

# Relativistic CFT Hydrodynamics from the Membrane Paradigm

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## Abstract

We use the membrane paradigm to analyze the horizon dynamics of a uniformly boosted black brane in a  $(d+2)$ -dimensional asymptotically Anti-de-Sitter space-time and a Rindler acceleration horizon in  $(d+2)$ -dimensional Minkowski space-time. We show that in these cases the horizon dynamics is governed by the relativistic CFT hydrodynamics equations. The fluid velocity and temperature correspond to the normal to the horizon and to the surface gravity, respectively. The second law of thermodynamics for the fluid is mapped into the area increase theorem of General Relativity. The analysis is applicable, in general, to perturbations around a stationary horizon, when the scale of variations of the macroscopic fields is much larger than the inverse of the temperature. We show that the non-relativistic limit of our analysis yields the incompressible Navier-Stokes equations.

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## I. INTRODUCTION

The hydrodynamics of relativistic conformal field theories (CFTs) has attracted much attention recently, largely in view of the AdS/CFT correspondence between gravitational theories on asymptotically Anti-de-Sitter (AdS) spaces and CFTs [1] (for a review see [2]). Hydrodynamics is an effective description of the long distance field theory dynamics and applies under the condition that the correlation length of the fluid  $l_{cor}$  is much smaller than the characteristic scale  $L$  of variations of the macroscopic fields. The AdS/CFT correspondence suggests that the long wavelength dynamics of gravity provides a dual description of the CFT hydrodynamics.

It has been shown in [3] that the  $(d + 1)$ -dimensional CFT hydrodynamics equations are the same as the equations describing the evolution of large scale perturbations of the  $(d + 2)$ -dimensional black brane. The derivation of this result parallels the conventional derivation of hydrodynamics equations from the Boltzmann equation [4], where the “thermal equilibrium” solution is the boosted black brane. Thus, the equations of gravity play for the hydrodynamic equations of a strongly coupled CFTs the same role as the Boltzmann equation plays at a weak coupling.

The limit of non-relativistic macroscopic motions in CFT hydrodynamics leads to the non-relativistic incompressible Navier-Stokes (NS) equations [5, 6]. Since we can obtain the NS equations in the non-relativistic limit of CFT hydrodynamics, the AdS/CFT correspondence implies that these equations have a dual gravitational description, which can be found by taking the non-relativistic limit of the geometry dual to the relativistic CFT hydrodynamics [6].

In this duality picture the dynamics of the fluid entropy of the field theory at the asymptotic boundary can be expressed directly in terms of the black brane horizon geometry [7]. This behavior of the horizon is reminiscent of the membrane paradigm in classical General Relativity, where the dynamics of the black hole event horizon is analogous to that of a fictitious fluid [8, 9, 10]. Indeed, the membrane paradigm has already been an important tool for calculating transport coefficients of the boundary gauge theory [11, 12, 13, 14, 15].

In [16] we used the membrane paradigm formalism and an expansion in powers of the Knudsen number  $l_{cor}/L$  to show that the dynamics of a membrane defined by the event horizon of a black brane in asymptotically AdS space-time is described by the incompressible Navier-Stokes equations of non-relativistic fluids. Moreover, the analysis performed in [16]

holds for a non-singular null hypersurface, provided a large scale hydrodynamic limit exists.

The purpose of this paper is to generalize the analysis of [16] to the relativistic CFT hydrodynamics. Our starting point is an equilibrium  $(d+2)$ -dimensional solution containing a timelike Killing vector field and a stationary  $(d+1)$ -dimensional causal horizon. This solution is associated with a thermal state at uniform temperature. When a hydrodynamic limit exists, we can expand the solution in the neighborhood of the causal horizon in powers of  $l_{cor}/L$ . We consider two specific examples in this paper: a black brane in asymptotically AdS and a Rindler acceleration horizon in Minkowski space-time. Assuming there is no singularity at the horizon, we show that at lowest orders in  $l_{cor}/L$  the set of Einstein equations projected into the horizon surface is equivalent to the  $(d+1)$ -dimensional relativistic CFT Navier-Stokes equations. Our results imply that the analogy between horizon dynamics and hydrodynamics in the membrane paradigm is in fact an identity in certain cases. Since the non-relativistic incompressible NS equations arise in the slow motion limit of CFT hydrodynamics [5, 6], we will also obtain results of [16] in this limit.

The paper is organized as follows. In Section 2 we will briefly review some of the basics of CFT hydrodynamics that we will need, including the expansion in the Knudsen number and the form of the stress-energy tensor in the ideal and dissipative cases. In Section 3 we will outline the membrane paradigm formalism and consider the geometry and dynamics of null hypersurfaces and of the stretched horizon. We also clarify the relationship between the membrane dynamics in general and real hydrodynamics. In Section 4 we will apply the membrane paradigm to a uniformly boosted black brane in a  $(d+2)$ -dimensional asymptotically AdS space-time and show that the horizon dynamics is governed by the CFT hydrodynamics equations. In Section 5 we will show that our analysis is applicable to cases other than the black branes in AdS by considering the example of Rindler space associated with accelerated observers in  $d+2$  Minkowski space-time. We will show that the Rindler horizon dynamics is also governed by the CFT hydrodynamics equations. In Section 6 we will consider the non-relativistic limit of our analysis and re-derive the non-relativistic hydrodynamics results of [16]. Along the way we will clarify the relation between the surface gravity and the fluid pressure. We conclude in Section 7 with a discussion of open problems. In the following, unless explicitly stated, we will use the convention  $G = c = \hbar = k_B = 1$ .

## II. RELATIVISTIC CFT HYDRODYNAMICS

Conformal hydrodynamics in  $(d+1)$ -dimensional space-time ( $d \geq 2$ ) is described by  $d+1$  fields: temperature  $T(x)$  and the  $(d+1)$ -velocity vector field  $u^\mu(x)$ ,  $\mu = 0, \dots, d$ , satisfying  $u_\mu u^\mu = -1$ . The stress-energy tensor of the CFT obeys

$$\partial_\nu T^{\mu\nu} = 0, \quad T^\mu{}_\mu = 0, \quad (1)$$

and the equations of relativistic hydrodynamics are determined by the constitutive relation expressing  $T^{\mu\nu}$  in terms of the temperature and the four-velocity field. The constitutive relation has the form of a series in the small parameter (Knudsen number)

$$Kn \equiv l_{cor}/L \ll 1, \quad (2)$$

where  $l_{cor}$  is the correlation length of the fluid and  $L$  is the scale of variations of the macroscopic fields. Since the only dimensionfull parameter is the characteristic temperature of the fluid  $T$ , one has by dimensional analysis that  $l_{cor} \sim \frac{1}{T}$ . The constitutive relation reads

$$T^{\mu\nu}(x) = \sum_{l=0}^{\infty} T_l^{\mu\nu}(x), \quad T_l^{\mu\nu} \sim (Kn)^l, \quad (3)$$

where  $T_l^{\mu\nu}(x)$  is determined by the local values of  $u^\mu$  and  $T$  and their derivatives of a finite order. Keeping only the first term in the series gives ideal hydrodynamics, while dissipative hydrodynamics arises when one keeps the first two terms in the series.

