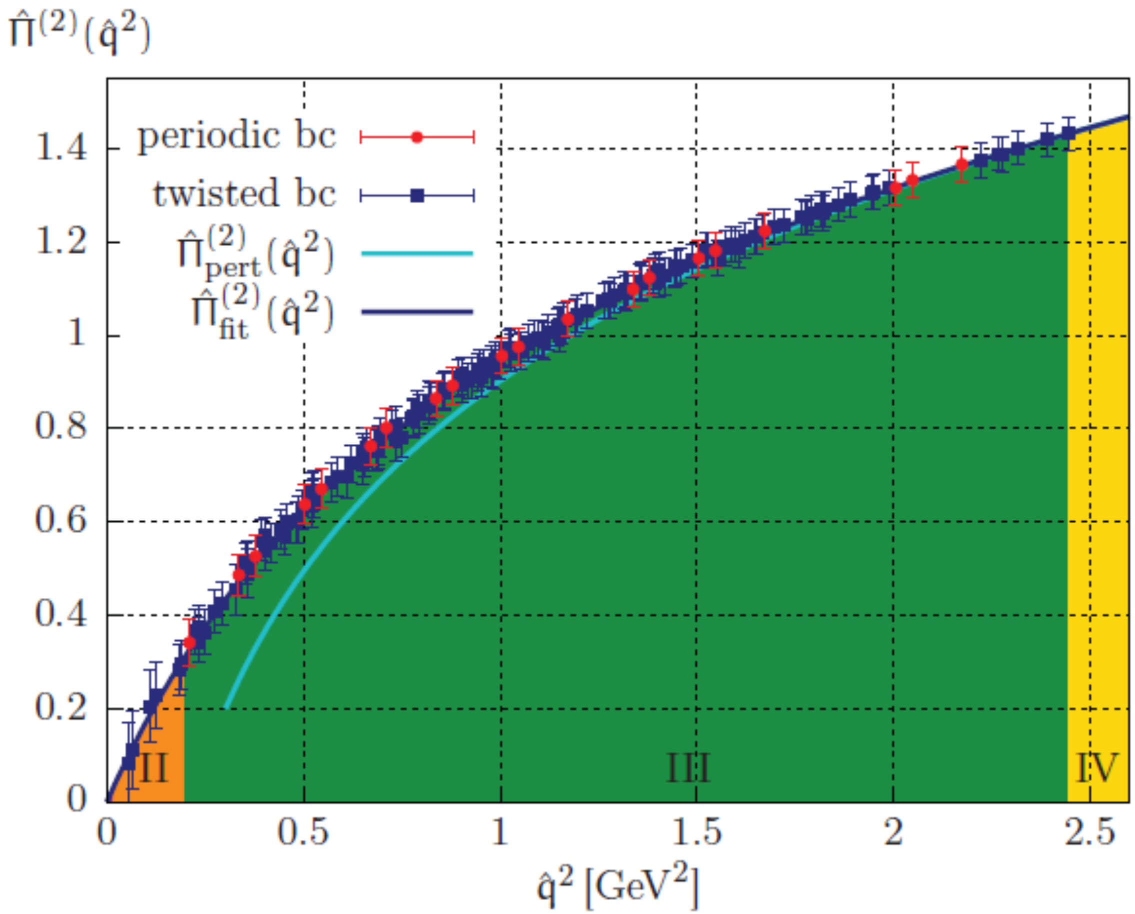
- degree 2 / degree 3 Padé
- Vector dominance model

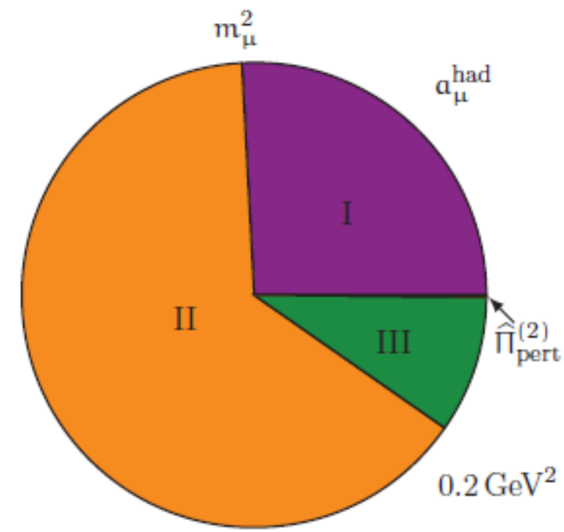
$$\Pi^{N_{\rm f}}(p^2) = a + \frac{b}{c^2 + p^2} ,$$

