Introduction to Numerical Relativity

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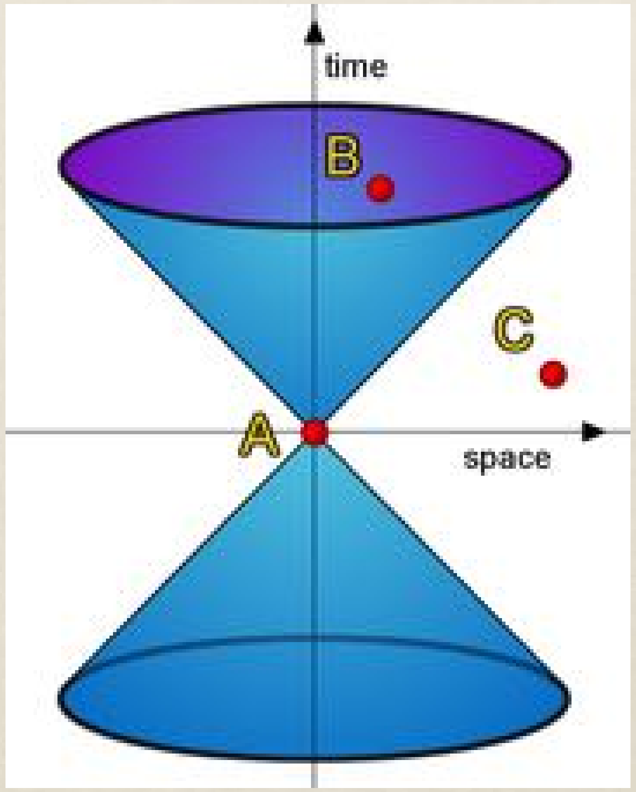
Lecture 1

Not just an academic exercise

Special Relativity

- maximun speed for signal propagation
- It is not possible to send or receive signal at a speed greater than c
- EXPERIMENTAL FACT: THE SPEED OF LIGHT IS INDEPENDENT OF THE OBSERVER
- A speed limit is not compatible with GALILEAN RELATIVITY.
- space and time cannot be seen as independent concept: we must think in term of space-time
- SPECIAL RELATIVITY



Light Cone

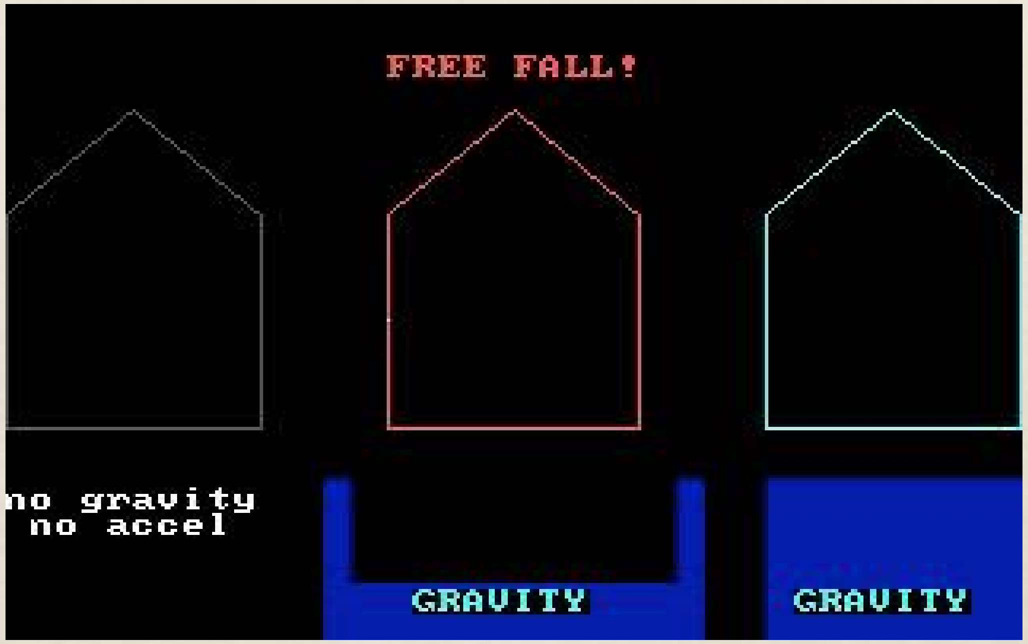
Equivalence Principle

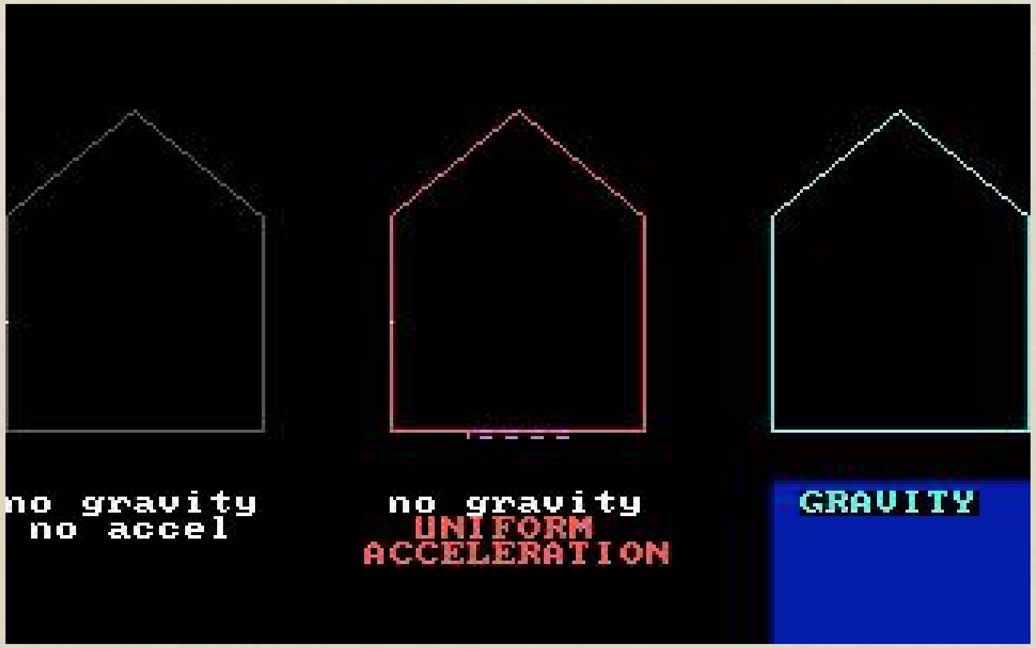
- It is not possible to distinguish INERTIAL MASS from GRAVITATIONAL MASS

- Inertial Mass
  = Gravitational Mass

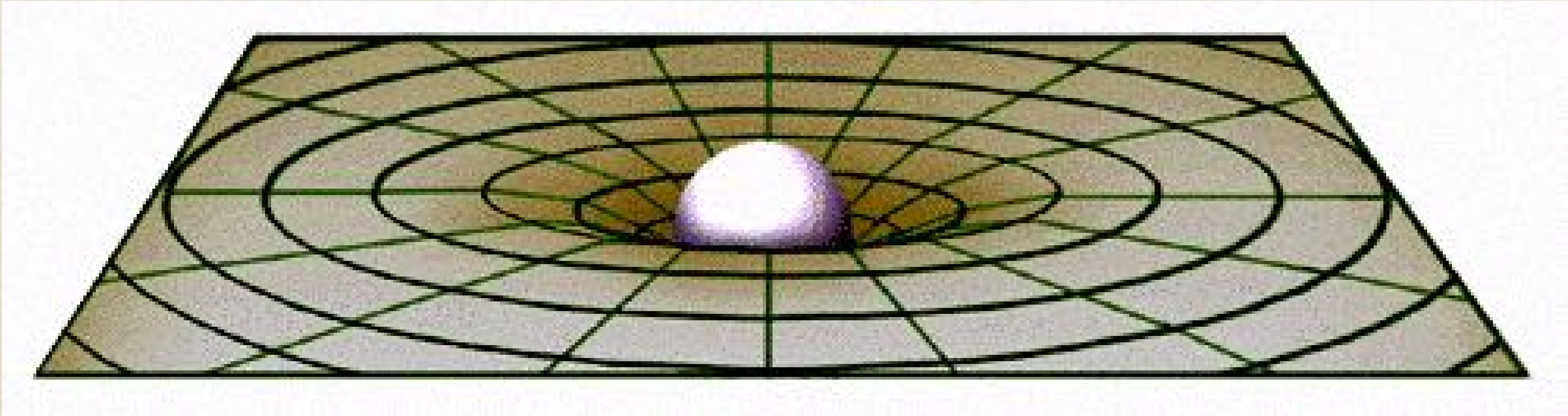
$$\cancel{m_\alpha} \frac{d^2 x^i_\alpha(t)}{dt^2} = \sum_{\alpha \neq i} \cancel{m_\alpha} \frac{G\, m_\beta \left(x^i_\alpha(t) - x^i_\beta(t)\right)}{\left|x^j_\alpha(t) - x^j_\beta(t)\right|^3}$$

- Gravity must be described in geometrical terms

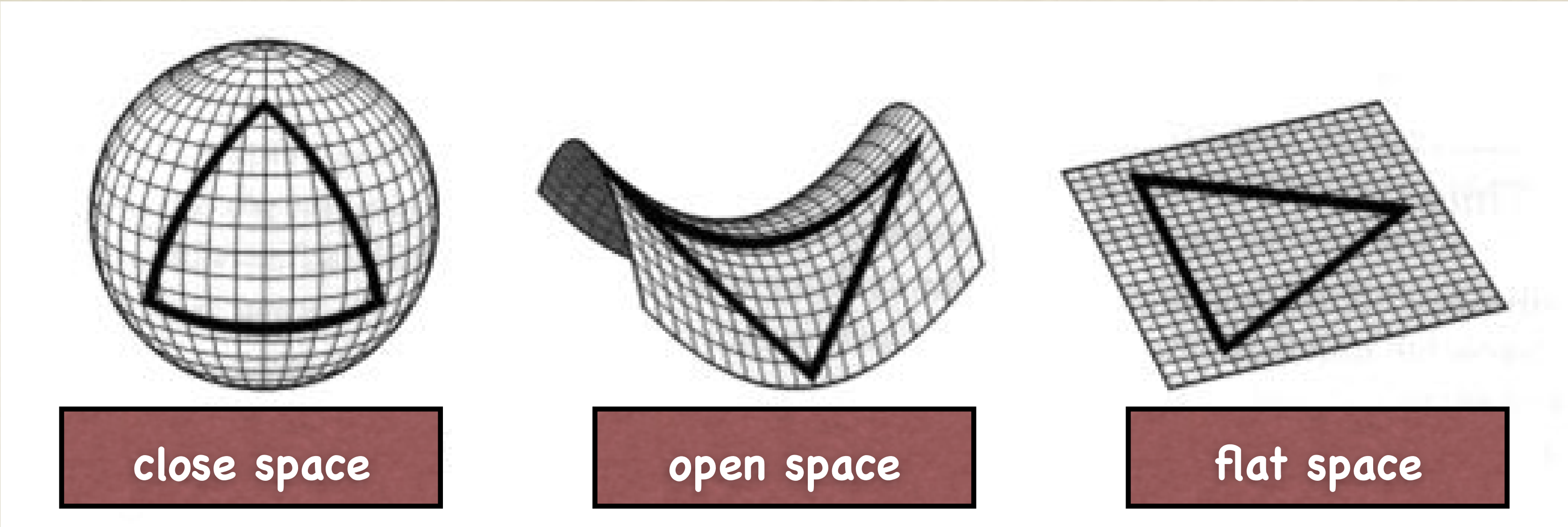




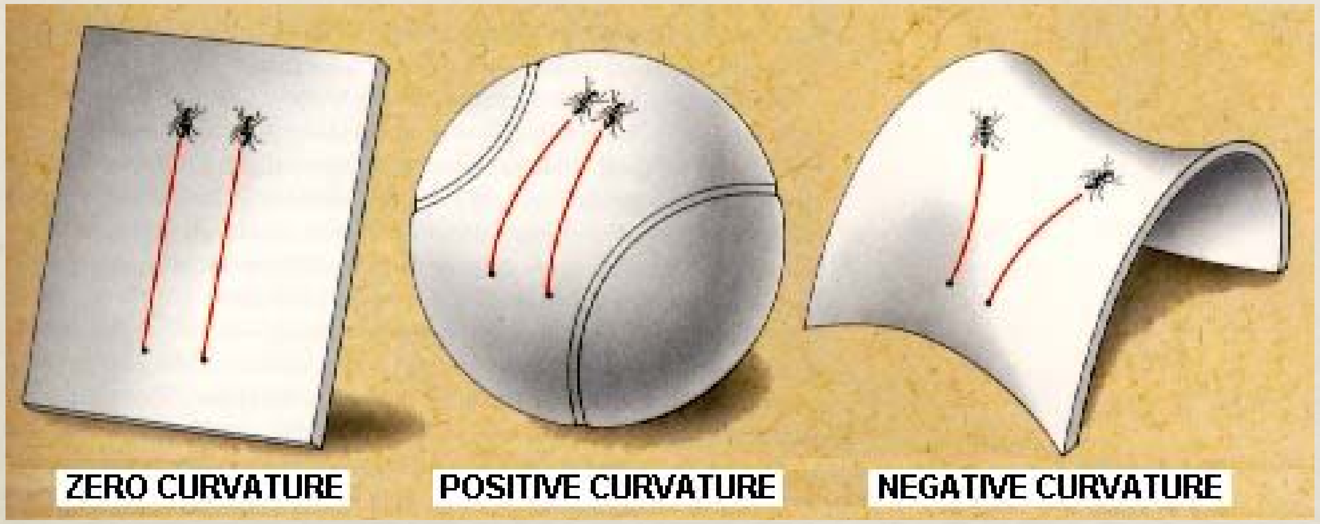
Dynamical space-time



- Space time curvature will dynamically fix length and time



The world as seen by an ant

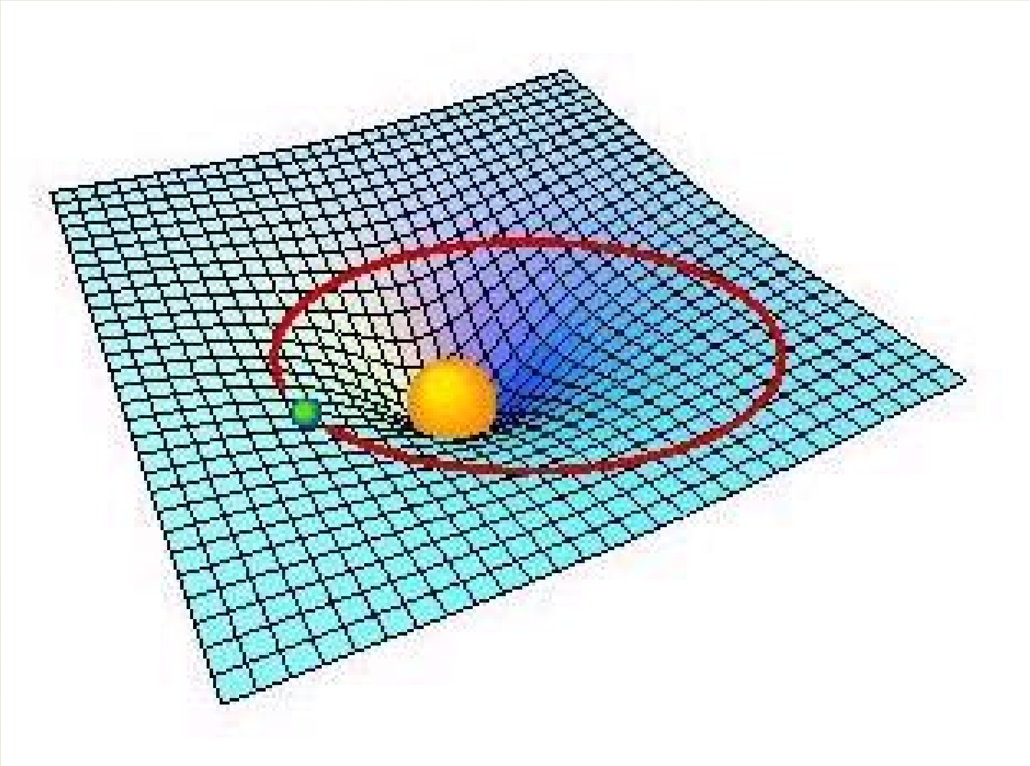


BI-DIMENSIONAL VIEW

BUT (special relativity) WE MUST THINK IN SPACE-TIME TERMS ! The curvature is not the curvature of SPACE but the 4-dimensional curvature of SPACE-TIME

Space and matter

- One can visualize Einstein's general relativity as a sheet that in the absence of matter is flat but that in the presence of matter is not.
- The trajectory of particles will be the geodesics of this curved space



MATTER =

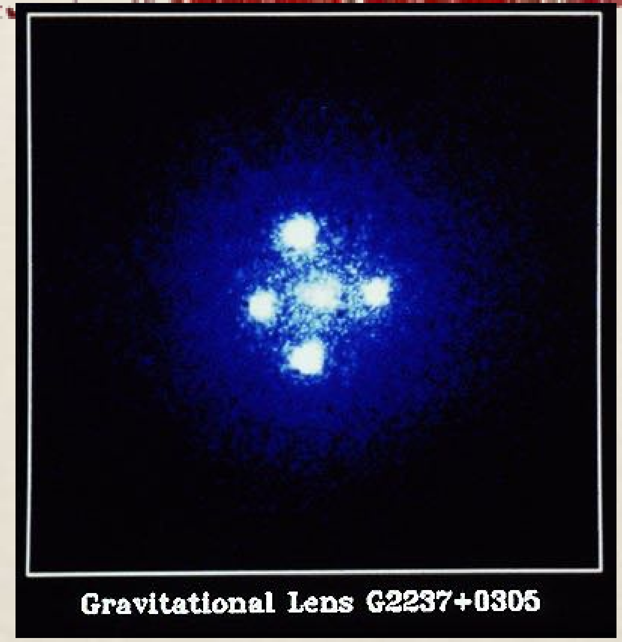
source of the curvature

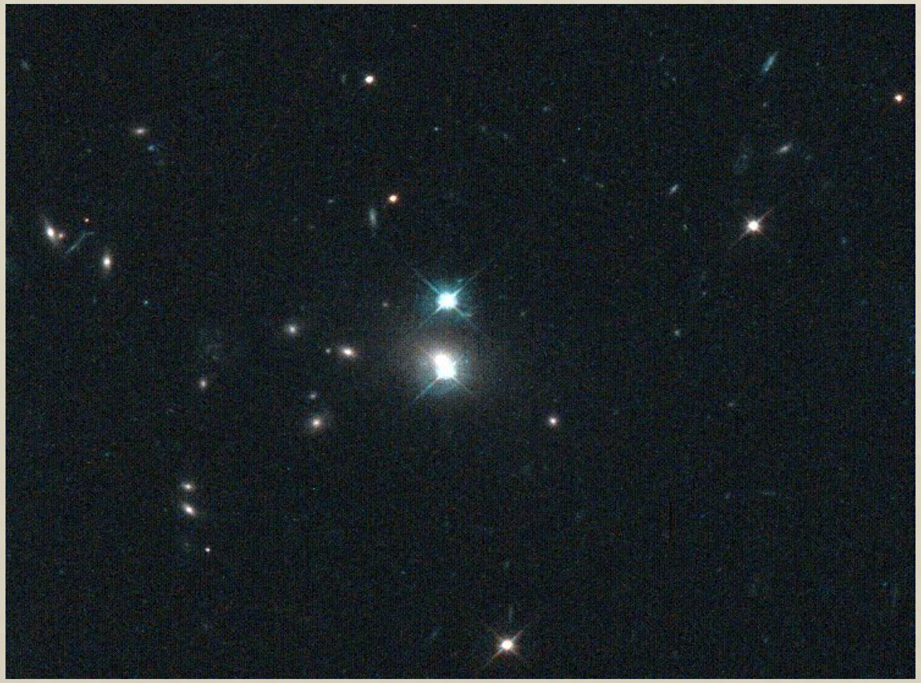
of space-time

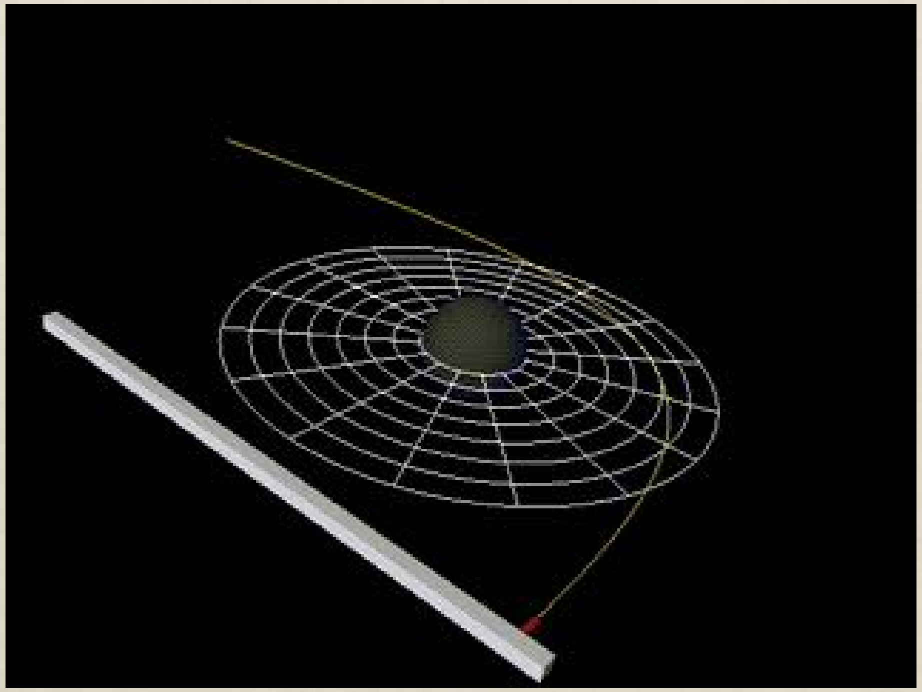
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \, T_{\mu\nu}$$

Light deflection (lensing)

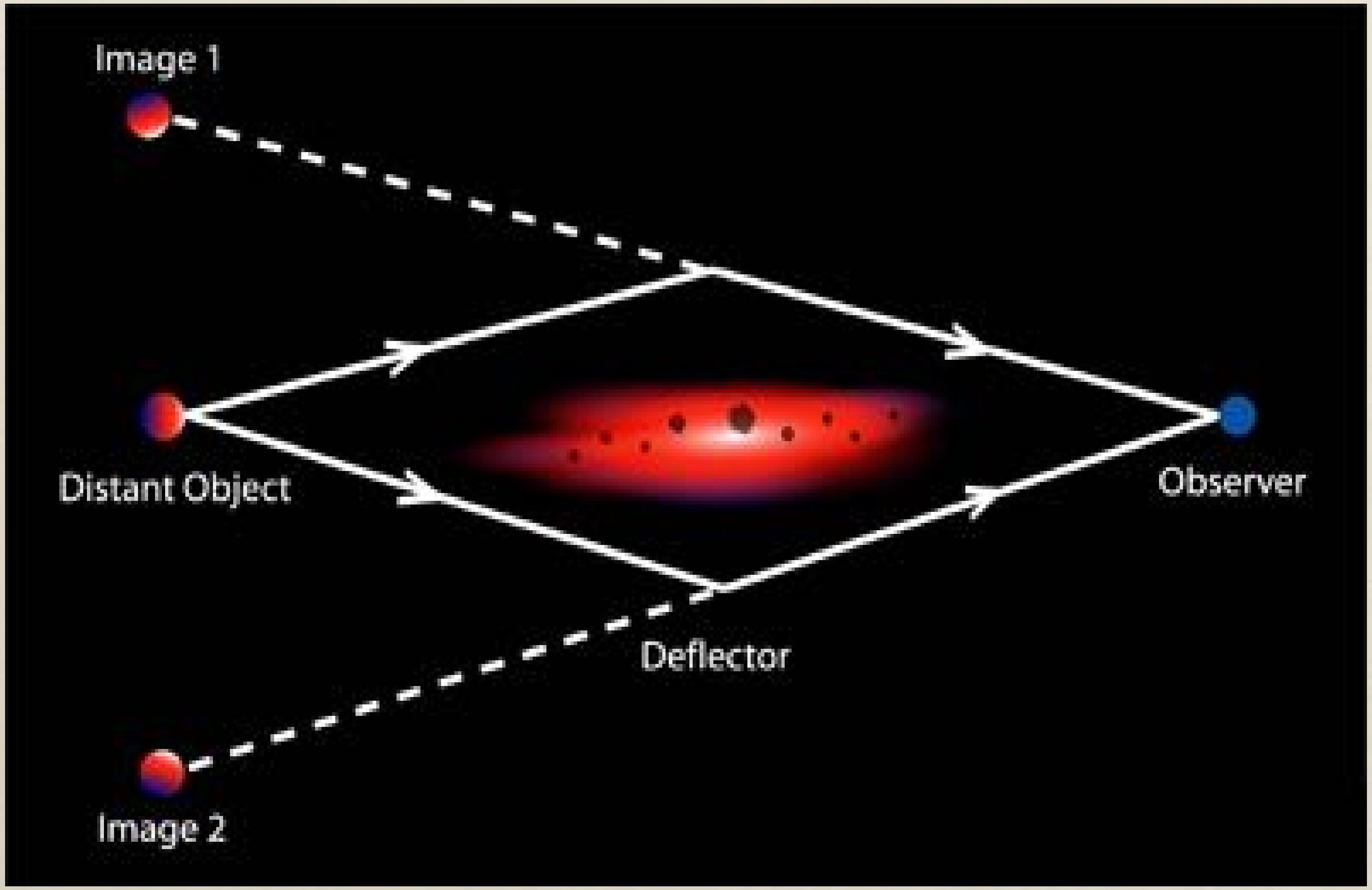
- Gravity will modify the space-time texture
- Light will be deflected and/or focalized by matter-distribution

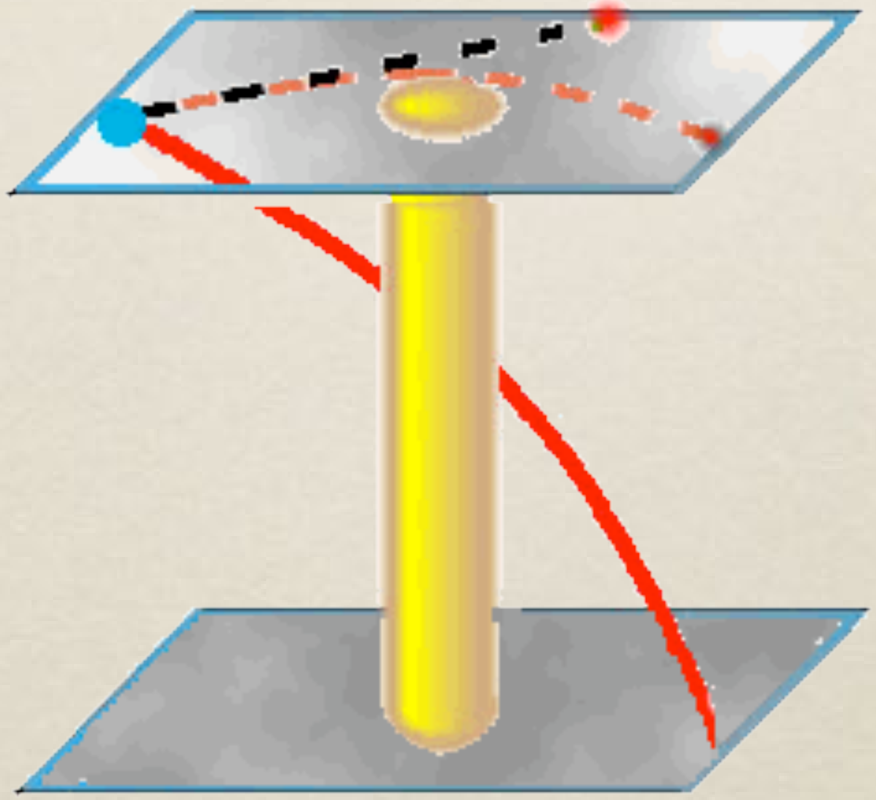




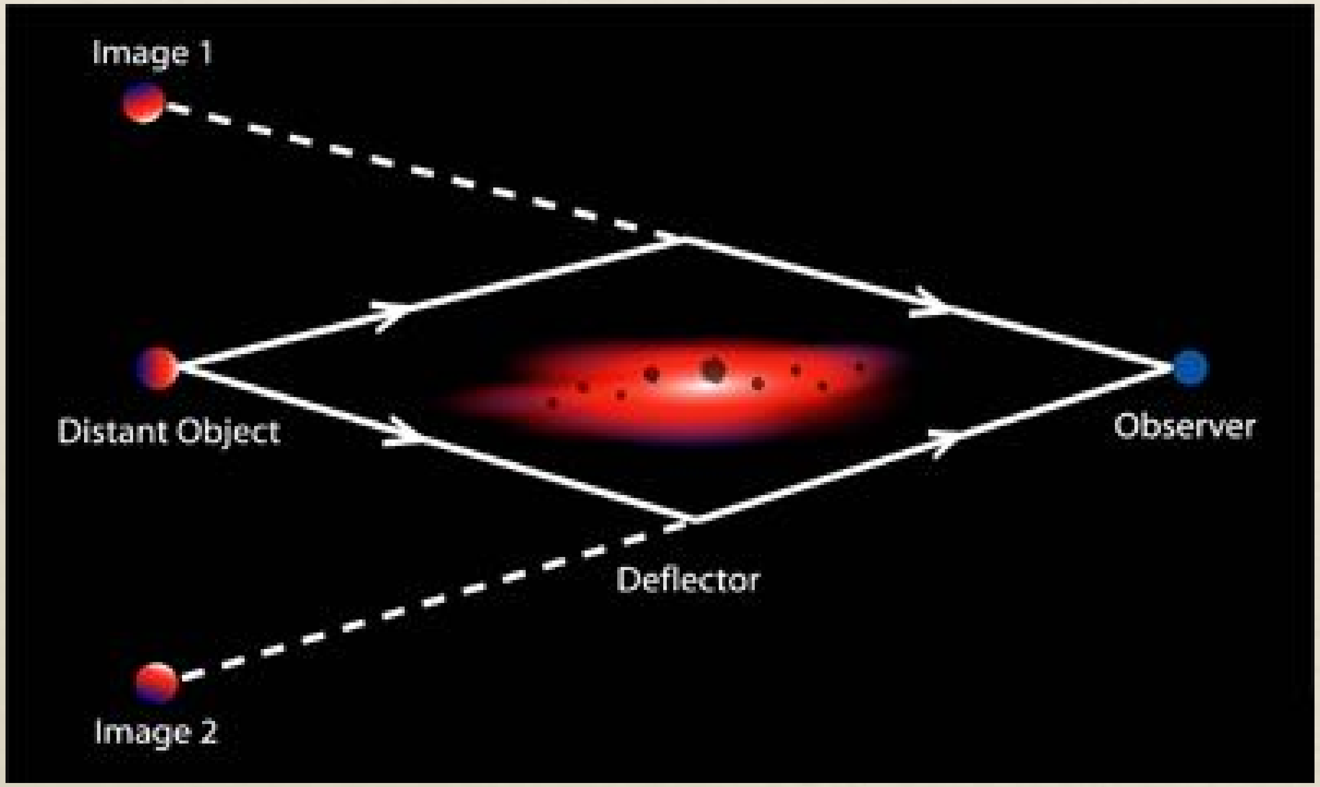


Space-time view of light deflection



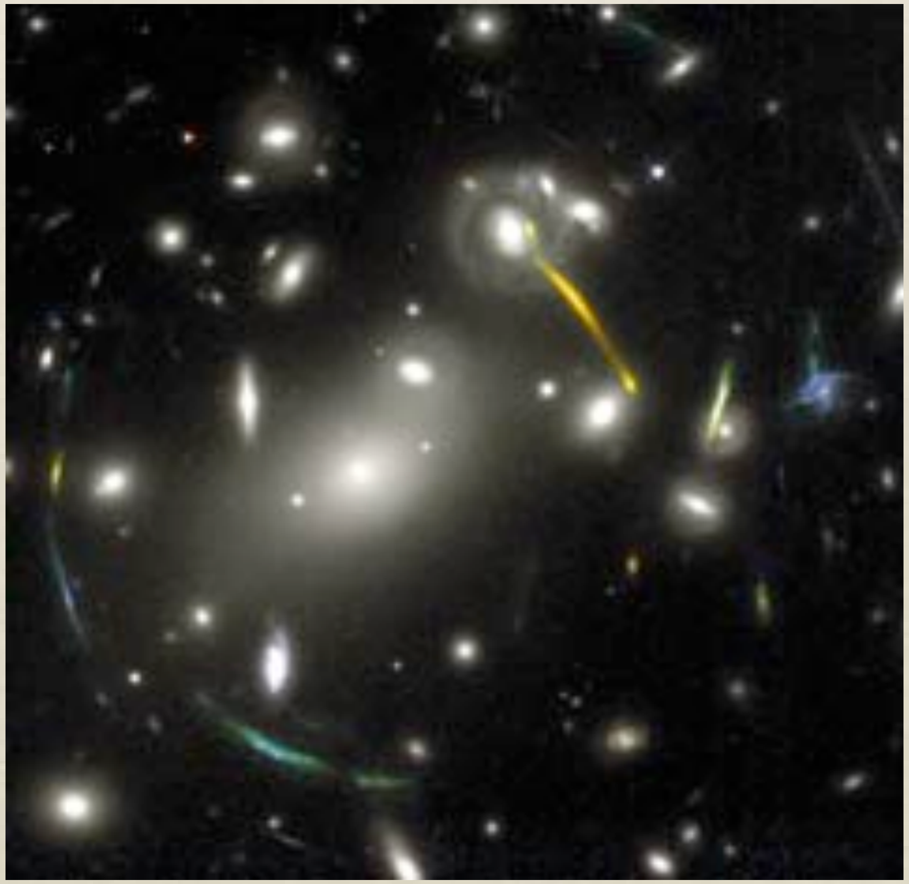


Curved space time effects: Gravitational lensing



mass of the halo

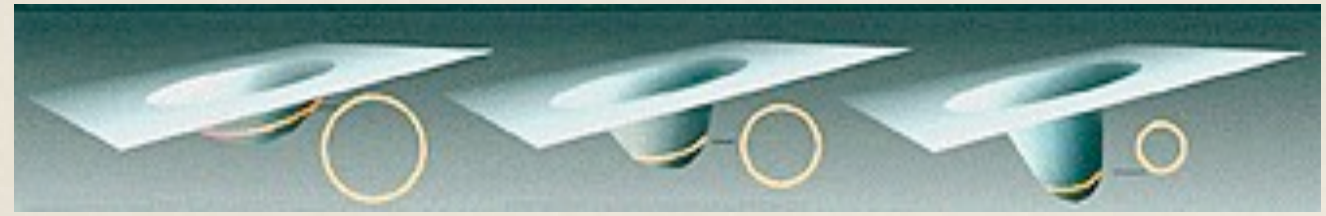
⇒ arcs form a size



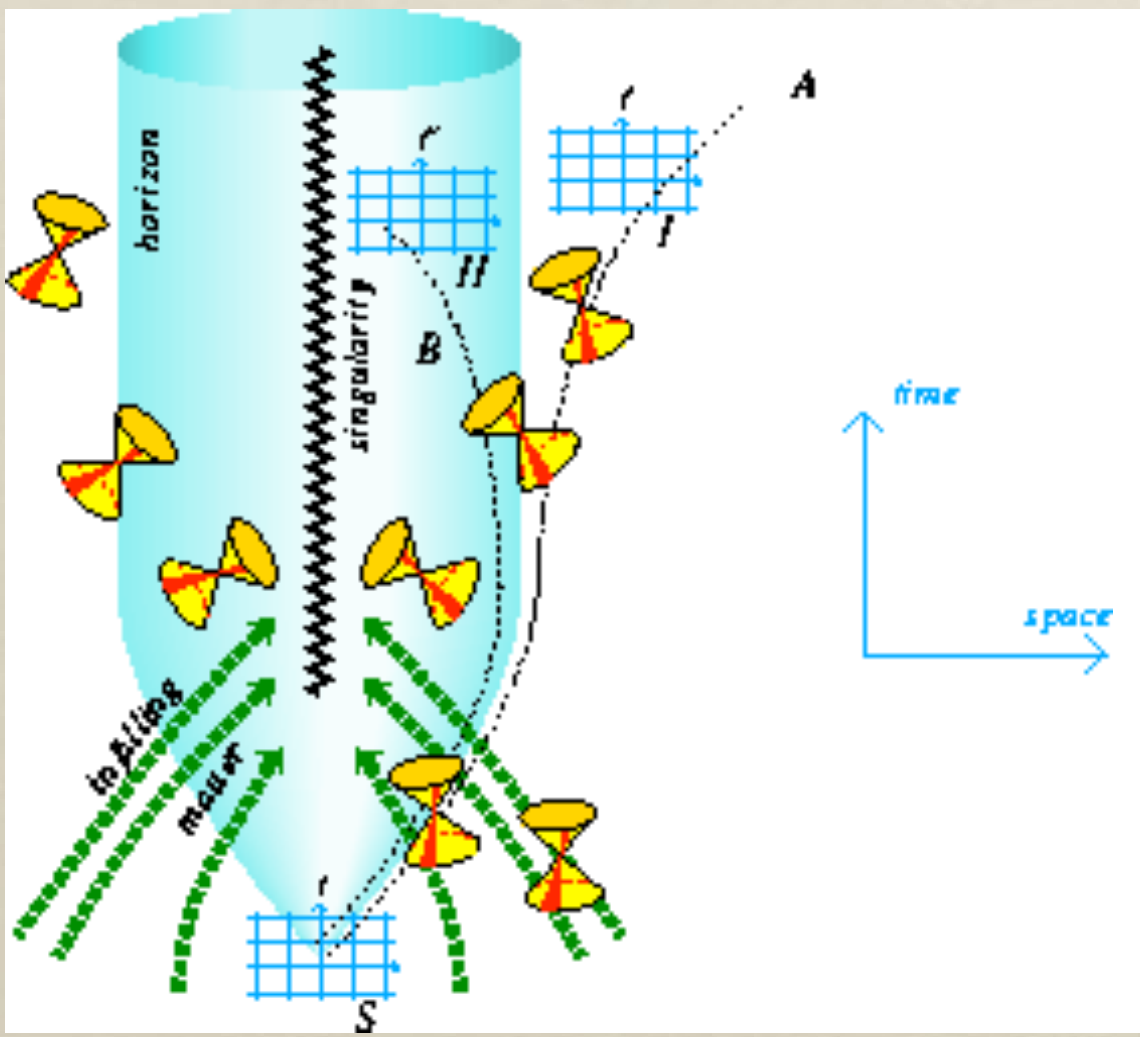
Abell 2218, a galaxy cluster at 3 billion light year, deflect the light coming from other galaxies creating apparent arcs.