The ideal hydrodynamics approximation for  $T^{\mu\nu}$  does not contain the spatial derivatives of the fields. The  $l = 0$  term in (3) gives the stress-energy tensor that reads (up to a multiplicative constant)

$$T_{\mu\nu} = T^{d+1} [\eta_{\mu\nu} + (d+1)u_\mu u_\nu], \quad (4)$$

where  $\eta_{\mu\nu} = \text{diag}[-, +, +, \dots, +]$ .

The dissipative hydrodynamics is obtained by keeping the  $l = 1$  term in the series in Eq. (3). In the Landau frame [3, 17, 18], that fixes the ambiguity in the form of the stress-energy tensor under a field redefinition of the temperature and velocity, the stress-energy tensor reads (up to a multiplicative constant)

$$T_{\mu\nu} = T^{d+1} [\eta_{\mu\nu} + (d+1)u_\mu u_\nu] - 2\eta\sigma_{\mu\nu}, \quad (5)$$

where the shear tensor  $\sigma_{\mu\nu}$  obeys  $\sigma_{\mu\nu}u^\nu = 0$  and is given by

$$\sigma_{\mu\nu} = \frac{1}{2} (\partial_\mu u_\nu + \partial_\nu u_\mu + u_\nu u^\rho \partial_\rho u_\mu + u_\mu u^\rho \partial_\rho u_\nu) - \frac{1}{d} \partial_\alpha u^\alpha [\eta_{\mu\nu} + u_\mu u_\nu]. \quad (6)$$

The dissipative hydrodynamics of a CFT is determined by only one kinetic coefficient - the shear viscosity  $\eta$ . The bulk viscosity  $\zeta$  vanishes for the CFT, while the absence of the particle number conservation and the use of the Landau frame allow one to avoid the use of heat conductivity [19]. The dimensional analysis dictates  $\eta = F(\lambda)T^d$  where  $F(\lambda)$  is a function of the dimensionless parameters that characterize the CFT. For strongly coupled CFTs described by an AdS gravity dual one gets  $F = 1/\pi$ .

### III. THE MEMBRANE PARADIGM

The four laws of black hole thermodynamics are global statements derived from the Einstein equations restricted to quasi-stationary perturbations near equilibrium. In the late 1970's and 1980's Damour [8] and later Price, Thorne, and collaborators [9, 10] developed a very general analogy between the local, non-equilibrium physics of any horizon and a fluid membrane. In this picture the horizon fluid membrane is governed by the Einstein equations, which correspond to fictitious Navier-Stokes equations with universal shear and bulk viscosities. In this section we will first review Damour's approach to the membrane paradigm, which involves the geometry and dynamics of null hypersurfaces. Although this formalism is elegant, in some cases it may be more convenient or conceptually useful to work instead with Price and Thorne's notion of a *stretched horizon*, a timelike surface located just outside the true null horizon. We will briefly discuss how in a particular limit the stretched horizon becomes the true horizon and the two approaches yield identical results.

#### A. Geometry and dynamics of null hypersurfaces

We will consider a  $(d + 2)$ -dimensional bulk space-time  $M$  with coordinates  $X^A$ ,  $A = 0, \dots, d + 1$  with a Lorentzian metric  $g_{AB}$ . Let  $H$  be a  $(d + 1)$ -dimensional null hypersurface in this space-time defined by a restriction of the bulk coordinates  $F(X^A) = 0$ . The normal vector to this hypersurface,  $\ell^A = g^{AB}\partial_B F$ , by definition satisfies

$$g_{AB}\ell^A\ell^B = g^{AB}\partial_A F\partial_B F = 0. \quad (7)$$

This condition implies that for a null hypersurface the normal vector is also a tangent vector.

We can choose a set of adapted coordinates so that hypersurface in the bulk space-time is given by  $x^{d+1} \equiv r = r_H = \text{const.}$ , and denote the remaining coordinates on the horizon surface as  $x^\mu, \mu = 0, \dots, d$ . In this coordinate system  $\ell^r = 0$ , so  $\ell^A \rightarrow \ell^\mu$ .

The first fundamental form of the horizon is the pullback of the space-time metric to the horizon surface. In the adapted coordinate system,

$$\gamma_{\mu\nu} = g_{AB} e_\mu^A e_\nu^B, \quad (8)$$

where  $e_\mu^A$  represents a horizon basis. It follows from (7) that the horizon metric  $\gamma_{\mu\nu}$  is degenerate:  $\gamma_{\mu\nu} \ell^\nu = 0$ . We next introduce the auxiliary null ‘‘rigging’’ vector  $m^A$ , which is everywhere transverse to the horizon and normalized such that  $m^A \ell_A = 1$ . A vector satisfying these conditions is

$$m^A = m^r = 1, \quad (9)$$

with all other components zero. Using this vector, one can write a completeness relation

$$\gamma_{AB} = g_{AB} - \ell_A m_B - \ell_B m_A, \quad (10)$$

where on the horizon the  $(d+2)$  tensor  $\gamma_{AB}$  reduces to the degenerate induced metric.

In order to construct the generalized second fundamental form for a null surface, consider the space-time covariant derivative  $\nabla_A$  projected into the horizon using the transverse projector  $\Pi_B^A = \delta_B^A - m^A \ell_B$  [21] and acting on the normal vector  $\ell^A$ ,

$$\Pi_A^C \nabla_C \ell^B. \quad (11)$$

Since  $\ell^B \ell_B = 0$  (7), we have in the adapted coordinate system

$$\ell_A \nabla_\mu \ell^A = 0. \quad (12)$$

This implies that  $\nabla_\mu \ell^A$  is tangent to the horizon and can be expanded in the horizon basis

$$\nabla_\mu \ell^A = \Theta_\mu{}^\nu e_\nu^A. \quad (13)$$

The mixed index object  $\Theta_\mu{}^\nu$  acts as a ‘‘Weingarten map’’ from the horizon tangent space onto itself and therefore is the generalized extrinsic curvature of the horizon. Together, these first and second fundamental forms provide a description of the embedding of the null hypersurface in the bulk space-time.

Consider Lie transport of  $\gamma_{AB}$  along the null normal vector  $\ell$  which is given by the Lie derivative  $\mathcal{L}_\ell \gamma_{AB}$ . This expression can be split into its trace part (the horizon expansion  $\theta$ ) and trace free part (the horizon shear  $\sigma_{AB}^{(H)}$ ):

$$\theta = \nabla_A \ell^A , \quad (14)$$

$$\sigma_{AB}^{(H)} = \gamma_A^C \gamma_B^D \nabla_{(C} \ell_{D)} - \theta \gamma_{AB} / d . \quad (15)$$

Here  $\gamma_A^B$  is the projector onto the  $d$ -dimensional spacelike cross-sections of the horizon transverse to  $\ell^A$ . The components of the Weingarten map are

$$\gamma_D^A \gamma_B^C \Theta_C^D = \sigma^{(H)D}_C + \theta \gamma_C^D / d \quad (16)$$

$$\Theta_A^B \ell^A = \kappa(x) \ell^B \quad (17)$$

$$\Theta_A^B m_B \gamma_C^A \equiv \Omega_C \quad (18)$$

Note that in these formulas we have used the covector  $m_A$ , which is a one-form tangent to the horizon, just as  $\ell^A$  is a tangent basis vector.  $\kappa(x)$  is the surface gravity and, as we will see below,  $\Omega_C$  is a covector whose components can be associated with a horizon ‘‘momentum’’. Eqn. (17) follows just from the null geodesic equation

$$\ell^B \nabla_B \ell^A = \kappa(x) \ell^A , \quad (19)$$

with the surface gravity as the ‘‘non-affinity’’ coefficient.

