INTRODUCTION TO STRING THEORY

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1 Historical Introduction

After the formulation of quantum mechanics the quantization rules were also applied to the electromagnetic field and its interaction with electrons and positrons. After the elimination of the infinities through the renormalization procedure one could compute quantities as for instance the Lamb shift and the anomalous magnetic moment of the electron and muon obtaining results that are in very good agreement with the experimental data [1]. In order to give an idea of the spectacular agreement between theory and experiments we give here the number for the theoretical predictions and the experimental data for both the anomalous magnetic moment of the electron and the Lamb shift.

The interaction energy of an electron in an external magnetic field is given by

$$E = -\frac{e}{2mc} g \vec{S} \cdot \vec{H} \tag{1.1}$$

where g is the gyromagnetic ratio that is equal to $g = 2$ in the Dirac theory of the electron, but that gets modified from the quantum corrections in QED. If we write g as

$$g = 2(1 + a_e) \tag{1.2}$$

the theoretical predictions for a_e up to four loops are

$$a_e^{TH} = \frac{\alpha}{2\pi} - 0.328478445 \left(\frac{\alpha}{\pi}\right)^2 + 1.181241456 \left(\frac{\alpha}{\pi}\right)^3 - 1.557(70) \left(\frac{\alpha}{\pi}\right)^4 \tag{1.3}$$

Using the value of the inverse of the fine structure constant:

$$\frac{1}{\alpha} = 137.0359979(32) \tag{1.4}$$

we get

$$a_e^{TH} = 1159652201.2(2.1)(27.1)10^{-12} \tag{1.5}$$

to be compared with the experimental result:

$$a_e^{EX} = 1159652188.4(4.3)10^{-12} \tag{1.6}$$

Concerning the Lamb shift the Dirac theory predicts that in the hydrogen atom the levels $2S_{1/2}$ and $2P_{1/2}$ have the same energy. The experiments give for the difference of energy between the two levels

$$(E_{2S_{1/2}} - E_{2P_{1/2}})^{EX} = 1057.862 \pm 0.020 MHZ \tag{1.7}$$

to be compared with the result of the theoretical calculations:

$$(E_{2S_{1/2}} - E_{2P_{1/2}})^{TH} = 1057.864 \pm 0.014 MHZ \tag{1.8}$$

The successful application of the quantization procedure to the electromagnetic phenomena open the way to apply them also to the strong interactions. Yukawa formulated the hypothesis that the pion, as the photon in QED, is the meson responsible for the interaction between protons and neutrons. However the use of a field theory for describing the interaction among hadrons met pretty soon serious problems:

1. Because of the strength of the strong coupling constant (pion-nucleon coupling constant) it was clear that perturbation theory, so successful in QED, would not work for strong interactions.

2. The experiments showed the existence of a big number of hadrons and if we associate an elementary field to each one of them we would obtain a very complicated theory.

Mainly because of these problems, in the sixties it was proposed to forget field theory and Lagrangians and to concentrate on the S-matrix elements that are the quantities that are directly observed in the experiments [2]. In field theory the fundamental objects are the fields and the Lagrangian describing their interaction. However what one observes in the experiments are particles and their S-matrix elements. The spectrum of particles and their interaction can be in principle computed from the original Lagrangian. However because of the problems 1. and 2. this may be very difficult.

It was therefore proposed to construct directly the S-matrix elements starting from a number of postulates, that they were supposed to satisfy. They were the following:

1. Invariance under the Poincaré group and under T, C and P.

2. Unitarity

3. Crossing Symmetry

4. Analyticity of first kind implying that a particle corresponds to a simple pole in the scattering amplitude, while two or many particle thresholds correspond to branch points.

5. Analyticity of second kind or Regge behaviour. This implies that at high energy ($s \to \infty$) and small transverse momentum (t small) the scattering amplitude behaves as $s^{\alpha(t)}$.

The physical S-matrix was supposed to satisfy the previous rules, but unlike field theory, where the S-matrix is constructed in a very clear way from the Lagrangian, no precise rule was given on how to implement its construction in the S-matrix theory. It was mentioned the principle of bootstrap, but the procedure for constructing the S-matrix remained always very vague.