- Vector dominance model with two vectors

$$\Pi^{N_{\rm f}}(p^2) = a + \frac{b}{m_{\rm V}^2 + p^2} + \frac{c}{d^2 + p^2} .$$

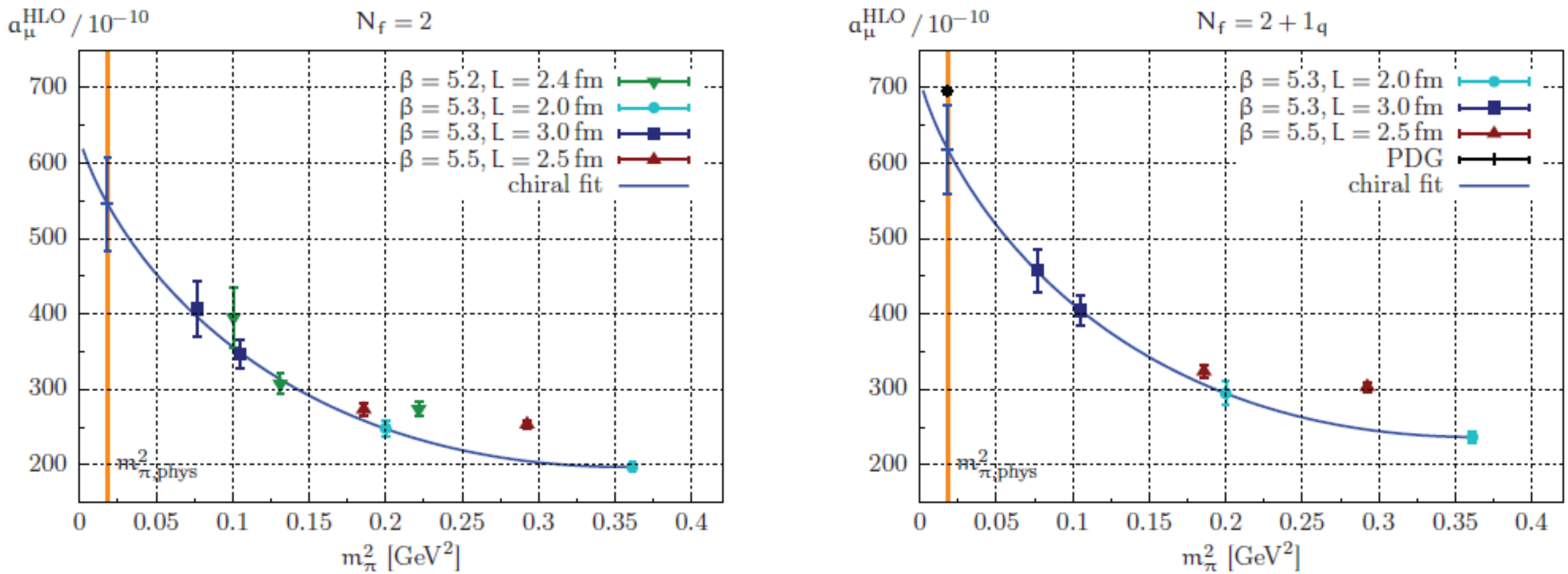
and we smoothly (up to 1st derivative) connect it to 2 loop PT [Chetyrkin, Kuhn, Steinhauser, 1996] in the region $p^2_{\rm max}$ to ∞. This reduces by 1 the parameters in the fit.





We (and everybody else) have no direct lattice measurement of $\Pi(p^2)$ in region I, which contributes around 25% to $a_\mu^{\rm HLO}$. We inevitably end up with a O(5%) statistical error on $a_\mu^{\rm HLO}$.

We do the same for each ensemble an then we need to do a chiral extrapolation



Fits to $\beta = 5.3$ data. [χPT inspired form with free coeffs.]:
$a_\mu(m_\pi^2) = a + b m_\pi^2 (1 + c \log(m_\pi^2))$.

The inclusion of a quenched strange makes the result larger by 15-20% as expected.