We assume that the dynamics of the horizon geometry perturbations are governed by the Einstein equations, which are (with a non-zero cosmological constant  $\Lambda$ )

$$R_{AB} - (1/2) R g_{AB} + \Lambda g_{AB} = 8\pi T_{AB}^{matt} . \quad (20)$$

The Ricci tensor contracted with the normal vector and projected transversely into horizon  $R_{AB} \ell^A \Pi_C^B$  can be expressed solely in terms in terms of the intrinsic horizon metric and extrinsic curvature using a generalization of the contracted Gauss-Codazzi equation for a null surface [8]. In our adapted coordinates it has the form

$$R_{AB} \ell^A e_\nu^B = \bar{D}_\mu \Theta_\nu^\mu - \partial_\nu \Theta . \quad (21)$$

Since the horizon metric is degenerate, one cannot define a unique connection compatible with it. However there is a well-defined rigged covariant derivative operator on the horizon  $\bar{D}_\mu$ , which is defined in terms of the bulk connection projected transversely into the

horizon[22]. In the adapted coordinate system it has the form

$$\bar{D}\bar{e}_\mu e_\nu^A = \Gamma^\sigma{}_{\mu\nu} e_\sigma^A. \quad (22)$$

The right hand side of Eqn. (21) can be expressed as the covariant divergence of a horizon stress tensor

$$T_{(H)\nu}{}^\mu = \Theta_\nu{}^\mu - \delta_\nu^\mu \Theta. \quad (23)$$

From (16)-(18), we see that the horizon stress tensor has the general form

$$T_{(H)\nu}{}^\mu = \kappa m_\nu \ell^\nu + \Omega_\nu \ell^\mu + \sigma_{\nu}^{(H)\mu} + \frac{1}{d} \gamma_\nu^\mu \theta - \delta_\nu^\mu (\theta + \kappa), \quad (24)$$

where we have used  $\Theta = \theta + \kappa$ .

Consider first the component of (21) along  $\ell$ , which is the contraction of the Ricci tensor with  $\ell^A \ell^B$ . This yields the null geodesic focusing equation

$$R_{AB} \ell^A \ell^B = -\ell^\mu \nabla_\mu \theta + \kappa(x) \theta - \theta^2/d - \sigma_{\mu\nu}^{(H)} \sigma_{(H)}^{\mu\nu}. \quad (25)$$

Imposing the Einstein equation, there is no contribution from the Ricci scalar and cosmological constant terms proportional to the metric due to (7) and we have

$$-\ell^\mu \nabla_\mu \theta + \kappa(x) \theta - \theta^2/d - \sigma_{\mu\nu}^{(H)} \sigma_{(H)}^{\mu\nu} - 8\pi T_{AB}^{matt} \ell^A \ell^B = 0. \quad (26)$$

The other  $d$  components of the Gauss-Codazzi equation can be obtained by projecting (21) orthogonal to  $\ell$  using the projection tensor  $\gamma_B^A$ . When the Einstein equations are imposed, the terms proportional to the metric again do not contribute because by construction  $\gamma_B^A$  and  $\ell$  are orthogonal. To write this equation Damour splits space and time ( $t, x^i$ ) by introducing a horizon basis  $\ell = \partial_t + v^i \partial_i$  and  $\partial_i$ . The coordinate  $t$  parameterizes a slicing of space-time by spatial hypersurfaces and  $x^i$  are coordinates on  $d$ -dimensional sections of the horizon with constant  $t$ . As a result the equations take the form

$$-\mathcal{L}_\ell \Omega_i - \theta \Omega_i = -\partial_i \kappa(x) + D_j \sigma^{(H)i}_j + \frac{1-d}{d} \partial_i \theta - 8\pi T_{AB}^{matt} \ell^A e_i^B, \quad (27)$$

where  $\Omega_i = \Theta_i{}^\ell$ . Together the focusing equation (26) and (27) describe the dynamics of any null hypersurface.

Although the horizon system is an intrinsically relativistic system, (27) looks just like a  $d$ -dimensional non-relativistic Navier-Stokes equation

$$\mathcal{L}_\ell \mathcal{P}_i + \theta \mathcal{P}_i = -\partial_i p + 2\eta D_j \sigma^{(H)i}_j + \xi \partial_i \theta - T_{AB}^{matt} \ell^A e_i^B, \quad (28)$$



where  $\mathcal{P}_i = -\Omega_i/8\pi$  is the membrane's momentum,  $p = \kappa/8\pi$  the fluid pressure,  $\eta = 1/16\pi$  the shear viscosity,  $\xi = \frac{1-d}{8\pi d}$  the bulk viscosity, and  $\ell^A e_i^B T_{AB}^{matt}$  an external forcing term. Moreover, using the formula for the expansion as the fractional rate of change in the horizon's cross-sectional area  $A$ ,

$$\theta = \mathcal{L}_\ell \ln \sqrt{\gamma}, \quad (29)$$

the focusing equation (26) can be written like a non-equilibrium entropy balance law

$$\frac{dS}{dt} - \frac{1}{\kappa} \frac{d^2 S}{dt^2} = \frac{dA}{T} (\xi \theta^2 + 2\eta \sigma_{\mu\nu}^{(H)} \sigma^{\mu\nu}_{(H)} + T_{AB}^{matt} \ell^A \ell^B), \quad (30)$$

where the Bekenstein-Hawking entropy  $S = A/4$  and Hawking temperature  $T = \kappa/2\pi$ . Therefore there is an complete analogy between the dynamics of a null hypersurface and the dynamics of a non-relativistic fluid.

The viscous entropy production term due to the shear does appear here as one would expect. It is important to note, however, that the general horizon fluid does not actually correspond to a real fluid because it possesses an unphysical negative “bulk viscosity”. Moreover, the second term on the left hand side of the entropy balance law does not appear in hydrodynamics and reflects the general non-local and teleological (as opposed to causal) character of a globally defined null surface.

These discrepancies with hydrodynamics arise because the general membrane paradigm formalism is valid regardless of the size of the Knudsen number  $Kn \equiv l_{cor}/L$ . For example, the membrane equations (26) and (27) describe the dynamics of a black hole with spherical topology in an asymptotically flat spacetime (e.g. Schwarzschild). In this case the correlation length of a fluid will scale as  $l_{cor} \sim T^{-1}$ , where  $T$  is the Hawking temperature, while  $T^{-1} \sim r_H$ , where  $r_H$  the horizon radius. Since the horizon is compact,  $L$  can be no greater than  $\sim r_H$ . Thus the dimensionless Knudsen number  $Kn$  in these cases is of order unity and hydrodynamics is not an appropriate effective description.

## B. The stretched horizon

The above membrane paradigm results can also be obtained via the stretched horizon formalism (see [9, 10, 20]). Since this approach employs the familiar formalism of a  $(d+1)$ -dimensional timelike surface we can avoid the mathematical complications of dealing with a hypersurface whose normal vector is also a tangent vector. A timelike surface is also

physically advantageous as a boundary since a null horizon is an infinite redshift/blueshift surface.