The very great result of the S-matrix theory was the construction of a model for the process $\pi\pi \to \pi\omega$ performed by Veneziano [3] in the attempt to implement the previous postulates of the S-matrix theory with an infinite number of narrow width resonances lying on linearly rising Regge trajectories.

The scattering amplitude for the process $\pi\pi \to \pi\omega$ that Veneziano proposed is given by

$$T(\pi\pi \to \pi\omega) = \epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu} p_1^{\nu} p_2^{\rho} p_3^{\sigma} A(s,t,u) \tag{1.9}$$

where p_i are the momenta of the three pions, ϵ^{μ} is the polarization vector of the ω particle and

$$A(s,t,u) = A(s,t) + A(s,u) + A(t,u) \tag{1.10}$$

with

$$A(s,t) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)} \tag{1.11}$$

s, t, u are the Mandelstam variables

$$s = -(p_1+p_2)^2 \quad , \quad t = -(p_3+p_2)^2 \quad , \quad u = -(p_1+p_3)^2$$

and

$$\alpha_s = \alpha_0 + \alpha' s \tag{1.12}$$

is a linearly rising Regge trajectory. This feature is in very good agreement with the high energy data in a wide range of energies.

The scattering amplitude (1.9) provides a realization of the postulates of S-matrix theory with only one particle narrow width states. Only unitarity was not satisfied; it was supposed to be implemented only at a later stage.

Immediately after its discovery the Veneziano model was extended to the scattering of four scalar particles:

$$A(s,t,u) = A(s,t) + A(s,u) + A(t,u) \tag{1.13}$$

with

$$A(s,t) = \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t)} \tag{1.14}$$

and later to the scattering of N external scalar particles [4] including also an internal flavour symmetry through the multiplication with Chan-Paton factors [5].

The enthusiasm for the construction of the dual resonance model, that reproduced many important features of hadron physics, led many people to believe that it was possible to construct the scattering amplitude for N pions incorporating current algebra with the right physical parameters. But unfortunately up to now only a partial success was obtained in this direction through the construction by Lovelace and Shapiro [6] of the $\pi\pi \to \pi\pi$ scattering amplitude. For the $\pi^+\pi^+$ elastic scattering they proposed:

$$A(\pi^+\pi^+ \to \pi^+\pi^+) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \tag{1.15}$$

that has the good feature of incorporation the Adler zeroes if, in the chiral limit, the intercept of the ρ Regge trajectory is taken to be $\alpha_0 = \frac{1}{2}$. This value is in good agreement with the experimental value. Another very interesting feature of (1.15) is the presence of ghosts if the space-time dimension $D > 4$ [7]. Although this model has interesting physical features, a realistic model for pions is still lacking because nobody has been able to extend it to an arbitrary number of external pions keeping $\alpha_0 = \frac{1}{2}$. Actually an extension of (1.14) for an arbitrary number of external particles corresponds to the Neveu-Schwarz-Ramond model [8], that is consistent however only if $\alpha_0 = 1$ and $D = 10$.

After few years of research in the dual resonance models essentially two models, i.e. the Veneziano model and the Neveu-Schwarz-Ramond model, were constructed, that although not quite matching the experimental data, presented a high degree of consistency deserving an intensive study.

In this investigation it was realized that a relativistic string [9] was the structure underlying the dual resonance model. This observation makes much easier today to teach the dual resonance models to those who did not partecipate in the early days of duality. In fact it is now possible to describe their properties starting from the fundamental string action.

After so many attempts toward more realistic models it became clear in the middle of the seventies that it would be very difficult to construct completely realistic models for the strong interacting particles.

The following diseases were present in the internally consistent models:

1. The lowest state of the spectrum was always a tachyon.

2. All kind of massless particles (photon, graviton etc.) were present in the spectrum. The only massless hadron expected in the chiral limit, the pion, was impossible to accommodate in the spectrum as already discussed.