Final estimate

$a_\mu^{HLO} = 618(64) \times 10^{-10}$ [MDM, Jäger, Jüttner and Wittig ,2011] $N_f = 2 + 1_q$.

We combine statistical and systematic (chiral extrapolation, fitting procedure, uncertainty on a) in quadrature.

$a_\mu^{HLO} = 674(21)(18) \times 10^{-10}$ [ETMC ,2013] $N_f = 2 + 1 + 1$.
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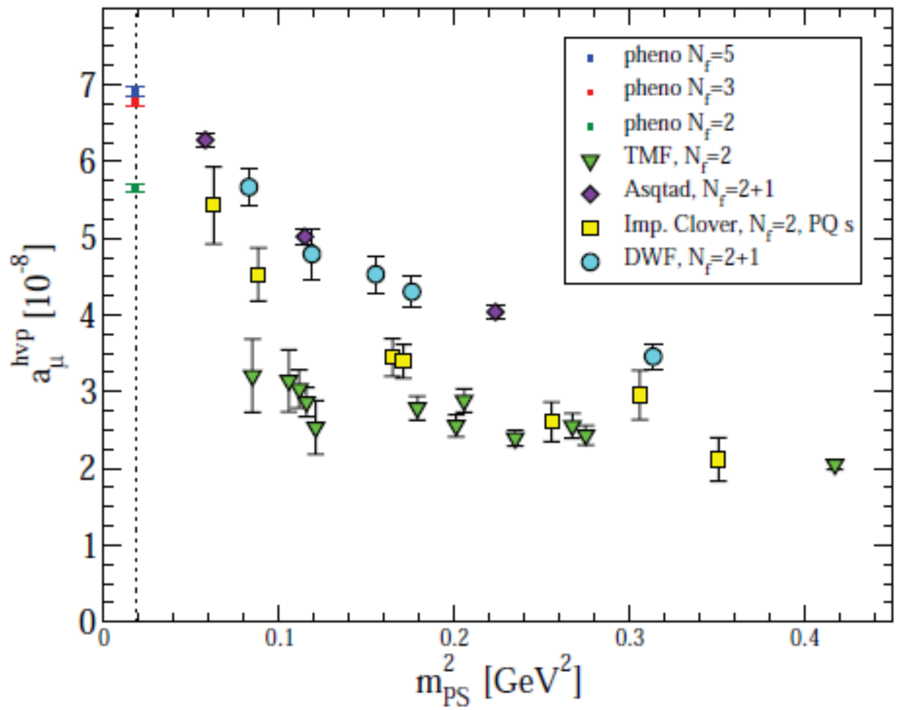
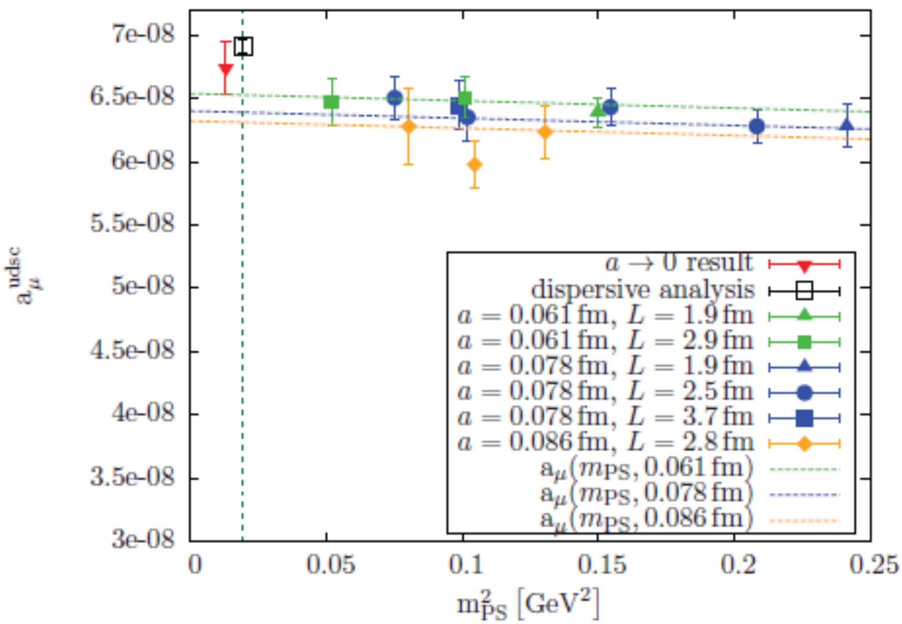