To start, we imagine that the causally complete region of the bulk space-time  $M$  outside the general horizon discussed in the previous section is foliated by a set of timelike hypersurfaces with spacelike unit normal vector  $n^A n_A = 1$ . As the previous section we can use an adapted coordinate system so that the foliation is given by surfaces of  $x^{d+1} = r = \text{const.}$  and the stretched horizon coordinates are  $x^\mu$ . The induced metric on these surfaces is given by

$$h_{AB} = g_{AB} - n_A n_B \quad (31)$$

and the horizon extrinsic curvature is

$$K_A^B = h_A^C \nabla_C n^B \quad (32)$$

One can also consider a unit timelike vector field  $U^A U_A = -1$  normal to spacelike  $d$  sections of the timelike surfaces. The induced metric on these horizon cross-sections is

$$s_{AB} = h_{AB} + U_A U_B. \quad (33)$$

The “distance” between a given timelike hypersurface and the true null horizon can be parameterized by the affine parameter (or a function thereof,  $\alpha$ ) along a congruence of ingoing null geodesics. For example, in the previous section, the set of null geodesics with tangent vector  $m^A$ , and affine parameter  $r$ . The true horizon is at  $\alpha = 0$ . To approximate the true horizon one considers a timelike surface  $\alpha \ll 1$  and takes the limit  $\alpha \rightarrow 0$  at the end of calculations. The combinations  $\alpha n^A$  and  $\alpha U^A$  are fixed in this limit such that

$$\alpha n^A \rightarrow \ell^A \quad (34)$$

$$\alpha U^A \rightarrow \ell^A. \quad (35)$$

In the horizon limit some kinematic quantities will diverge like inverse powers of  $\alpha$  and need to be renormalized so they are fixed in the limit. For example, one can show that

$$\alpha K_A^B U^A U_B \rightarrow \kappa(x) \quad (36)$$

$$\alpha s_A^C s_D^B K_C^D \rightarrow \sigma^{(H)B}_A + \frac{1}{d} \gamma_A^B \theta \quad (37)$$

where  $\kappa$  is the horizon surface gravity and  $\sigma^{(H)}$  and  $\theta$  are the true horizon shear and expansion respectively. In contrast,

$$K_i^A U_A \rightarrow \Omega_i, \quad (38)$$

so the horizon's  $d$ -momentum with respect to  $(t, x^i)$  can be obtained without any  $\alpha$  renormalization.

The dynamics of the stretched horizon is determined by the usual contracted Gauss-Codazzi equations. When the Einstein equation is imposed we have

$$D_\nu T_{(S)}^{\mu\nu} = T_{\text{matt}}^{AB} e_A^\mu n_B \quad (39)$$

where  $D_\mu$  is the intrinsic covariant derivative and the stretched horizon stress tensor is

$$T_{(S)}^{\mu\nu} = K^{\mu\nu} - h^{\mu\nu} K. \quad (40)$$

Note that this same quantity appears in the literature as a quasi-local energy-momentum stress tensor. When evaluated in an asymptotically AdS space-time it is the Balasubramanian-Kraus stress-tensor [23]. After introducing appropriate counterterms to (UV) regularize at the AdS boundary,  $\alpha \rightarrow \infty$ , this stress tensor is equivalent to the stress tensor of the CFT.

In the opposite (IR) limit, at the horizon, it turns out that (40) is ill-defined. In this limit the non-degenerate induced metric  $h_{\mu\nu}$  on timelike surfaces becomes the degenerate horizon metric  $\gamma_{\mu\nu}$ . Therefore the inverse of the horizon metric does not exist and there is no unique canonical way to raise and lower indices. However, one can avoid this problem by always working with the mixed index stress tensor

$$T_{(S)\mu}^\nu = K_\mu^\nu - \delta_\mu^\nu K. \quad (41)$$

The Kronecker delta is well-defined in the horizon limit and Eqns. (36) and (37) imply that in the true horizon limit (41) agrees with the horizon stress tensor we defined in (24) from the Weingarten map. It can also be shown in the same limit that the timelike Gauss-Codazzi equations yield exactly the null focusing equation (26) and the Damour-Navier-Stokes equation (27).

In the following sections we will apply this general membrane paradigm formalism to two examples where a large scale hydrodynamic limit  $Kn \ll 1$  exists, black branes in AdS space-time and a Rindler horizon in flat Minkowski. We will show that in these cases the focusing and Damour-Navier-Stokes equations are exactly the relativistic Navier-Stokes equations for a real fluid. Hence the analogy between null surface dynamics and hydrodynamics is actually an identity in these cases.

#### IV. BLACK BRANES IN ASYMPTOTICALLY ADS SPACE-TIME

We will first apply the membrane paradigm to a uniformly boosted black brane in an  $(d+2)$ -dimensional asymptotically AdS space-time. This is a solution to the vacuum Einstein equations with negative cosmological constant

$$R_{AB} + (d+1)g_{AB} = 0 . \quad (42)$$

The bulk metric of this unperturbed, equilibrium solution in Eddington-Finkelstein (EF) coordinates is

$$ds^2 = -2u_\mu dx^\mu dr + \frac{\pi^4 T^4}{r^2} u_\mu u_\nu dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu, \quad (43)$$

where  $T$  is the Hawking temperature and  $u^\mu = (\gamma, \gamma v^i)$  ( $\gamma = (1 - v^2)^{-1/2}$ ) is the  $(d+1)$ -velocity. Note that the  $\mu$  index on the  $(d+1)$ -velocity is raised and lowered by the flat metric  $\eta_{\mu\nu}$  and its norm with respect to this metric  $u^\mu u_\mu = -1$ . The black brane horizon is located at  $r_H = \pi T$  and its normal vector is given by  $\ell^r = 0$  and

$$\ell^\mu = u^\mu. \quad (44)$$

We want to consider perturbations of this black brane horizon parameterized by allowing  $u^\mu(x^\mu)$ ,  $T(x^\mu)$ . The horizon location is  $r_H(x^\mu)$  and  $x^\mu$  are the coordinates on the horizon surface. The resulting non-uniform black brane is no longer a solution to the Einstein equation. However if the velocity and temperature are slowly varying functions of  $x^\mu$  (long wavelength, long time perturbations) we can solve the Einstein equations order by order in a derivative expansion and calculate the corrected metric and stress tensor order by order in Knudsen number. In what follows we suppose that  $u^\mu(\varepsilon x^\mu)$  and  $T(\varepsilon x^\mu)$  and use  $\varepsilon$  as a parameter to keep track of the number of derivatives.

Using (43) we find the induced metric on the horizon at zeroth order in  $\varepsilon$  is

$$ds_H^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = (\pi T)^2 (\eta_{\mu\nu} + u_\mu u_\nu) dx^\mu dx^\nu = (\pi T)^2 P_{\mu\nu} dx^\mu dx^\nu \quad (45)$$

where  $P_{\mu\nu}$  is the projection tensor onto the  $d$ -dimensional subspace orthogonal to  $u^\mu$ . Taking the Lie derivative along  $\mathbf{u}$  we get the  $O(\varepsilon)$  expressions

$$\theta = \partial_\mu u^\mu + d \mathcal{D}\xi \quad (46)$$

$$\sigma_{\mu\nu}^{(H)} = (\pi T)^2 (P^\alpha{}_\mu P^\beta{}_\nu \partial_{(\alpha} u_{\beta)} - \partial_\gamma u^\gamma P_{\mu\nu} / d) . \quad (47)$$

We have defined  $\mathcal{D} = u^\mu \partial_\mu$  and  $\xi = \ln T$ . The horizon shear is equivalent to (up to an overall factor) the usual fluid shear in hydrodynamics (6). Note that (15) implies  $\sigma^{(H)\mu}_\nu = \sigma^\mu_\nu$ .