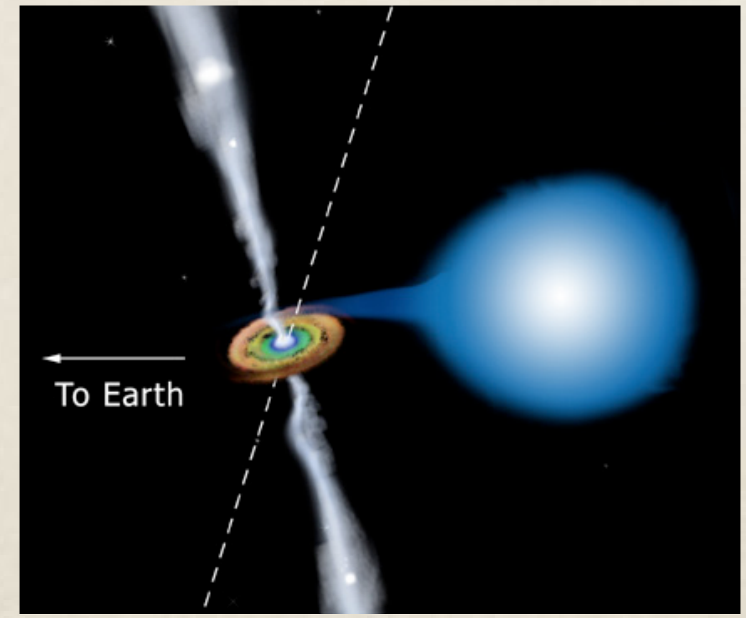
http://ngst.gsfc.nasa.gov/science/gravlens.htm

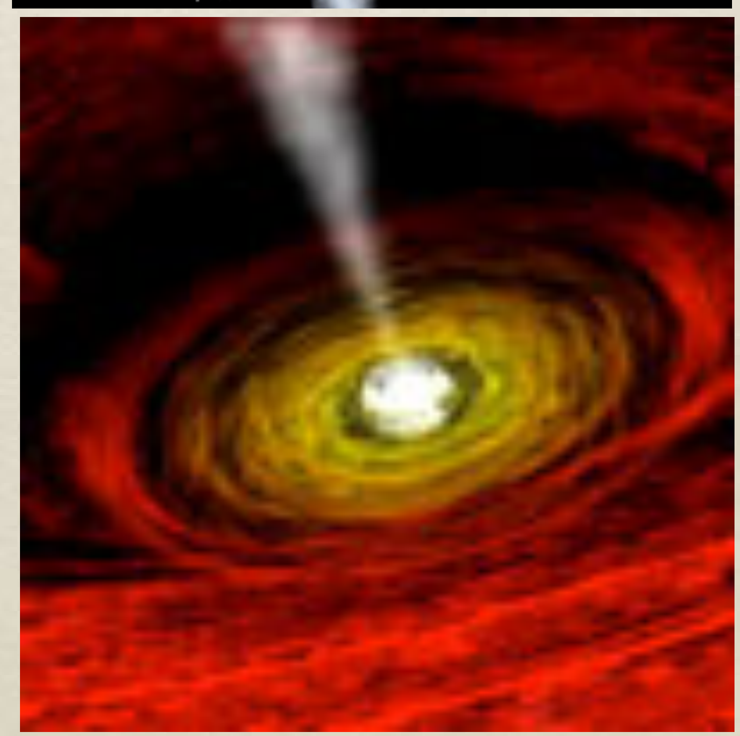
Black Holes



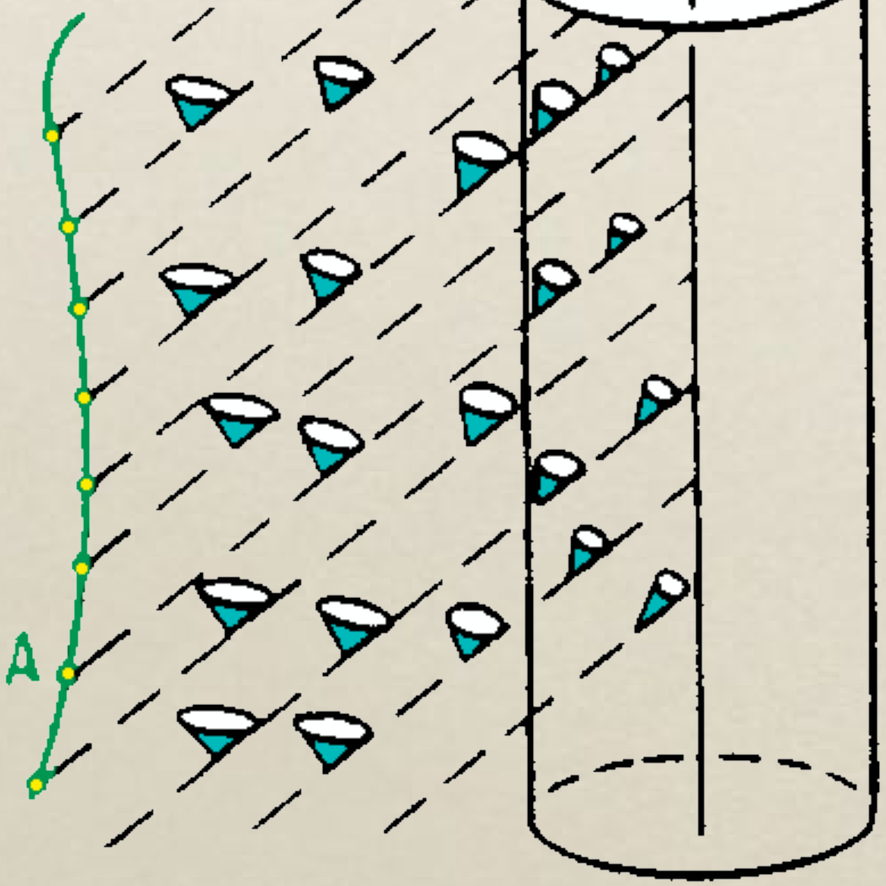
- Space-time region ... that .... traps light



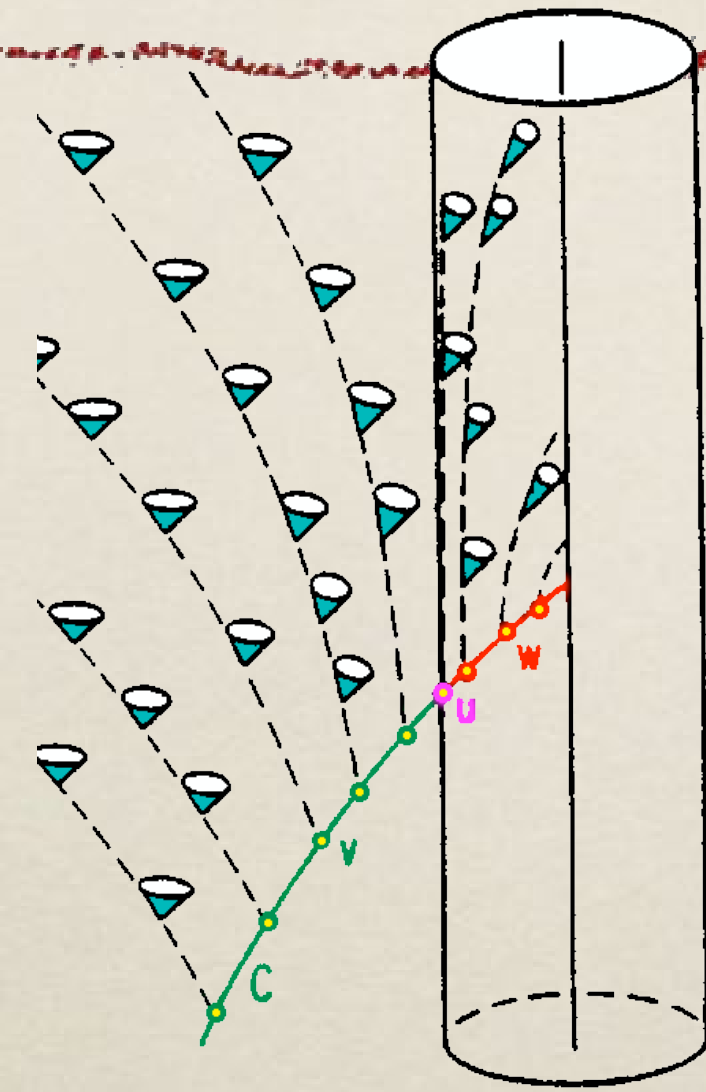




Light signal around a Black Hole

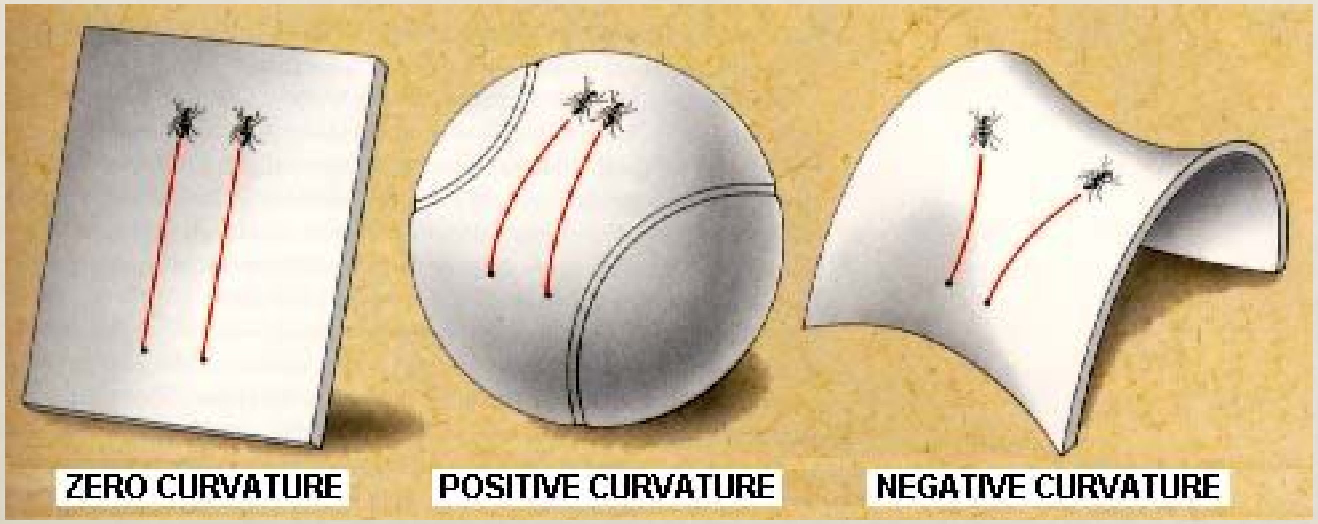


Signal from an outside source



Signal from a source falling into a Black Hole

Equivalence principle



The trajectory of freely falling particles follow the geodetic of a curved space-time

The curvature of the space-time is determined by the distribution of energy (The Energy-Momentum tensor is the source of the Einstein's equations).

The Energy-Momentum tensor is conserved as a consequence of the Einstein's equations).

Relativistic Stars and matter evolution

- To construct stellar models in General Relativity or to study matter evolution in the relativistic regime it is necessary to chose a specific form of the energy-momentum tensor that describe the matter inside the star.
- Perfect fluid is a medium in which the pressure is isotropic in the rest frame of each fluid elements and where shear stress and heat transport are absent.

EM-Tensor in local Lorentz frame

- For such a system any point-like-observer co-moving with the fluid will observe the fluid, in its neighborhood, as isotropic with an energy density e and a pressure p. In this local frame the energy-momentum tensor is:

$$T^{\mu\nu} = \begin{pmatrix} \mathbf{e} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- Its expression in any other frame can be obtained by performing a suitable Lorenz transformation. If now the fluid element is moving with respect to the laboratory frame with velocity: $u^{\mu} = 1/\sqrt{1 - v^i v_i}(1, v^i)$

$$T^{\mu\nu} = (\mathbf{e} + p)u^{\mu}u^{\nu} + p\eta^{\mu\nu}$$

Perfect Fluids in GR

- **FULL SET OF EQs:**
  - Einstein's equations:
    $$G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}$$
  - Conservation of energy:
    $$T^{\mu}_{\nu;\nu} = 0$$
  - Equation of state (EOS): $p = p(\mathbf{e})$
- **Where:**
  - The fluid four velocity is: $u^{\mu} = W(1, v^{i})$
  - The expression for the Energy-Momentum-tensor is:
    $$T^{\mu\nu} = (\mathbf{e} + p)u^{\mu}u^{\nu} + p g^{\mu\nu}$$

Einstein's Equations

- 10 partial differential equations for the 10 metric function (in a coordinate frame)
- Main difficulty: the 4 coordinates have no physical meaning !
- Indeed we have 4 gauge function and indeed only 2 out of the 10 metric function will have any physical meaning.
- Like in the case of EM where of the 4 potential there are only 2 physical degree of freedom because we have 1 gauge function.

Gravitational WAVES

- 10 metric function $g_{\mu\nu}$
- vacum Einstein's equations (10 non-linear PDEs)

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\gamma}\Gamma^\gamma_{\nu\mu} - \Gamma^\gamma_{\alpha\mu}\Gamma^\alpha_{\nu\gamma} = 0$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\alpha'}\left(\partial_\gamma g_{\beta\alpha'} + \partial_\beta g_{\gamma\alpha'} - \partial_{\alpha'} g_{\beta\gamma}\right)$$

- Expand around the Minkowsky background:

$$g_{\mu\nu} = \eta_{\mu\nu} + \left(\bar{h}_{\mu\nu} - 1/2\, \eta_{\mu\nu}\bar{h}\right)$$

- Impose De-Donge gauge: $\eta^{\alpha\beta}\partial_\alpha \bar{h}_{\beta\mu} = 0$
- Go to transverse trace-less gauge

$$\bar{h}_{0\mu} = 0 \quad \text{and} \quad \eta^{\mu\nu}\bar{h}_{\mu\nu} = 0$$

Gravitational Waves (2)

- The Einstein's equations become:

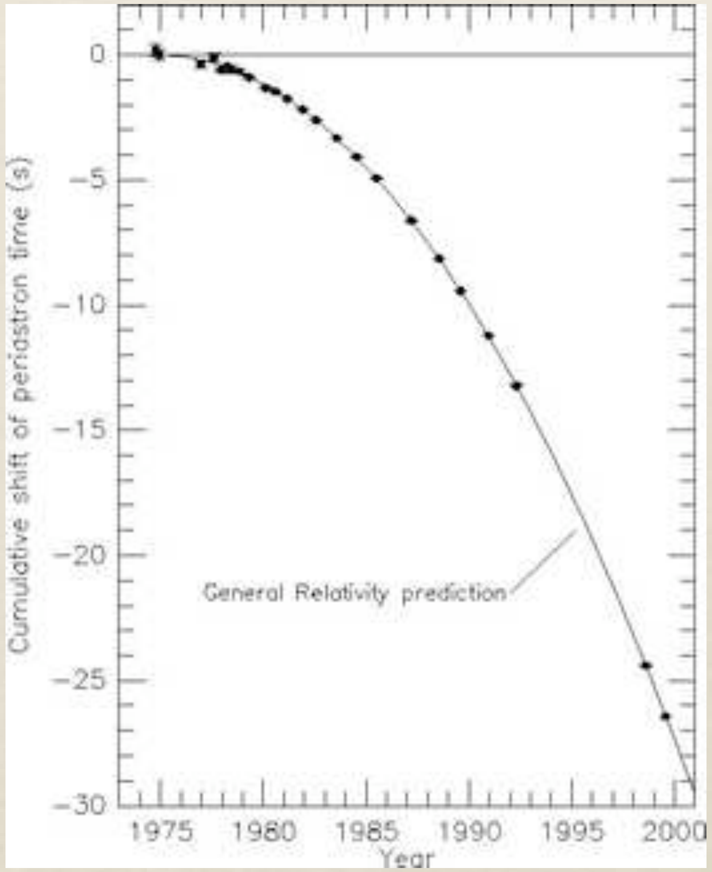
$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{\alpha\beta} \quad = \quad 16\pi G\, T_{\alpha\beta}$$

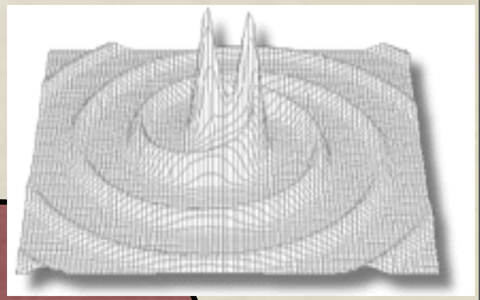
- They becomes (where there is no matter) a wave equation for the two independent degree of freedom of the metric perturbation.
- Gauge Invariance $\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\alpha\xi^\alpha$
- On shell ... transverse-traceless gauge

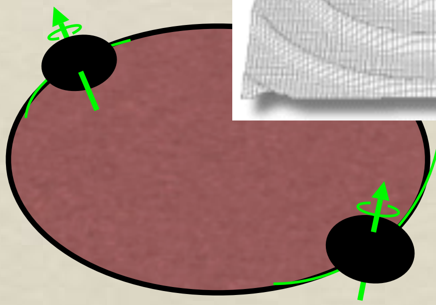
$$\bar{h}_{\mu\nu} = H_{\mu\nu}\, e^{ik_\mu x^\mu} \qquad k^\mu H_{\mu\nu} = 0,\, H_{0\mu} = 0,\, \eta^{\mu\nu} H_{\mu\nu} = 0$$

Experimental evidence for GWs

- PSR B1913+16 (also known as J1915+1606) is a pulsar in a binary star system, in orbit with another star around a common center of mass. In 1974 it was discovered by Russell Alan Hulse and Joseph Hooton Taylor, Jr., of Princeton University, a discovery for which they were awarded the 1993 Nobel Prize in Physics
- Nature 277, 437 - 440 (08 February 1979), J. H. TAYLOR, L. A. FOWLER & P. M. MCCULLOCH:
  Measurements of second- and third-order relativistic effects in the orbit of binary pulsar PSR1913 + 16 have yielded self-consistent estimates of the masses of the pulsar and its companion, quantitative confirmation of the existence of gravitational radiation at the level predicted by general relativity, and detection of geodetic precession of the pulsar spin axis

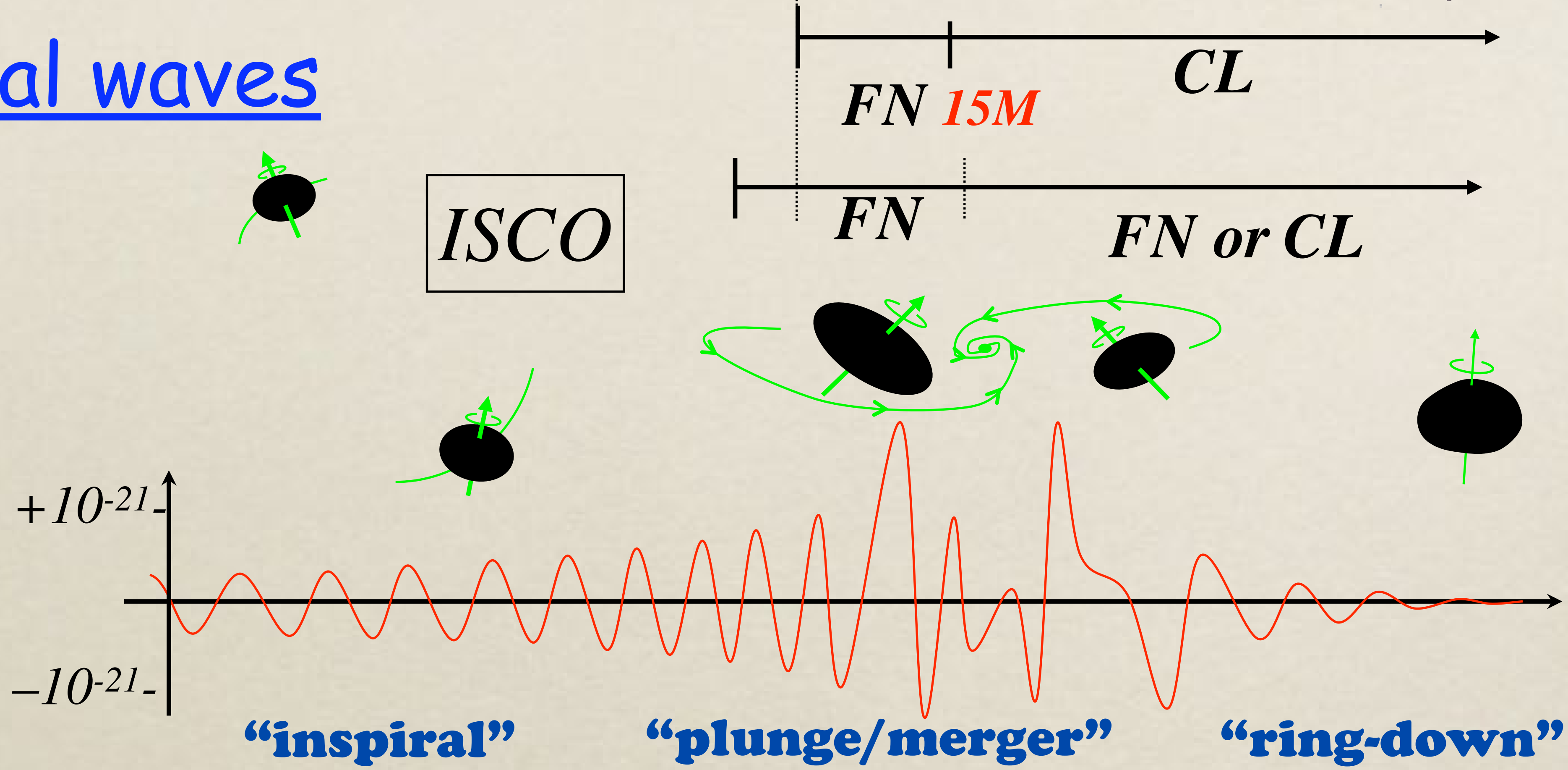


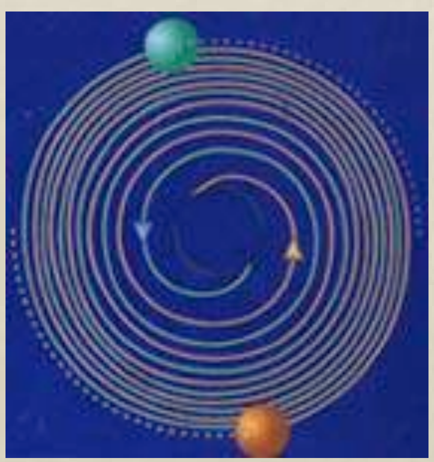


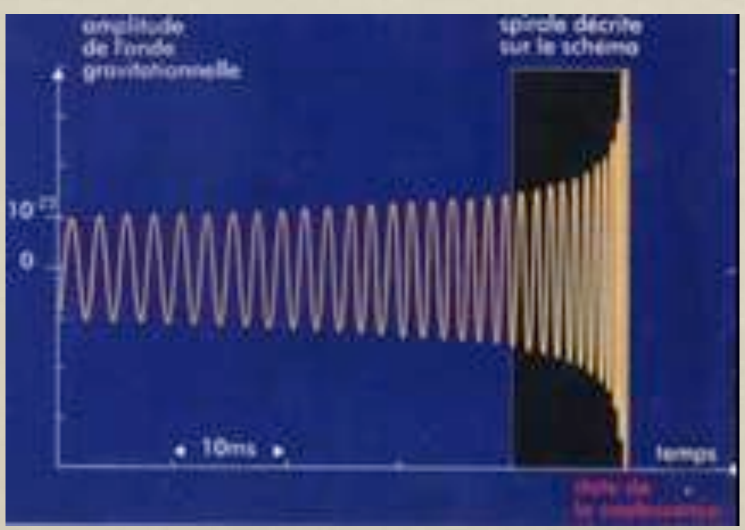


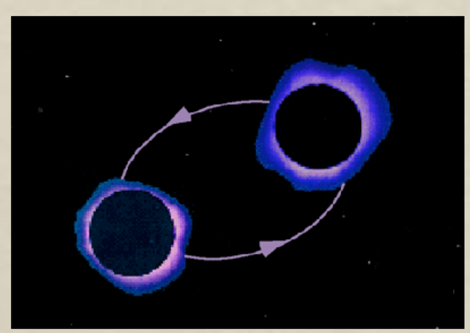
Two-bodies problem and GWs

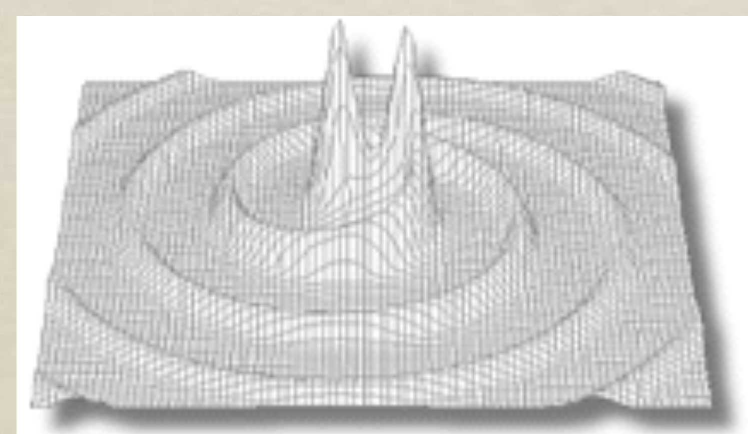
Gravitational waves



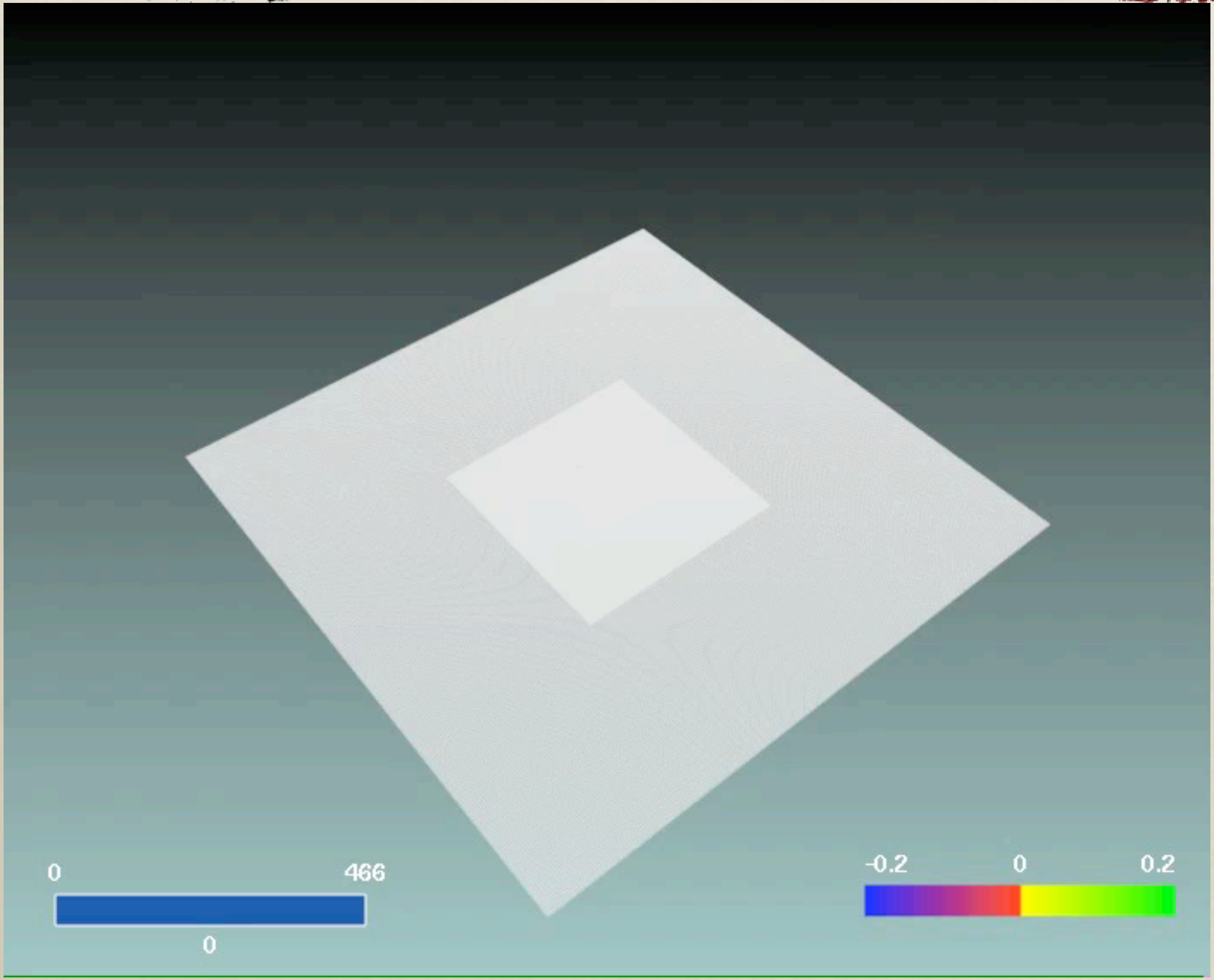








Is it possible to numerical study the merger of 2 Black-Hole ? Yes it is!



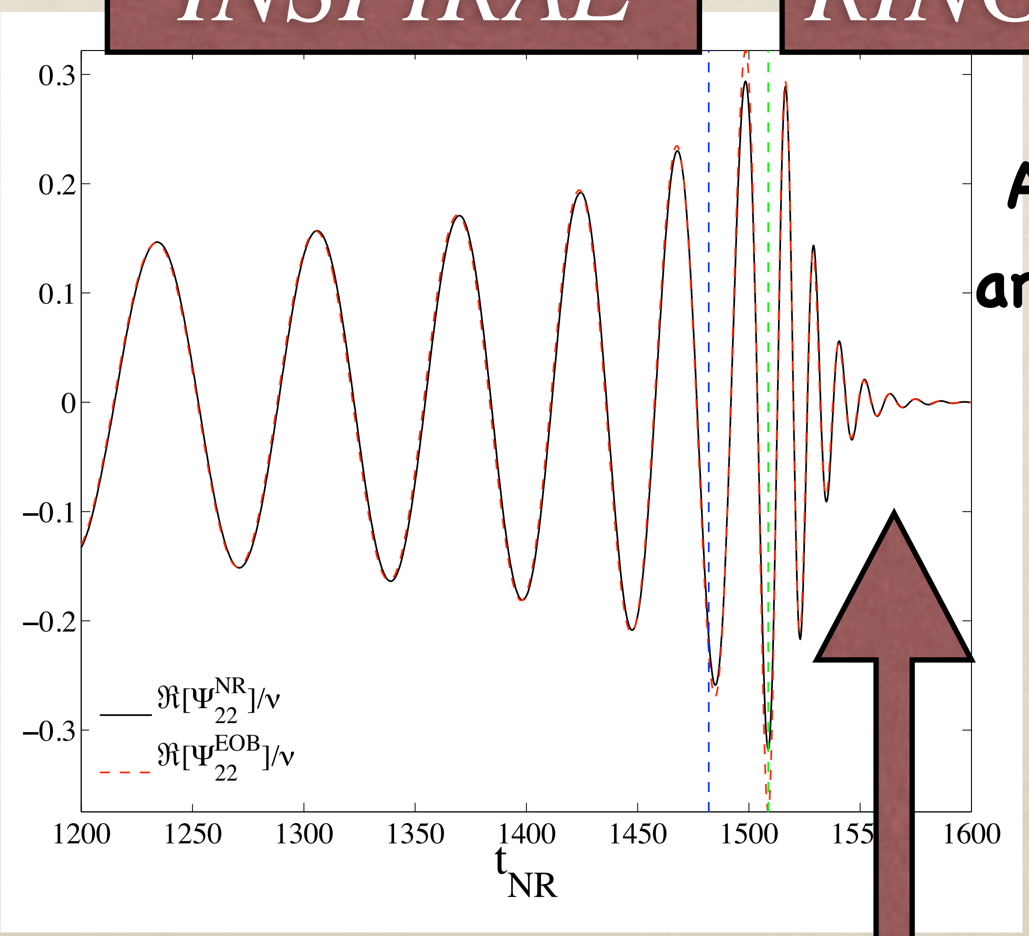
Credits: R. Kaehler & L. Rezzolla

http://arxiv.org/pdf/0707.2559

RUN R7: equal-mass, spinning bhs, different spins.

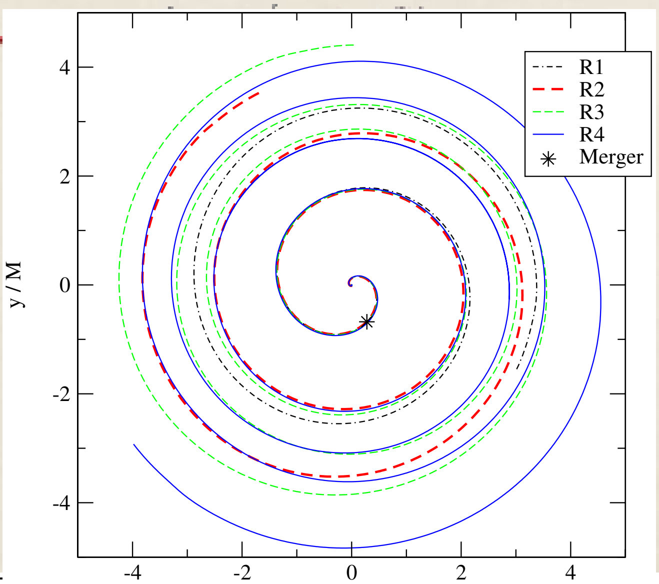
Black Hole Merger results

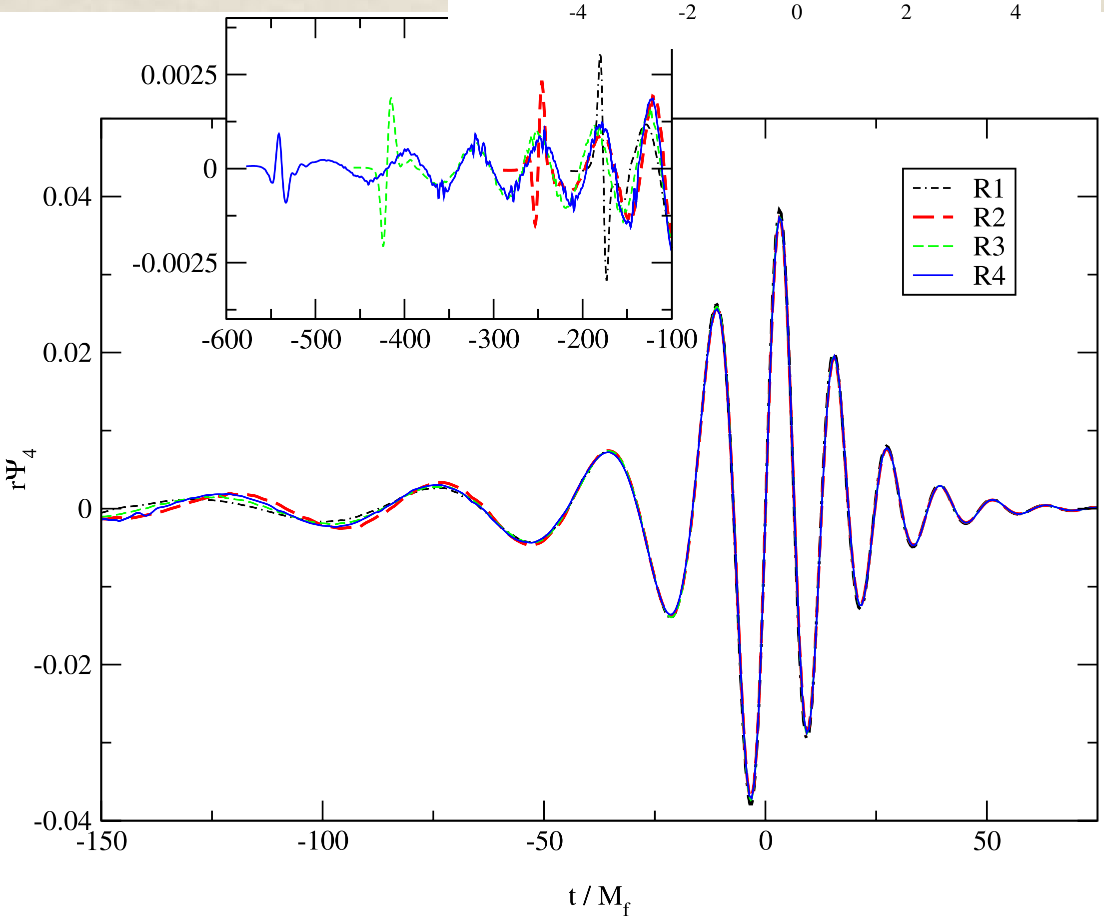
INSPIRAL *RING DOWN*



AEI simulation
arXiv:0712.3003

PHYSICAL REVIEW D **73**, 104002 (2006)

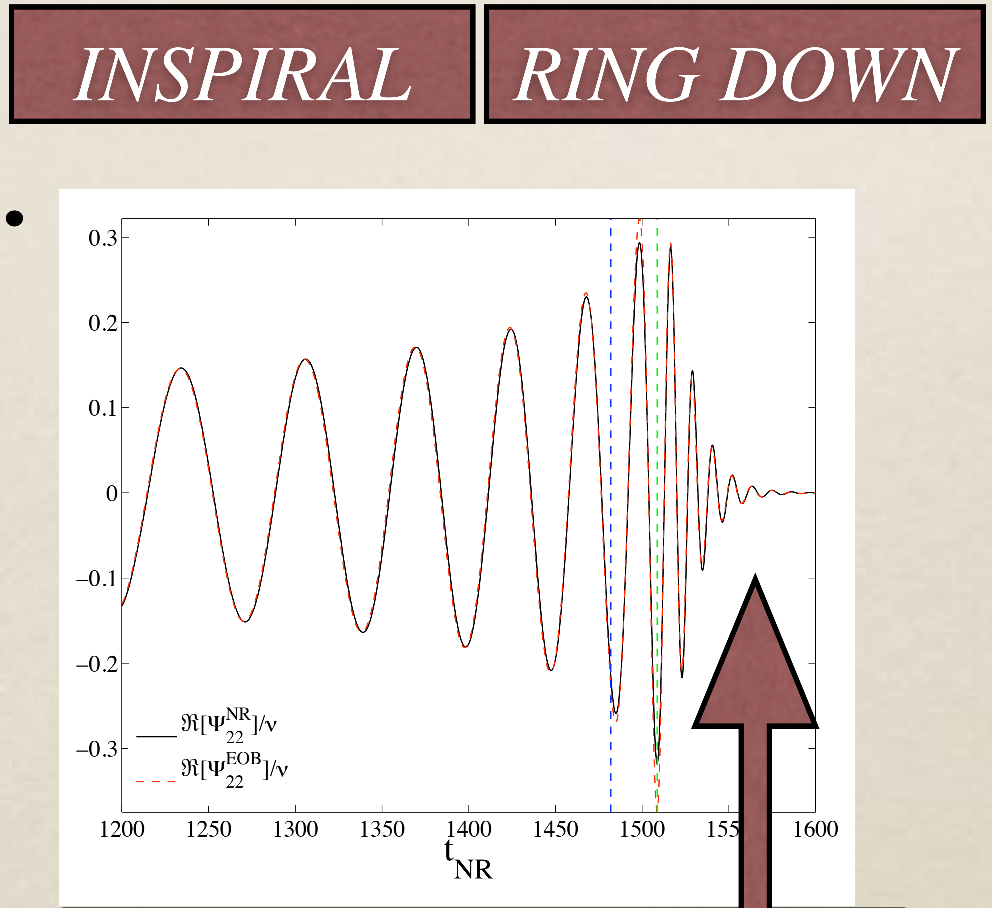




Exponentially dumped oscillation (QNM)

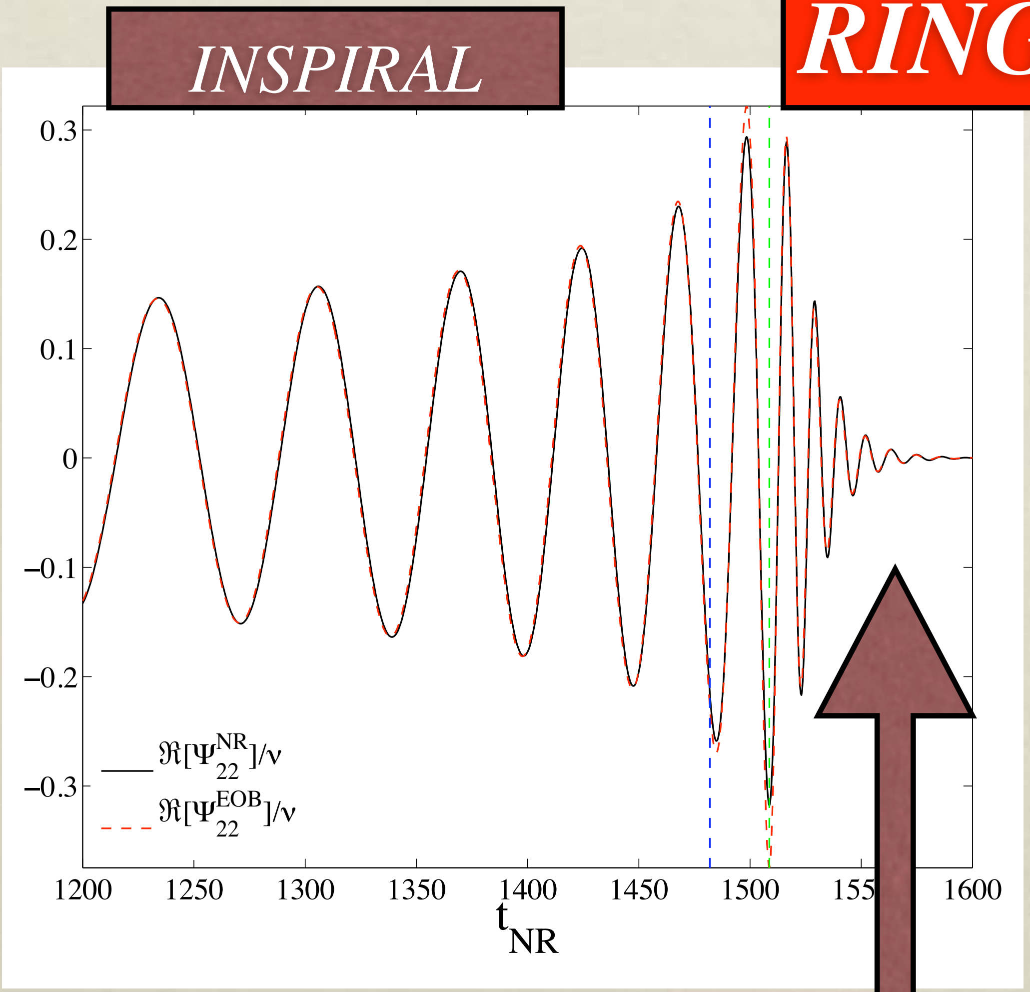
Inspiral part of the signal: .....