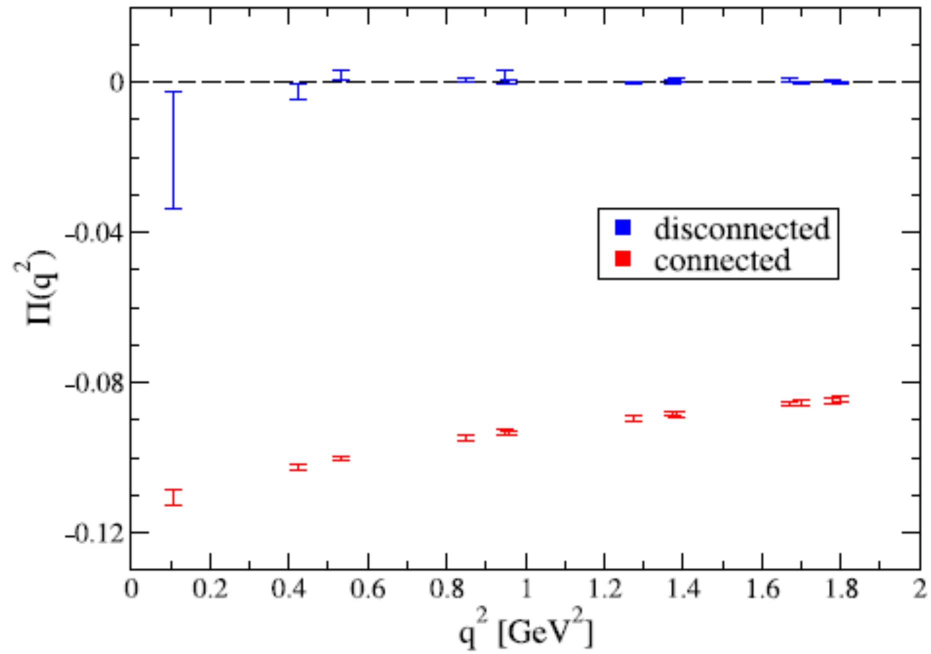
[Dru Renner, LATTICE 2011 and Burger, ETMC, '2013]

- ETM, $N_f = 2$, results are rather flat even without h-rescaling.
- A curvature is clearly visible in Mainz data for smaller masses
- The N_f-trend seems to produce consistency with the phenomenological number
- The most recent results include continuum extrapolation.

Conclusions and outlook

- The anomalous magnetic moment of the muon is a phenomenologically very interesting quantity. Lattice determinations of the leading hadronic vacuum polarization can play an important role in producing a solid theoretical number.
- The appearance of disconnected diagrams and the poor momentum resolution represent the main difficulties in a lattice computation.
- PqχPT can be used to estimate the size of the disconnected contribution. Interesting way to look also at other quantities where disconnected diagrams appear.
- The χPT result can be used to estimate the size of the disconnected piece, but eventually those will have to be numerically computed.
- However this estimate tells us we should control the disconnected contribution to a 10-20% level to have a final result with a precision of a few percent. That is challenging.

- Numerically these disconnected contributions seem to be even smaller than predicted by PqχPT at NLO. That is good.



[D. Renner, LAT10]

- A target accuracy of 5% appears to be reachable for the connected contribution. Many ideas helped there in bringing the problem from hopeless to tough in a few years
  - twisting
  - modified chiral extrapolations
  - momentum expansion of propagators
- There is clear need for new ideas, also towards a solid estimate of the light-by-light contribution. Only one approach and one group working on that.

Further readings

- On $g-2$: F. Jegerlehner and A. Nyffeler, arXiv:0902.3360.
- On $g-2$ on the lattice: T. Blum, PoS (Lattice 2012) 022.
- On Partially Quenched χPT: S. Sharpe, hep-lat/0607016, and M. Golterman, arXiv:0912.4042.

again, and references therein.

I tried to avoid technical details, although sometimes are instructive. If you need more just write to me: dellamor@cp3-origins.net

THANK YOU !

Now it is time to go back to normal food and leave this beautiful place !

THANKS MARISA AND FRANCESCO FOR THE WONDERFUL WEEK !!