3. It was impossible to quantize the various string models for values of the space-time dimensions that are not the critical ones 26, 10, 2.[1]

These problems together with the discovery of pointlike structures in deep inelastic experiments giving a hard structure at large transverse momentum and with the proof of renormalizability of non abelian gauge theories made many people to go back to field theory and in particular to non abelian gauge theories and to try to use them for strong interactions. As a result of these attempts QCD was proposed in 1973, that has been up to now very successful in explaining the physics of strong interactions. In the subsequent years its perturbative and nonperturbative properties were studied. These results brought back people to field theory. In the beginning of the seventies pratically everybody went back to work in field theory by using it both

[1]An attempt to quantize the string for non critical values of the space-time dimension is due to Polyakov [10].

for strong interactions, building up QCD, and for the weak and electromagnetic interactions, building up the Weinberg-Salam model. This brought to the formulation of the standard model of strong, electromagnetic and weak interactions, that is a gauge field theory based on the gauge group:

$$SU(3) \otimes SU(2) \otimes U(1) \tag{1.16}$$

with coupling constants g_3, g_2 and g_1 respectively. It has been very successful in predicting the existence of neutral currents, of the W and Z gauge bosons, of jets etc.. All experiments performed up to an energy of about $210GeV$ at LEP are in perfect agreement with it. The success of the standard model shows that, at least up to an energy of $210GeV$, the world is successfully described by a field theory. The introduction in the quantum theory of the Planck constant

$$h = 6.62510^{-27} erg \cdot gr \tag{1.17}$$

that appears in the De Broglie relation between the momentum and energy of a massless particle and the wave lenght and frequency of a wave

$$p = \frac{h}{\lambda} \qquad\qquad E = h\nu \tag{1.18}$$

gives the possibility of relating, through the relation $E = pc$, lenghts with energies

$$\lambda = \frac{hc}{E} \tag{1.19}$$

If we express energies E in eV and lenghts L in cm we get the relation

$$L(cm) = \frac{1.95 \cdot 10^{-5} cm \cdot eV}{E(eV)} \tag{1.20}$$

Using the previous relation it is easy to check that an energy of $200GeV$ corresponds to a distance of about $10^{-16}cm = 10^{-3}Fermi$. This means that we have checked the validity of field theory up to a distance of $10^{-3}Fermi$. Remember that the dimension of a proton is of about $1Fermi$ corresponding to an energy of about $200MeV$.

In the standard model, if we neglect quark masses, there are two fundamental scales:

1. The QCD scale $\Lambda \sim 200MeV$ that corresponds to the dimension of the hadrons $\sim 1Fermi$.

2. The Fermi scale given by the v.e.v. of the Higgs field

$$< \Phi > \sim 246GeV \tag{1.21}$$

   This is both the scale in which the original group $SU(2) \otimes U(1)$ gets broken into the electromagnetic $U(1)$ and the scale for which the elementary particles get a mass. For energies higher than the Fermi scale they can be treated as massless.

In conclusion the hadrons of QCD get a mass given by $c\Lambda$, while the particles of the standard model get a mass given by $\alpha < \Phi >$, where c and α are dimensionless constants.

In the meantime some dualists continued to work in the dual string models trying to eliminate the problems 1. , 2. and 3., persuaded that such a rich and interesting structure could be of physical relevance in some future.

In order to bypass problem 2. it was proposed in 1974 by Scherk and Schwarz [11] to use the dual models not as models for hadrons but as a unified theory for all interactions including gravity. In this case the Regge slope does not correspond anymore to the dimension of an hadron 10^{-13} cm; but it is related to the Planck mass being $10^{-33} cm$. Then in order to get rid of problem (3) it was resurrected the old Kaluza-Klein mechanism [12].

Finally in 1976 Gliozzi, Olive and Scherk [13] proposed to consider a subsector of the Neveu-Schwarz-Ramond model, that is consistent by itself obtaining the first dual string model without a tachyon in the spectrum. The consistency and the supersymmetry properties of this submodel were shown only later by Green and Schwarz [14].

By the end of seventies it was clear that superstring was a consistent theory; it was not describing strong interactions, but rather a unified theory including gauge theories and the Einstein's theory of general relativity.

But what about gravity from the field theory point of view?