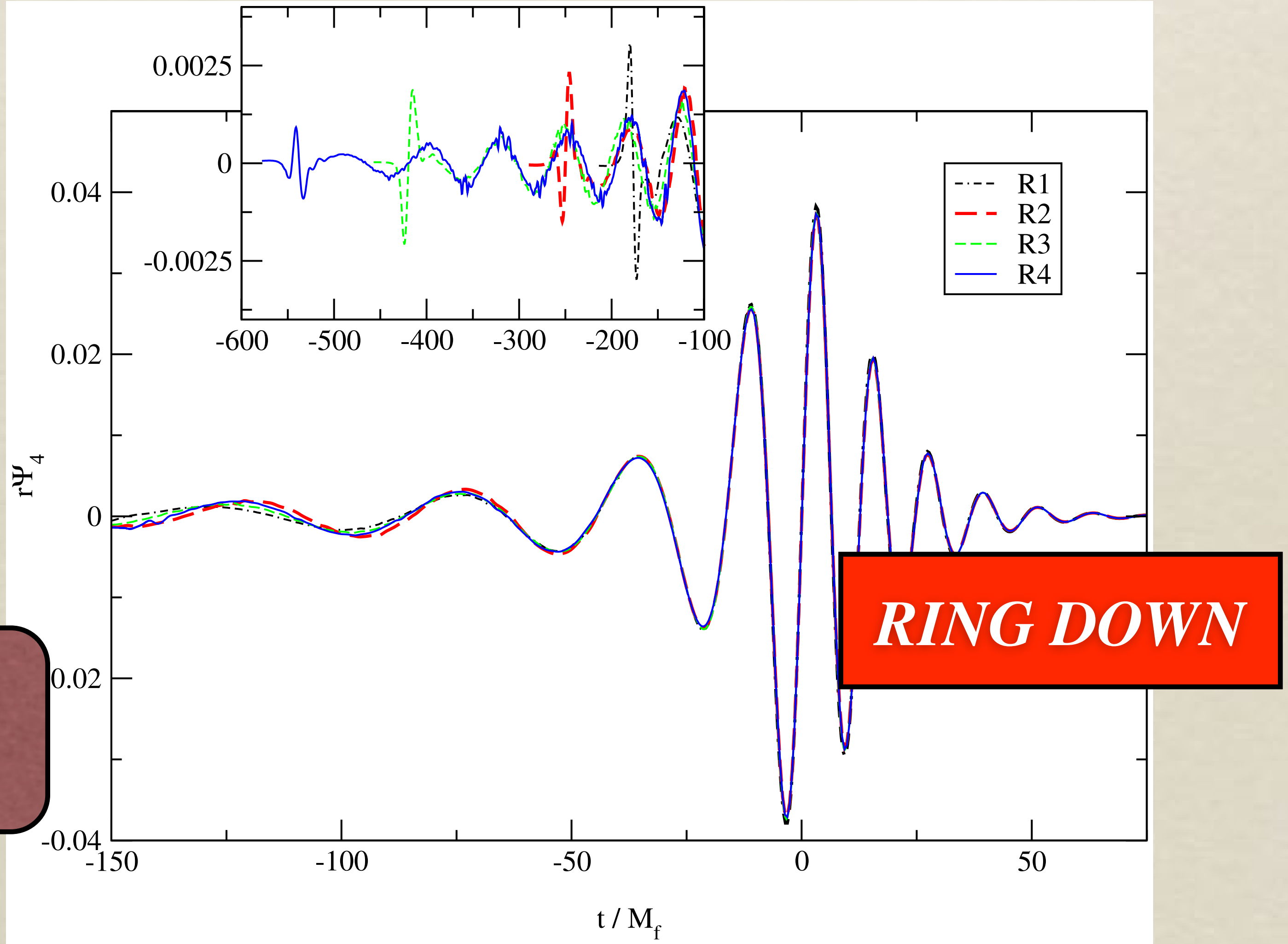
- Post Newtonian approximation ....
- ... Damour EOB (Effective one body) waveforms for the two bodies problem.
- See Damour-Nagar about matching Numerical-Relativity waveform and EOB ones.



Exponentially dumped oscillation (QNM)

QNM part of the signal: 1D codes





Exponentially dumped oscillation (QNM)

Not a characteristic of just BH merger

- Works of Vishveshwara [Nature, 227, 936 (1970)], Press [Astrophys. J. Letts. 170, L105 (1971)] and Davis, Ruffini and Tiomno [Phys. Rev. D 5, 2932 (1972)], unambiguously showed that a non-spherical gravitational perturbation of a Schwarzschild Black Hole is radiated away via exponentially damped harmonic oscillations.
- These dumped oscillation are the QNM first studied by Regge and Wheeler [Phys. Rev. 108 1063 (1957)]

Metric perturbations

- **Einstein Linearized Eqs.** $\bar{g}_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu},$
- **Spherical symmetry:**

$$\mathrm{d}s^2 = -\left(1 - \frac{2M_\bullet}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M_\bullet}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2,$$

$$h = \begin{pmatrix} \boxed{\mathrm{S}} & \boxed{\mathrm{S}} & \boxed{\mathrm{V}} \\ \boxed{\mathrm{S}} & \boxed{\mathrm{S}} & \boxed{\mathrm{V}} \\ \boxed{\mathrm{V}} & \boxed{\mathrm{V}} & \boxed{\mathrm{T}} \end{pmatrix}.$$

- **Better expressed in terms of tensor Harmonics**

$$\left(\overset{1}{V}_{LM}\right)_a = (S_{LM})_{;a} = \frac{\partial}{\partial x^a} Y_{LM}(\theta, \varphi)$$

$$\left(\overset{2}{V}_{LM}\right)_a = \epsilon_a{}^b\,(S_{LM})_{;b} = \gamma^{bc}\epsilon_{ac}\frac{\partial}{\partial x^b} Y_{LM}(\theta, \varphi)$$

$$\left(\overset{1}{T}_{LM}\right)_{ab} = (S_{LM})_{;ab}$$

$$\left(\overset{2}{T}_{LM}\right)_{ab} = S_{LM}\,\gamma_{ab}$$

$$\left(\overset{3}{T}_{LM}\right)_{ab} = \tfrac{1}{2}\left[\epsilon_a{}^c(S_{LM})_{;cb} + \epsilon_b{}^c(S_{LM})_{;ca}\right].$$

S_{LM}	polar	$(-1)^L$
$\overset{1}{V}_{LM}$	polar	$(-1)^L$
$\overset{2}{V}_{LM}$	axial	$(-1)^{L+1}$
$\overset{1}{T}_{LM}$	polar	$(-1)^L$
$\overset{2}{T}_{LM}$	polar	$(-1)^L$
$\overset{3}{T}_{LM}$	axial	$(-1)^{L+1}$.

Axial perturbations

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -h_0(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{LM}}{\partial\varphi} & h_0(t,r)\sin\theta\frac{\partial Y_{LM}}{\partial\theta} \\ 0 & 0 & -h_1(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{LM}}{\partial\varphi} & h_1(t,r)\sin\theta\frac{\partial Y_{LM}}{\partial\theta} \\ * & * & \frac{1}{2}h_2(t,r)\frac{1}{\sin\theta}X_{LM} & -\frac{1}{2}h_2(t,r)\sin\theta W_{LM} \\ * & * & * & -\frac{1}{2}h_2(t,r)\sin\theta X_{LM} \end{pmatrix},$$

- Where:

$$X_{LM}(\theta,\varphi) = 2\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}Y_{LM} - \cot\theta\frac{\partial}{\partial\varphi}Y_{LM}\right)$$

$$W_{LM}(\theta,\varphi) = \left(\frac{\partial^2}{\partial\theta^2}Y_{LM} - \cot\theta\frac{\partial}{\partial\theta}Y_{LM} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y_{LM}\right).$$

- Use gauge freedom to set h2=0

- The linearized equation are:
- Apparently 3 equations with two unknown

$$\delta R_{23}:\quad 0 = R_1(h_0,h_1,t,r) = \frac{1}{B(r)}\frac{\partial}{\partial t}h_0 - \frac{\partial}{\partial r}(B(r)h_1)$$

$$\delta R_{13}:\quad 0 = R_2(h_0,h_1,t,r)$$

$$= \frac{1}{B(r)}\left(\frac{\partial^2 h_1}{\partial t^2} - \frac{\partial^2 h_0}{\partial t\partial r} + \frac{2}{r}\frac{\partial h_0}{\partial t}\right) + \frac{1}{r^2}(L(L+1)-2)h_1$$

$$\delta R_{03}:\quad 0 = R_3(h_0,h_1,t,r)$$

$$= \frac{1}{2}B(r)\left(\frac{\partial^2 h_0}{\partial r^2} - \frac{\partial^2 h_1}{\partial t\partial r} - \frac{2}{r}\frac{\partial h_1}{\partial t}\right) + \frac{1}{r^2}\left(r\frac{\partial}{\partial r}B(r) - \frac{1}{2}L(L+1)\right)h_0,$$

where $B(r) = (1 - 2M_\bullet/r)$.

The Regge Wheeler equation

- Eliminating h_0 and defining Q_L

$$Q_L(t,r) := \frac{1}{r}B(r)(h_1)_{\rm RW}(t,r) = \frac{1}{r}B(r)k_1(t,r),$$

$$B(r) = (1 - 2M_\bullet/r).$$

$$k_1 = h_1 + \frac{1}{2}\left(h_{2,r} - 2\frac{h_2}{r}\right),$$

- using the tortoise coordinate x

$$x = r + 2M_\bullet \ln\left(\frac{r}{2M_\bullet} - 1\right)$$

$$V_{\rm RW}(x) = \left(1 - \frac{2M_\bullet}{r(x)}\right)\left[\frac{L(L+1)}{r(x)^2} - \frac{6M_\bullet}{r(x)^3}\right]$$

- We get the equation

$$\frac{\partial^2}{\partial t^2}Q_L(t,x) - \frac{\partial^2}{\partial x^2}Q_L(t,x) + V_{\rm RW}(x)Q_L(t,x) = 0,$$

- That is a wave equation .... for Q_L

Solving RW in the frequency domain

- Since the potential is positive we will not have bounded solution and we have to look to a wave like solution that at the two boundary behave like outgoing waves. In the frequency domain:

$$Q_L(t,x) = e^{-i\omega t}\chi(\omega,x)$$

- The function should be a solution of the eigenvalues equation:

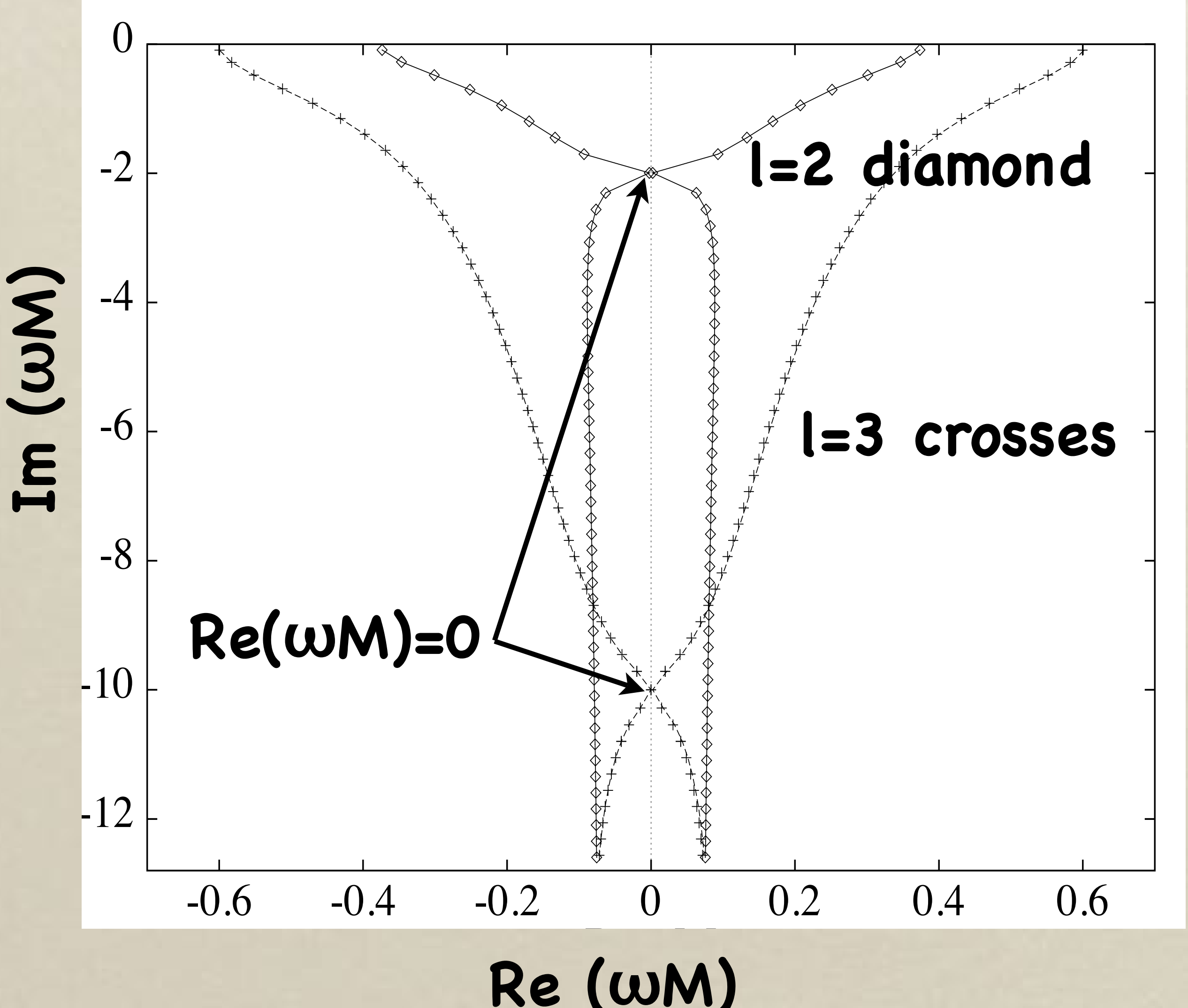
$$\left(-\frac{\partial^2}{\partial x^2} + V_{RW}(x)\right)\chi(\omega,x) = \omega^2\chi(\omega,x)$$

- Fulfilling the boundary condition:

$$\begin{array}{lll} \chi(\omega,x) \rightarrow e^{-i\omega x} & \text{for} & x \rightarrow +\infty \\ \chi(\omega,x) \rightarrow e^{+i\omega x} & \text{for} & x \rightarrow -\infty \end{array}$$

QNM of Schwarzschild BH

- The solution of the above problem are by definition the QNM of a Schwarzschild BH



n	$\ell = 2$		$\ell = 3$		$\ell = 4$	
0	0.37367	-0.08896 i	0.59944	-0.09270 i	0.80918	-0.09416 i
1	0.34671	-0.27391 i	0.58264	-0.28130 i	0.79663	-0.28443 i
2	0.30105	-0.47828 i	0.55168	-0.47909 i	0.77271	-0.47991 i
3	0.25150	-0.70514 i	0.51196	-0.69034 i	0.73984	-0.68392 i

$$Q_L^{(n)}(t,x) = e^{-i\omega_n t}\chi^{(n)}(x)$$

Where:

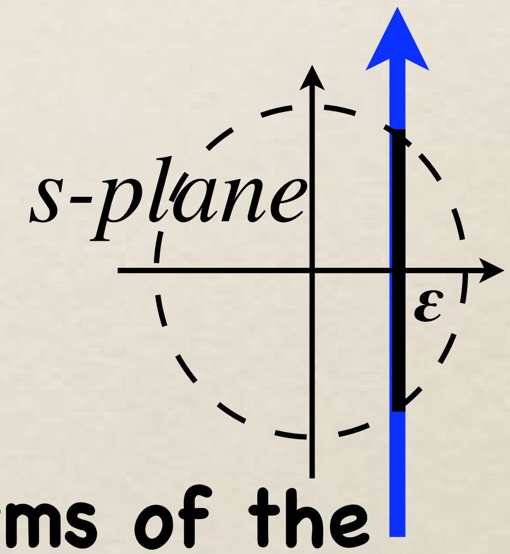
$$\chi(\omega_n, x) \rightarrow e^{-i\omega_n x} \quad \text{for} \quad x \rightarrow +\infty$$
$$\chi(\omega_n, x) \rightarrow e^{+i\omega_n x} \quad \text{for} \quad x \rightarrow -\infty$$

Why the name quasi-NM

- At first sight we have reduced the problem of the evolution of a perturbation to a mode-expansion.
- Unfortunately this is not the case since it is not possible to write a generic solution fulfilling an initial condition just in terms of Quasi-Normal-Modes.

Laplace Transform:

- The technique that allows us the possibility to analise the problem is the use of the Laplace transform instead of the Fourier one: $$\hat{f}(s,x) = \int_0^\infty \mathrm{e}^{-st}\, Q(t,x)\,\mathrm{d}t,$$



- Its inverse is: $$Q(t,x) = \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}\infty}^{\varepsilon + \mathrm{i}\infty} \mathrm{e}^{st}\, \hat{f}(s,x)\,\mathrm{d}s,$$
- And the solution of the problem is given in terms of the solution of the following homogeneous equations:

$$\hat{f}''(s,x) + \left(-s^2 - V(x)\right)\hat{f}(s,x) = \mathcal{I}(s,x),$$

- with initial data given by: $$\mathcal{I}(s,x) = -s\, Q\big|_{t=0} - \left.\frac{\partial Q}{\partial t}\right|_{t=0}.$$

Laplace Transform:

- **There is a very standard technique to solve for an in-homogenous equation.**
  - Find two independent solution of the homogenous:

$$f''(s,x) + \left(-s^2 - V(x)\right) f(s,x) = 0$$

$$x_< \equiv \min(x', x),$$
$$x_> \equiv \max(x', x)$$

  - Denote this two solution as f_+ and f_-
  - Construct the Wronksian W(s): $f_-(s,x) f'_+(s,x) - f'_-(s,x) f_+(s,x)$
  - The Green-Function is:

$$G(s,x,x') = \frac{1}{W(s)} f_-(s,x_<) f_+(s,x_>),$$

- **The final solution is:**

$$\hat{f}(s,x) = \int_{-\infty}^{\infty} G(s,x,x') \mathcal{I}(s,x') \, \mathrm{d}x',$$

f_- stays bounded as $x \to -\infty$

f_+ stays bounded as $x \to +\infty$

Inverse Laplace transforming

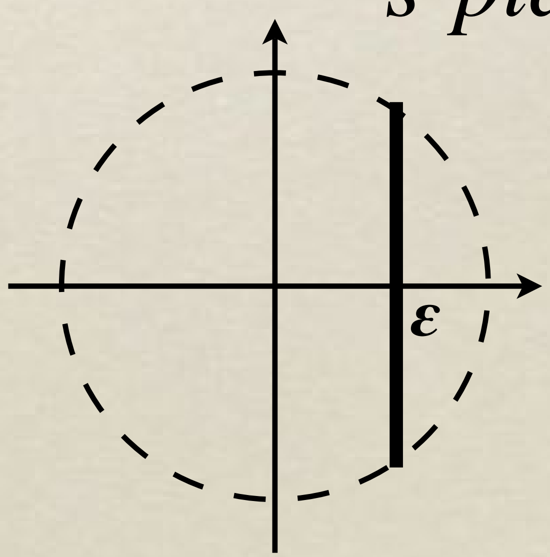
$$G(s,x,x') = \frac{1}{W(s)} f_-(s,x_<) f_+(s,x_>),$$

$$x_< \equiv \min(x',x),$$
$$x_> \equiv \max(x',x)$$

$$\begin{aligned} Q(t,x) &= \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon-\mathrm{i}\infty}^{\varepsilon+\mathrm{i}\infty} \mathrm{e}^{st} \int_{-\infty}^{\infty} G(s,x,x')\,\mathcal{I}(s,x')\,\mathrm{d}x'\,\mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{e}^{st} \frac{1}{W(s)} \int_{-\infty}^{\infty} f_-(s,x_<) f_+(s,x_>)\,\mathcal{I}(s,x')\,\mathrm{d}x'\,\mathrm{d}s \\ &= \sum_q \mathrm{e}^{s_q t}\,\mathrm{Res}\left(\frac{1}{W(s)}, s_q\right) \int_{-\infty}^{\infty} f_-(s_q,x_<) f_+(s_q,x_>)\,\mathcal{I}(s_q,x')\,\mathrm{d}x'. \end{aligned}$$

$$Q(t,x) = \sum_q c_q u_q(t,x),$$

s-plane

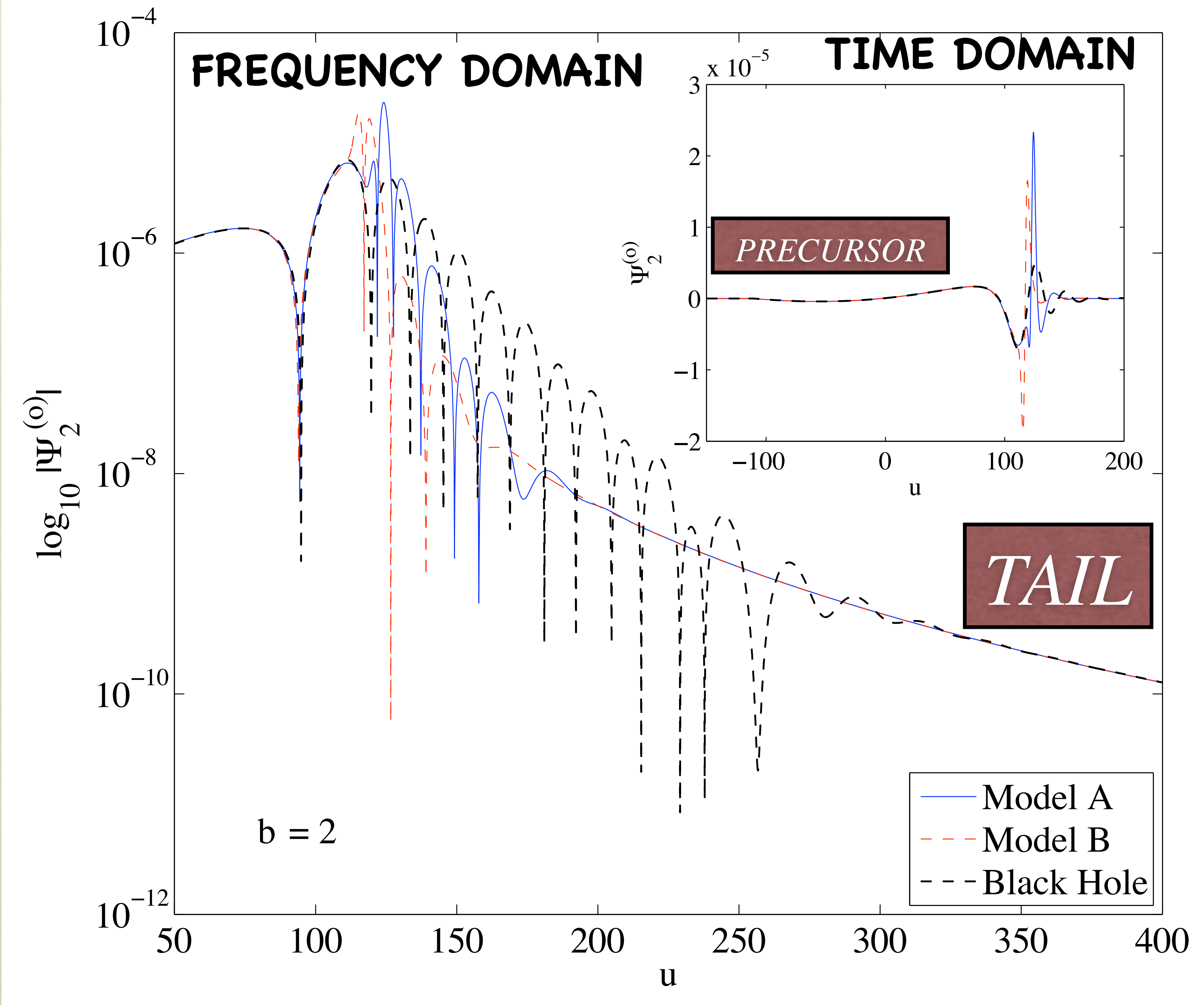


$$c_q = \frac{1}{\mathrm{d}W(s_q)/\mathrm{d}s} \int_{x_\mathrm{l}}^{x_\mathrm{r}} f_-(s_q,x')\,\mathcal{I}(s_q,x')\,\mathrm{d}x'$$

$$u_q(t,x) = \mathrm{e}^{s_q t} f_+(s_q,x),$$

A real case from numerics....

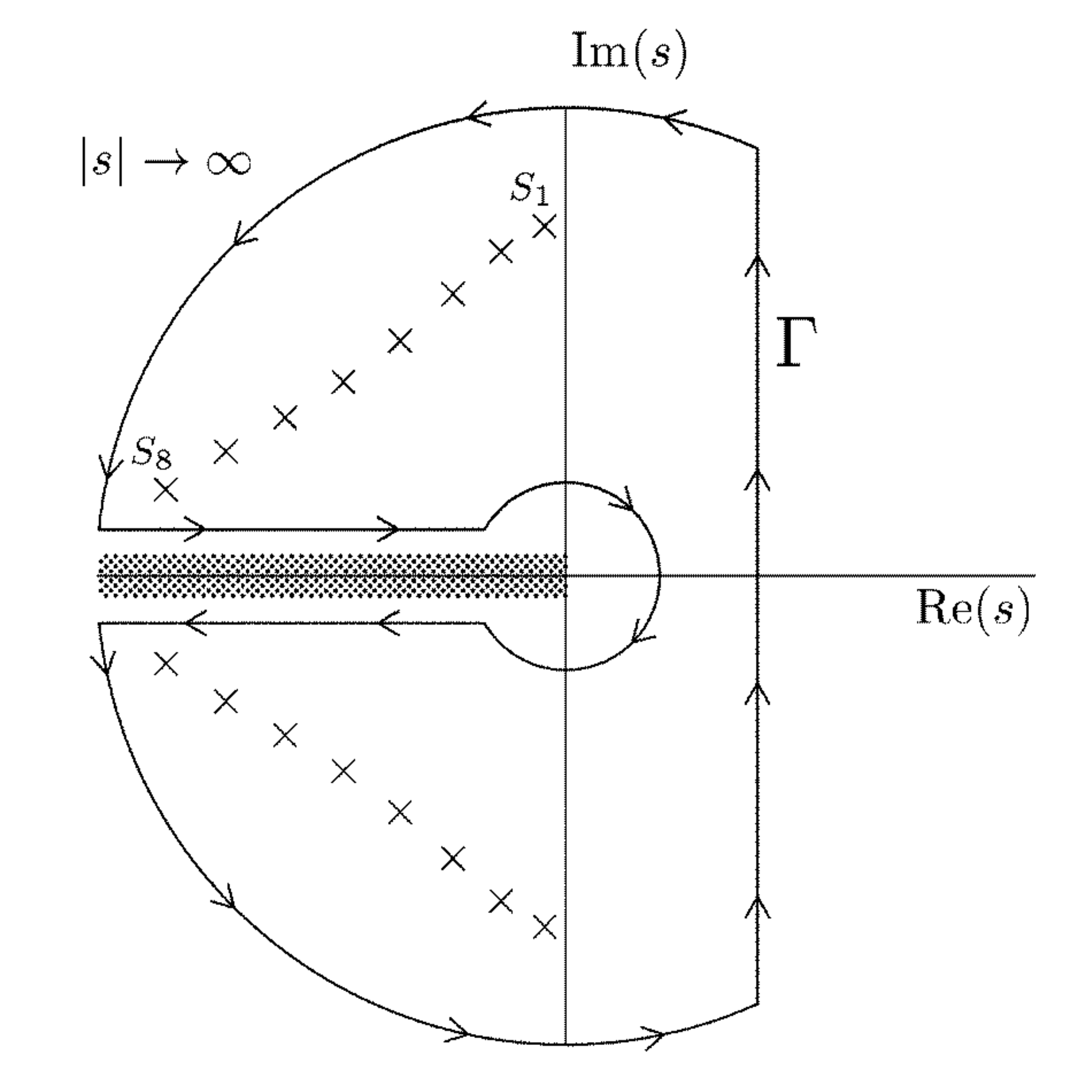
Scattering of a Gaussian pulse of odd-parity metric perturbation by black-hole and relativistic stars models



S. Bernuzzi, A. Nagar, R. De Pietri, Dynamical excitation of space-time modes of compact objects, **arXiv:0801.2090**

Unfortunately not so easy...

- For Black Hole perturbation f_- is analytic
- Unfortunately f_+ is not:
  - Due to the fall-off property of the potential
    - f_+ has an essential singularity at s=0.
    - f_+ has isolated singularity along the negative s-axis



- Other contributions to the inverse Laplace transformation

$$Q(t,x) = \sum_q c_q u_q(t,x) + (\text{other contributions}).$$

QNM of Stars: do the same of BHs

- **Linearize Einstein equation:**

$$\delta\left(G^{\mu}_{\nu}-\frac{8\pi G}{c^4}T^{\mu}_{\nu}\right)=0,$$

- **Around a solution of the TOV equations.**

$$\delta\left(T^{\mu}_{\nu;\mu}\right)=0,$$

- **We will have two cases:**
  - Axial perturbations:

$$-\frac{1}{c^2}\frac{\partial^2 X}{\partial^2 t}+\frac{\partial^2 X}{\partial^2 r_*}+\frac{e^v}{r^3}\left[\ell(\ell+1)r+r^3(\rho-p)-6M\right]=0.$$

  - Polar Perturbations:
    - S and F metric perturbation
    - H density perturbation

$$-\frac{1}{c^2}\frac{\partial^2 S}{\partial^2 t}+\frac{\partial^2 S}{\partial^2 r_*}+L_1(S,F,\ell)=0,$$

$$-\frac{1}{c^2}\frac{\partial^2 F}{\partial^2 t}+\frac{\partial^2 F}{\partial^2 r_*}+L_2(S,F,H,\ell)=0,$$

$$-\frac{1}{(c_s)^2}\frac{\partial^2 H}{\partial^2 t}+\frac{\partial^2 H}{\partial^2 r_*}+L_3(H,H',S,S',F,F',\ell)=0,$$

  - + a constraint:

$$\frac{\partial^2 F}{\partial^2 r_*}+L_4(F,F',S,S',H,\ell)=0.$$

- **Where:**

$$\frac{\partial}{\partial r_*}=e^{(v-\lambda)/2}\frac{\partial}{\partial r}.$$

QNM of stars

- Clearly, outside we should only consider metric perturbations.
- Explicit form in: Kind, S., Ehlers, J., and Schmidt, B.G., "Relativistic stellar oscillations treated as an initial value problem", Class. Quantum Grav., 10, 2137- 2152, (1993).
- QNM mode problem even more complicate:
  - Outgoing wave condition only for $r \to \infty$
  - We have to impose boundary condition at the origin and at the boundary of the star
- S=0 is the Cowling approximations. .... Not working very well ...... see: Dimmelmeier et all. Mon. Not. of the Royal Astron. Society, 368, (2006) 1609-1630.

W-modes (gravitational modes)

- Since in General Relativity the metric is a field we will have gravitational modes that have no counterpart in Newtonian Physics.
- These Space-time modes are called W-modes
- They are different from the QNM of a Black Hole of the same mass.

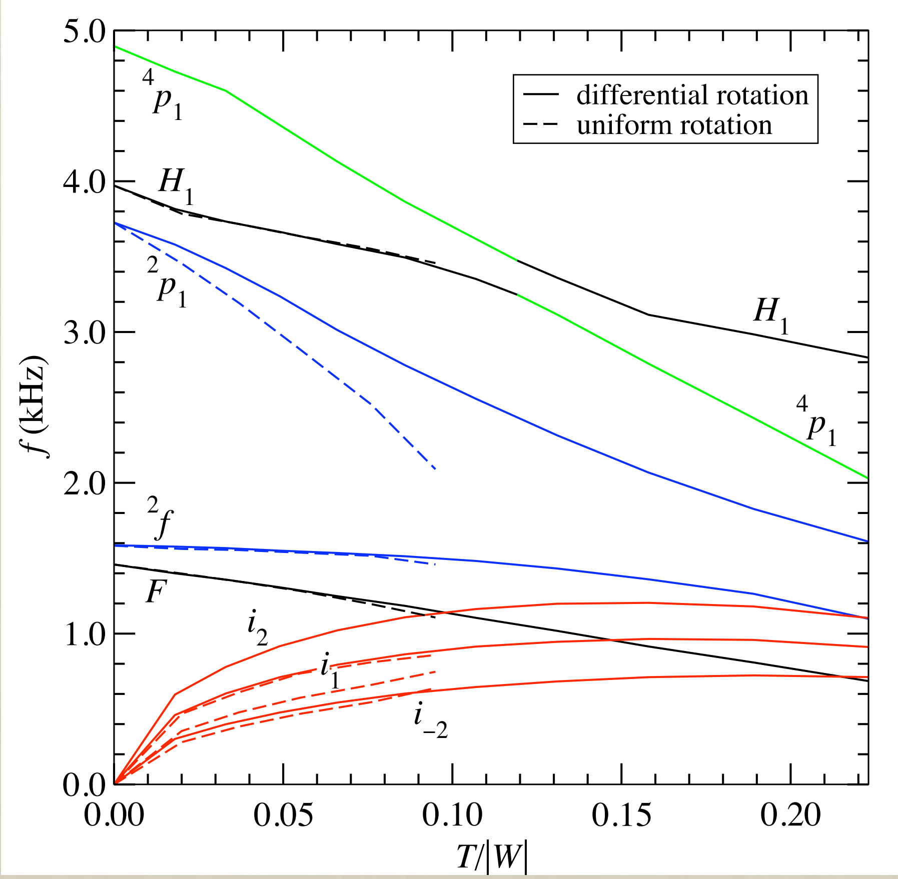
n	ν_{n2} [Hz]	τ_{n2} [μs]	ω_{n2}	α_{n2}
0	9497	32.64	0.29393	0.15091
1	16724	20.65	0.5176	0.23853
2	24277	17.21	0.75136	0.28621
3	32245	15.43	0.99796	0.31923

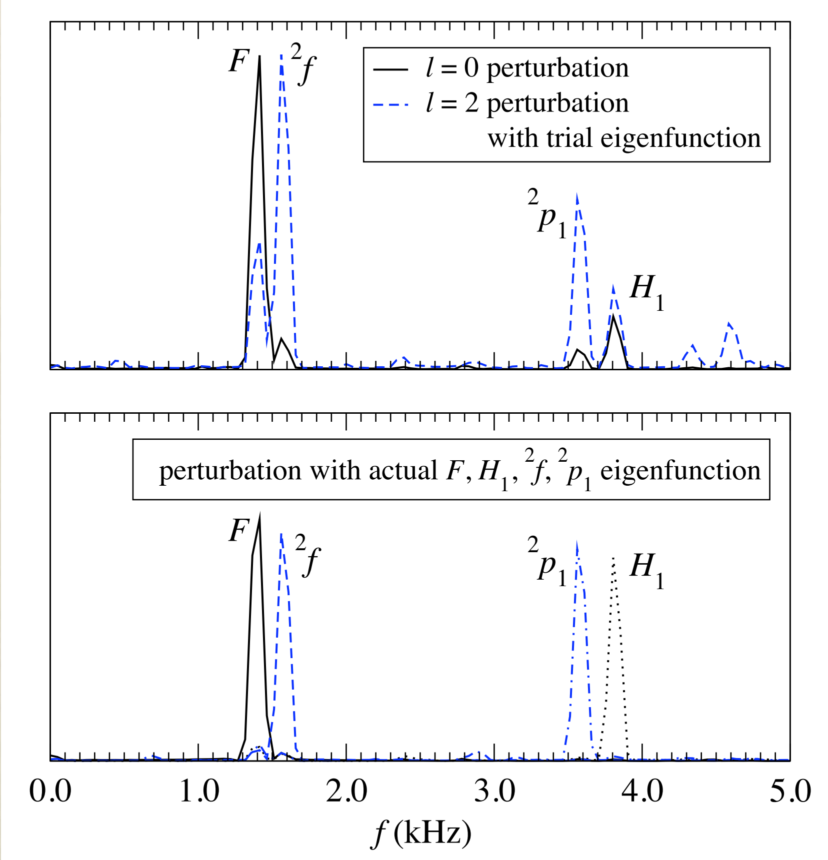
n	ν_{n2} [Hz]	τ_{n2} [μs]	ω_{n2}	α_{n2}
0	8624	77.52	0.2669	0.0635
1	8002	25.18	0.2477	0.1956
2	6948	14.42	0.2150	0.3416
3	5804	9.78	0.1796	0.5037

- K=56.16 Γ=2 polytrope vs a same mass BH ...
- It is possible to discriminate BH from Neutron stars.

QNM of Rotating stars... .... an example

- Better way to compute modes (instead of the Cowling approximation) is to use the CFC approximation (no gravity) making full 3D time simulations [*].
- One can study the dependence on the rotation state of the frequency.