We can also use (43) to calculate the other horizon geometrical quantities. From  $\Omega_\nu = P^\sigma_\nu m_\mu \nabla_\sigma \ell^\mu$ , we find  $\Omega_\mu = -\frac{1}{2}a_\mu$ , where  $a_\mu = \mathcal{D}u_\mu$ . Using  $u^\nu \nabla_\nu u^\mu = \kappa u^\mu$  we find at zeroth order the surface gravity is

$$\kappa(x) = 2\pi T(x). \quad (48)$$

With the kinematical quantities defined, we now consider the dynamics of horizon perturbations in a derivative expansion in  $\varepsilon$ . The dynamics are described by the generalized Gauss-Codazzi equation (21) in vacuum

$$\bar{D}_\mu \Theta_\nu{}^\mu - \partial_\nu \Theta = 0. \quad (49)$$

The Weingarten map can be calculated directly from the bulk covariant derivative along the horizon  $\mu$  coordinates

$$\Theta_\nu{}^\mu = \nabla_\nu u^\mu. \quad (50)$$

We start by calculating the zeroth order (in Knudsen number) part of this horizon stress tensor. Derivatives of this stress tensor should yield the equations of ideal hydrodynamics. At this lowest order we find

$$\Theta_\nu{}^\mu = -\kappa(x) u_\nu u^\mu. \quad (51)$$

Note that this is consistent with the first term (24) since  $m_\mu = -u_\mu$ . Expanding out (49) gives

$$\partial_\nu \Theta_\mu{}^\nu + \Gamma^\beta_{\beta\lambda} \Theta_\nu{}^\lambda - \Gamma^\lambda_{\mu\nu} \Theta_\lambda{}^\mu - \partial_\nu \Theta = 0, \quad (52)$$

where, following the definition (53), we define the connection as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}), \quad (53)$$

evaluated on the horizon. Plugging in (48) and using the formula  $\Gamma^\beta_{\beta\lambda} = \partial_\lambda \ln \sqrt{\gamma}$  we find at  $O(\varepsilon)$

$$-\mathcal{D}T u_\nu - T \partial_\mu u^\mu u_\nu - T a_\nu - d\mathcal{D}\xi T u_\nu - \partial_\nu T = 0. \quad (54)$$

Projecting along  $u^\nu$  yields

$$\partial_\mu u^\mu + d\mathcal{D}\xi = 0. \quad (55)$$

This equation is equivalent to the vanishing of the horizon expansion,  $\theta = 0$ . Note that we could have read off this equation from the lowest  $O(\varepsilon)$  part of the horizon focusing equation (26).

Since we are after the relativistic NS equations we will not introduce an explicit split of space and time  $(t, x^i)$  as done previously in the membrane paradigm, which would correspond to hydrodynamic equations written in terms of the  $d$ -velocity  $v^i$  ( $u^\mu = (\gamma, \gamma v^i)$ ). Instead, we just project transverse to  $\mathbf{u}$  with the operator  $P_\sigma^\nu$  to obtain the other set of  $d$  equations. The result is

$$a_\sigma + P_\sigma^\nu \partial_\nu \xi = 0. \quad (56)$$

Together this set of membrane equations is identical to the equations of relativistic ideal CFT hydrodynamics

$$\partial_\nu T^{\mu\nu} = 0 \quad (57)$$

with traceless perfect fluid stress tensor given in (4). The focusing equation is equivalent to the contraction along  $\mathbf{u}$ ,  $u_\mu \partial_\nu T^{\mu\nu} = 0$ , while the second equation is equivalent to the projection orthogonal,  $P_{\alpha\mu} \partial_\nu T^{\alpha\nu} = 0$ .

An important characteristic of ideal hydrodynamics is that the fluid entropy current is a conserved quantity  $\partial_\mu J_s^\mu = 0$ . In our case the horizon expansion is the fractional rate of change in the horizon cross-sectional area  $\theta = \mathcal{L}_\mathbf{u} \ln \sqrt{\gamma}$ , so our results imply that at  $O(\varepsilon)$  the area is unchanged. Indeed, using the equivalent definition

$$\theta = \frac{1}{\sqrt{\gamma}} \partial_\mu (\sqrt{\gamma} u^\mu) = 0, \quad (58)$$

and the Bekenstein-Hawking entropy formula  $S = A/4$ , we find

$$J_s^\mu = \frac{1}{4} \sqrt{\gamma} u^\mu. \quad (59)$$

As expected, and in agreement with the entropy current derived in [7], the fluid entropy density is proportional to the horizon area density  $(\pi T)^d$ .

### A. Viscous hydrodynamics

We now consider  $O(\varepsilon)$  terms in the generalized extrinsic curvature and the structure of the membrane equations to  $O(\varepsilon^2)$ . An immediate difficulty is that a priori the location of

the horizon  $r_H$  is modified at  $O(\varepsilon)$ . To compute this correction it seems one would have to know the full  $O(\varepsilon)$  metric (the solution to all the Einstein equations up to  $O(\varepsilon^2)$ [32]). However, using horizon geometric variables, we can make the following simple argument that the location is unchanged at lowest order. Since the only scalar horizon variable at  $O(\varepsilon)$  is the expansion  $\theta$ , imposing the ideal equation  $\theta = 0$  means that  $r_H = \pi T(x^\mu) + O(\varepsilon^2)$ .

Next, using the zeroth order metric (43), and (50) we obtain

$$\Theta_\nu{}^\mu = -2\pi T u_\nu u^\mu - \frac{1}{2} a_\nu u^\mu + \sigma_\nu^\mu + \frac{1}{d} P_\nu^\mu \theta. \quad (60)$$

There are several important things to discuss pertaining to this result. First, the ideal equations can also be imposed in terms of  $O(\varepsilon^2)$  in the membrane equations, so the last term involving  $\theta$  above effectively will not contribute at this order. Second, the question again arises whether the  $O(\varepsilon)$  corrected metric can affect the remaining three terms in (60). If they do, can we even proceed without knowing the details of these corrections? It turns out  $O(\varepsilon)$  corrections in the metric can introduce  $O(\varepsilon^2)$  corrections to the shear tensor, but these can only appear at higher order,  $O(\varepsilon^3)$ , in the membrane equations. On the other hand, the corrections to the near-horizon metric will contribute and modify the first two terms of (60) at the order we are considering. In particular, from (24) we see that at  $O(\varepsilon)$  the non-zero contributions to the horizon stress tensor are

$$T_{(H)\nu}{}^\mu = -\kappa^{(1)} u_\nu u^\mu + 2\pi T (m_\nu^{(1)} u^\mu - u_\nu \ell_{(1)}^\mu) - \frac{1}{2} a_\nu u^\mu + \sigma_\nu^\mu + \frac{1}{d} P_\nu^\mu \theta - \delta_\nu^\mu \kappa^{(1)}. \quad (61)$$

Here  $\kappa^{(1)}$ ,  $m_\nu^{(1)}$ , and  $\ell_{(1)}^\mu$  are the first order parts of the surface gravity, covector, and null normal respectively.