It was known that gravity is described by the Einstein's theory of general relativity through the Einstein's action

$$S = \frac{c^3}{16\pi G_N} \int d^4x \sqrt{-g} R \tag{1.22}$$

G_N is the Newton constant equal to $6.67 \cdot 10^{-8} cm^3 gm^{-1} sec^{-2}$. In the Newtonian limit the gravitational interaction between two particles with the same mass M is given by the Newton force:

$$\vec{F} = -\frac{GM^2}{R^2} \frac{\vec{R}}{R} \tag{1.23}$$

For small masses the gravitational interactions are weak. They become strong when the dimensionless constant becomes of the order 1:

$$\frac{G_N M^2}{\hbar c} \sim 1 \rightarrow M_P = \sqrt{\frac{\hbar c}{G_N}} = 1.22 \cdot 10^{19} GeV \tag{1.24}$$

This is happening when the two particles have a mass equal to the Planck mass M_P corresponding to the Planck lenght:

$$L_P = \frac{\hbar}{M_P c} = 1.6 \cdot 10^{-33} cm \tag{1.25}$$

Gravitational interactions become relevant in subatomic physics for particles having a mass of the order of the Planck mass and having a dimension of the order of the Planck lenght.

Apart from the QCD scale we have therefore two scales, the Fermi scale corresponding to an energy of $\sim 10^3 GeV$ and the Planck scale corresponding to an energy of $\sim 10^{19} GeV$. What is happening in between? Do we have other scales? Is the region in between described by a quantum field theory?

Let us say what we know and what are our prejudices about what is going to happen at an energy higher than the Fermi scale. First of all, if field theory provides also a consistent description of gravity, we must be able to construct a quantum theory for gravity. But gravity has a dimensional constant, the Newton constant G_N, that has dimension of a $[mass]^{-2}$, as the Fermi constant G_F of weak interaction, and therefore exactly as the Fermi four-fermion theory of weak interaction is not renormalizable. In weak interactions the problem was solved by introducing the gauge bosons and later on also the Higgs field in order to arrive at a renormalizable theory. But, unlike the gauge interactions coupled to a charge, gravity couples to energy and although we introduce the gravitons we are not able to construct a renormalizable theory for gravity. But, if a theory is nonrenormalizable, we must introduce a cutoff $\Lambda \sim M_P$ at high energy and the theory can only be used for energy much smaller than the cutoff. When we reach the energy of the cutoff, new physics must show up. A theory with cutoff is only an effective theory valid for energy much smaller than the cutoff. In conclusion the first fact is that nobody has yet been able to construct a renormalizable field theory including gravity.

The second point is that by using the fact that coupling constants in a gauge theory run with the energy at which we perform our experiments, if we extrapolate the three couplings of the standard model using the low-energy particle spectrum, we see that they have the tendency of meeting together at an energy of about $10^{16} GeV$. This brought Georgi and Glashow to construct a unified theory of the strong and electro-weak interaction with a single gauge coupling constant based on the gauge group $SU(5)$. But, in order to break the original $SU(5)$ symmetry, one must introduce additional Higgs fields that break the symmetry by acquiring a non-zero v.e.v $< \Phi > \sim M_{GUT} \sim 10^{16} GeV$. In such a theory or in any other grand-unified theory one has the scale of grand-unification as an intermediate scale between the Fermi scale and the Planck scale. In this way we have not explained the Planck scale at all. In addition we have introduced some other problems that are strongly related to each other. They are:

1. **Hierarchy problem**

   Why do we have two scales so different as the Fermi and the grand-unification scale? Not to talk about the Planck scale that is even higher.

2. **Naturalness problem**

   In the grand-unified theory we need two kinds of Higgs particles. The first kind are those that break the original grand-unified theory and get a mass of the order of the scale of grand-unification. In addition we need also other scalar Higgs particles, that remain massless after the breaking of grand unification and that break the symmetry of the standard model. But it is unnatural to have massless scalar particles in a theory with a grand-unified scale. Scalars are in this respect different from fermions because there is no symmetry protecting them from getting a mass.

3. **Fine tuning problem**

   Theories with scalars have quadratic divergences and therefore at each order of perturbation theory, if we want them to be massless, we need a fine tuning.

One possible way out of these problems is to extend the standard model in the supersymmetric standard model by introducing a supersymmetric partner for each particle of the standard model. But, there is no evidence of it in the experiments and therefore it must be broken. One can see that the previous problems can be solved also in the case of broken supersymmetry provided that the scale of supersymmetry breaking is of the same order of the Fermi scale. In addition, in the minimal supersymmetric standard model the unification of the three gauge coupling can be made to happen at a single point. Therefore the new prediction is that we expect to find the supersymmetric particles in the near future at LEP 200 or at LHC. After that we have a desert up the scale of grand unification and after that we have also to explain gravity. We have introduced supersymmetric particles and another scale, but we have not learned anything about gravity.