- Dimmelmeier et all. Mon. Not. of the Royal Astron. Society, 368, (2006) 1609-1630

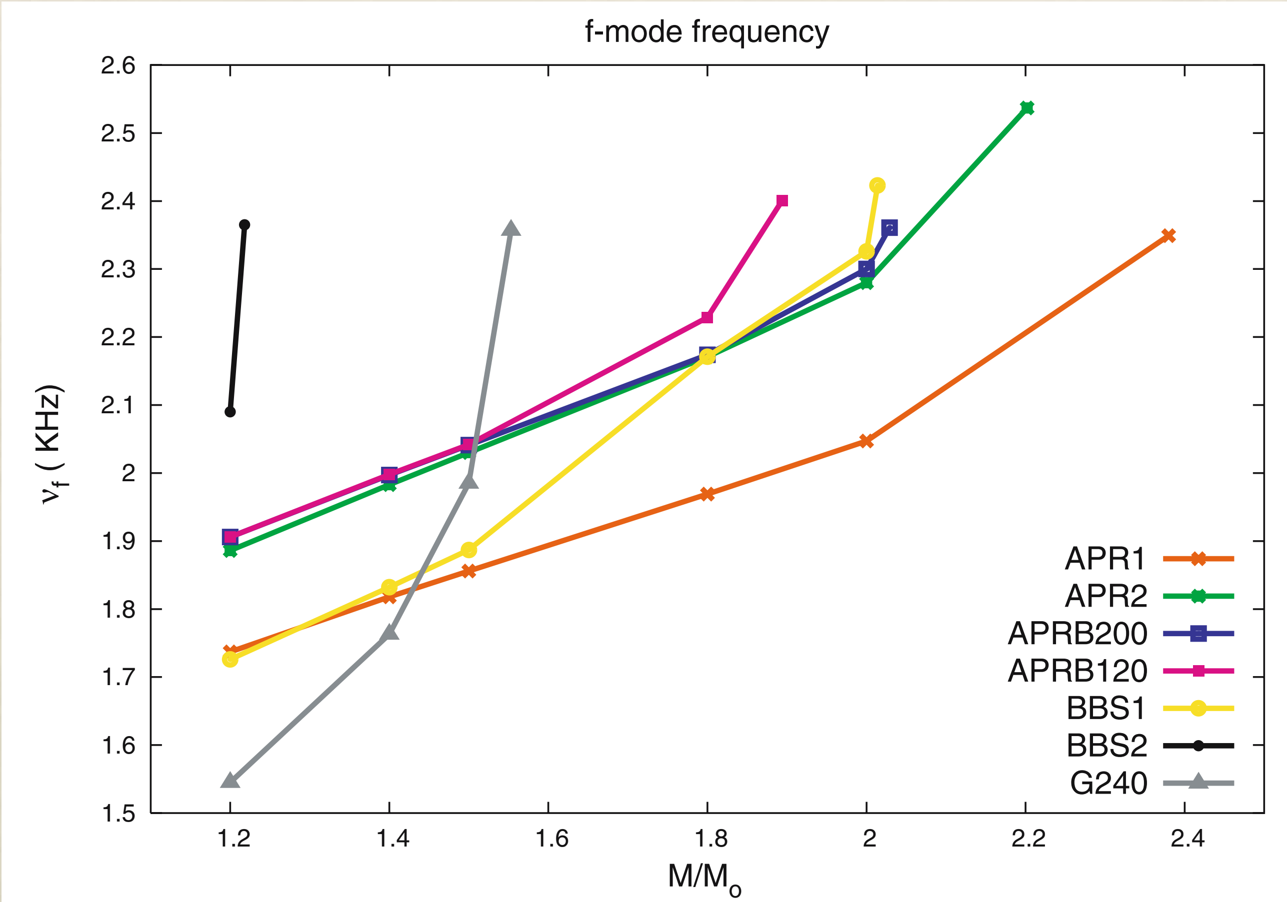
Issue related to QNMs

- **Completeness:**
  - We expect that QNM do not form a complete basis for the perturbation. For Schwarzschild [Leaver 62] this is due to a branch cut in the Green function. Power-law tail
- **Stability of Schwarzschild:**
  - Vishveshvara '70 showed that the imaginary part of the QNM frequency is always negative
  - Wald '79 showed that if the imaginary part of the QNM frequency is always negative all perturbation remains bounded
- **Some evidence for Kerr Black Hole**

Valeria Ferrari & Leonardo Gualtieri: Quasi-normal modes and gravitational wave astronomy: Gen. Relativ. Gravit. 0001-7701 (Print) 1572-9532 (Online)

Frequency of mode depends on the EOS

- **Frequency of the fundamental mode for different realistic EOS**
  - G240 Relativistic Mean Field Theory
  - Non relativistic Hamiltonian describing the Electroweak equilibrium of neutron, proton, muon, electron
    - APR1...APRB120: three body Urbana IX
    - BBS1, BBS2: three body Urbana VII



- **STRONG DEPENDENCE ON THE USED EQUATION OF STATE**

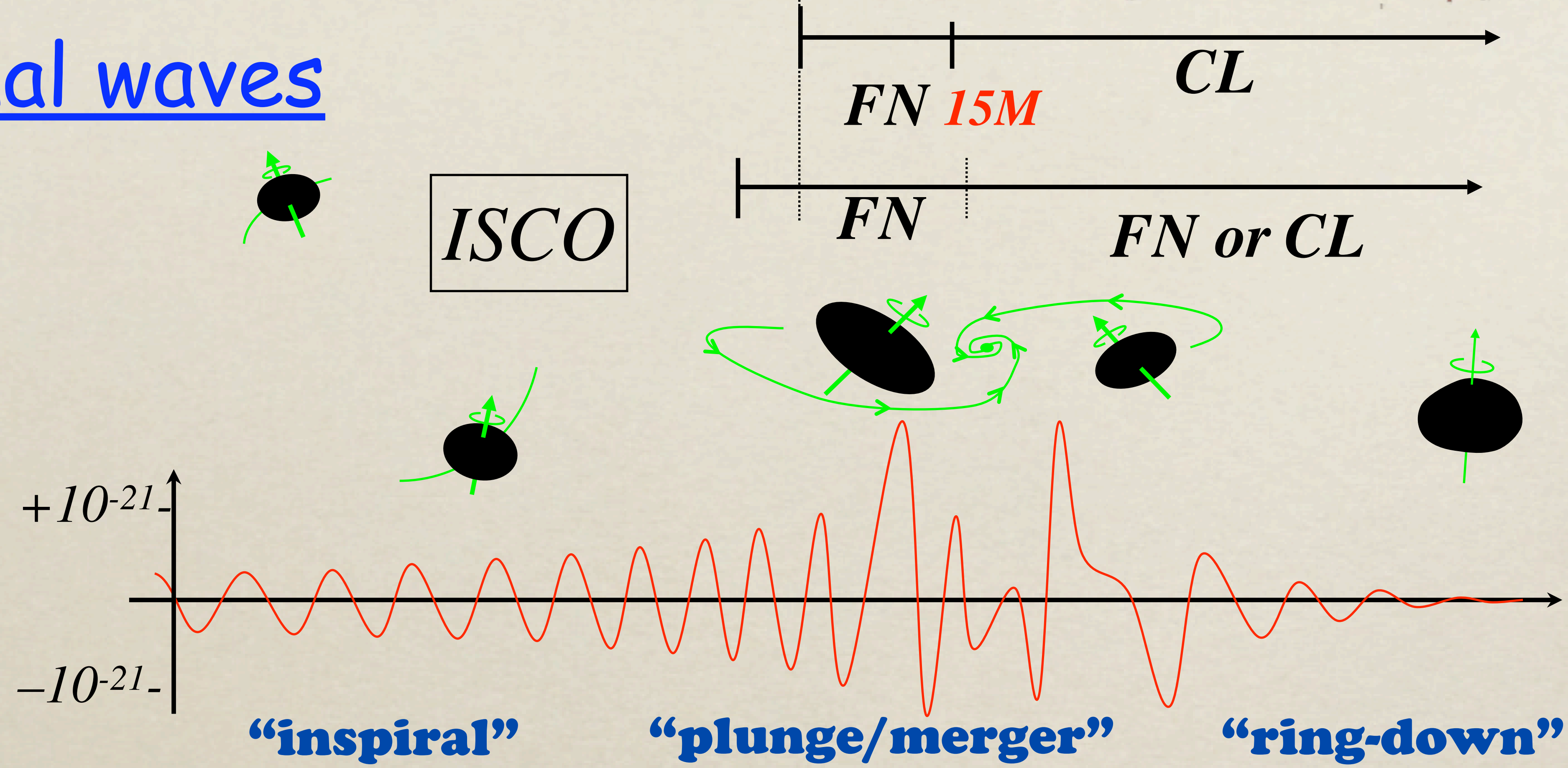
Introduction to Numerical Relativity

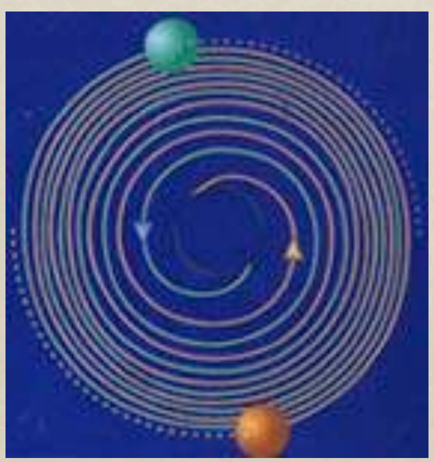
Roberto De Pietri
(Parma University)

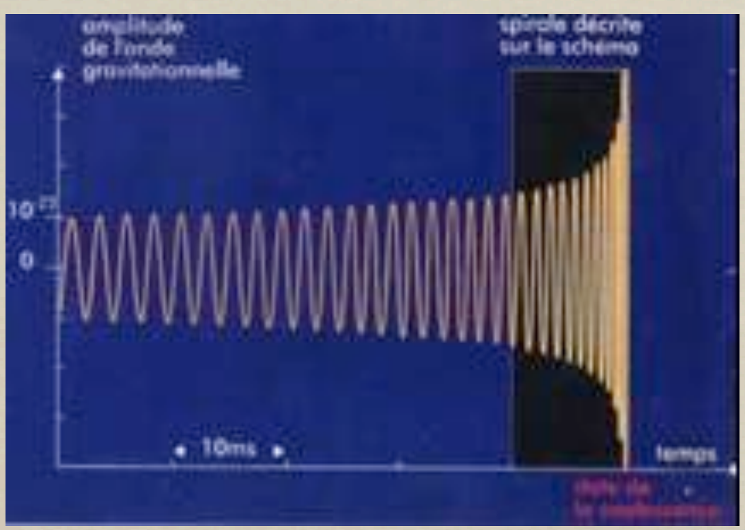
Lecture 2

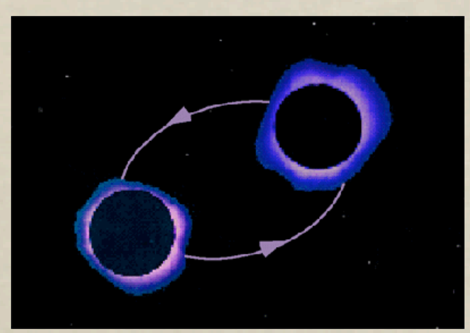
Two-bodies problem and GWs

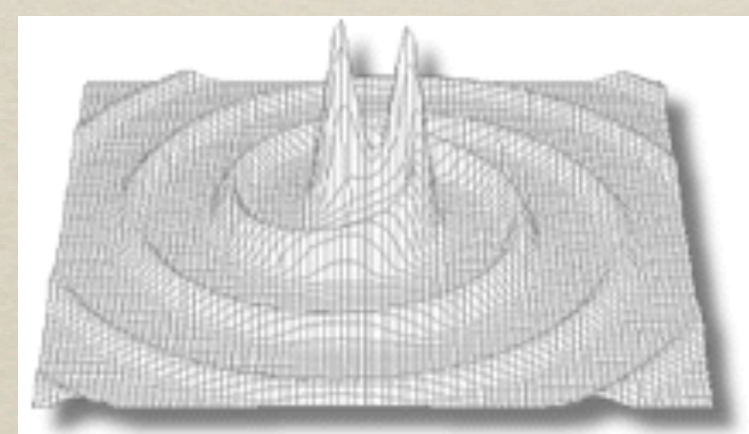
Gravitational waves











Black Hole Merger results

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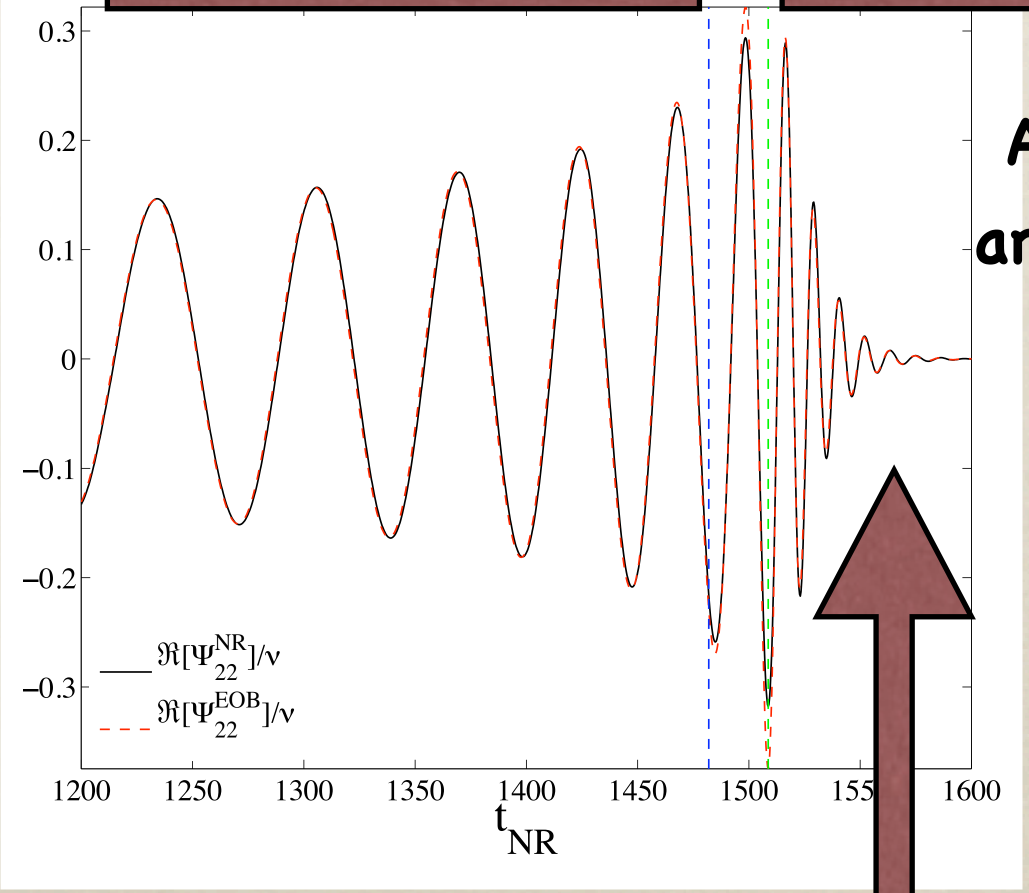
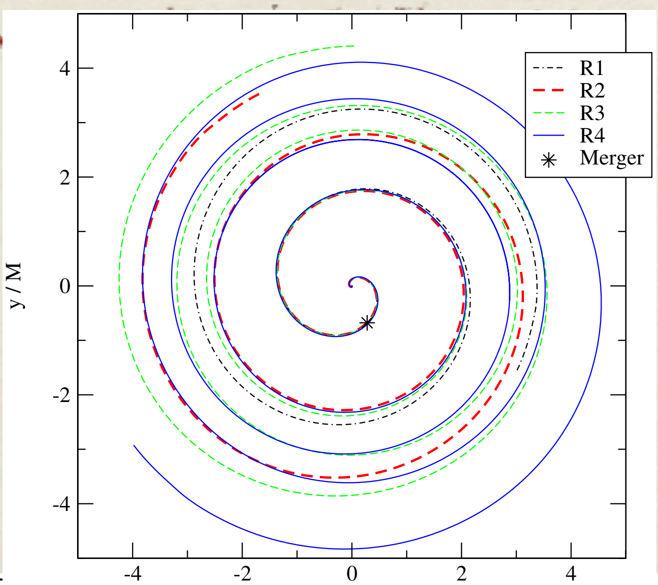


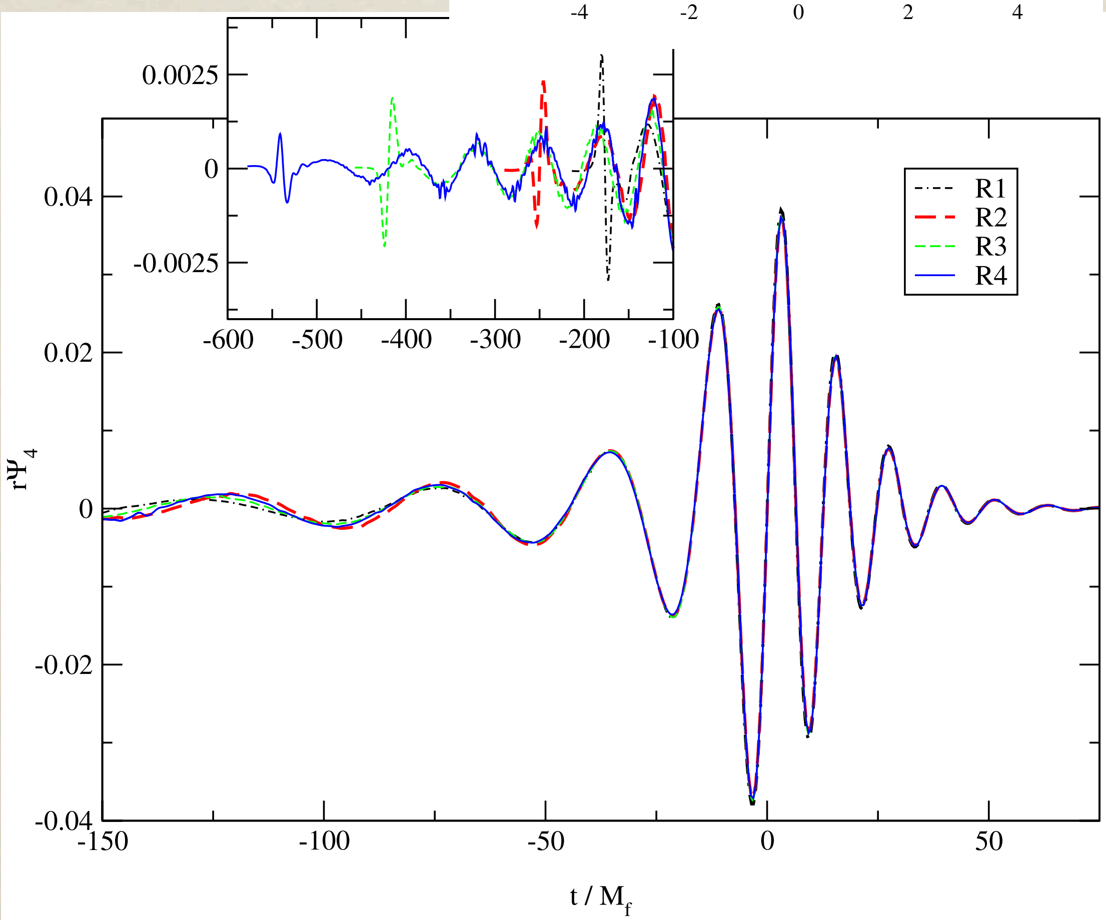
AEI simulation

arXiv:0712.3003

Exponentially dumped oscillation (QNM)

PHYSICAL REVIEW D **73,** 104002 (2006)





Inspiral part of the signal: .....

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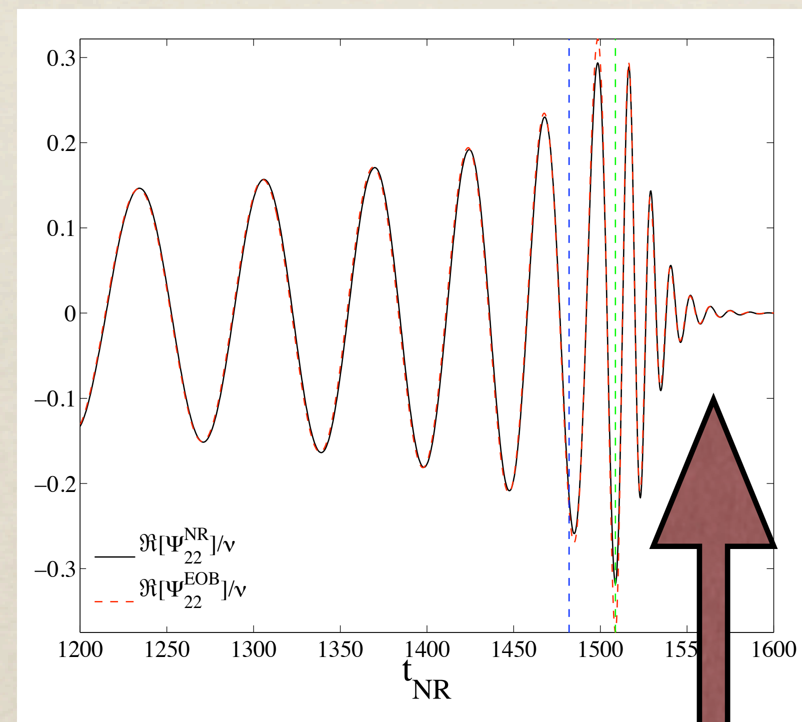
- Post Newtonian approximation ....
- ... Damour EOB (Effective one body) waveforms for the two bodies problem.
- See Damour-Nagar about matching Numerical-Relativity waveform and EOB ones.



Exponentially dumped oscillation (QNM)

QNM part of the signal: 1D codes

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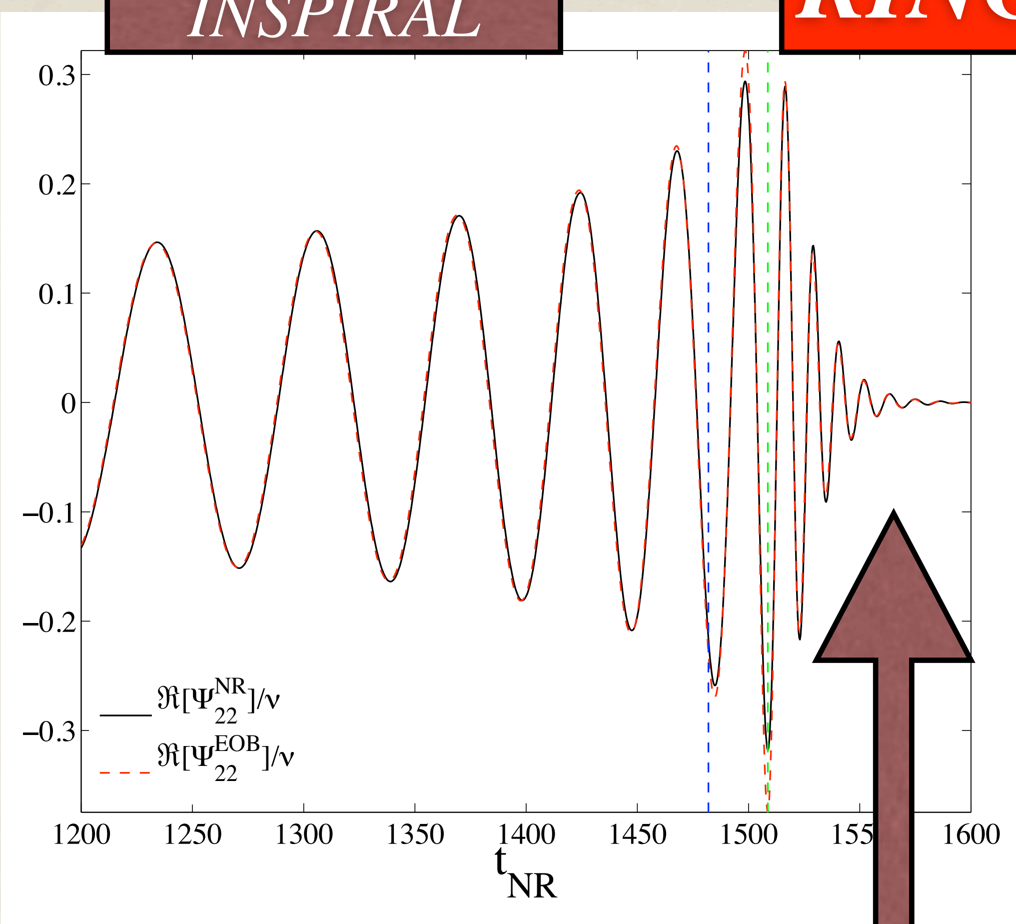
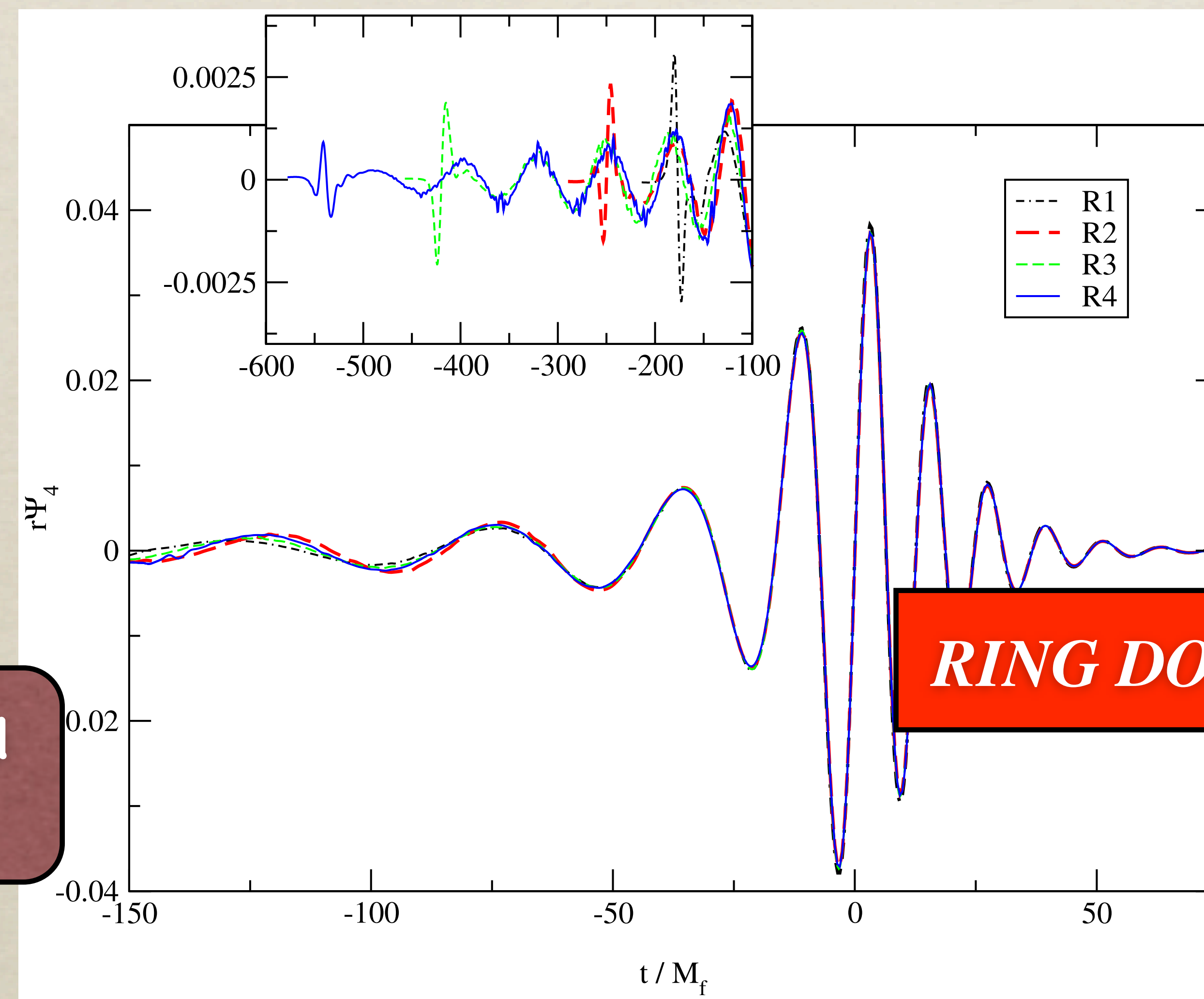


Exponentially dumped oscillation (QNM)



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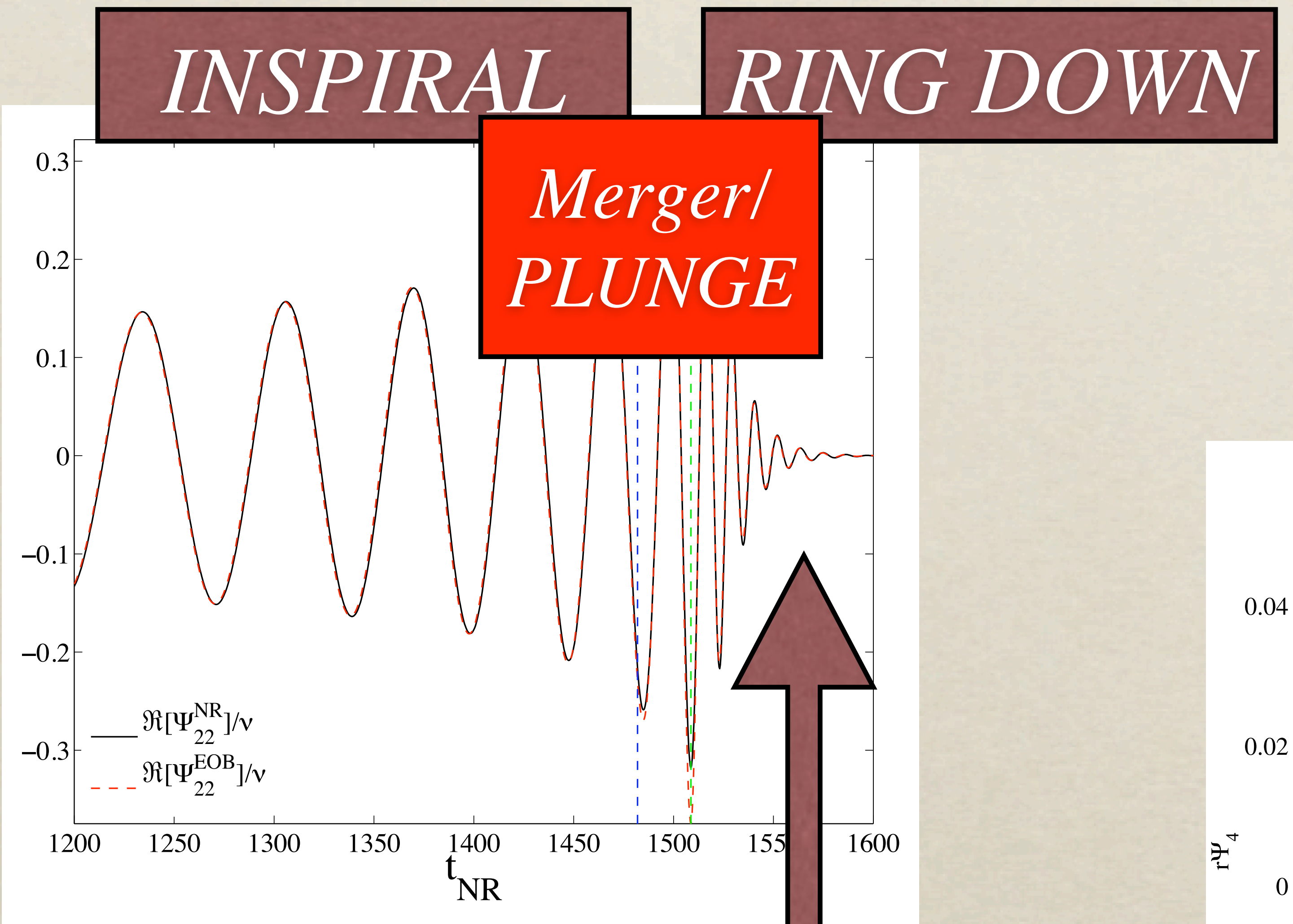
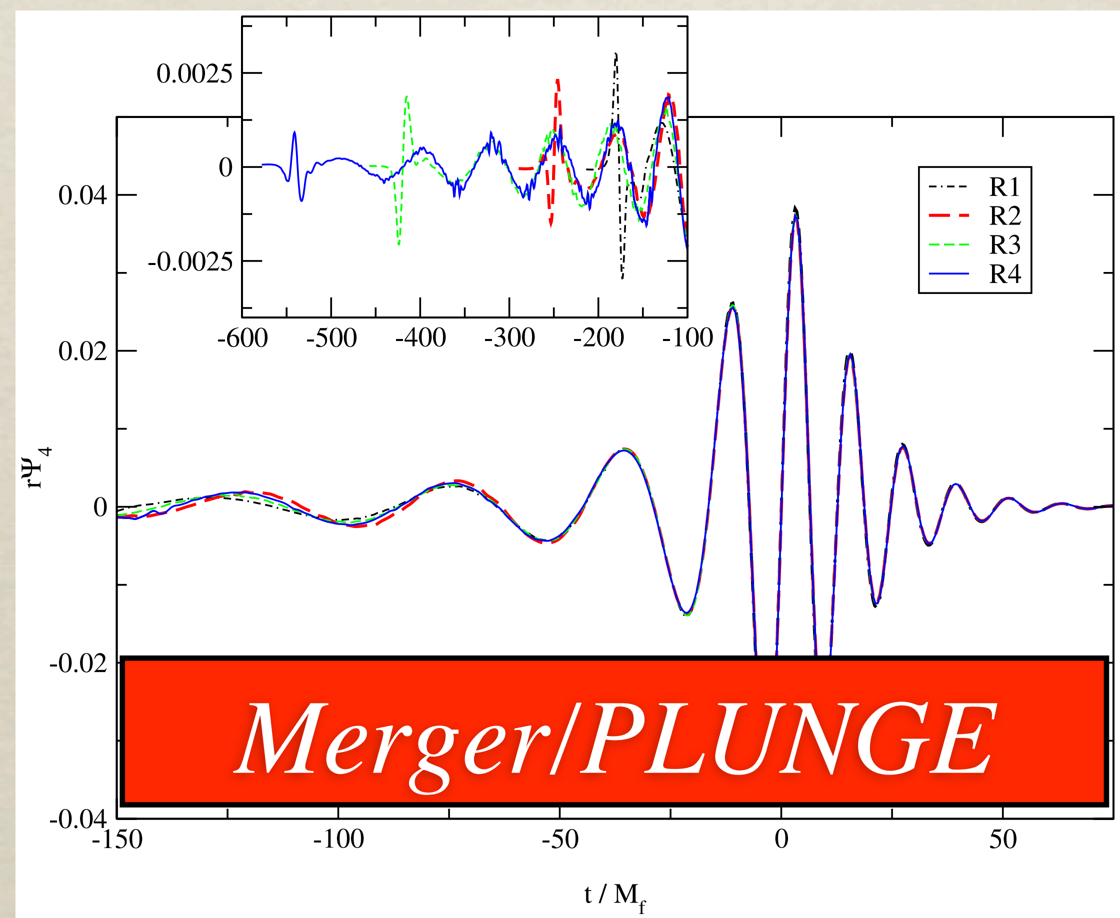
MERGER part of the signal: 3D codes



Exponentially dumped oscillation (QNM)



Numerical General Relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$ Einstein Equations

$$\nabla_\mu T^{\mu\nu} = 0$$ Conservation of energy momentum

$$\nabla_\mu(\rho\, u^\mu) = 0$$ Conservation of baryon density

$$p = p(\rho, \epsilon)$$ Equation of state

- Introduce a foliation of space-time
- write as a 3+1 evolution equation
- solve them on a computer !

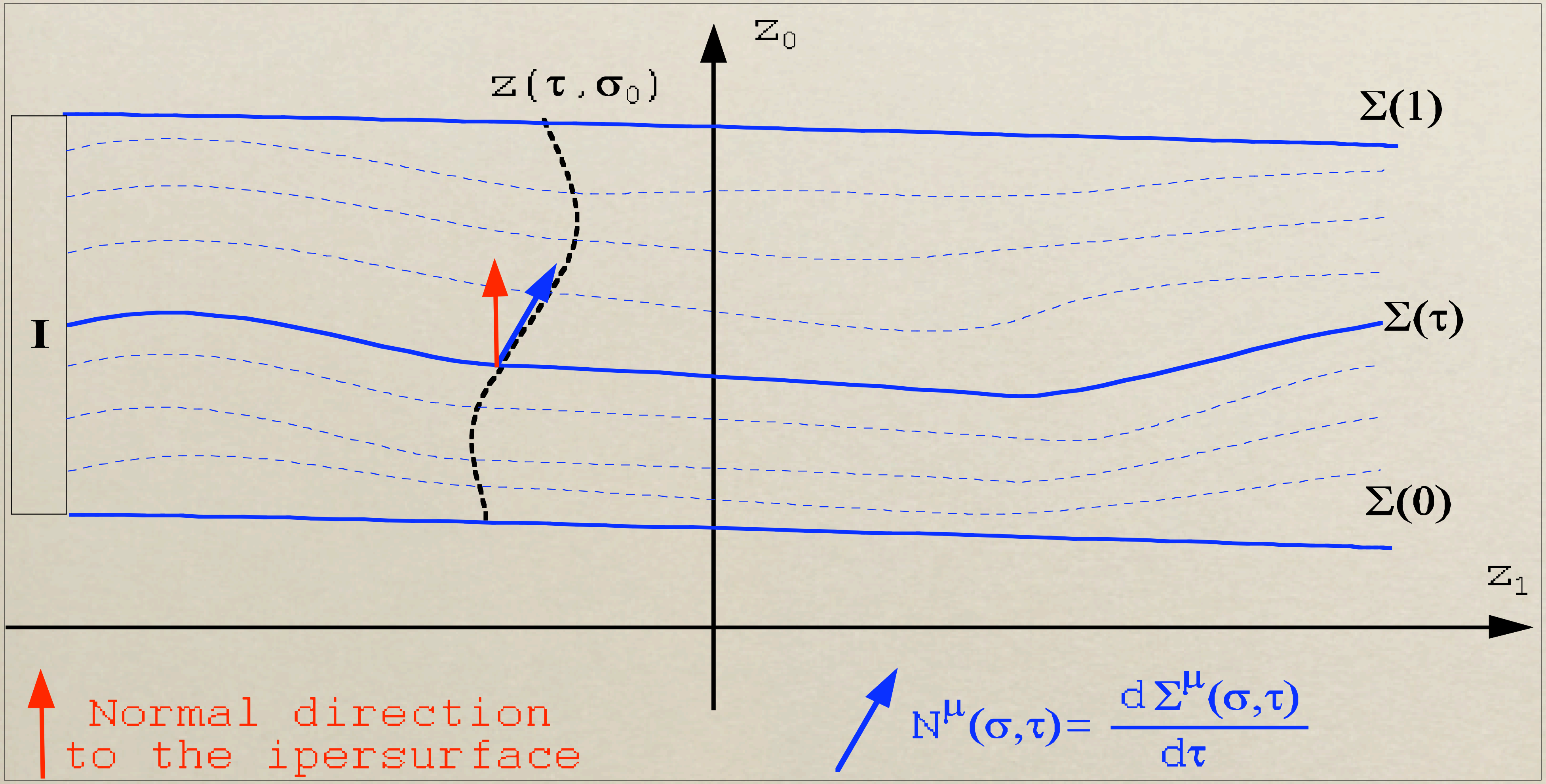
$$T^{\mu\nu} = (\rho(1+\epsilon) + p) u^\mu u^\nu + p g^{\mu\nu}$$

Why Numerical Relativity is hard!