These variables are associated with various ambiguities in the geometrical description that need to be fixed. The first ambiguity arises from the fact that the horizon null normal vector is not unique; any vector obtained by an overall scaling of the original one is still a null normal vector:

$$\ell^\mu \rightarrow f(x) \ell^\mu. \quad (62)$$

This freedom means that unlike the non-null cases, the generalized horizon extrinsic curva-

ture is not unambiguously defined (and can be non-symmetric). Under this scaling [21]

$$m_\mu \rightarrow f^{-1}m_\mu \quad (63)$$

$$\kappa' \rightarrow f(\kappa + \mathcal{D} \ln f) \quad (64)$$

$$\sigma' \rightarrow f\sigma \quad (65)$$

$$\theta' \rightarrow f\theta. \quad (66)$$

At zeroth order in  $\varepsilon$ ,  $\ell^\mu = u^\mu$  (i.e.  $f = 1$ ) is the natural scaling fixed by the equilibrium solution, but at  $O(\varepsilon)$  there is again an ambiguity. Since there is a zeroth order  $\kappa$  (the temperature), (64) implies that the scaling is equivalent to the freedom to define  $\kappa^{(1)}$ . We choose a  $f = 1 + O(\varepsilon)$  such that the  $O(\varepsilon)$  correction to  $\kappa$  is zero; that is

$$\kappa = 2\pi T + O(\varepsilon^2). \quad (67)$$

Thus, fixing this ambiguity determines  $\kappa$  at first order and relates the surface gravity to the temperature.

The second ambiguity corresponds to the freedom in the form of the bulk metric evaluated at the horizon, i.e. the choice of  $(d + 1)$  of the  $(d + 2)$  functions  $Y_B$  in

$$g_{AB} \rightarrow g_{AB} + \partial_{(A} Y_{B)}. \quad (68)$$

One can also see this from the fact that since  $m^r = 1$  exactly (9), the covector is  $m_\nu = g_{\nu r}$ , evaluated at the horizon. We make the gauge choice that

$$P_\sigma^\nu m_\nu^{(1)} = \frac{1}{2}(2\pi T)^{-1} a_\sigma. \quad (69)$$

This sets the  $O(\varepsilon)$  correction to  $m_\nu$  such that the  $a_\nu$  term in (60) is eliminated.

The third ambiguity is analogous to the ambiguity in the definition of relativistic viscous hydrodynamics. This ambiguity corresponds to the choice in the definition of the fields  $T(x)$  and  $u^\mu(x)$ . One choice of a hydrodynamic frame is the so called Landau frame, defined by the  $d + 1$  conditions imposed on the first order viscous correction to the symmetric fluid stress tensor  $T_{(1)}^{\mu\nu}$

$$u_\mu T_{(1)}^{\mu\nu} = 0. \quad (70)$$

These conditions are equivalent to the statement that  $u^\mu$  is an eigenvector of the full stress tensor. Physically the Landau choice is that  $T(x)$  and  $u^\mu(x)$  are such that in the local rest



frame at each point the fluid momentum is zero and energy density can be expressed in terms of equilibrium quantities, without dissipative corrections.

We will fix this ambiguity in the gravitational description by the requirement that

$$u^\nu T_{(H)\nu}^{(1)\mu} = 0, \quad (71)$$

as the analog of the Landau frame choice at the horizon. Imposing this condition, we find the requirement that the first order null normal obey

$$P_\mu^\sigma \ell_{(1)}^\mu = 0. \quad (72)$$

Note that the frame choice (71) implies the constraint  $m_\mu \ell^\mu = 1$  also at first order, i.e.

$$\ell_{(1)}^\mu u_\mu = m_\mu^{(1)} u^\mu, \quad (73)$$

where we use the zeroth order relations  $m_\mu^{(0)} = -u_\mu$  and  $\ell_{(0)}^\mu = u^\mu$ .

With these gauge and frame conditions (67), (69), and (71) we can proceed to calculate the membrane equations using (52). Imposing the lower order ideal hydrodynamics equations where they are applicable, we find

$$\begin{aligned} -2\pi T \mathcal{D}\xi u_\nu - 2\pi T \partial_\mu u^\mu u_\nu - 2\pi T a_\nu + \partial_\mu \sigma_\nu^\mu - 2\pi T \partial_\nu \xi - da_\lambda \sigma_\nu^\lambda - 2\pi T d\mathcal{D}\xi u_\nu \\ + \Gamma^{(2)\beta}_{\beta\lambda} \Theta^{(0)}_{\nu}{}^\lambda - \Gamma^{(2)\lambda}_{\mu\nu} \Theta^{(0)}_{\lambda}{}^\mu = 0. \end{aligned} \quad (74)$$

The last two terms represent corrections to the connection from the first order metric  $\gamma_{\mu\nu}^{(1)}$ , which is composed of first derivatives of  $u_\mu$  and  $T(x)$ . Since the zeroth order part of the extrinsic curvature is  $2\pi T u^\mu u^\nu$ , these two terms reduce to

$$2\pi T \left( u_\nu \mathcal{D} \ln \sqrt{\gamma^{(1)}} - (1/2) u^\lambda u^\mu \partial_\nu \gamma_{\mu\lambda}^{(1)} \right). \quad (75)$$

The horizon metric must satisfy  $\gamma_{\mu\nu}^{(1)} u^\mu = 0$ . Therefore, in terms of the null hypersurface geometric variables it has the general form (up to an overall factor)

$$\gamma_{\mu\nu}^{(1)} \sim \sigma_{\mu\nu}^{(H)} + \theta \gamma_{\mu\nu}^{(0)}. \quad (76)$$

In (75) contributions from traceless shear will be zero identically and in addition we can impose the ideal hydrodynamics equation  $\theta = 0$ . Therefore we find both terms vanish.

Contracting the remaining terms in (74) with  $u^\nu$  yields the scalar equation

$$\partial_\mu u^\mu + d\mathcal{D}\xi = \frac{1}{2\pi T} \sigma_{\mu\nu} \sigma^{\mu\nu}. \quad (77)$$

One can readily show this equation is just the null focusing equation up to  $O(\varepsilon^2)$ . The first order focusing equation implied  $\theta = 0 + O(\varepsilon^2)$ . Imposing this result means that the  $u^\mu \partial_\mu \theta$  and  $\theta^2$  terms in (26) are  $O(\varepsilon^3)$  and  $O(\varepsilon^4)$  respectively. Therefore at  $O(\varepsilon^2)$ ,

$$\kappa \theta = \sigma_{\mu\nu} \sigma^{\mu\nu}. \quad (78)$$

Using (58) this equation can also be expressed as an entropy balance law

$$T \partial_\mu (J_s^\mu) = \frac{\sqrt{\gamma}}{8\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} = 2\eta \sigma_{\mu\nu} \sigma^{\mu\nu} \quad (79)$$

where  $\eta = \pi^{d-1} T^d / 16$  is a shear viscosity. This agrees at  $O(\varepsilon^2)$  with Eqn. B. 27 in [7] and the shear viscosity is the Kovtun-Son-Starinets [11] universal viscosity to entropy density ratio  $\eta/s = 1/4\pi$  since  $s = (\pi T)^d / 4$ . Note also that since the right hand side of the above equation is positive definite, the fluid entropy always increases. Therefore the second law of thermodynamics for the fluid is mapped into the area increase theorem of General Relativity.