The problems of quantum gravity are due to the short distance infinities that we have in most field theories and that are due to the pointlike structure of the fundamental constituents. This problem appears already in classical electromagnetism and is solved by introducing the classical electron radius:

$$\frac{e^2}{r_0} = mc^2 \Longrightarrow r_0 = \frac{e^2}{mc^2} = \alpha \frac{\hbar}{mc} \tag{1.26}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant.

We could therefore think of solving this problem by introducing a theory in which the fundamental objects are not pointlike, but have an extension. Is it possible to extend field theory to another theory that reduces to field theory in some limit? We could think for instance of having a theory with an additional dimensional parameter that acts as a cutoff, but it is not a cutoff because it has a physical meaning and when this cutoff goes to infinity we recover field theory. In this way we would have a theory that would extend field theory and at the same time we could hope in this theory to solve the problems of quantum gravity. We will see that a fundamental theory based on one-dimensional objects, called strings, is precisely a theory that is able to solve the short distance problems of gravity and at the same time to provide an extension of field theory by having a parameter, the string tension T that is an energy per unit lenght. When $T \to \infty$, string theory reduces to field theory pretty much in the same way as quantum mechanics reduces to classical mechanics when $h \to 0$ or as special relativity reduces to gallilean mechanics when $c \to \infty$.

There is strong evidence nowdays that the fundamental theory is not a pure string theory, but rather what is now called M-theory. Why should then we discuss string theory? As we will see later M-theory is not a clearly defined theory and it reduces to one of the consistent string theories in some limit. For this reason many of the quantities that one computes in M-theory are actually computed in string theory. Finally string theory is a non trivial extension of field theory that sometimes can be used to compute field theoretical quantities in a simpler way.

2 Free bosonic string

2.1 Spinless point particle

A spinless free point particle is described by the coordinates of its position $x_\mu(\tau)$ in Minkowski space and by an action that is proportional to the length of its world line:

$$S = -mc \int \sqrt{-dx_\mu dx^\mu} = -mc \int d\tau \sqrt{-\dot{x}^2} \tag{2.27}$$

τ is an arbitrary parameter describing the motion of the particle. It does not have any physical meaning since S is invariant under an arbitrary reparametrization $\tau \to f(\tau)$. This follows from the fact that, if we perform a reparametrization $x^\mu(\tau) \to x^\mu(f(\tau))$ then the Lagrangian in eq.(2.27) will transform as:

$$d\tau \sqrt{-\dot{x}^2(\tau)} \to df(\tau) \sqrt{-\left(\frac{dx^\mu}{df(\tau)}\right)^2} \tag{2.28}$$

that implies that the action is invariant.

The identification of τ with some physical parameter corresponds to a gauge choice. A possible and often used gauge corresponds to taking τ proportional to the time: $\tau \sim x^0 \equiv ct$. In this gauge the action (2.27) becomes:

$$S = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}} \qquad \vec{v} = \frac{d\vec{x}}{dt} \tag{2.29}$$

Since we have treated the time component of $x^\mu(\tau)$ differently from the space components the action (2.29) is not anymore manifestly Lorentz covariant. But from it we can derive the dynamics of a free relativistic particle by introducing the three-momentum and the hamiltonian:

$$\vec{p} = \frac{dL}{d\vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ; \quad H = E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.30}$$

that satisfy the mass-shell relation:

$$\frac{E^2}{c^2} - p^2 = m^2 c^2 \tag{2.31}$$

In order to keep the manifest Lorentz covariance of the original theory one can choose the proper time gauge characterized by $\dot{x}^2 = -1$ or one can work directly with the gauge invariant action (2.27).

Starting from (2.27) one can compute the momentum of the particle:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{mc\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{2.32}$$

that implies the following primary constraint:

$$p^2 + m^2c^2 = 0 \tag{2.33}$$

corresponding to the mass-shell condition for the particle.

Because of the reparametrization invariance the canonical Hamiltonian is identically vanishing and therefore, following the Dirac procedure of quantization of a system with constraints, the Hamiltonian of our particle is given by:

$$H = -\frac{1}{2}e(\tau)(p^2 + m^2c^2) \tag{2.34}$$

where $e(\tau)$ is an arbitrary function of τ, that reflects the reparametrization invariance of (2.27). A choice of $e(\tau)$ corresponds to a gauge choice in the Hamiltonian formalism. For instance in the proper time gauge $e(\tau) = constant$.