- No obviously "better" formulation of Einstein's equations
  - ADM, conformal decomposition, first-order hyperbolic form,.... ???
- Coordinates (spatial and time) do not have a special meaning
  - this gauge freedom need to be carefully handled
  - gauge conditions must avoid singularities
  - gauge conditions must counteract "grid-stretching"
- Einstein's Field equations are highly non-linear
  - Essentially unknown in this regime
- Physical singularity are difficult to deal with

3+1 formulation

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$



α :: lapse

β^i :: shift vector

γ_{ij} :: 3-metric

$$N^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\alpha}\left(\frac{\partial}{\partial t} - \beta^j \frac{\partial}{\partial x^j}\right)$$

ADM evolution

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (2.1)$$

6 equations for the metric

$$\begin{aligned}\partial_t K_{ij} = & -\nabla_i \nabla_j \alpha + \alpha \Big[R_{ij} + K\, K_{ij} - 2 K_{im} K^m_j \\ & -8\pi \left(S_{ij} - \frac{1}{2}\gamma_{ij} S \right) - 4\pi \rho_{\rm ADM} \gamma_{ij} \Big] \\ & + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m . \end{aligned} \quad (2.2)$$

+6 equations for the time-coordinate derivative of the metric (extrinsic curvature)

Hamiltonian + Momentum constraints

$$^{(3)}R + K^2 - K_{ij} K^{ij} - 16\pi \rho_{\rm ADM} = 0$$

+1 constrain equation

$$\nabla_j K^{ij} - \gamma^{ij} \nabla_j K - 8\pi j^i = 0$$

+3 constrain equation

ADM evolution is not stable !

- Use BSSN rewriting of the evolution equation

$$\partial_t \varphi = -\tfrac{1}{6}\alpha K + \beta^i \partial_i \varphi + \tfrac{1}{6}\partial_i \beta^i$$

$$\partial_t K = -g^{ij}\nabla_i \nabla_j \alpha + \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \tfrac{1}{3}K) + \beta^i \partial_i K$$

$$\partial_t \tilde{g}_{ij} = -2\alpha K_{ij} + \tilde{g}_{jk}\partial_i \beta^k + \tilde{g}_{ik}\partial_j \beta^k - \tfrac{2}{3}\tilde{g}_{ij}\partial_k \beta^k$$

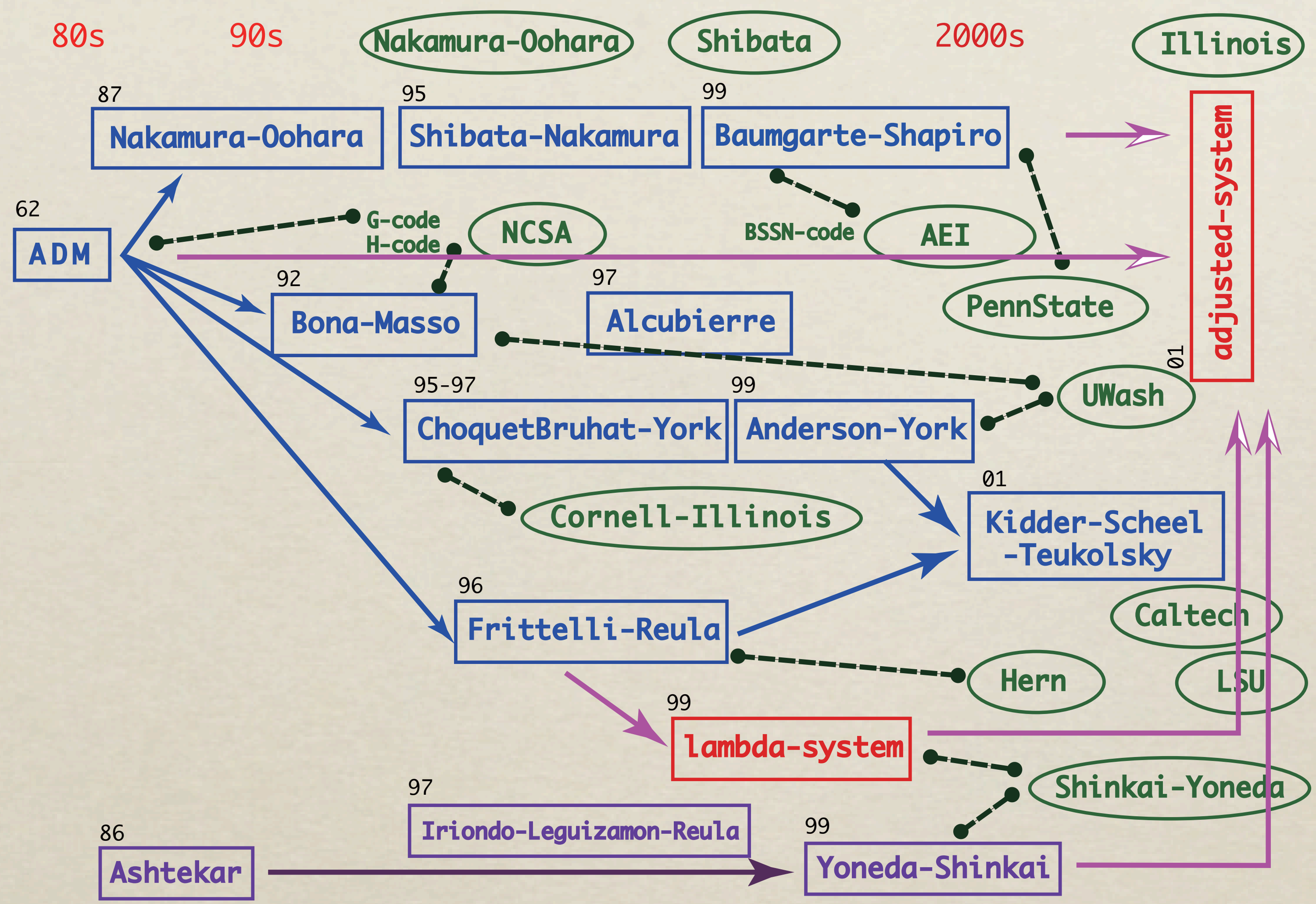
$$\partial_t \tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha(\Gamma^i_{jk}\tilde{A}^{jk} - \tfrac{2}{3}\tilde{g}^{ij}\partial_j K + 6\tilde{A}^{ij}\partial_j \varphi) +$$
$$+ \beta^k \partial_k \tilde{\Gamma}^i - \tilde{\Gamma}^k \partial_k \beta^i + \tfrac{2}{3}\tilde{\Gamma}^i \partial_k \beta^k + \tfrac{1}{3}\tilde{g}^{ij}\partial_j \partial_k \beta^k + \tilde{g}^{jk}\partial_j \partial_k \beta^i$$

$$\partial_t \tilde{A}_{ij} = e^{-4\varphi}(-(\nabla_i \nabla_j \alpha)^{TF} + \alpha R^{TF}_{ij}) + \alpha\left(\tilde{A}_{ij}K - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) - \partial_i \partial_j \alpha +$$
$$+ \beta^k \partial_k \tilde{A}_{ij} + \left(\tilde{A}_{ik}\partial_j + \tilde{A}_{jk}\partial_i\right)\beta^\kappa - \tfrac{2}{3}\tilde{A}_{ij}\partial_k \beta^k$$

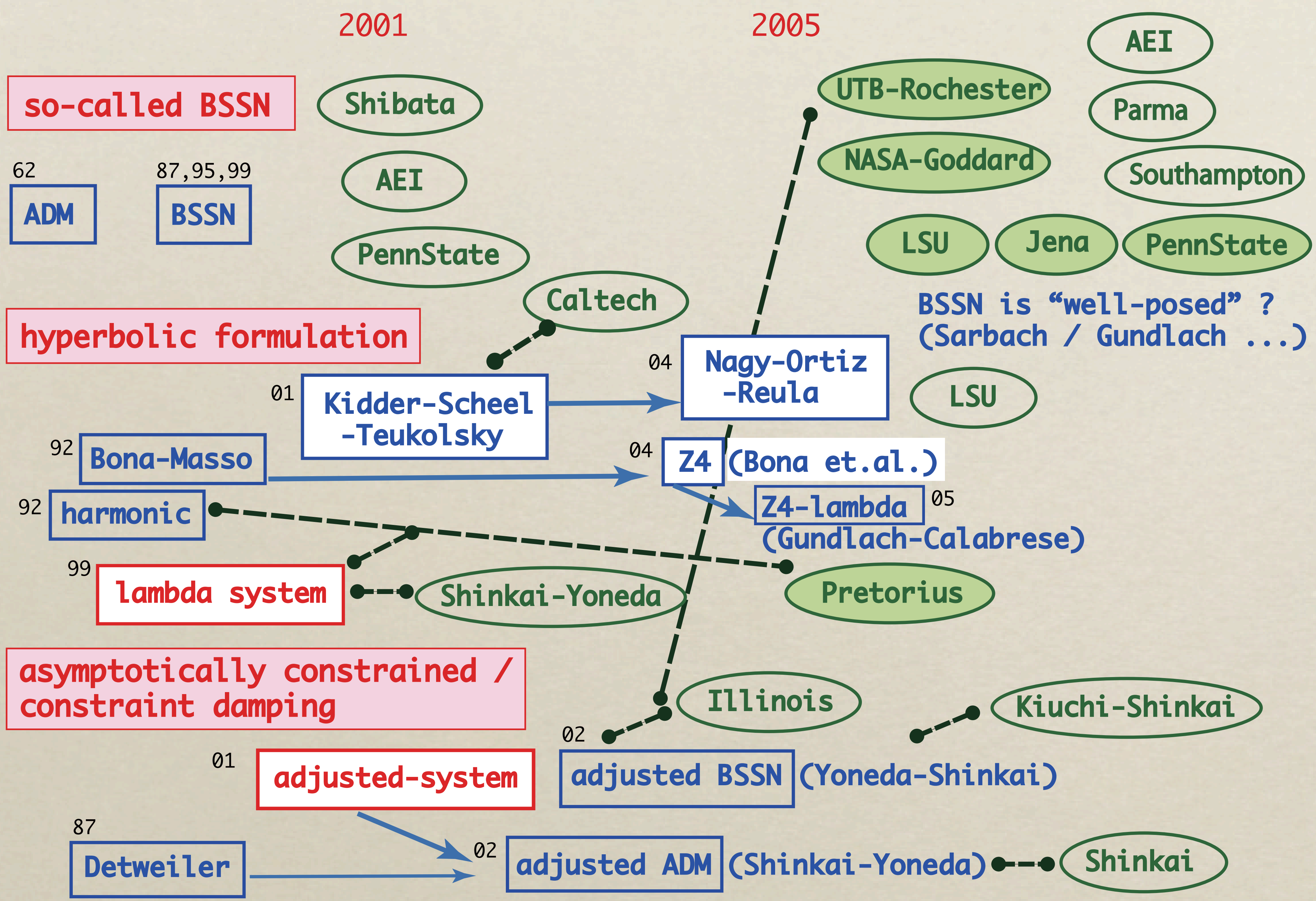
- or Use Harmonic evolution equations

Other schemes (beside BSSN)

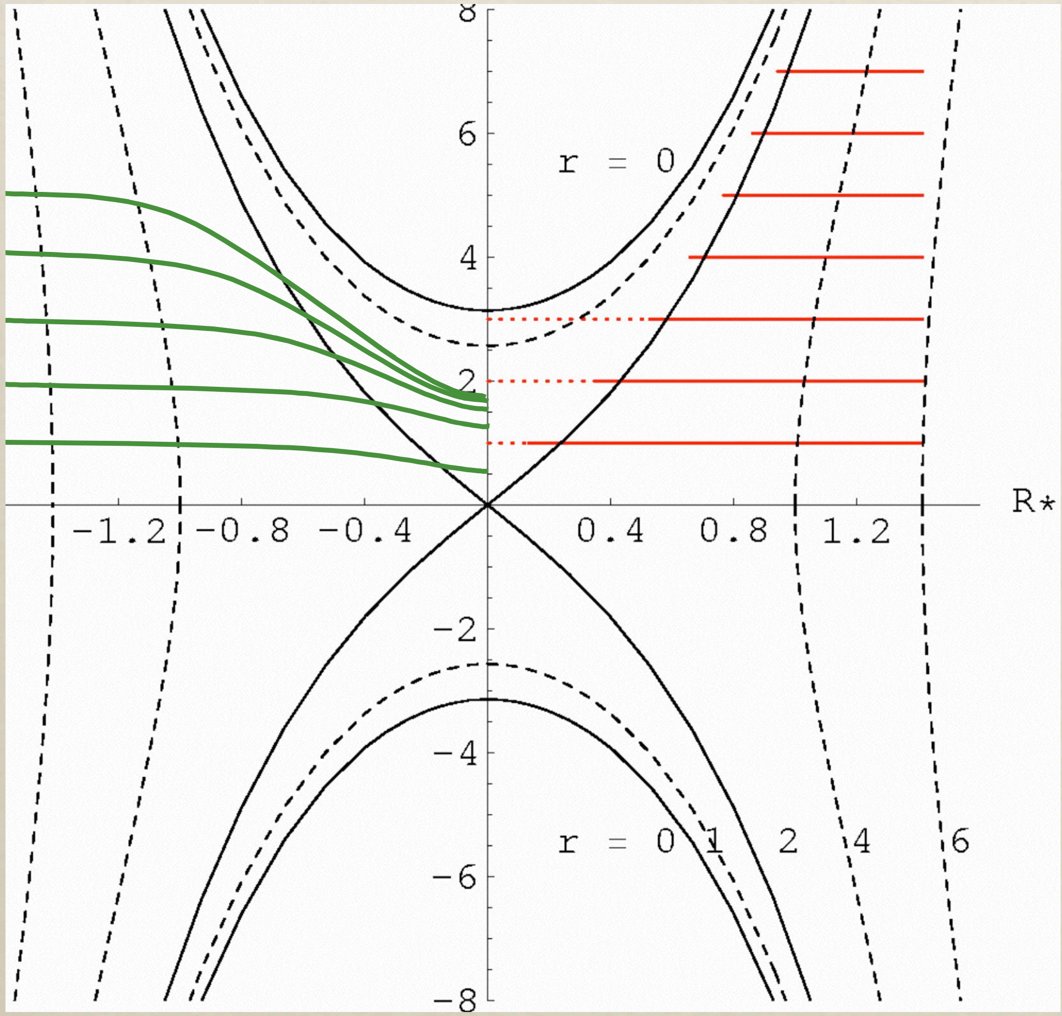
- See Hisa-aki Shinka, **Formulations of the Einstein equations for numerical simulations**, arXiv:0805.0068 for a review.



Situation NOW: from 0805.0068



The problem of foliations.....



- Schwarzschild in Novikov Coordinates
- Geodesic slicing $(\alpha = 1,\ \beta^i = 0)$
- Singularity avoiding $(\beta^i = 0)$
- excision/puncture evolution

1+log $\quad \partial_t \alpha = -2\alpha(K - K_0)$

Gamma-driver $\quad \partial_t^2 \beta^i = \frac{3}{4}\alpha\partial_t\tilde{\Gamma}^i - 2\partial_t\beta^i$

Code Used

- **CACTUS/BSSN:** (www.cactuscode.org)
  Mainly developed at AEI (Golm, Germany) and LSU (USA)

- **WHISKY:** (http://www.aei-potsdam.mpg.de/~hawke/Whisky.html)
  Whisky is a code to evolve the equations of hydrodynamics on curved space. It is being written by and for members of the EU Network on Sources of Gravitational Radiation and is based on the Cactus Computational Toolkit.

- **Gauge choice** for the lapse and shift variables:

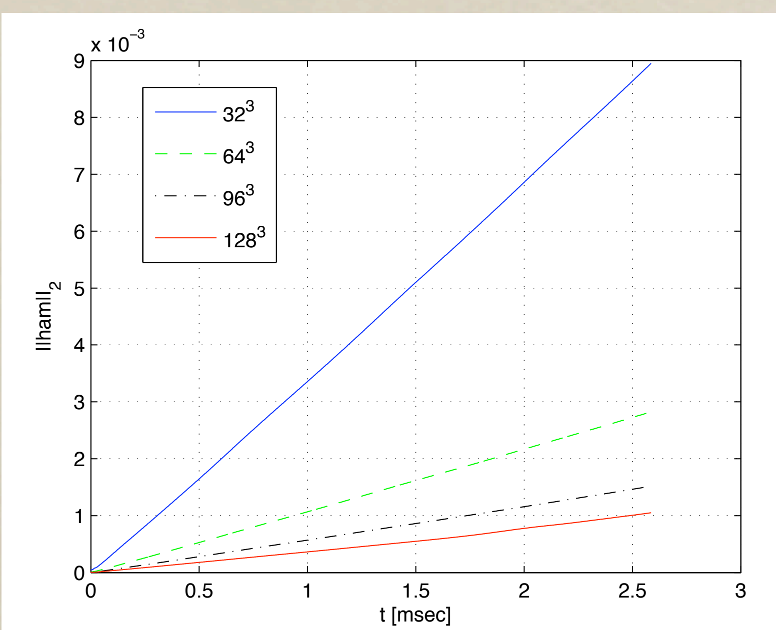
$$\text{1+log} \qquad \partial_t \alpha = -2\alpha(K - K_0)$$

$$\text{Gamma-driver} \qquad \partial_t^2 \beta^i = \frac{3}{4}\alpha \partial_t \tilde{\Gamma}^i - 2\partial_t \beta^i$$

Cactus ⇨ Infrastructure + GR

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (2.1)$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \Big[R_{ij} + K\, K_{ij} - 2K_{im} K^m_j - 8\pi \left(S_{ij} - \frac{1}{2}\gamma_{ij} S \right) - 4\pi \rho_{\rm ADM} \gamma_{ij} \Big] + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m . \quad (2.2)$$



Hamiltonian + Momentum constraints

$${}^{(3)}R + K^2 - K_{ij}K^{ij} - 16\pi \rho_{\rm ADM} = 0$$

$$\nabla_j K^{ij} - \gamma^{ij} \nabla_j K - 8\pi j^i = 0$$

WHISKY ⇨ Matter evolution

Write hydrodynamic equation in a flux conservative form [*] J. A. Font, Living Rev. Relativity 6, 4 (2003).

Use HRSC methods to solve the equations

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad \nabla_\mu(\rho u^\mu) = 0\,.$$

$$\Rightarrow \quad \partial_t \mathbf{q} + \partial_i \mathbf{f}^{(i)}(\mathbf{q}) = \mathbf{s}(\mathbf{q})$$

$$\mathbf{q} \equiv (D, S^i, \tau)$$

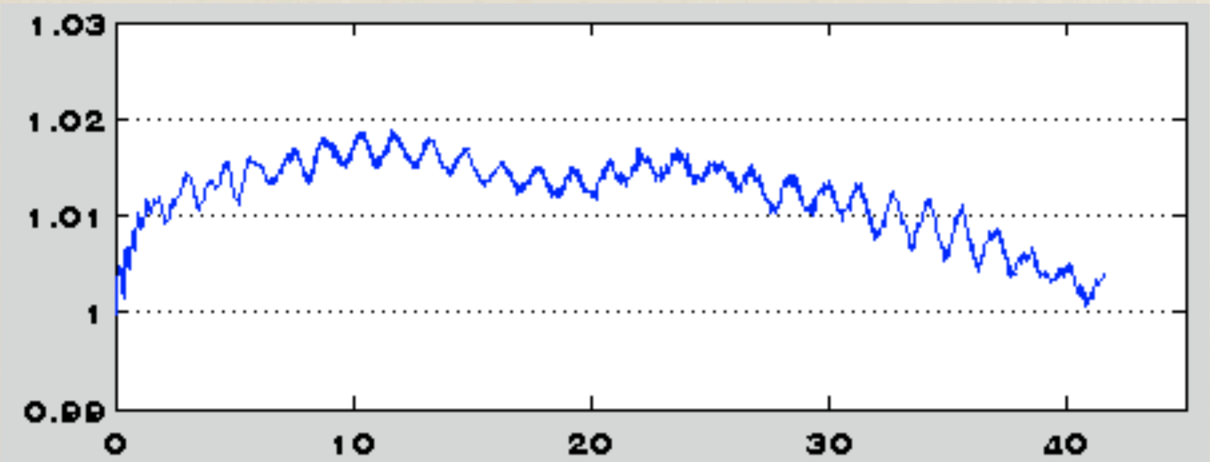
$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

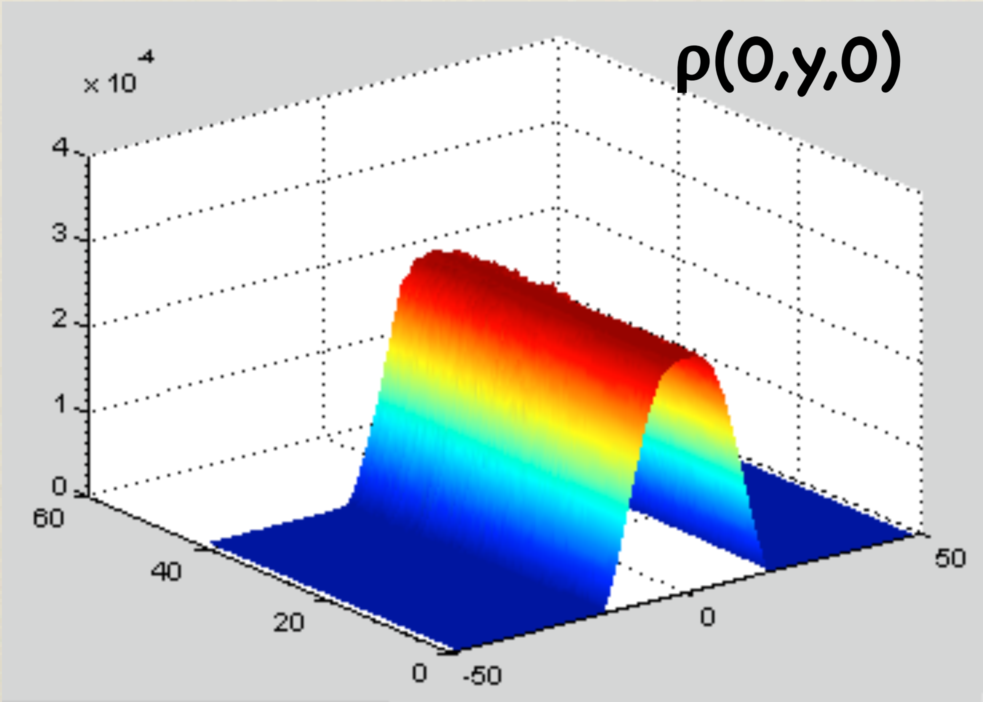
$$h = 1 + \varepsilon + \frac{p}{\rho}$$

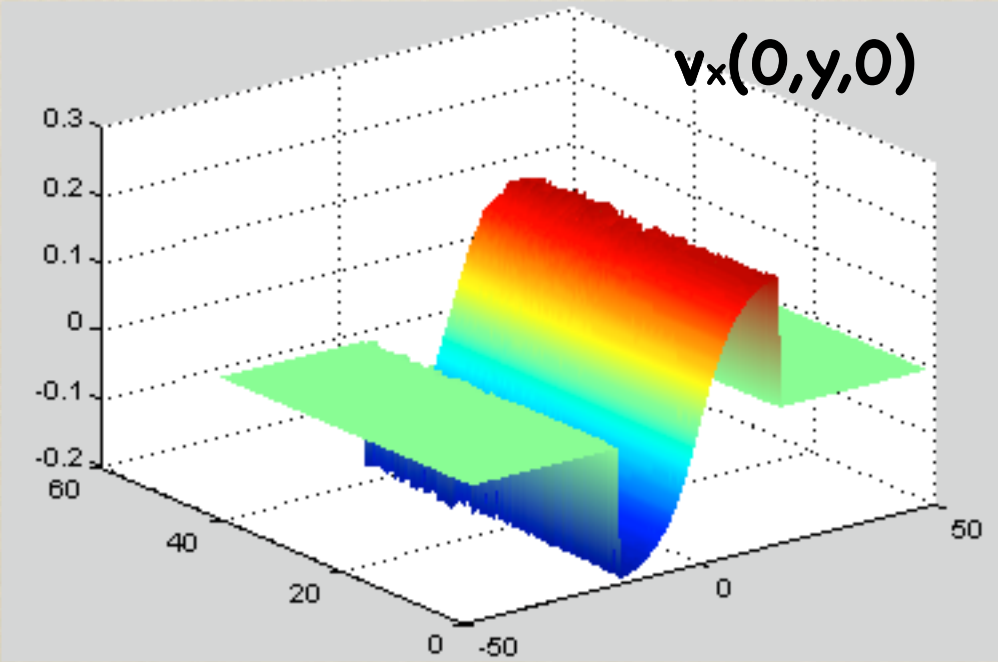
$$\begin{aligned} D &\equiv \rho^* = \sqrt{\gamma} W \rho\,, \\ S^i &\equiv \sqrt{\gamma} \rho h W^2 v^i\,, \\ \tau &\equiv \sqrt{\gamma}\left(\rho h W^2 - p\right) - D \end{aligned}$$

Stable evolutions of stable star!

$\rho_c(t)/\rho_c(0)$ as function of t for stable model A10







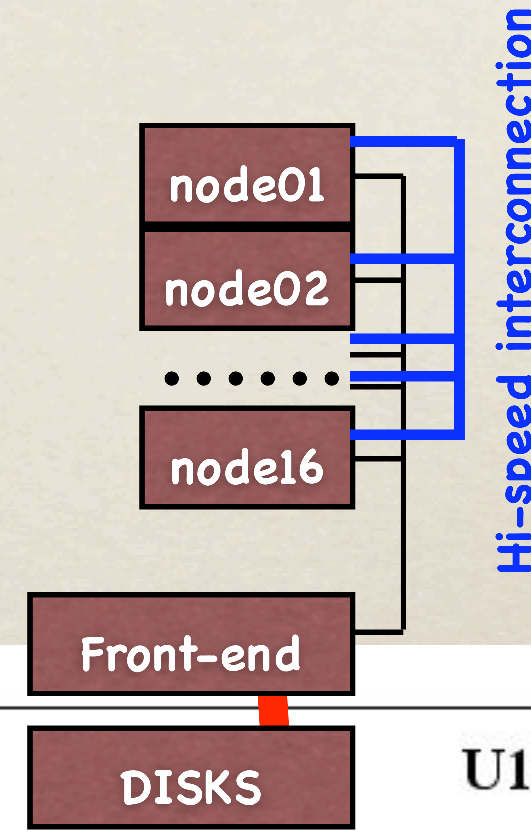
	β	Full GR	CFC [1]
A9	0.189	791 Hz	809 Hz
A10	0.223	674 Hz	685 Hz

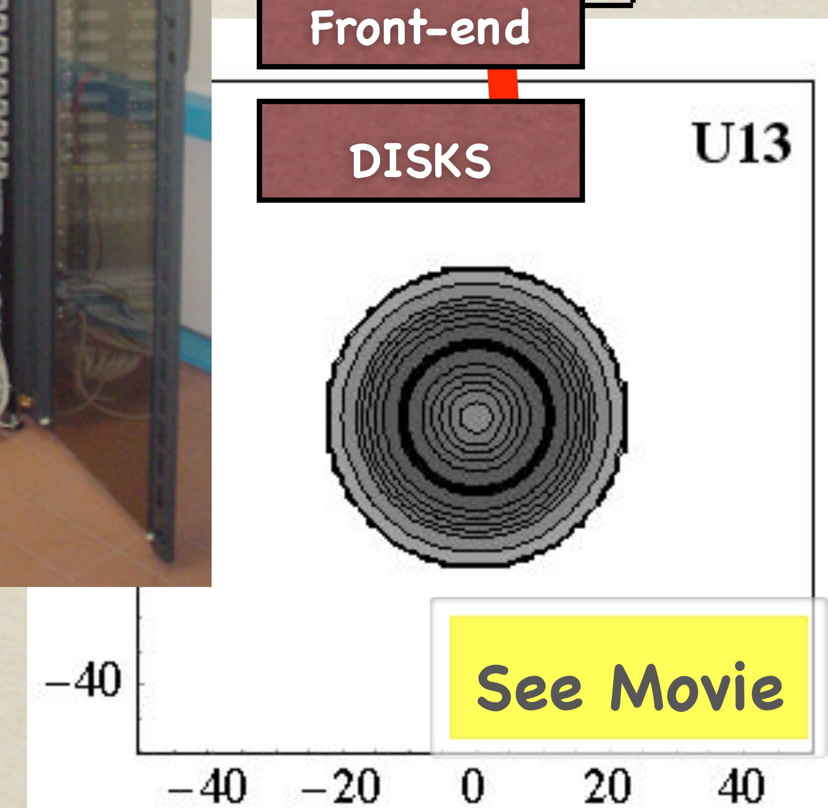
[1] Dimmelmeier, Stergioulas, Font: astro-ph/0511394

Computers for Numerical Relativity

- standard workstation nodes: e.g., biprocessor Opteron/Intel with 4-8 GBytes of RAM
- Fast interconnection, e.g., Infiniband
- A front-end workstation
- MPI communication Library
- Huge storage space to save results of the simulations

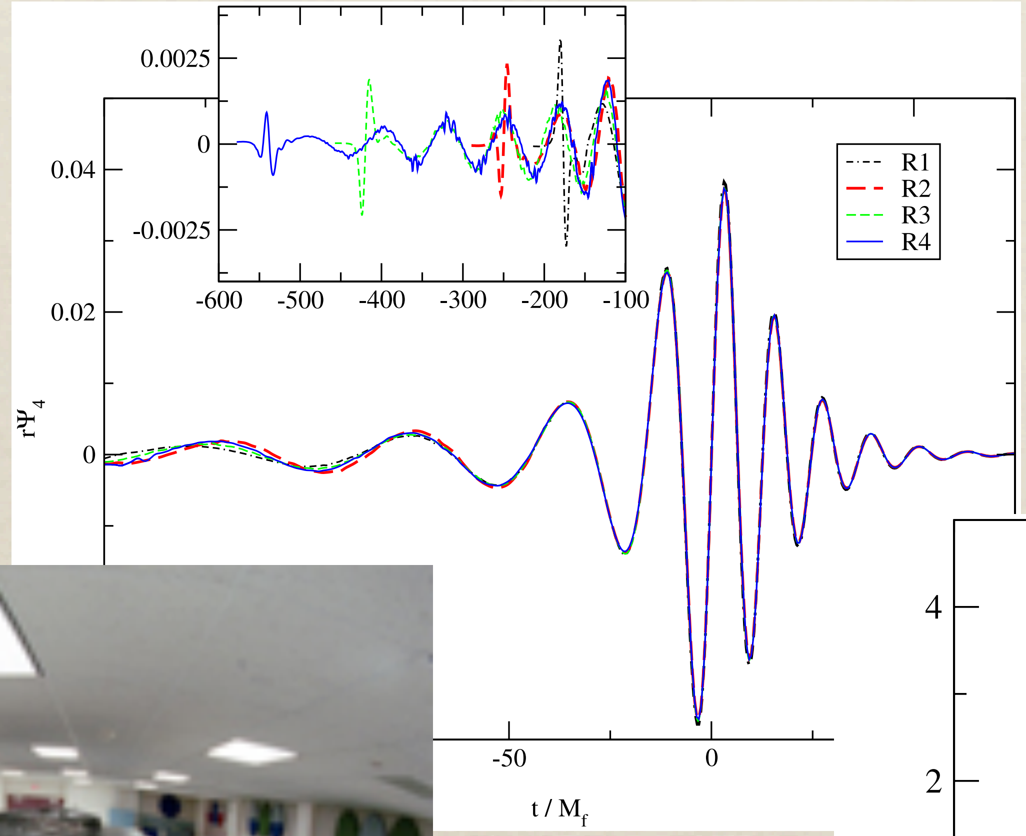
- WE NEED A LOT OT MEMORY !



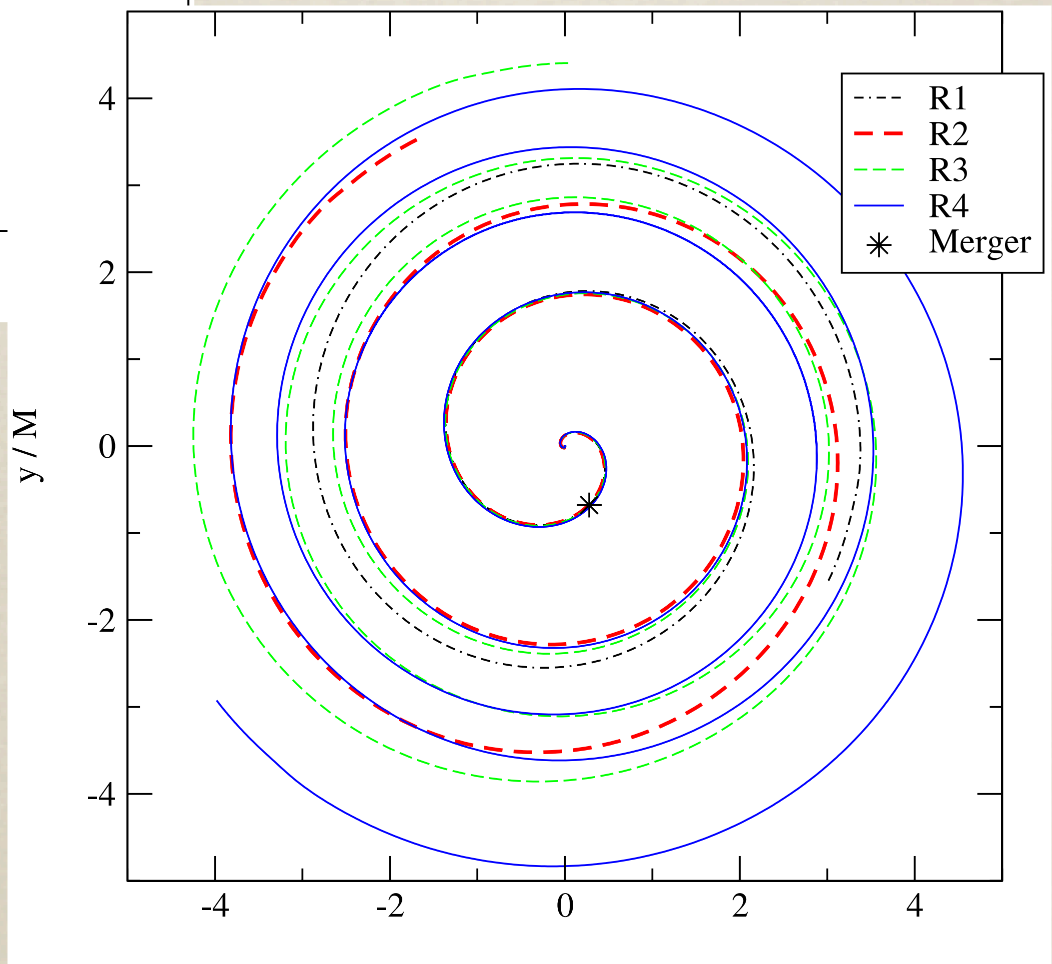


Computers for Numerical Relativity

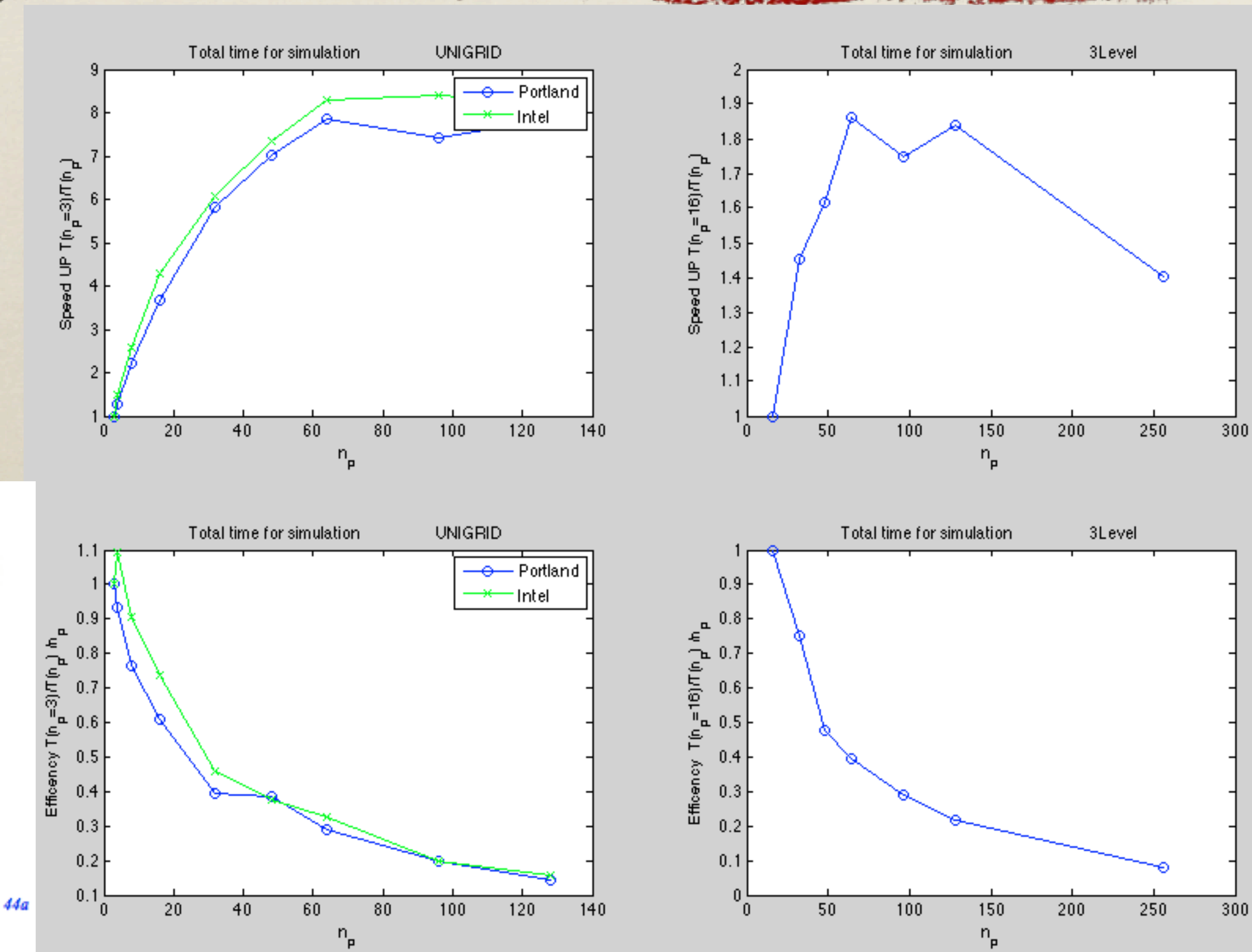
NASA Goddard (2005)



PHYSICAL REVIEW D **73,** 104002 (2006)