Projecting with  $P^\nu_\sigma$  we find the remaining  $d$  equations are

$$a_\sigma + P^\nu_\sigma \partial_\nu \xi = \frac{1}{2\pi T} P^\nu_\sigma (\partial_\alpha \sigma_\nu^\alpha - d \sigma_\nu^\alpha a_\alpha). \quad (80)$$

The set of equations (77) and (80) are equivalent to the projections along and orthogonal to  $\mathbf{u}$  of the equations of 1st order viscous CFT hydrodynamics  $\partial_\nu T^{\mu\nu} = 0$ , with traceless fluid stress tensor given by (5). These results show that a relativistic CFT fluid flow is encoded in the dynamics of a black brane horizon in AdS. Specifically, as long as a large scale hydrodynamic limit exists (so we can expand in derivatives)

$$\bar{D}_\nu T_{(H)\mu}^\nu = \partial_\nu T_\mu^\nu = 0, \quad (81)$$

at least up to  $O(\varepsilon^2)$ .

It is interesting to compare our results with the holographic *fluid-gravity correspondence* in [3]. In this case the black brane solution near the AdS boundary (or equivalently the  $(d+1)$ -dimensional boundary stress tensor) is expanded in Knudsen number. The Einstein equations at  $O(l_{cor}/L)$  projected into this timelike surface are the equations of ideal hydrodynamics and act as constraints on boundary data. In order to obtain the dual  $(d+2)$ -dimensional solution at  $O(l_{cor}/L)$  one can integrate the remaining “dynamical” Einstein equations into the bulk radial direction, subject to the condition of a regular event horizon. The horizon regularity condition fixes the boundary stress tensor and constraint equations

at next order to be those of viscous CFT hydrodynamics, with the particular shear viscosity to entropy density ratio  $1/4\pi$ . The procedure can be continued to higher orders in  $l_{cor}/L$  in the same way and it has already been used to derive the second order transport coefficients. Note, that a boundary stress tensor with a shear viscosity to entropy density ratio which is different than  $1/4\pi$  corresponds to a bulk background with a naked singularity (see for instance [24]).

In contrast, our local analysis at the horizon is only applicable at the lowest orders in the expansion in  $l_{cor}/L$ . At this level, we show that the details of the expanded bulk solution are not required in order to obtain the ideal and viscous hydrodynamics equations. The membrane equations and the implicit condition that the horizon is regular imply a boundary fluid stress tensor with the particular shear viscosity to entropy density ratio  $\eta/s = 1/4\pi$ . This is consistent with having a regular bulk.

The horizon dynamics is able to capture the leading long wavelength viscous dynamics of the finite temperature CFT on the boundary, but clearly cannot capture the all wavelength dynamics. In order to go to higher orders in  $l_{cor}/L$  using our method would likely require a knowledge of the bulk, possibly via integration out from the horizon into the radial direction.

## V. THE RINDLER HORIZON IN MINKOWSKI SPACE-TIME

To show that our analysis is applicable to cases other than the black branes in AdS, we consider the example of Rindler space associated with accelerated observers in  $d + 2$  Minkowski space-time. The metric is typically written in the form

$$ds^2 = -\kappa^2 \xi^2 d\tau^2 + d\xi^2 + \sum_{i=1}^d dx'^i dx'_i, \quad (82)$$

where  $\kappa$  is a constant surface gravity. This metric can be obtained from the standard Minkowski metric via the coordinate transformation  $x^0 = \xi \sinh(\kappa\tau)$  and  $x^{d+1} = \xi \cosh(\kappa\tau)$ . Therefore these Rindler coordinates cover only a “wedge” of the full Minkowski space-time. To a uniformly accelerated observer with worldline  $\xi = const.$ , the surface  $\xi = 0$  is a causal boundary that prevents the observer from an access to the entire space-time.

To employ the membrane paradigm conveniently, we make a coordinate change

$$\xi^2 = r/\kappa \quad (83)$$

$$x'^i = \kappa x^i, \quad (84)$$

where  $r$  is an affine parameter along ingoing null geodesics. In these coordinates the metric (82) has the Schwarzschild-like form

$$ds^2 = -\kappa r d\tau^2 + (4\kappa r)^{-1} dr^2 + \kappa^2 \sum_{i=1}^d dx^i dx_i. \quad (85)$$

When boosted uniformly in the  $x^\mu \equiv (\tau, x^i)$  directions the metric becomes

$$ds^2 = -\kappa r u_\mu u_\nu dx^\mu dx^\nu + (4\kappa r)^{-1} dr^2 + \kappa^2 P_{\mu\nu} dx^\mu dx^\nu, \quad (86)$$

with  $(d+1)$ -velocity  $u^\mu$  defined as in the previous section. Finally, in Eddington-Finkelstein-like coordinates  $x^\mu \equiv (t, x^i)$

$$ds^2 = -\kappa r u_\mu u_\nu dx^\mu dx^\nu - u_\mu dx^\mu dr + \kappa^2 P_{\mu\nu} dx^\mu dx^\nu. \quad (87)$$

The flat Rindler metric is a solution to the vacuum Einstein equation with zero cosmological constant

$$R_{AB} = 0. \quad (88)$$

To perturb this horizon we allow for, as in the black brane example, slowly varying (characteristic scale  $L \gg \kappa^{-1}$ ) fluid velocity  $u^\mu(x)$  and  $\kappa(x)$ .  $\kappa$  can be naturally identified with a temperature in the following way. Unruh [25] showed that accelerated observers feel the quantum vacuum to be a thermal state at temperature  $T = a/2\pi$ , where  $a$  the observer's proper acceleration. This is essentially a local temperature and can be expressed in a Tolman form  $T = \kappa/2\pi\chi$ , where  $\chi = \sqrt{-g_{\tau\tau}} = \sqrt{\kappa r}$  is the redshift factor. Given this form, we define  $\kappa = 2\pi T(x)$  as the location independent temperature of the system. The non-uniform Rindler metric is no longer a solution to the Einstein equation, but as before, one can solve order by order for the corrections in an expansion in derivatives of  $x^\mu$  (small Knudsen number).

Using (87) (or (86)), the zeroth order induced metric on the horizon ( $r = 0$ ) is

$$\gamma_{\mu\nu} = 4\pi^2 T^2 P_{\mu\nu}, \quad (89)$$

and taking the Lie derivative along  $\mathbf{u}$  one finds the horizon shear and expansion have the same forms (47) and (46) as in the black brane case. Imposing the membrane dynamical equations

$$\bar{D}_\nu T_{(H)\mu}^\nu = 0 \quad (90)$$

and following the analysis of the previous section, it is straightforward to find that these equations are again equivalent to the equations of relativistic CFT hydrodynamics

$$\partial_\nu T^{\mu\nu} = 0 \tag{91}$$

at least up to  $O(\varepsilon^2)$ . The shear viscosity  $\eta$  is the same in the black brane case. Therefore assuming the Rindler horizon has a Bekenstein-Hawking area entropy density, the shear viscosity to entropy density ratio in the hydrodynamics equations is again the Kovtun-Son-Starinets ratio  $\eta/s = 1/4\pi$ .

In this case though, the result cannot be understood as mirroring the hydrodynamics of a field theory fluid living on an asymptotic boundary of space-time. For example, the fluid stress tensor cannot be identified with a boundary CFT stress tensor. Essentially we have found that this membrane's dynamics can be re-expressed entirely in fluid mechanical language even without a notion of holography analogous to AdS/CFT. The fluid system here could be interpreted in terms of the near-horizon degrees of freedom as the vacuum thermal state (thermal atmosphere of Rindler particles analogous to Hawking radiation) perceived by accelerated observers [26].