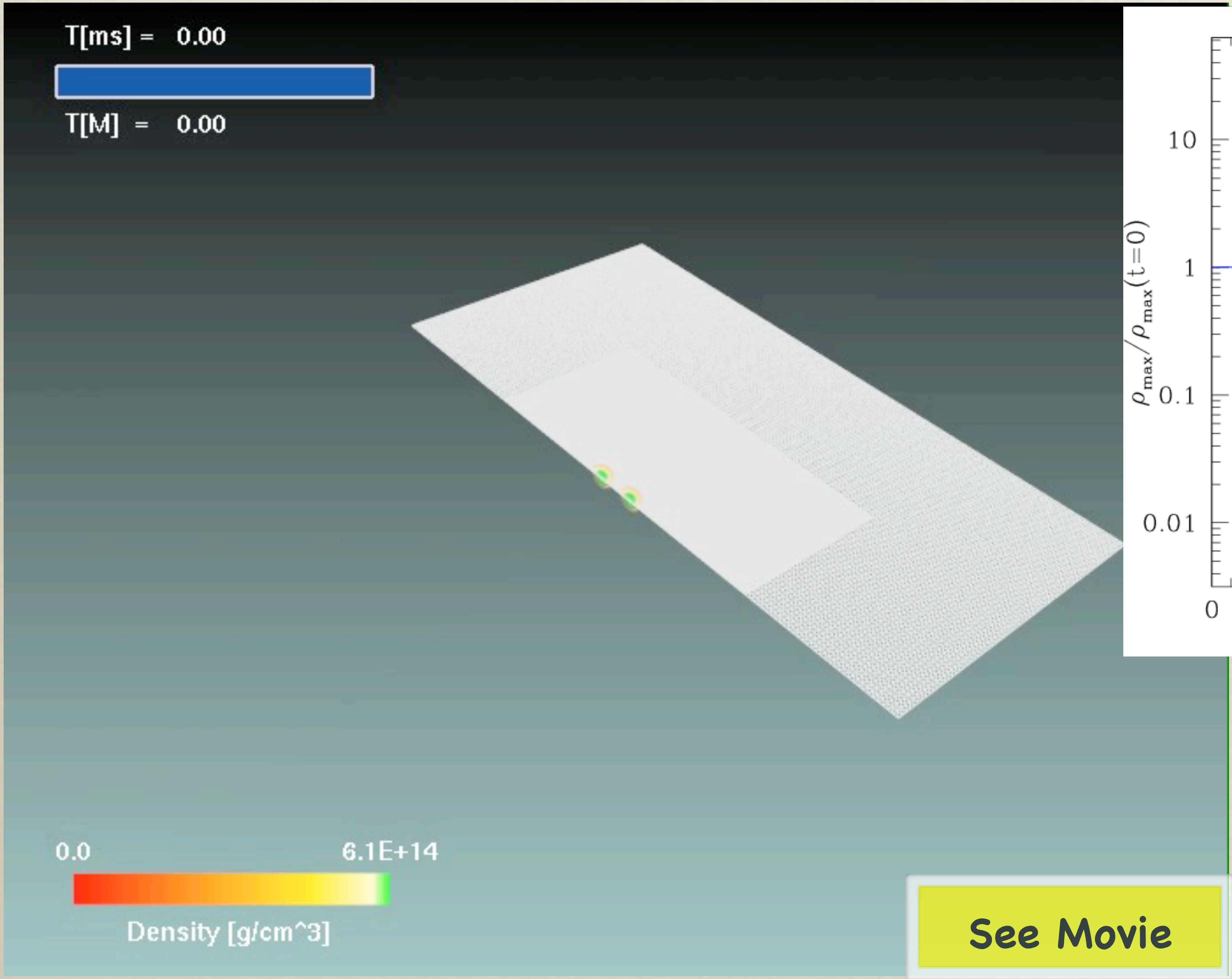
Code scaling on MPI clusters

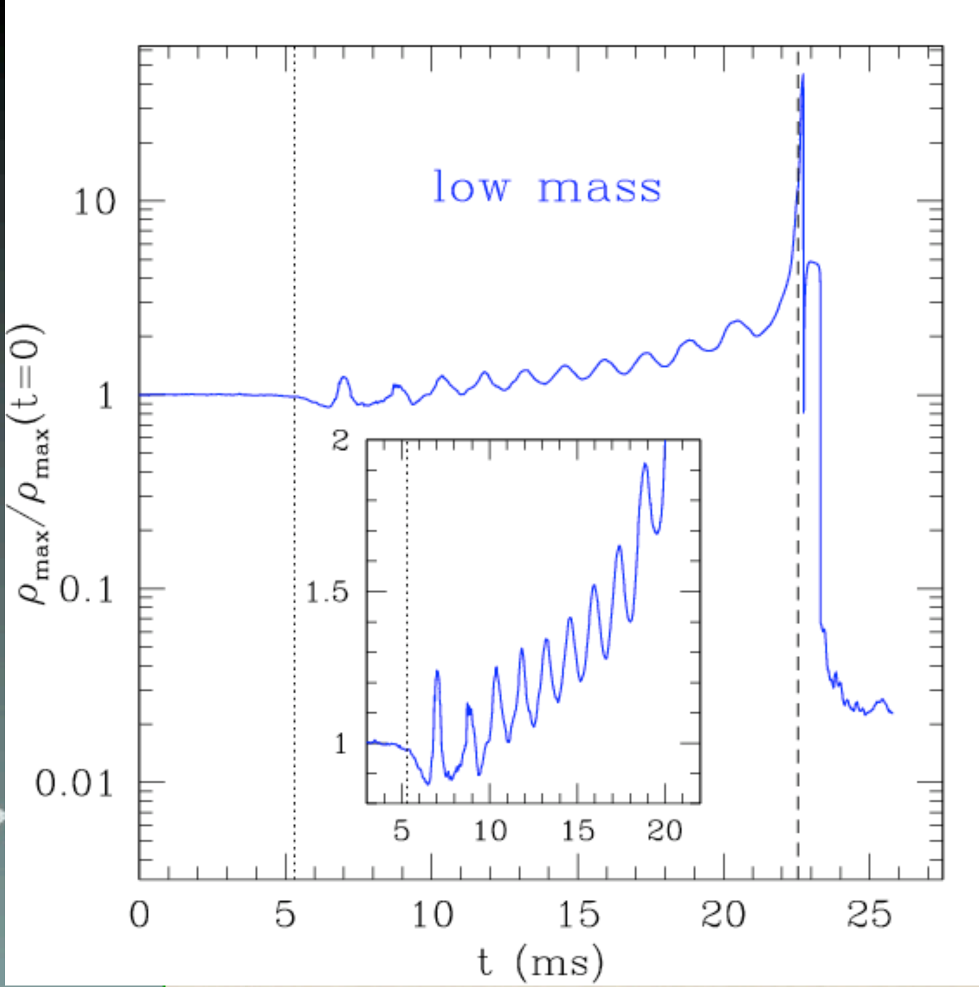


IBM BCX/5120, con 5120 processori
Lo scorso novembre, il sistema ha portato il Cineca alla 44a posizione nella prestigiosa lista TOP500 che raccoglie i server di calcolo più potenti al mondo.

Numerical relativity at work

Neutron star merger: low-mass merger to NS + disk





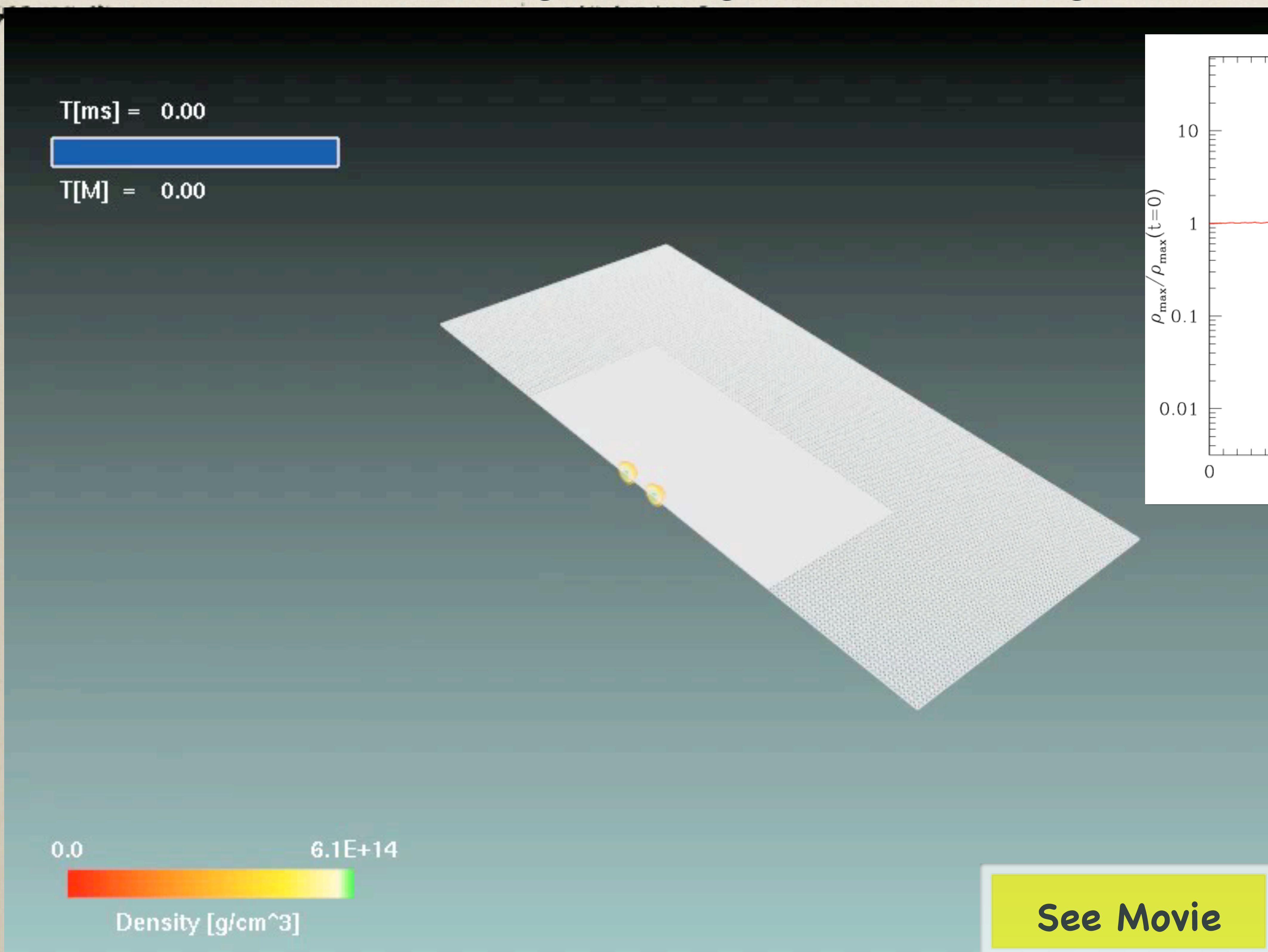
See Movie

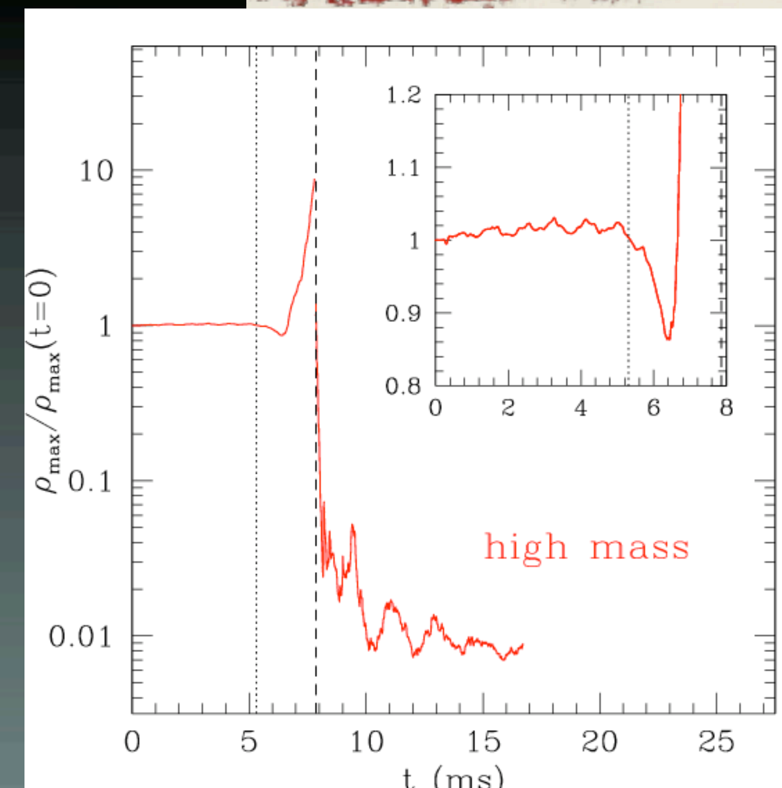
Credits: R. Kaehler & B. Giacommazzo & L. Rezzolla

http://arxiv.org/pdf/0804.0594

Numerical relativity at work

Neutron star merger: high-mass merger to BH + disk





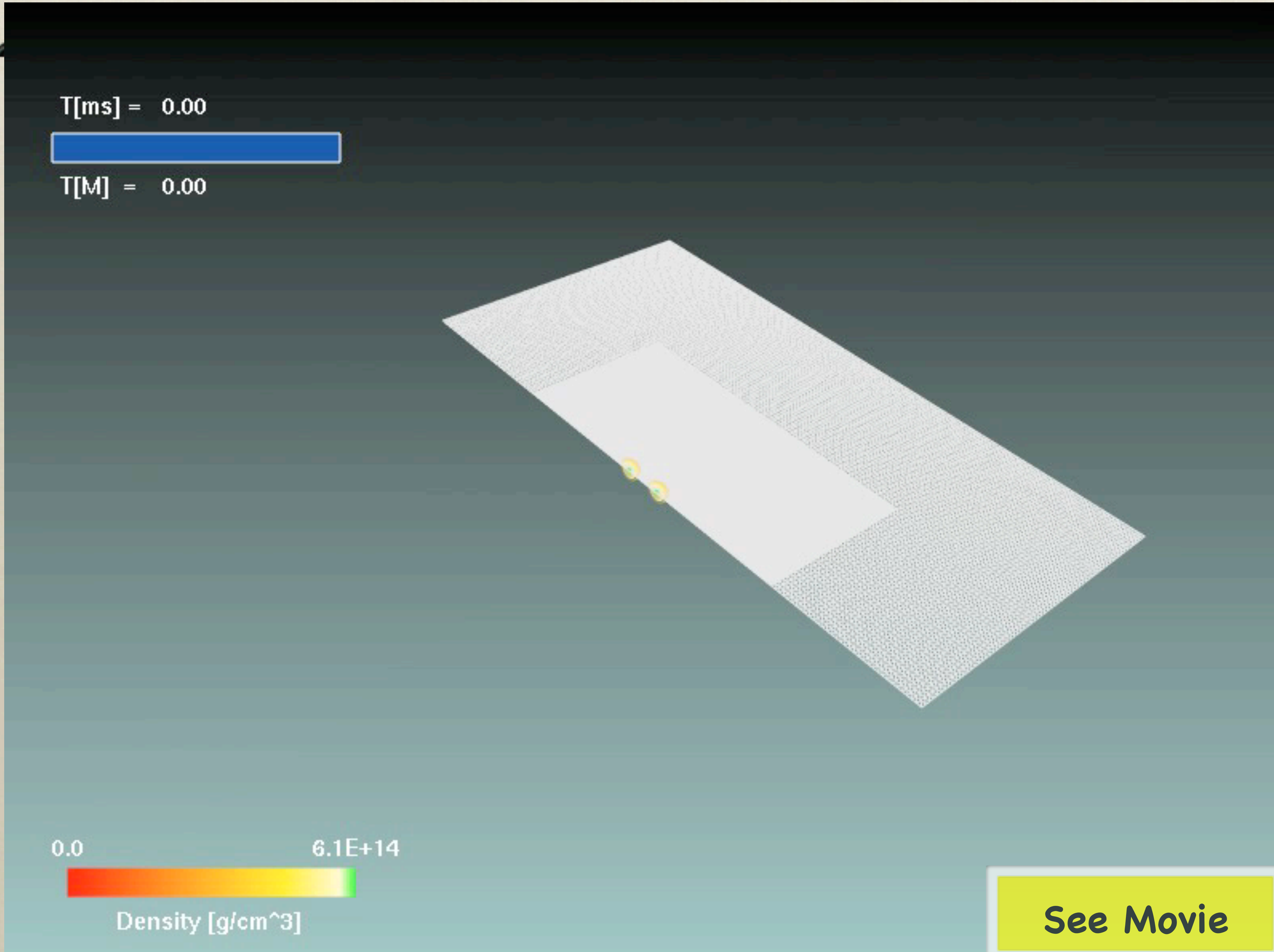
Credits: R. Kaehler & B. Giacommazzo & L. Rezzolla

http://arxiv.org/pdf/0804.0594

See Movie

Numerical relativity at work

Neutron star merger: high-mass merger to BS + disk



See Movie

(Ideal Fluid EOS)

Credits: R. Kaehler & B. Giacommazzo & L. Rezzolla

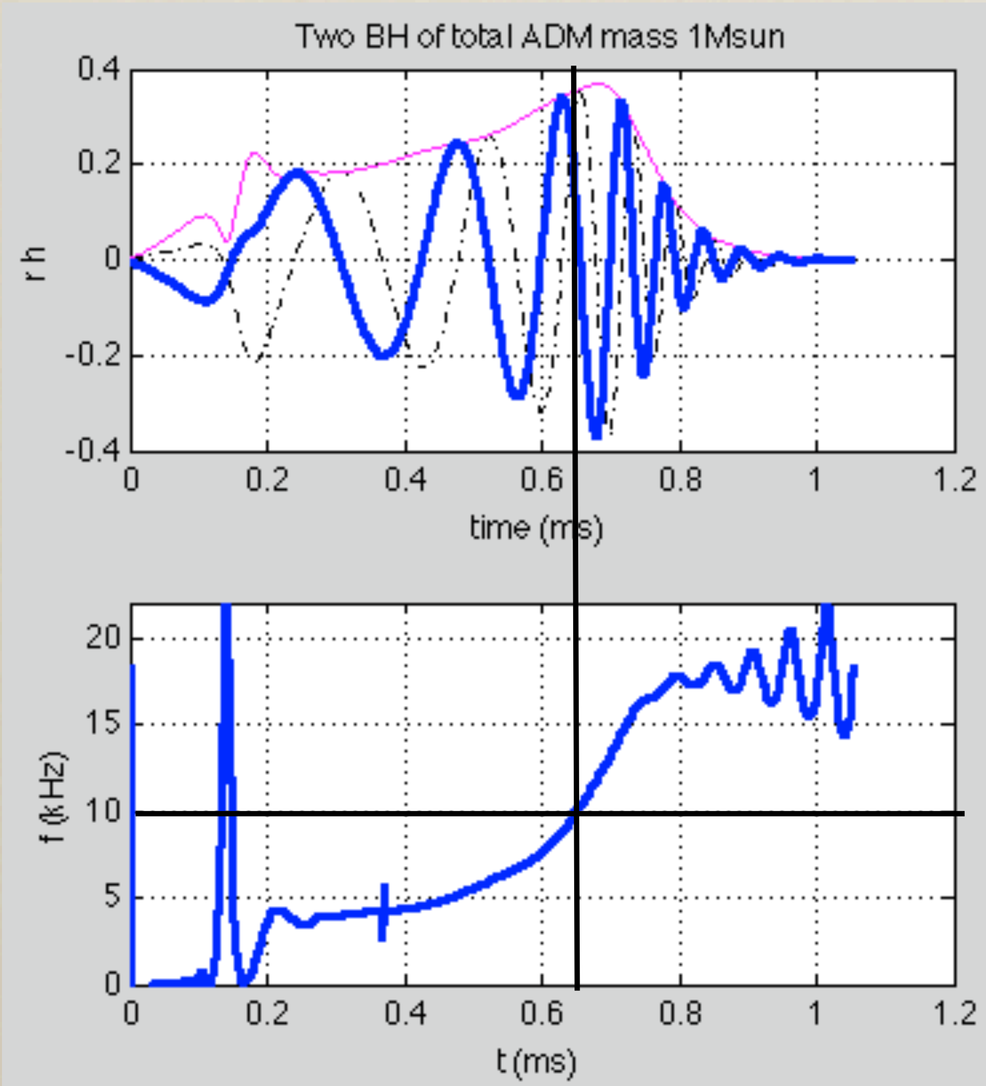
http://arxiv.org/pdf/0804.0594

Different EOS

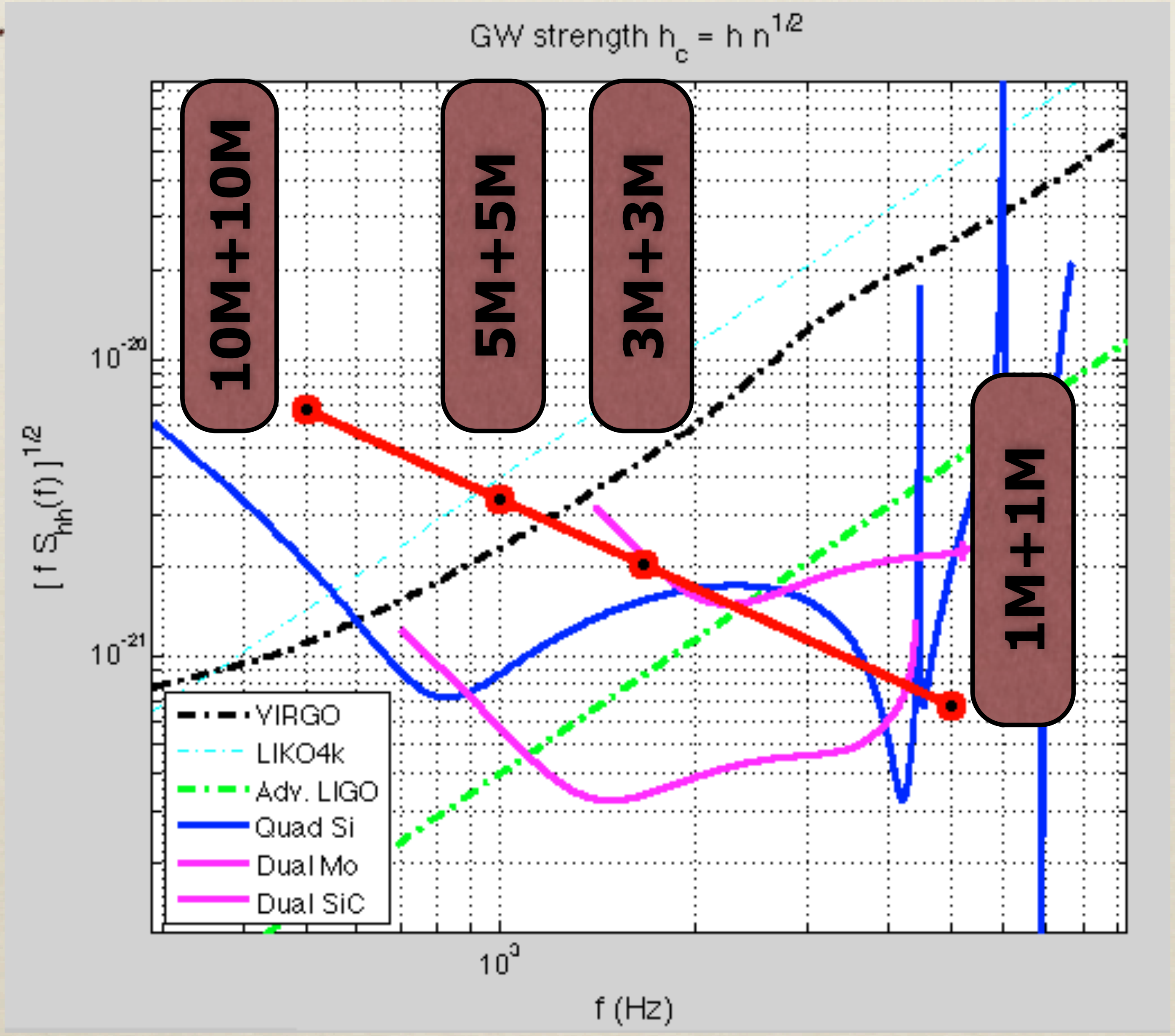
SNR BH-BH @ 100Mpc

At 100 Mpc to be scaled by:

$$\sim 10^{-21} M/M_{\odot}$$

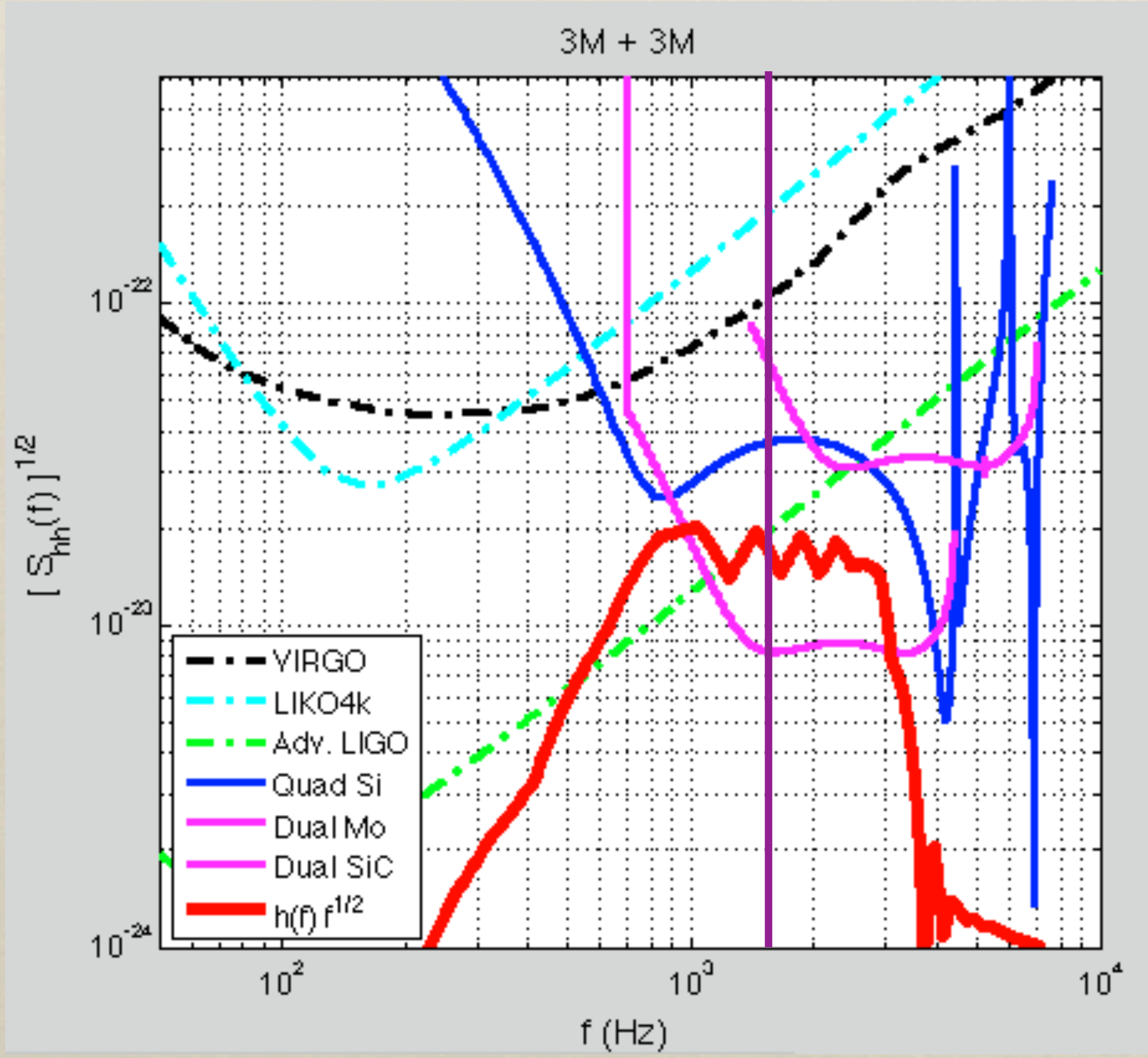


f to be scaled by: $0.5 M_{\odot}/M$



PHYSICAL REVIEW D 73, 061501(R) (2006) "Last orbit of binary black holes" M. Campanelli, C. O. Lousto, and Y. Zlochower

SNR 3M+3M @ 100Mpc



$$SNR^2 = 4\int df\ \frac{|\tilde{h}(f)|^2}{S_{hh}(f)}$$

SNR Dual SiC=1.4

SNR QUAD Si=3.9

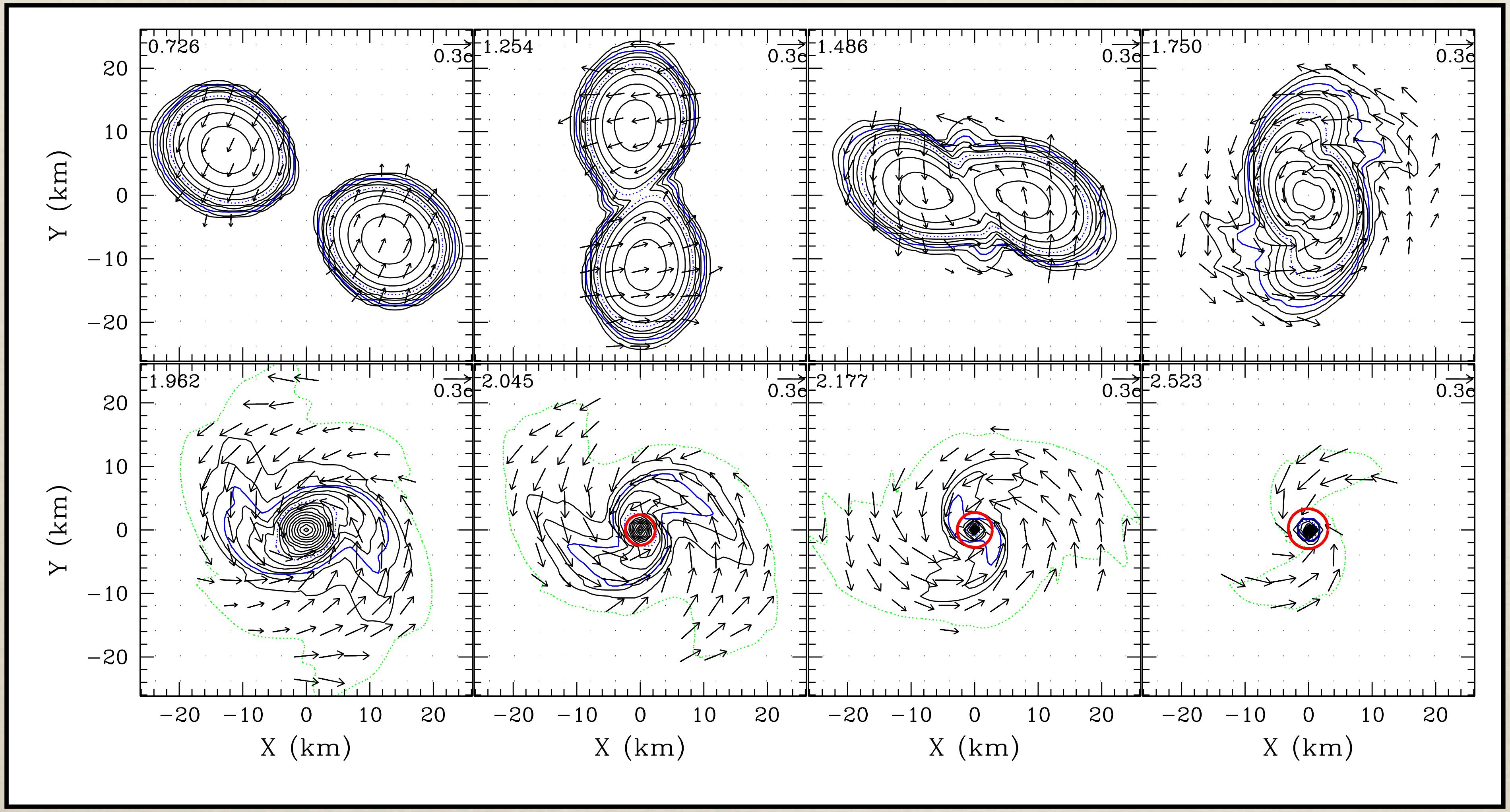
Two bodies merging of NS-NS

- Shibata et all., Phys.Rev. D71,084021 (2005)
- Shibata-Taniguchi, Phys.Rev. D73, 064027 (2006)

Model	$M_\infty (M_\odot)$
APR1313	1.30, 1.30
APR1214	1.20, 1.40
APR135135	1.35, 1.35
APR1414	1.40, 1.40
APR1515	1.50, 1.50
APR145155	1.45, 1.55
APR1416	1.40, 1.60
APR135165	1.35, 1.65
APR1317	1.30, 1.70
APR125175	1.25, 1.75
APR1218	1.20, 1.80
SLy1313	1.30, 1.30
SLy1414	1.40, 1.40
SLy135145	1.35, 1.45
SLy1315	1.30, 1.50
SLy125155	1.25, 1.55
SLy1216	1.20, 1.60

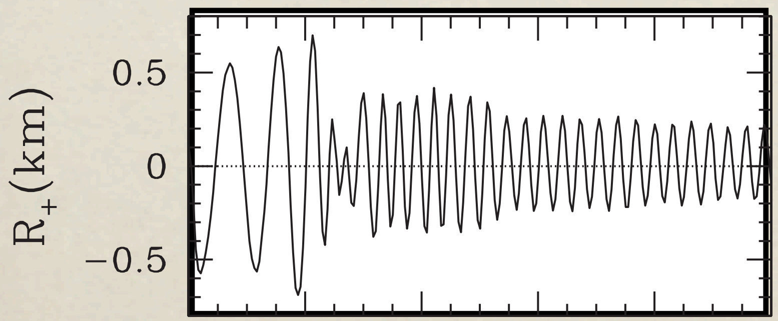
$$h_{\rm gw} \approx 10^{-22}\left(\frac{\sqrt{R_+^2 + R_\times^2}}{0.31\ \rm km}\right)\left(\frac{100\ \rm Mpc}{r}\right).$$

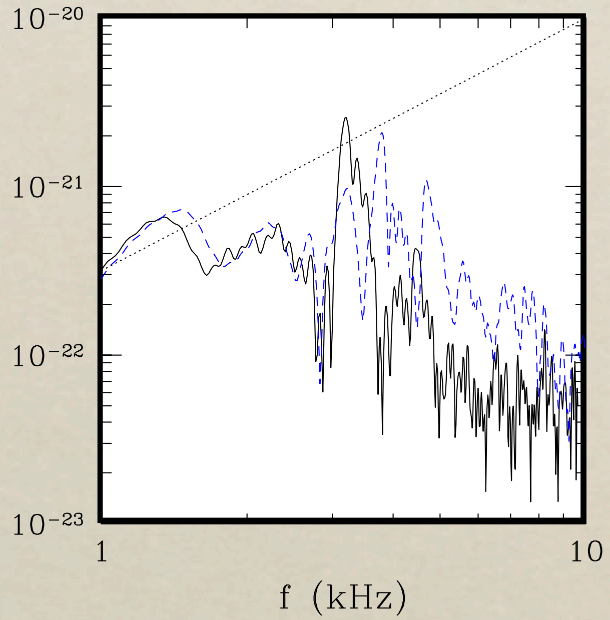
Simulation APR1515

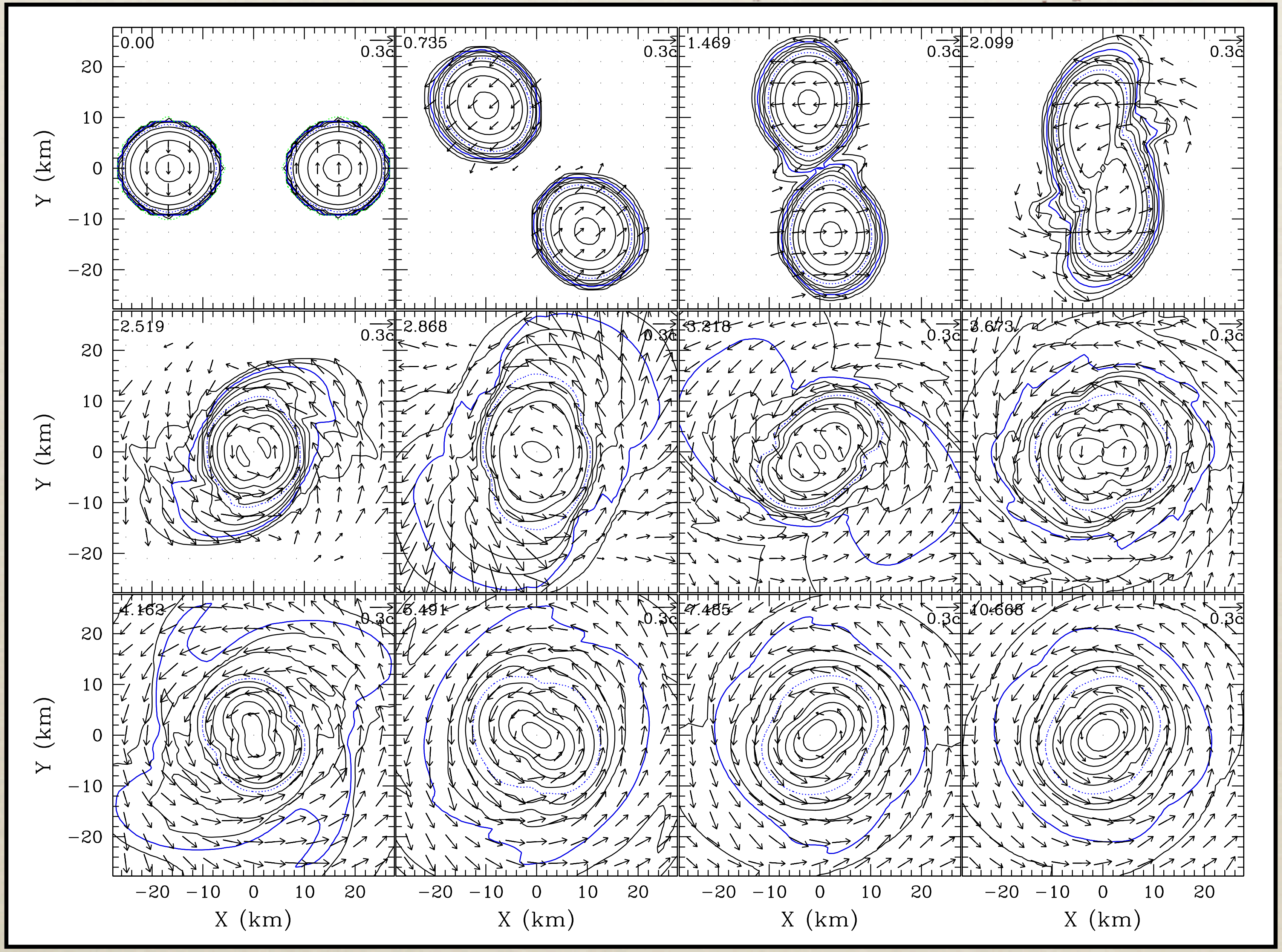


f_{merger} = 6.5 kHz

Simulation APR1313





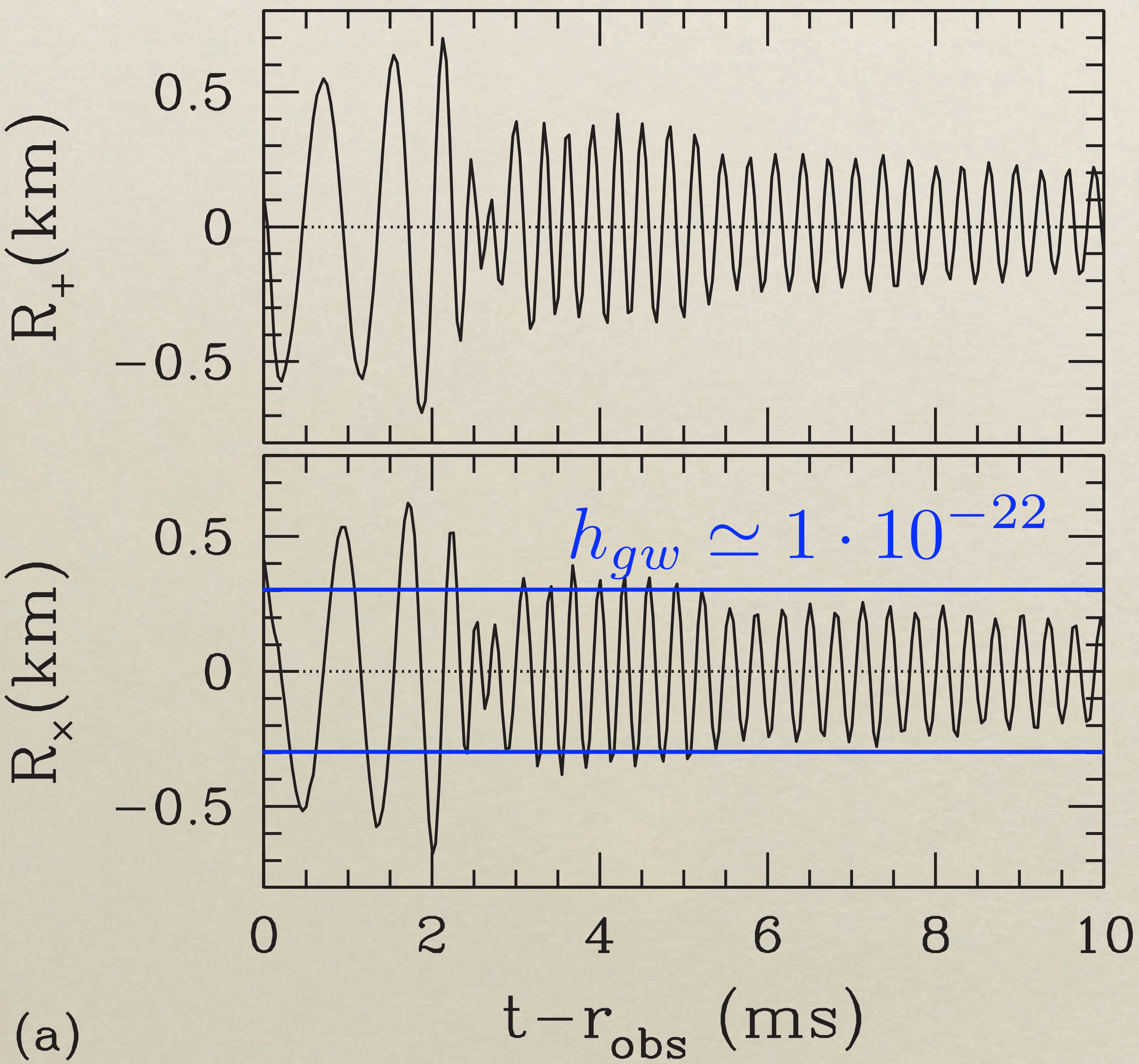


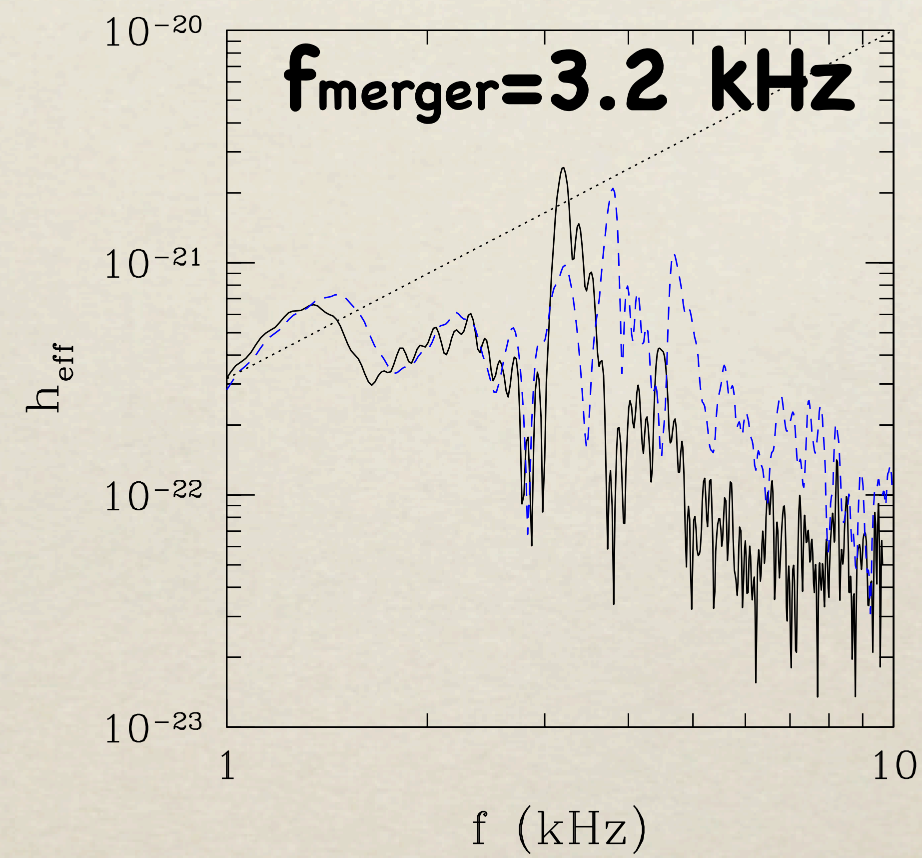
f_{merger}=3.2 kHz

$$h_{\text{eff}} \equiv \sqrt{|\bar{R}_+|^2 + |\bar{R}_\times|^2} f$$

$$= 1.8 \times 10^{-21} \left(\frac{dE/df}{10^{51} \text{ erg/Hz}} \right)^{1/2} \left(\frac{100 \text{ Mpc}}{r} \right),$$

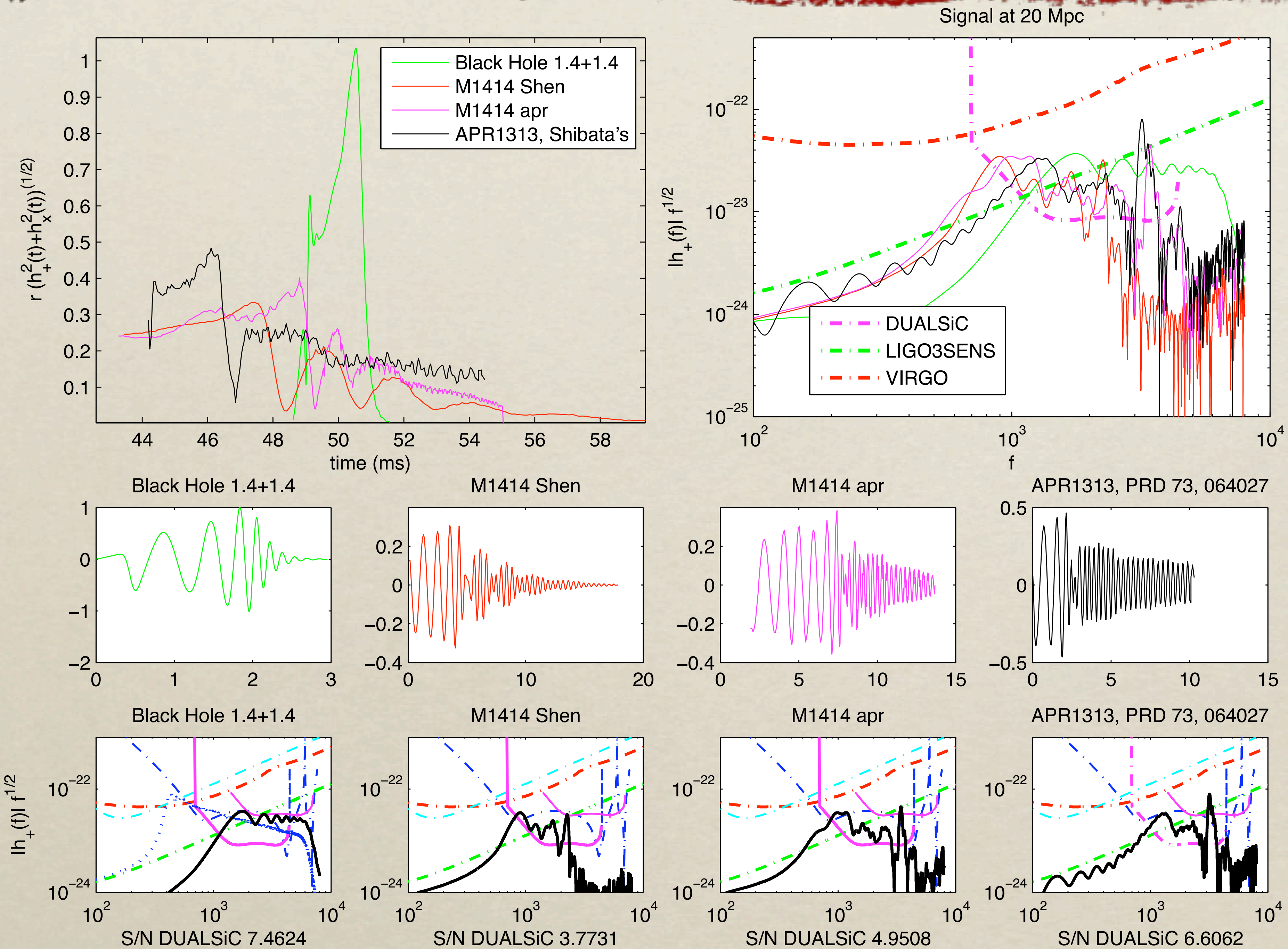
Simulation APR1313



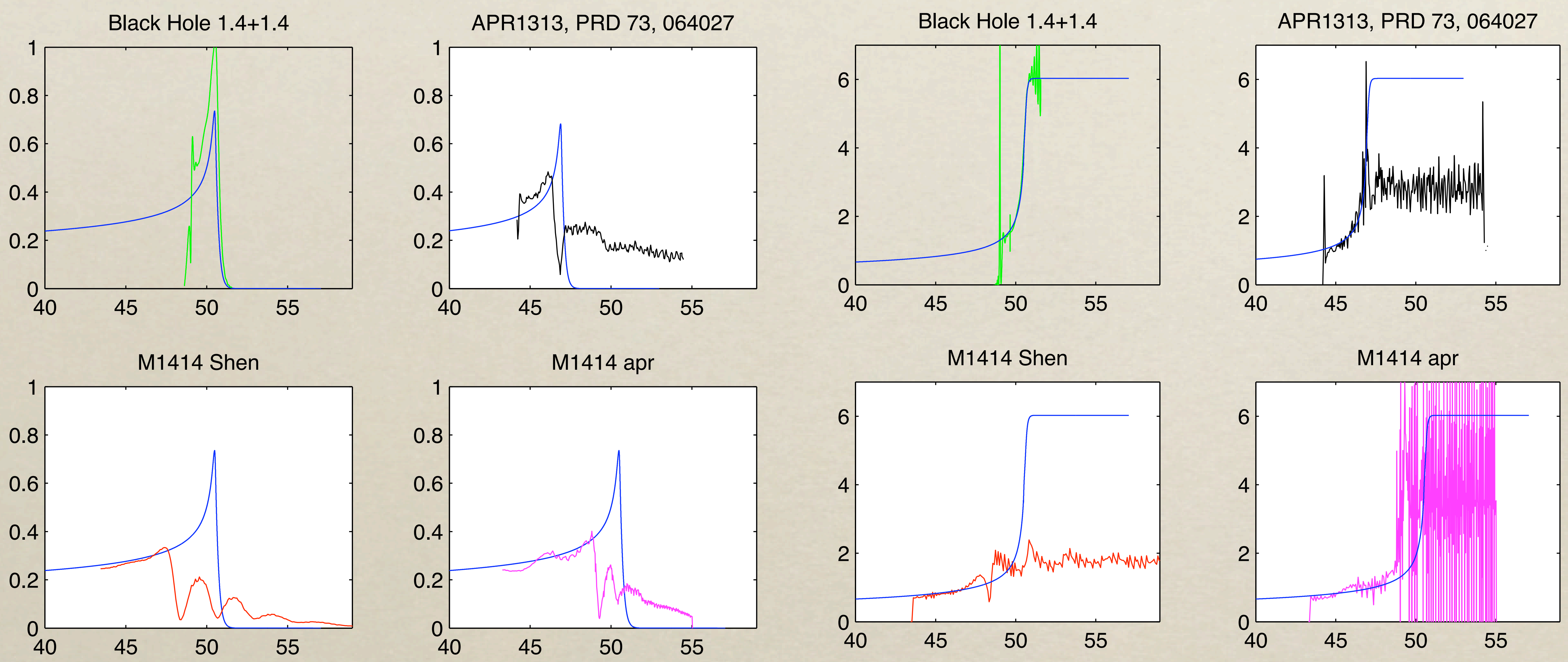


$$
\begin{aligned}
h_c &= h\sqrt{n} \\
&\simeq 2 \cdot 10^{-22}\sqrt{20 \times 3.2} \\
&= 1.6 \cdot 10^{-21}
\end{aligned}
$$

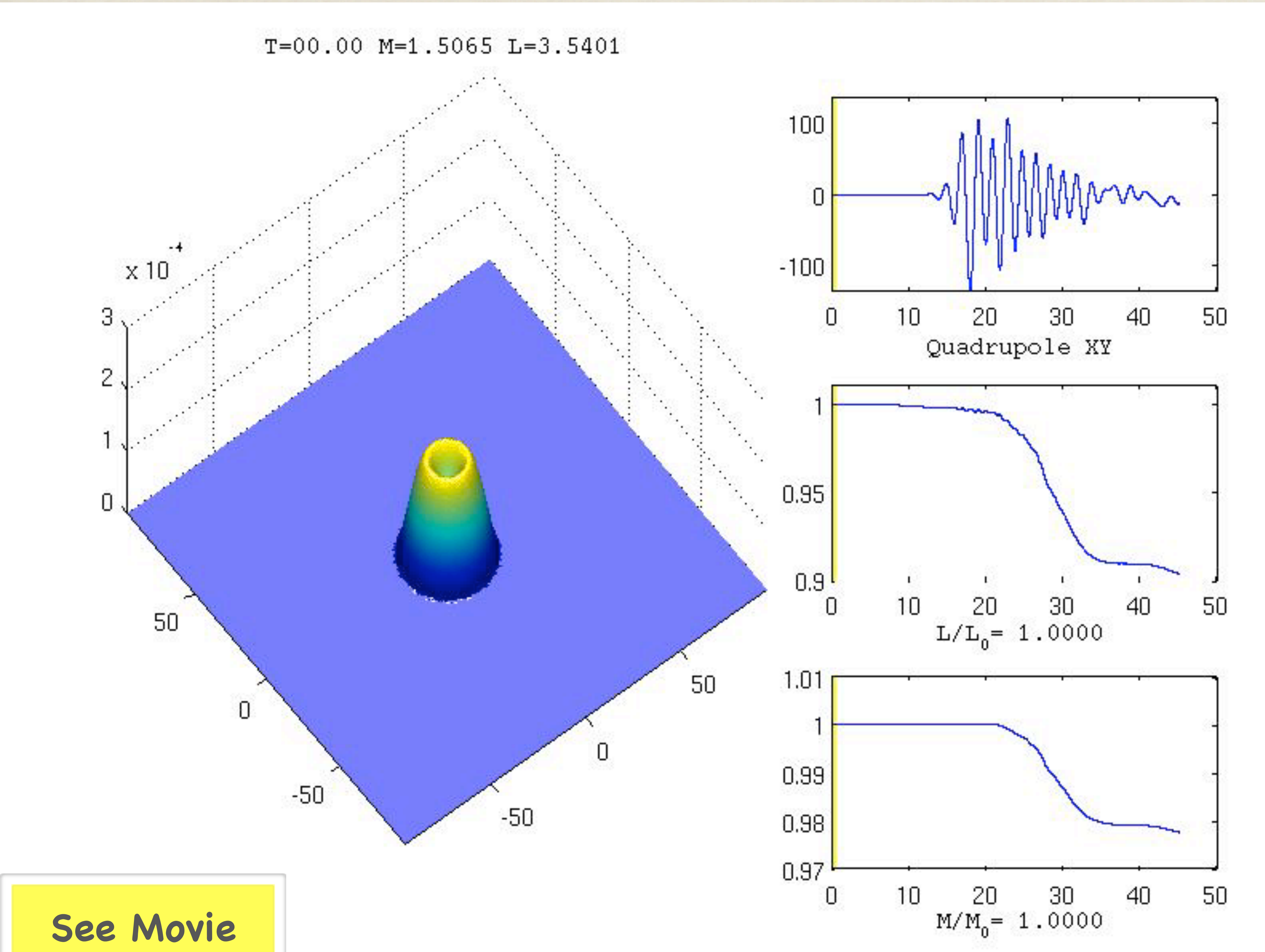
S/N for the merger phase



Blu-line is EOB



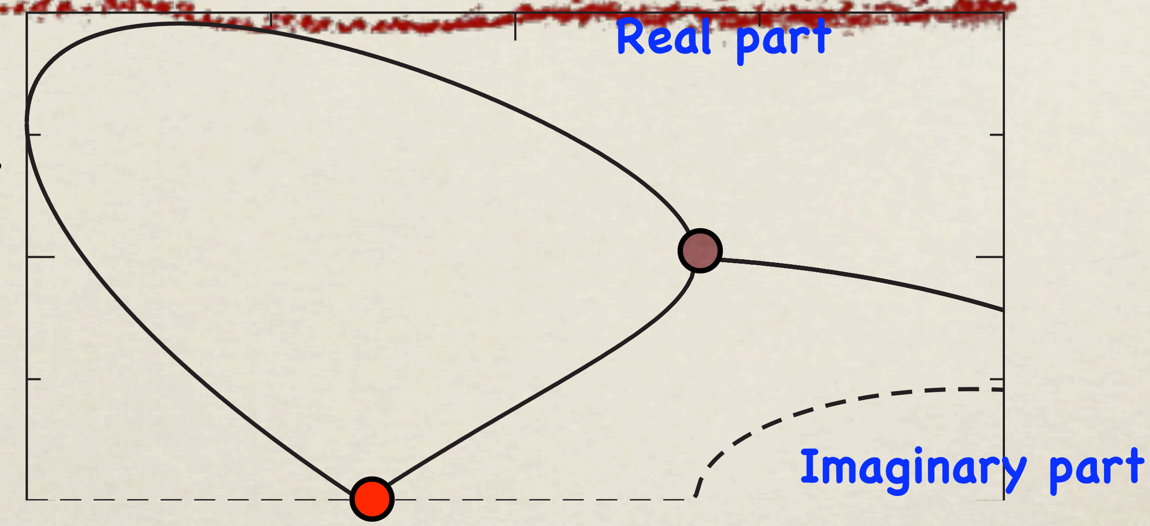
Numerical relativity at work



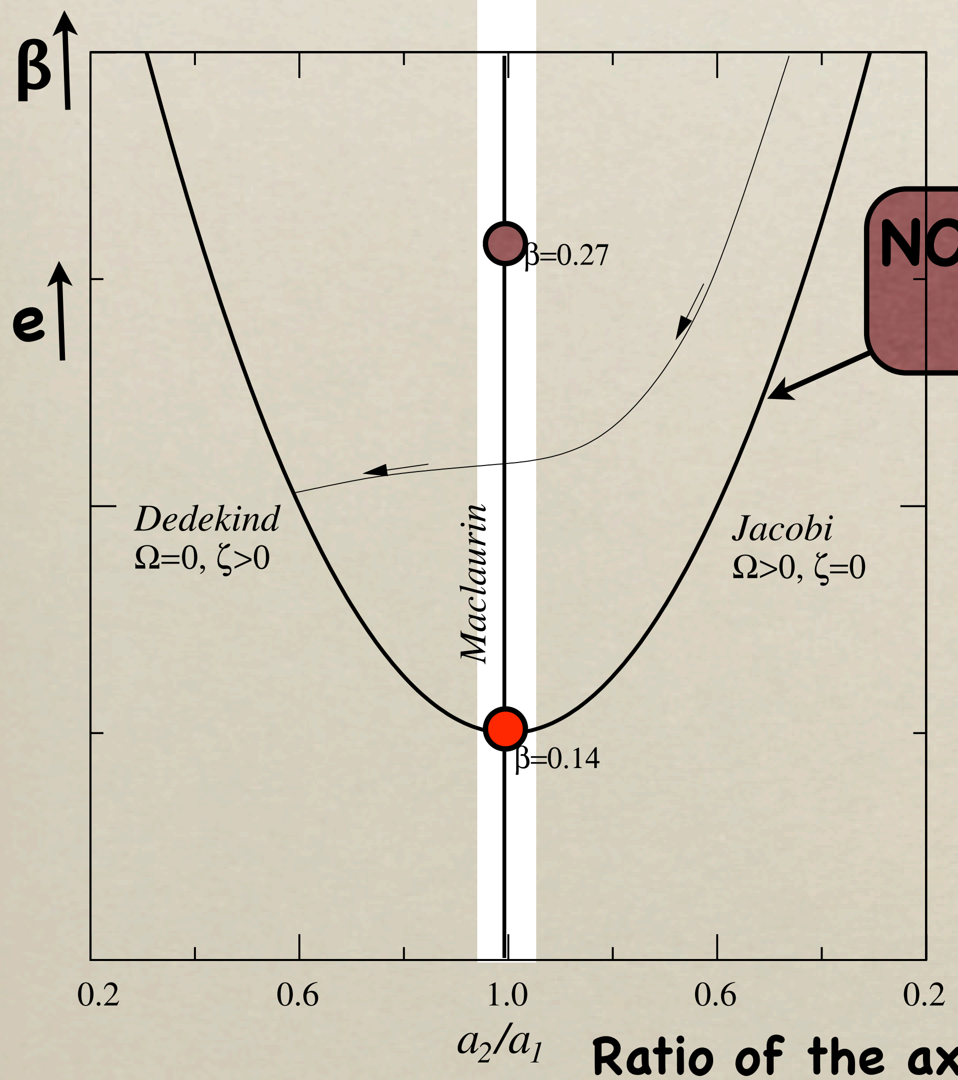
See Movie

Elipsoidal figures of equilibrium (Newtonian)

Eigenvalue of the m=2 mode



Axisymmetric configuration



Ratio of the axes on the xy plane

NON-Axisymmetric configuration

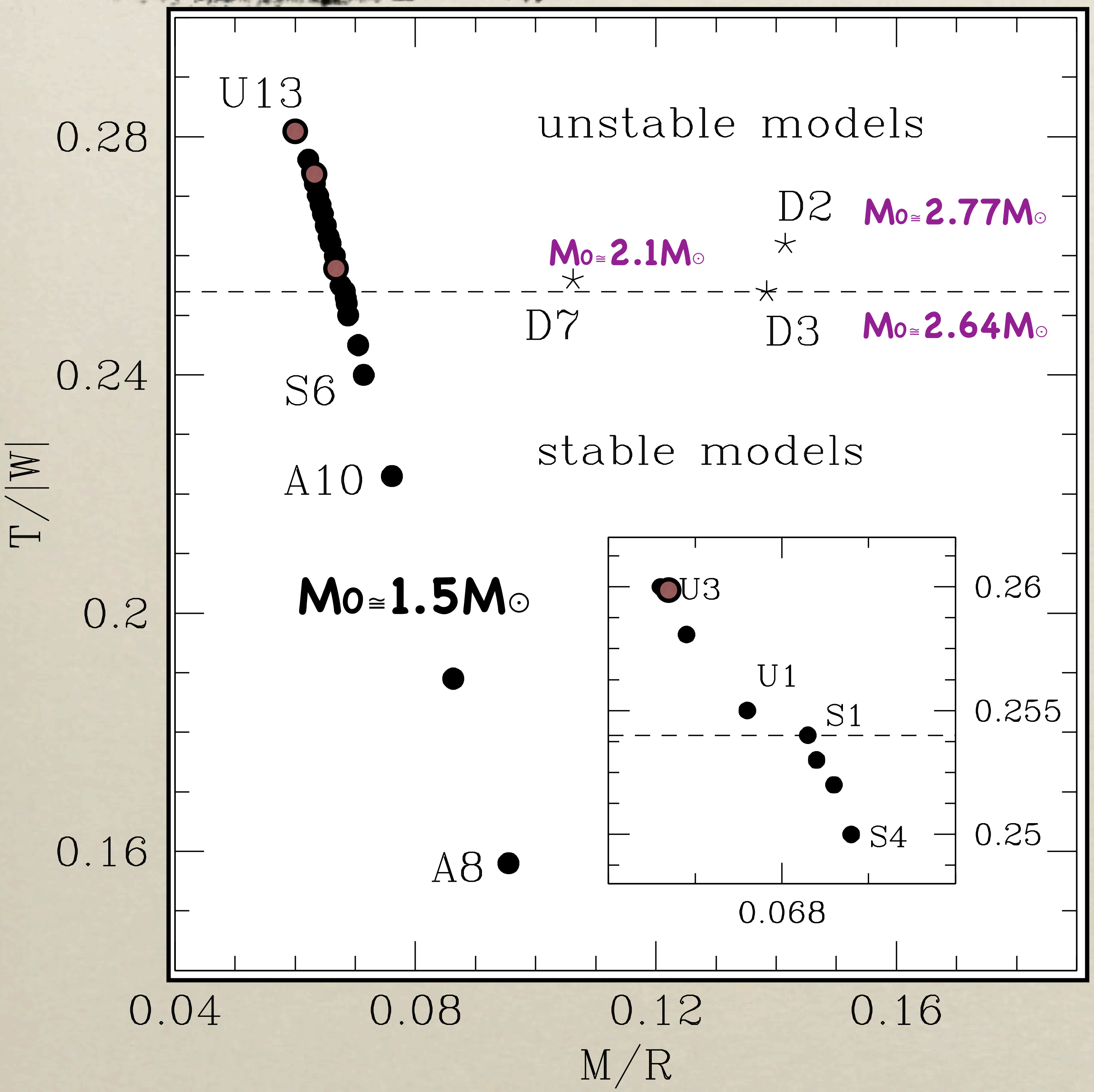
$$\sigma = \Omega(e) \pm \sqrt{4B_{11}(e) - \Omega^2(e)}$$

$$\beta(e) = \frac{\mathrm{T}}{|\mathrm{W}|} = -1 + \frac{3}{2e^2} - \frac{3\sqrt{1-e^2}}{2e\arcsin(e)}$$

$$\Omega^2 = \frac{-6\left(1-e^2\right)}{e^2} + \frac{2\left(3-2\,e^2\right)\sqrt{1-e^2}\,\arcsin(e)}{e^3}$$

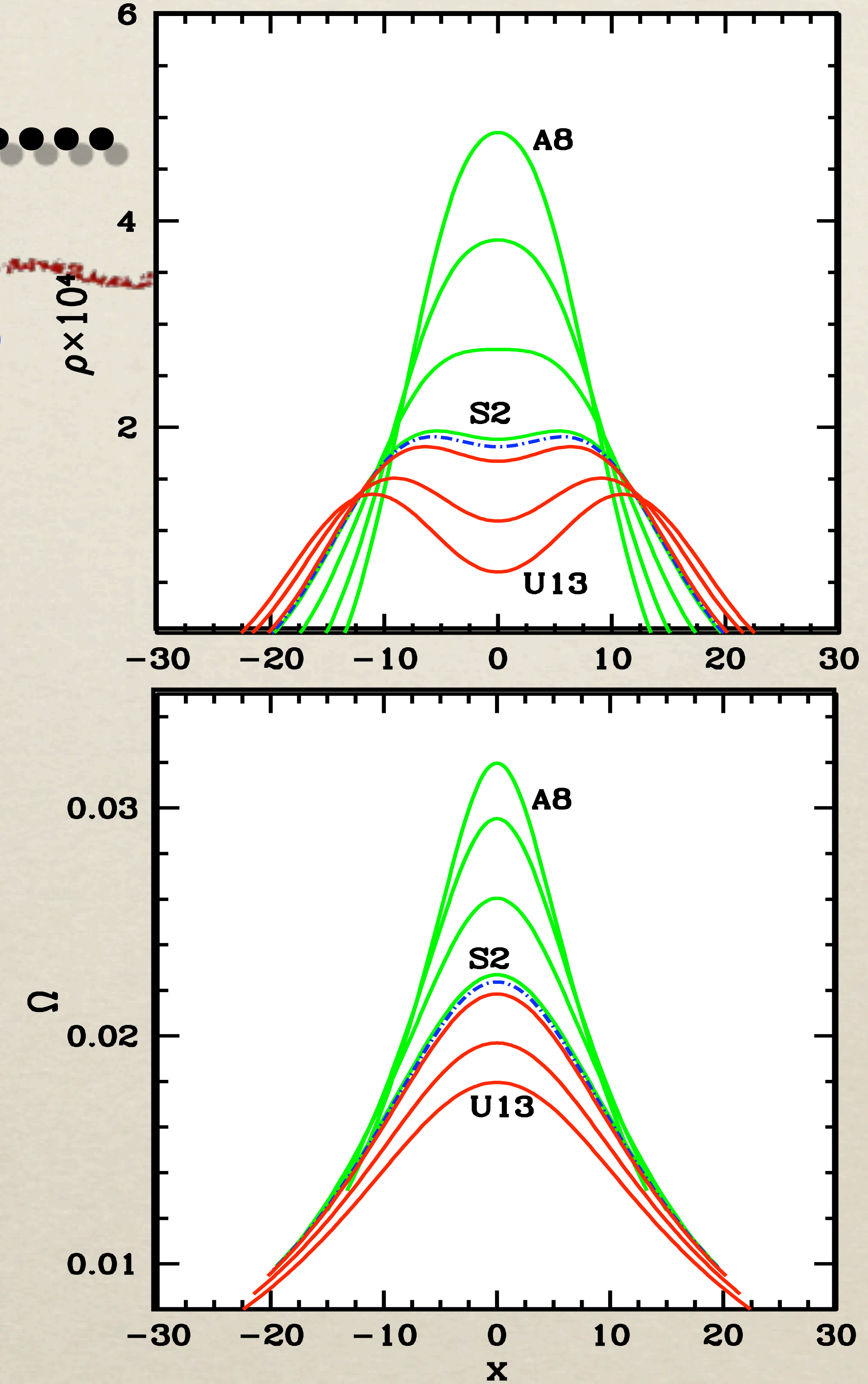
$$B_{11} = \frac{3\,e - 5\,e^3 + 2\,e^5 + \sqrt{1-e^2}\left(-3+4\,e^2\right)\arcsin(e)}{4\,e^5}$$

Simulated models .....



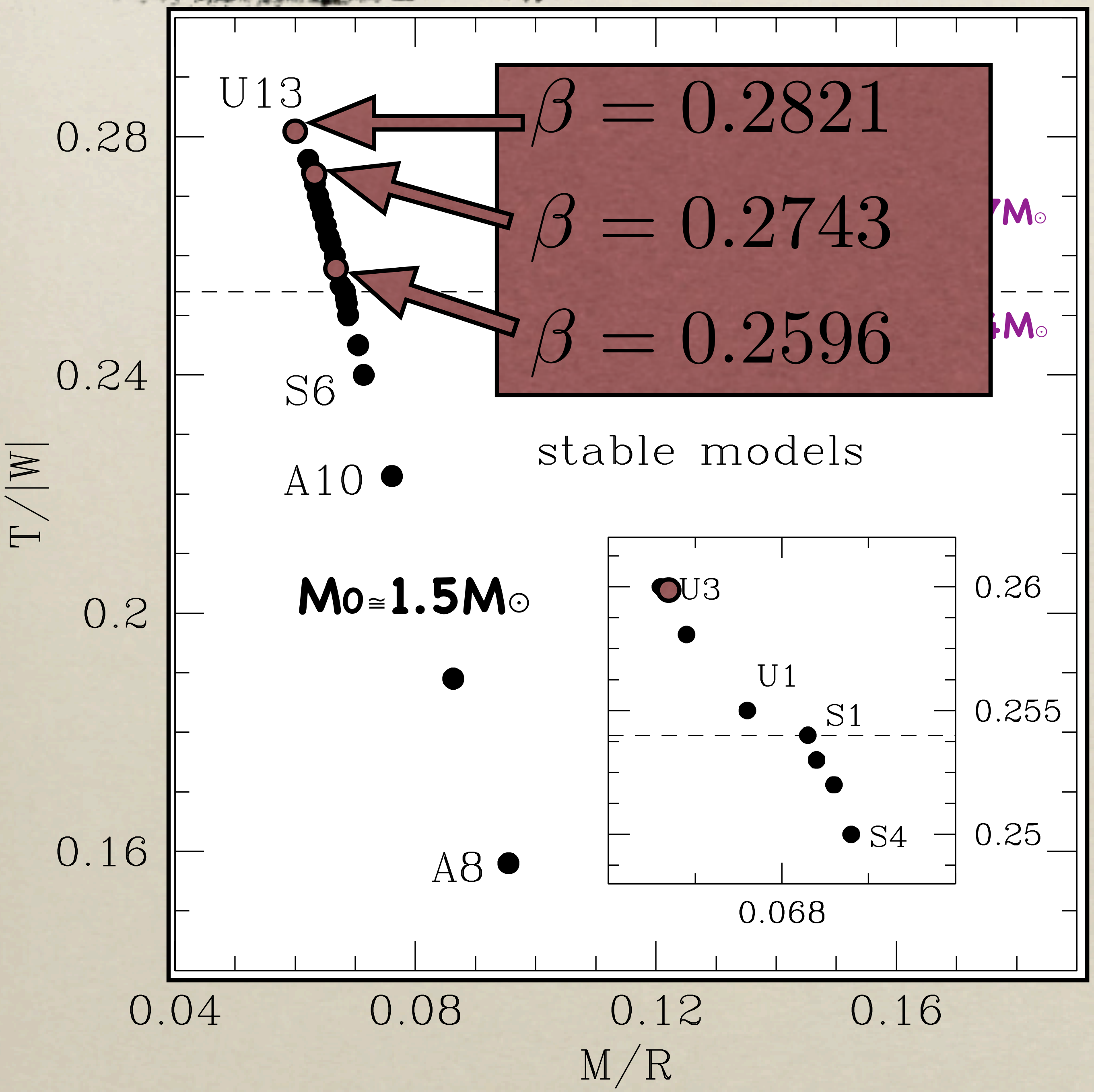
K=100

Γ=2

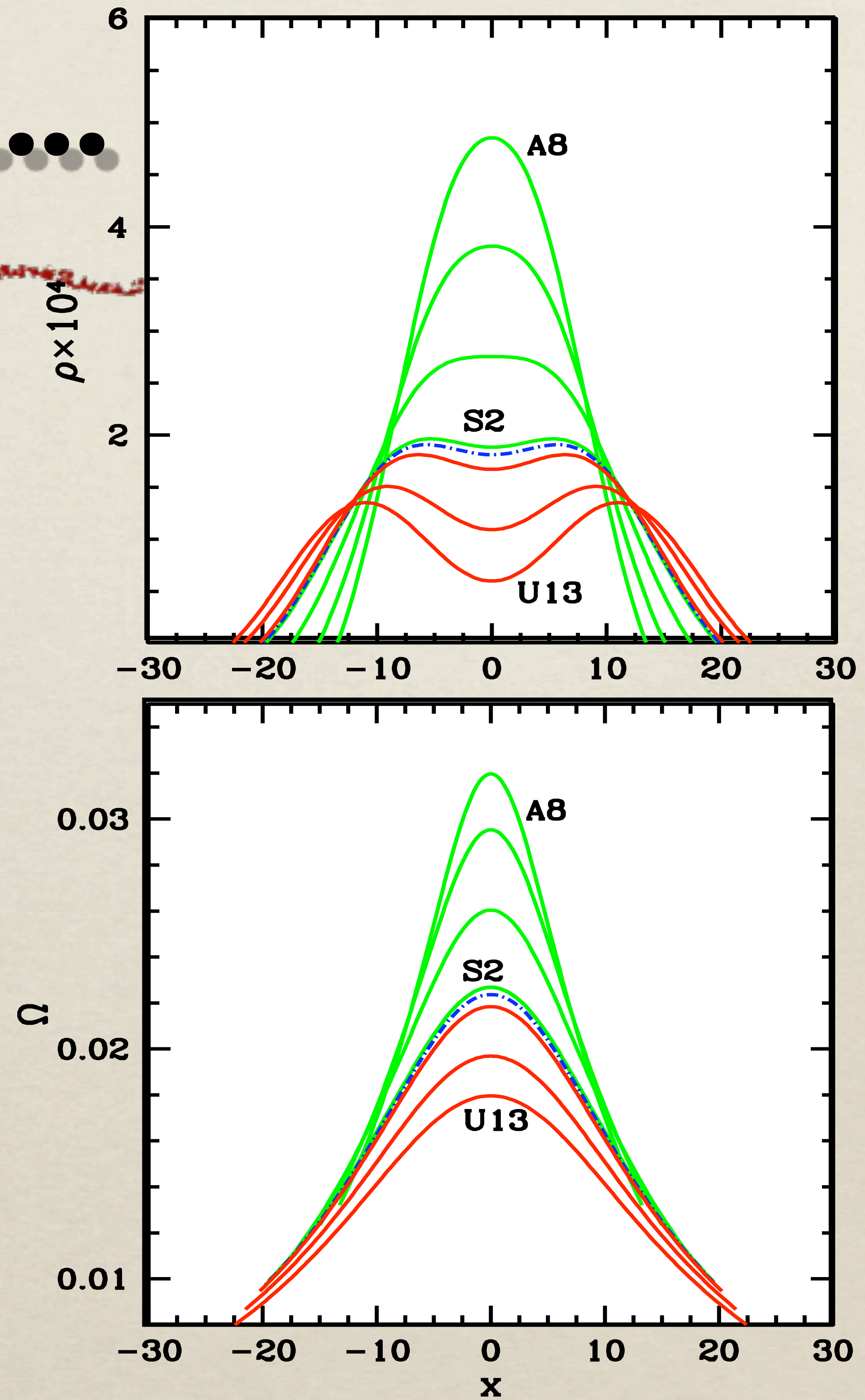


[1] Shibata, Baumgarte, Shapiro, ApJ. 542,(2000)453.
[2] Stergioulas, Apostolatos, Font: Mon. Not. R. Astron. Soc. 352(2004) 1089--1101

Simulated models .....