## VI. THE NON-RELATIVISTIC LIMIT AND INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

The hydrodynamics of relativistic conformal field theories is intrinsically relativistic as is the microscopic dynamics. In particular, the non-relativistic limit of a relativistic conformal hydrodynamics may not be well defined. Nevertheless, the limit of non-relativistic macroscopic motions of a CFT hydrodynamics is definable and leads to the non-relativistic incompressible NS equations [5, 6].

On the gravity side we implement this limit by considering the slow motion regime where the  $d$ -velocity  $v^i$  is a small perturbation. Temporarily restoring  $c$ , the horizon coordinates (now split into space and time) become  $(ct, x^i)$  and the non-relativistic slow motion limit corresponds to  $v^i/c \ll 1$ . In order to keep track of the different terms we impose the scaling  $\partial_t \sim c^{-2}$ ,  $v^i \sim \partial_i \sim c^{-1}$  and we consider  $c \rightarrow \infty$ . The temperature behaves as  $T(x) = T_0(1 + c^{-2}P(x))$ , where  $T_0$  is a constant and  $P(x)$  is the fluid pressure. Note this is not the only conceivable slow motion, long distance scaling limit. For example, one could

imagine  $v^i$  and  $\partial_i$  scaling differently from each other. However, it turns out our particular scaling is a natural choice because it is a symmetry of the incompressible NS equations [6].

Consider the scaling limit of the relativistic hydrodynamics/membrane dynamics equations derived in the previous sections, (77) and (80). At lowest order,  $O(c^{-2})$ , the first equation (the focusing equation) reduces simply to the fluid incompressibility condition

$$\partial_i v^i = 0. \quad (92)$$

The second equation is of  $O(c^{-3})$  at lowest order. Under the scaling the acceleration  $a_\sigma \rightarrow \partial_t v_i + v^j \partial_j v_i$ , while the  $\xi$  derivative reduces to just a derivative of the fluid pressure  $P$ . On the right hand side, the derivative of the shear tensor contributes  $(1/2)\nabla^2 v_i$  after imposing the incompressibility condition. The second shear times acceleration term does not contribute because it is of higher order. Therefore we arrive at the non-relativistic NS equation

$$\partial_t v_i + v^j \partial_j v_i + \partial_i P = \nu \nabla^2 v_i, \quad (93)$$

with kinematic viscosity  $\nu = (4\pi T_0)^{-1}$ .

In [16] we derived the same incompressible NS equations directly from the membrane focusing equation (26) and Damour-Navier-Stokes equation (27) discussed above in Section 3 without needing to know beforehand the fully relativistic hydrodynamics equations. Our two derivations are completely equivalent. Damour's special horizon adapted coordinate system and choice of horizon basis ( $\ell$ ,  $e_i = \partial_i$ ) gives the generic membrane equations their classic non-relativistic appearance, which earlier allowed us to easily show they are the incompressible NS equations. The only difference is that in [16] we required a surface gravity

$$\kappa = 2\pi T_0(1 + P - (1/2)v^2) \quad (94)$$

while here we must have  $\kappa = 2\pi T_0(1 + P)$  in the scaling limit. The discrepancy can be traced to Damour's parameterization of fluid velocity  $\ell = \partial_t + v^i \partial_i$ , while we work with  $\ell = u^\mu = (\gamma, \gamma v^i)$ . The null normals differ by an overall  $\gamma = 1 + (1/2)v^2 + \dots$  factor. As we discussed above in Section 4, an overall scaling in the null normal vectors just amounts to a difference in gauge, which at lowest order just affects how the surface gravity is to be defined.

## VII. DISCUSSION

In the paper we used the membrane paradigm and applied a Knudsen number expansion to analyze the  $(d+1)$ -dimensional horizon dynamics of a uniformly boosted black brane in a  $(d+2)$ -dimensional asymptotically Anti-de-Sitter space-time and to a Rindler acceleration horizon in  $(d+2)$ -dimensional Minkowski space-time. We showed that the horizon dynamics is governed by the relativistic CFT hydrodynamics equations. The fluid velocity and temperature correspond to the normal to the horizon and to the surface gravity, respectively. The second law of thermodynamics for the fluid is equivalent to the area increase theorem of General Relativity.

In the Rindler case, unlike the AdS one, there is no holographic screen at the asymptotic boundary of space-time. The result that the horizon dynamics is governed by the  $(d+1)$ -dimensional CFT hydrodynamics equations may imply that the near-horizon degrees of freedom behave as a conformal fluid.

The derivation of the CFT hydrodynamics equations required a knowledge of the horizon embedding and employed a local analysis near this horizon. The results apply to a general non-singular causal horizon, as long as there is a separation between the characteristic scale  $L$  of the macroscopic perturbations and some intrinsic microscopic  $l_c$  scale given by the inverse of the temperature. The non-singularity requirement was used when contracting the Einstein equations in order to obtain the membrane equations. The separation of scales was required in order to have a small Knudsen number and a valid derivative expansion. The separation of scales does not exist in general. This is the reason why, for example, in the general Damour-Navier-Stokes equation (27), the term  $\partial_i\theta$  does not vanish and leads to the assignment of an unphysical negative “bulk viscosity”.

Taking the non-relativistic limit of our analysis, we obtained the non-relativistic Navier-Stokes equations as found in the membrane paradigm approach in [16]. In this way we also clarified the relation between the surface gravity and the non-relativistic fluid pressure.

There are various issues to consider next in the membrane paradigm approach to relativistic CFT hydrodynamics. Most important is whether the geometrical formulation of the relativistic CFT hydrodynamics as a hypersurface dynamics can provide a new insight to the nonlinear fluid dynamics. This is of course relevant also to the non-relativistic limit, where we obtained the dynamics of non-relativistic incompressible fluid.

Another important question is the construction of the higher order derivative terms in the hydrodynamics equations. It is possible that these terms are still encoded in the membrane equations at higher orders in  $\varepsilon$ . However, it is not yet clear if the analysis can continue to be done locally near the horizon hypersurface. It is likely a more detailed knowledge of the bulk space-time will be required [27].

One can also consider various generalizations of gravity/fluid correspondence in the membrane paradigm formalism. As a simple example, since black holes in the presence of an electromagnetic field act like a charged membrane [8, 10], it should be possible to derive the additional hydrodynamic current conservation equation

$$\partial_\mu J^\mu = 0 \tag{95}$$

for gauge fields in black brane backgrounds [28].

Another interesting case to consider is non-conformal hydrodynamics. The conformal symmetry is broken by non-trivial background matter fields in the space-time, for example, a scalar field [29]. Therefore, one would need to consider the scalar field equation near the horizon in addition to the membrane Einstein equations with non-zero matter stresses. The analysis of the membrane equations should be modified at  $O(\varepsilon^2)$ , producing a positive bulk viscosity dependent on the bulk matter field content.

Finally, the membrane paradigm approach may offer an alternative route to the hydrodynamics equations in generalized gravity theories, where the Einstein field equations are corrected by higher curvature terms [30] and the shear viscosity to entropy density ratio of the fluid is no longer simply  $1/4\pi$  [31].

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