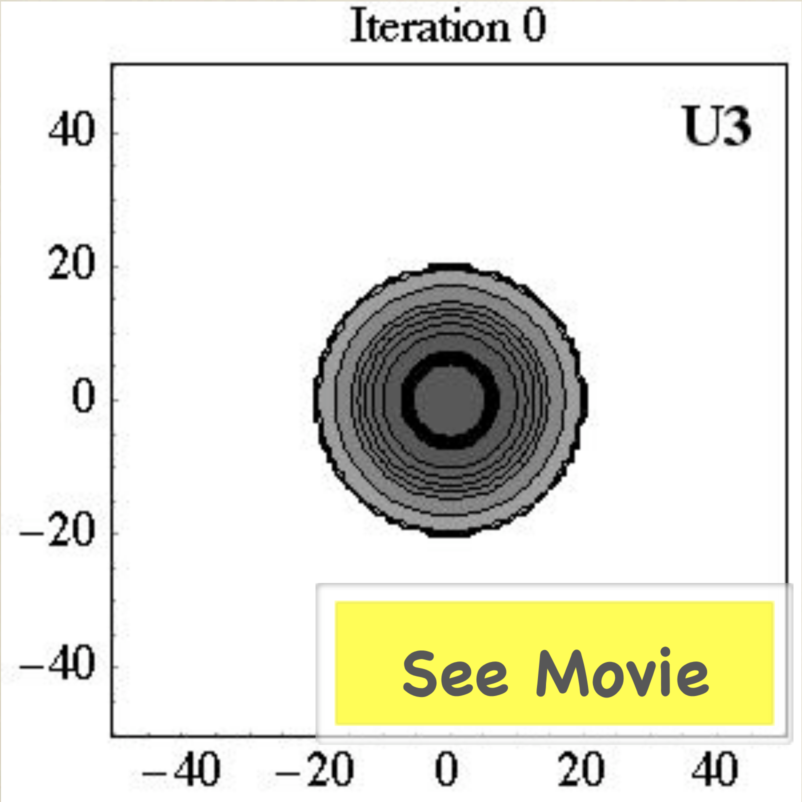
K=100
Γ=2

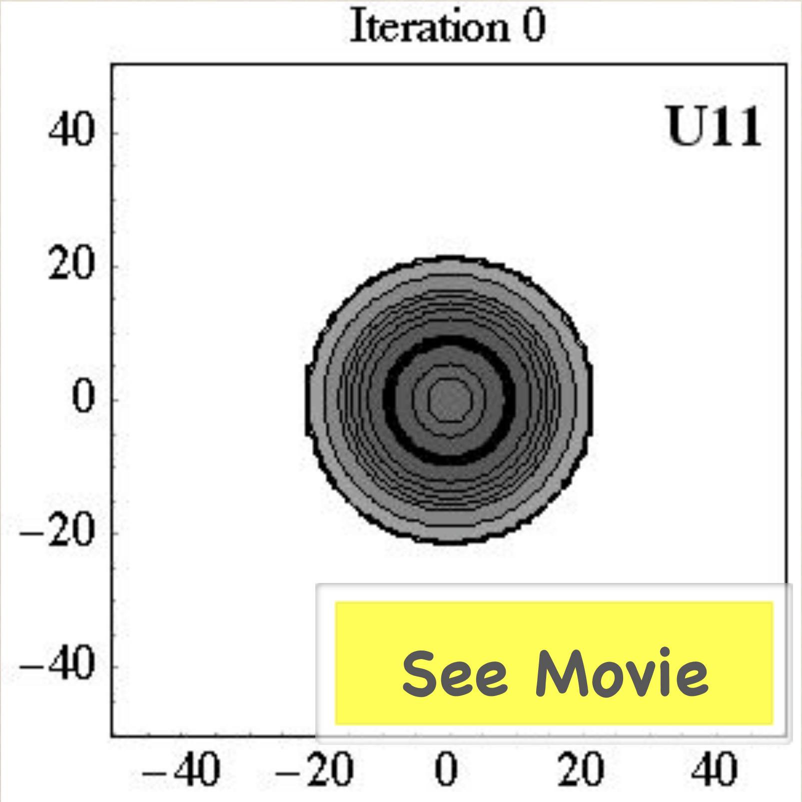


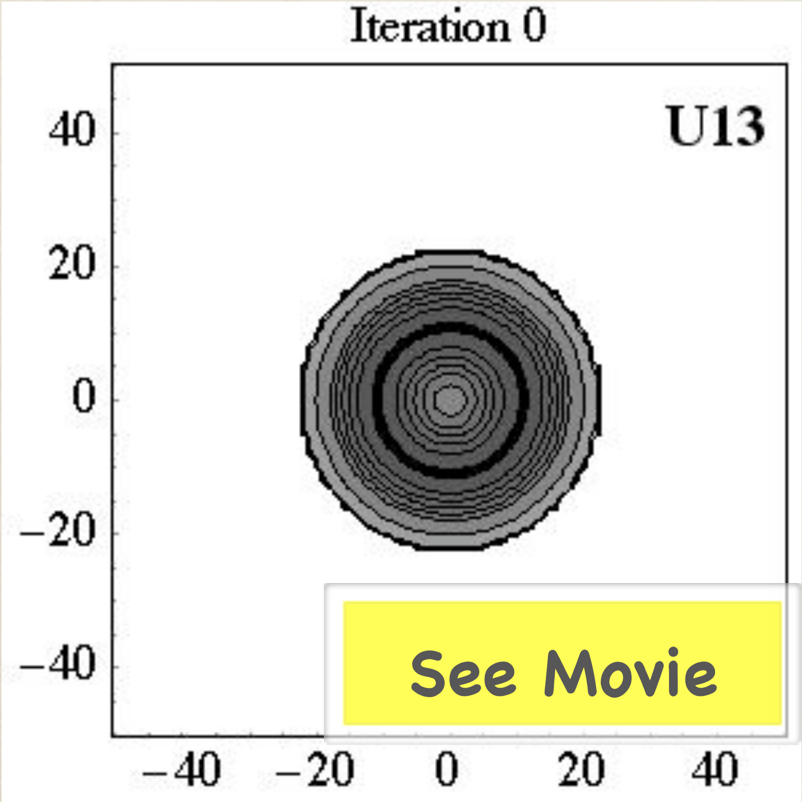
[1] Shibata, Baumgarte, Shapiro, ApJ. 542,(2000)453.
[2] Stergioulas, Apostolatos, Font: Mon. Not. R. Astron. Soc. 352(2004) 1089--1101

Movies

$\beta = 0.2596$ $\beta = 0.2743$ $\beta = 0.2812$



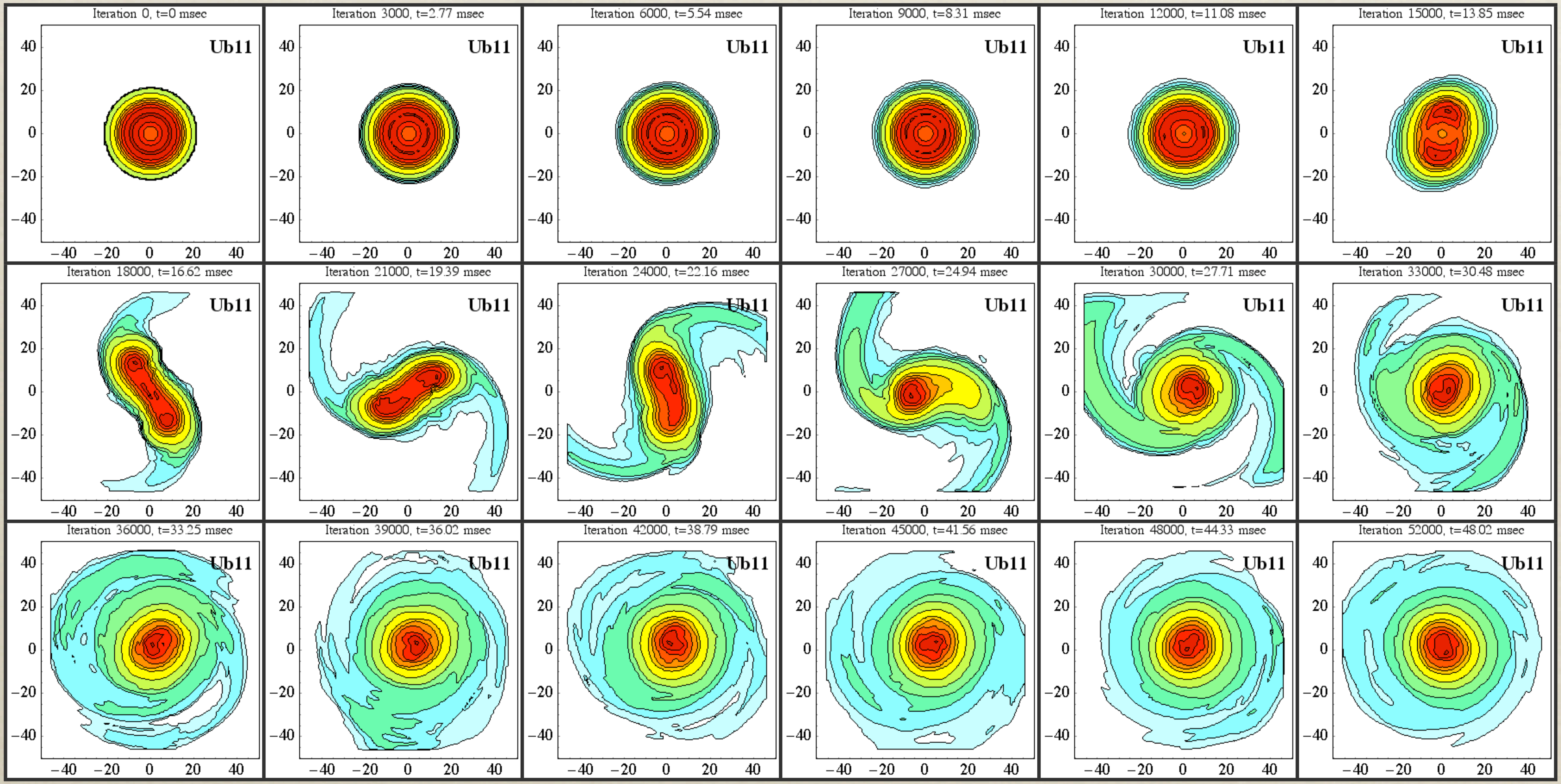




$\beta = 0.2743$

Simulation Ub11

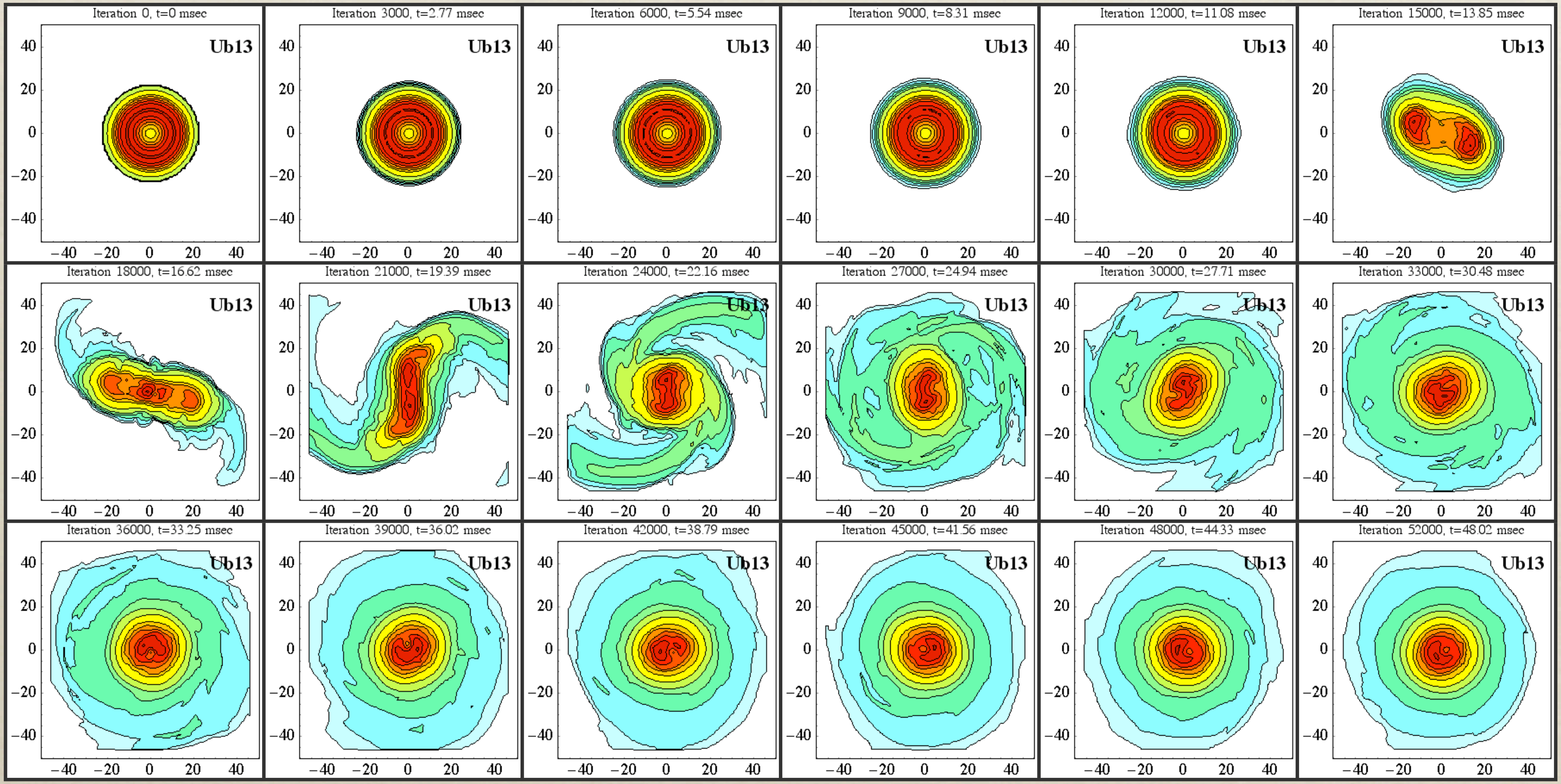
(Movies: http://www.fis.unipr.it/numrel/)



$\beta = 0.2821$

Simulation Ub13

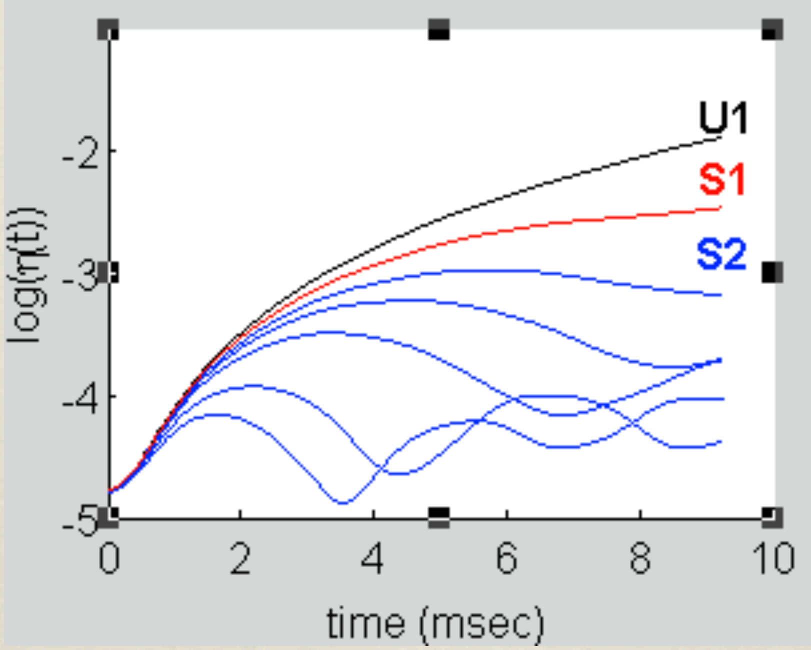
(Movies: http://www.fis.unipr.it/numrel/)

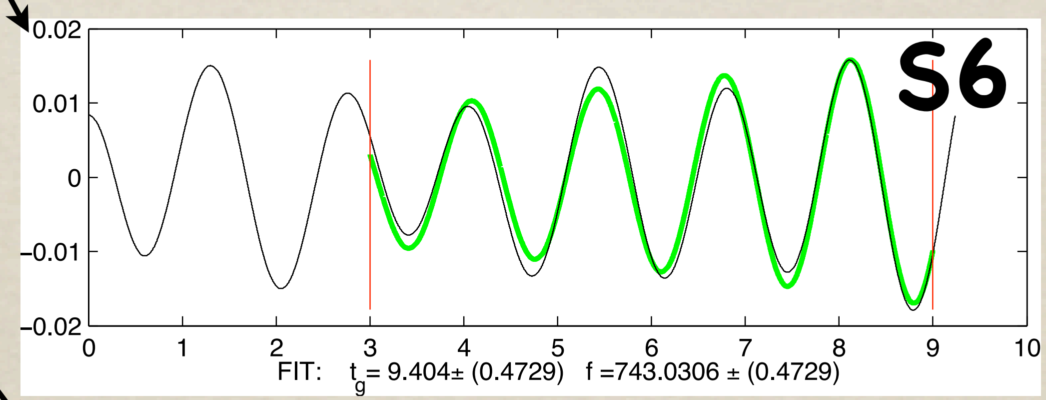


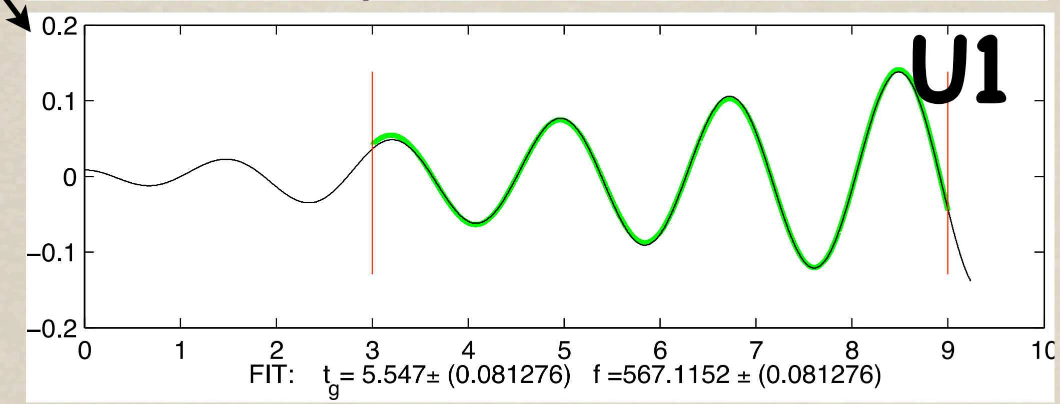
First Method

Model	β	notes	t_i ms	t_f ms	η (max)	τ_B (ms)	f_B Hz
S6	0.240	$\delta = .04$	3	9	0.02	——	740
S5	0.245	$\delta = .04$	3	9	0.02	——	705
S4	0.250	$\delta = .04$	3	9	0.03	——	656
S3	0.252	$\delta = .04$	3	9	0.04	——	611
S2	0.253	$\delta = .04$	3	9	0.05	——	588
S1	0.254	$\delta = .04$	3	9	0.09*	9.71	578
U1	0.255	$\delta = .04$	3	9	0.15*	5.26	567
S1	0.254		45	63	0.02	——	599
U1	0.255		45	63	0.13*	22.1	588

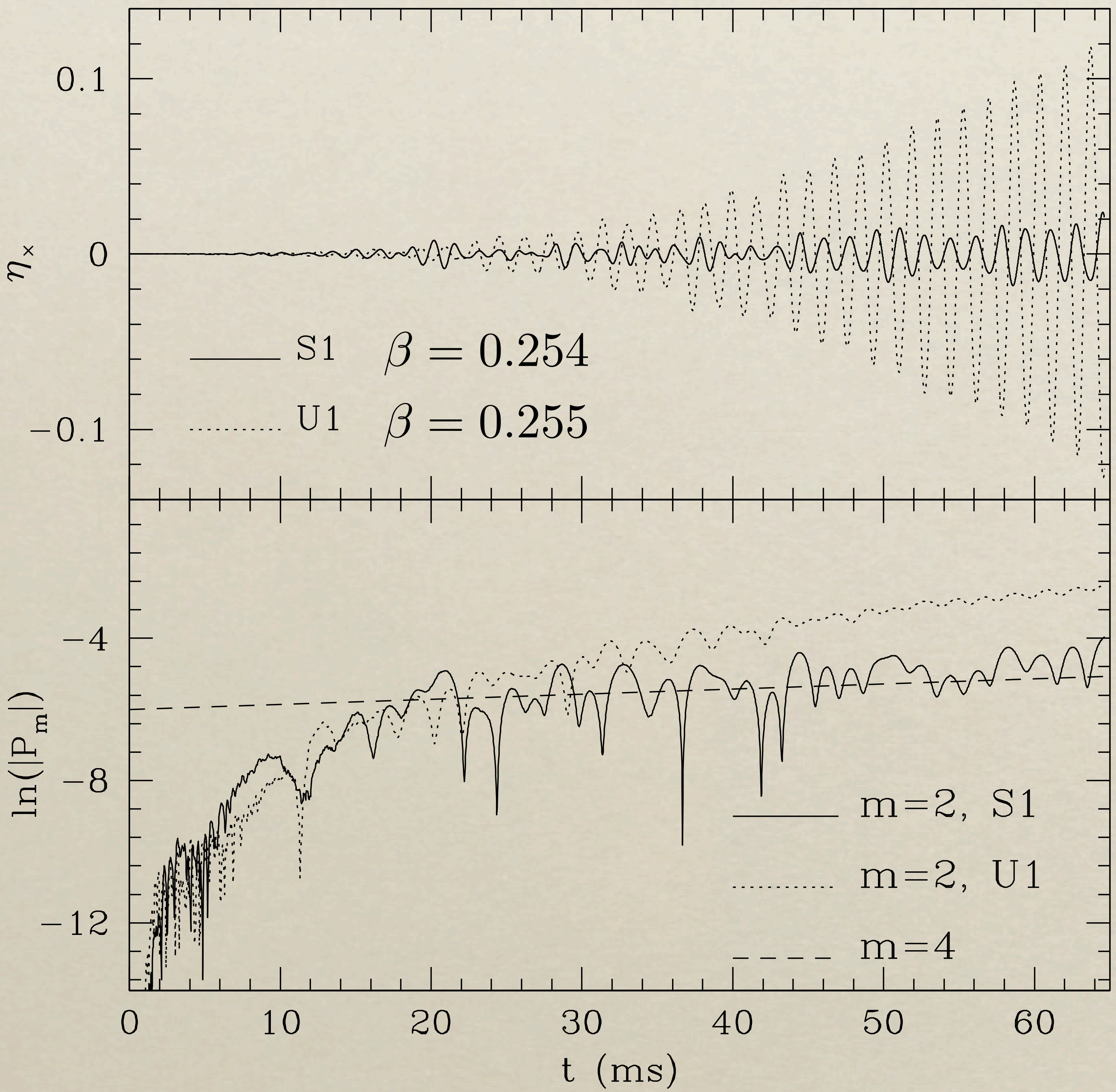
$$\delta\rho_2(x,y,z) = \delta_2 \left(\frac{x^2 - y^2}{r_e^2}\right) \rho \, ,$$







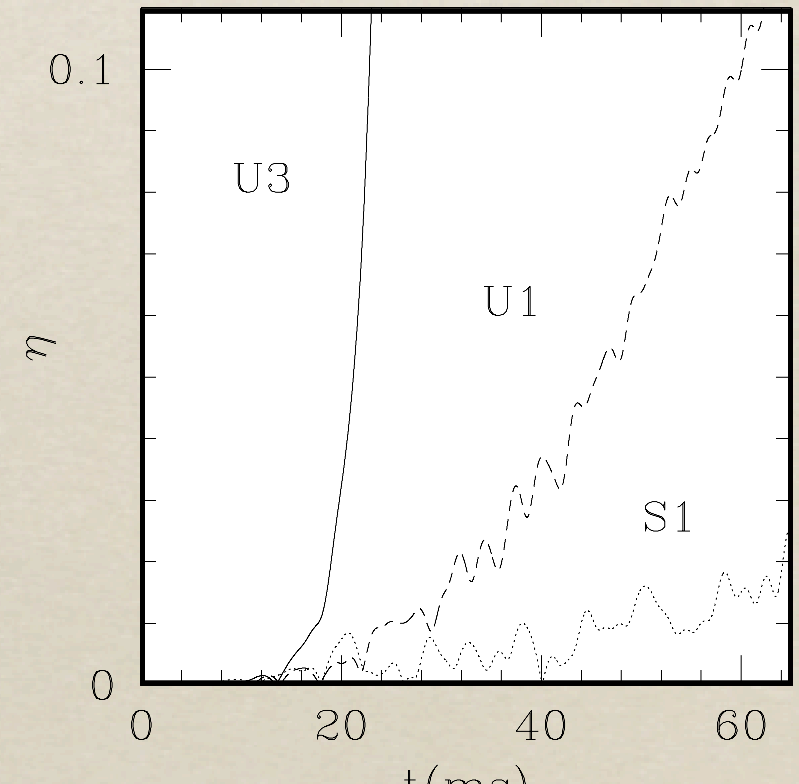
Un-perturbate dynamics at the threshold



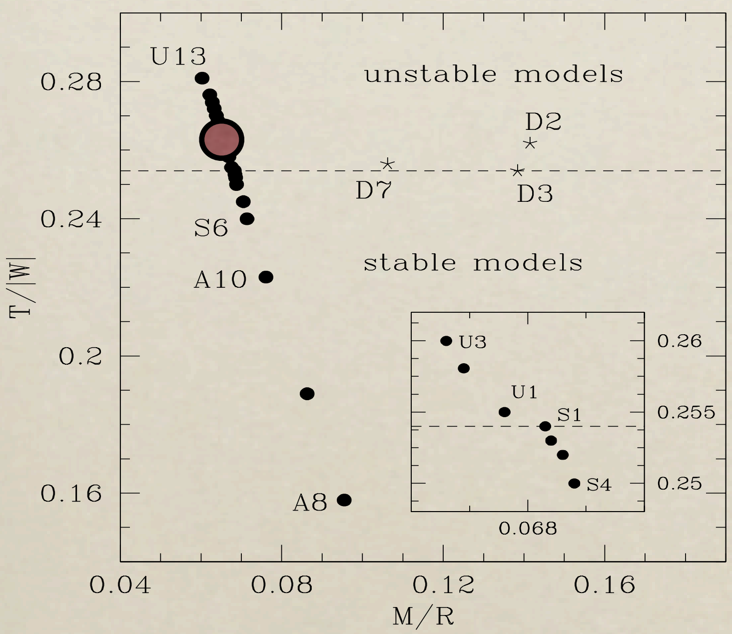
Unperturbed dynamics

Different value of β

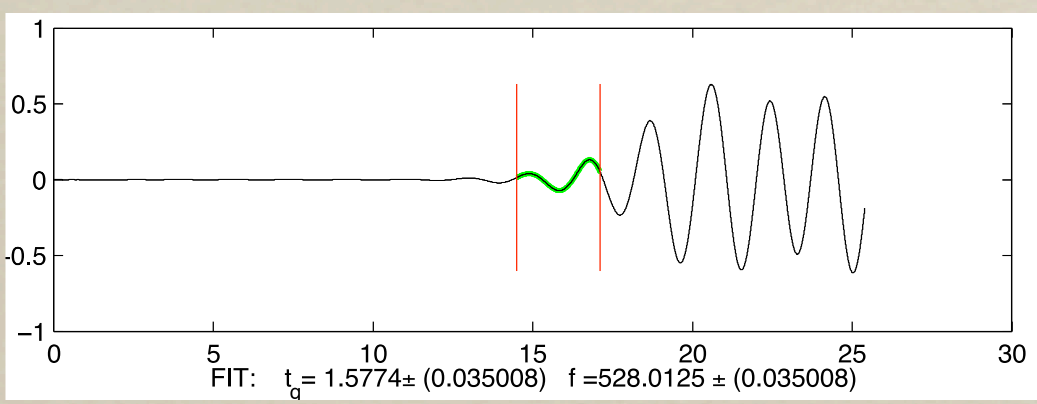
Importance of non linear coupling at the threshold

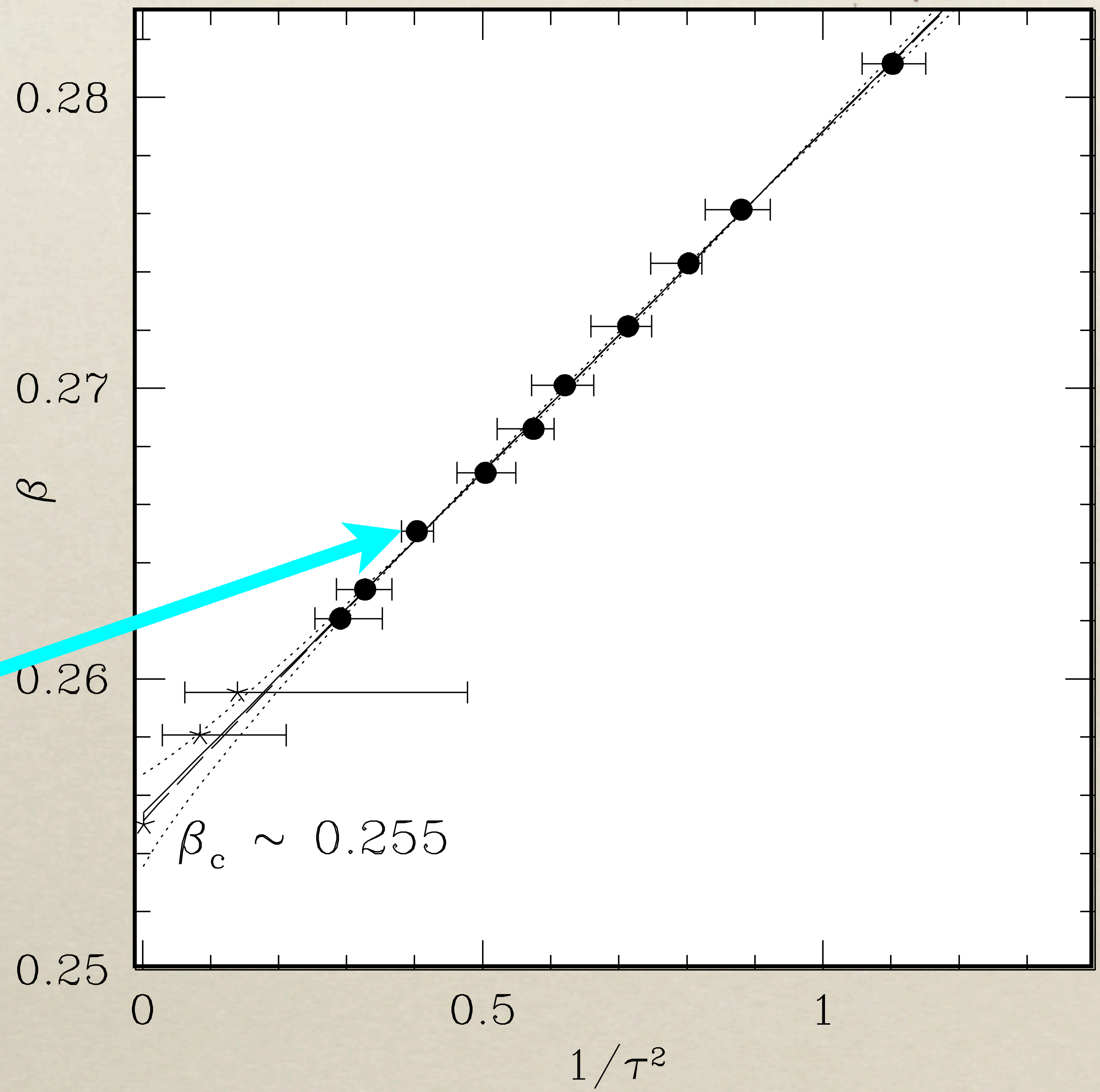


Second Method



U6 dx=0.625





$$\eta(t) = \eta_0 \, e^{t/t_g} \sin(2\pi f \; t + \phi)$$

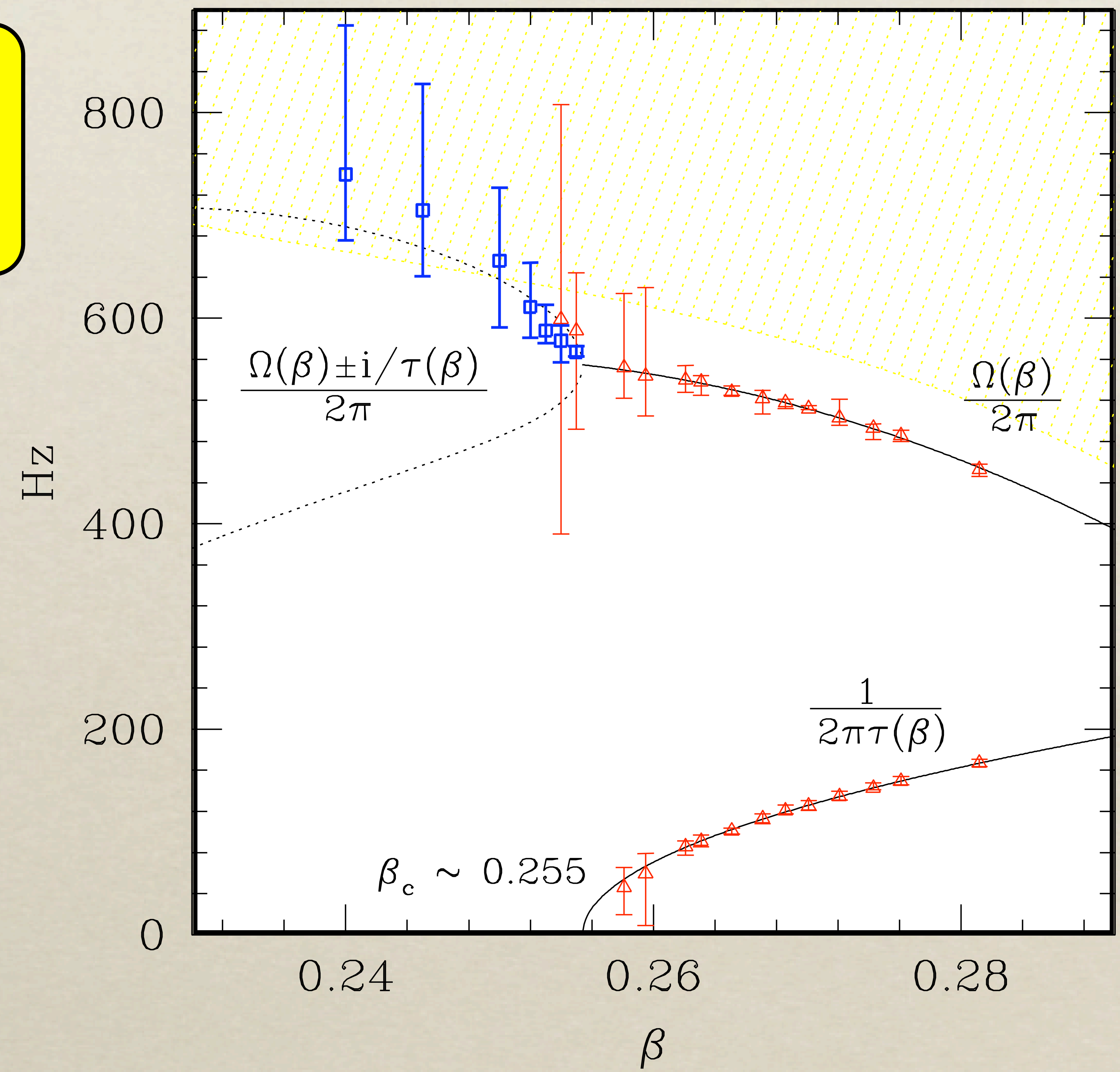
$$t_g = 1.5774(msec) \qquad f = 528Hz$$

Instability Diagram in full GR

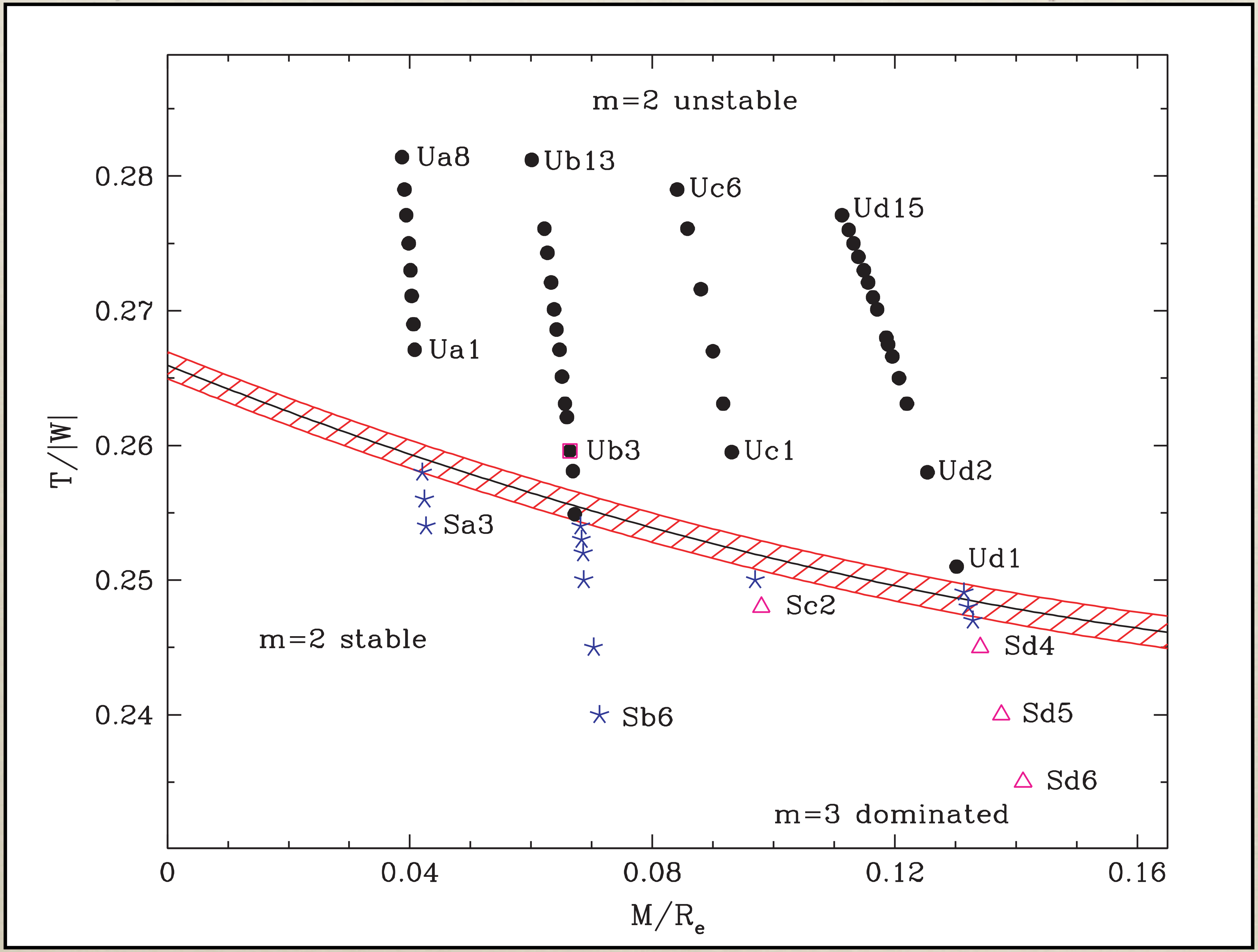
$$\Omega = f_c + f'_c(\beta - \beta_c) + f''_c(\beta - \beta_c)^2$$

$$\frac{1}{\tau} = \sqrt{k(\beta - \beta_c)}$$

Model	β	t_i ms	t_f ms	η (max)	τ_B (ms)	f_B Hz
U2	0.2581	16.9	22.4	0.3734	3.438	552
U3	0.2595	19.9	24.2	0.4241	2.678	544
U4	0.2621	15.3	18.3	0.5496	1.854	540
U5	0.2631	16.2	19.0	0.5788	1.748	538
U6	0.2651	14.5	17.1	0.6305	1.574	528
U7	0.2671	14.2	16.4	0.6694	1.408	522
U8	0.2686	12.2	14.3	0.7027	1.319	518
U9	0.2701	13.2	15.2	0.7223	1.269	512
U10	0.2721	13.7	15.6	0.7482	1.184	503
U11	0.2743	12.9	14.7	0.7749	1.116	493
U12	0.2761	12.0	13.7	0.7999	1.066	486
U13	0.2812	11.2	12.7	0.8551	0.952	453



Extending parameter space



Conclusions

- **Numerical relativity is ready to simulate real physics**
- **A lot of work to do:**
  - NS-NS merger with realistic EOS
  - MAGNETO-HYDRODYNAMICS
  - INSTABILITIES of isolated stars
  - Accretion driven collapse (of a NS to a BH)
  - ....