The theory is quantized by requiring the following commutation relations:

$$[x^\mu(\tau), p^\nu(\tau)] = i\hbar g^{\mu\nu} \tag{2.35}$$

Using them it is easy to see that H generates the equation of motion for our dynamical variable $x^\mu(\tau)$ and $p^\mu(\tau)$. A realization of (2.35) is obtained by choosing

$$p_\mu = -i\hbar\frac{\partial}{\partial x^\mu} \tag{2.36}$$

acting on the wave function $\Phi(x^\mu)$.

In the quantum theory the constraint (2.33) becomes a condition defining the physical states:

$$(-\partial^\mu\partial_\mu + \frac{m^2c^2}{\hbar^2})\Phi(x) = 0 \tag{2.37}$$

that is the Klein-Gordon equation valid for a scalar particle.

In conclusion we have shown that the quantization of the system described by eq. (2.27) reproduces the well known Klein-Gordon theory for a scalar particle.

The previous formulation works only for a massive particle. In the case of a massless particle we have to modify the action (2.27).

By introducing the additional dynamical variable $e(\tau)$ we can rewrite eq. (2.27) as follows [15]:

$$S = \int d\tau \left[-\frac{1}{2}\frac{\dot{x}^2}{e} + \frac{1}{2}m^2c^2e\right] \tag{2.38}$$

Eq. (2.27) follows from eq. (2.38) if we write the algebraic equation of motion for $e(\tau)$:

$$\dot{x}^2 = -m^2c^2e^2 \tag{2.39}$$

and if we insert it in eq. (2.38) after having chosen the negative root $e = -\sqrt{-\dot{x}^2}/(mc)$. Therefore action (2.38) is completely equivalent to the original action (2.27). Notice that, using the momentum of the particle $p_\mu = -\dot{x}_\mu/e$ obtained from eq. (2.38) in (2.39), one reproduces the mass shell condition (2.33).

The Hamiltonian corresponding to the action (2.38) can be easily computed. One gets:

$$H = -\frac{e}{2}(p^2 + m^2c^2) \tag{2.40}$$

that is identical to (2.34) provided that one identifies the two functions $e(\tau)$ appearing in the two equations. Again the appearance of the arbitrary function $e(\tau)$ corresponds to the reparametrization invariance of (2.38) under the following finite transformations:

$$\dot{x}^\mu(\tau) \rightarrow \dot{f}(\tau)\frac{dx^\mu(f(\tau))}{df(\tau)} \quad ; \quad e(\tau) \rightarrow \dot{f}(\tau)e(f(\tau)) \tag{2.41}$$

that reduce to

$$\delta x^\mu(\tau) = \epsilon(\tau)\dot{x}^\mu(\tau) \quad , \quad \delta e(\tau) = \dot{\epsilon}(\tau)e(\tau) + \epsilon(\tau)\dot{e}(\tau) \tag{2.42}$$

for infinitesimal ones ($f(\tau) = \tau + \epsilon(\tau)$).

The action in eq. (2.38) allows us to describe also massless particles for which the "cosmological term" in eq. (2.38) vanishes. In this case one gets:

$$S = -\frac{1}{2}\int d\tau \frac{\dot{x}^2}{e} \tag{2.43}$$

and it is not possible to eliminate $e(\tau)$ from S. The quantization proceeds as explained in the case of action (2.27) and the equation of motion for $e(\tau)$ [eq. (2.39)] becomes a constraint that must be imposed on the physical states giving the Klein-Gordon equation.

In the previous equations e is a dimensional quantity. It is convenient to introduce a dimensionless einbein:

$$E = m^2c^2e \tag{2.44}$$

The action in eq.(2.38) becomes:

$$S = \frac{1}{2}\int d\tau \left[-\frac{\dot{x}^2m^2c^2}{E} + E\right] \tag{2.45}$$

The proper time gauge corresponds to the choice where $E(\tau)$ is a constant, but this constant depends on the lenght of the path. If we parameterize it by taking τ

variable in the interval $0 \leq \tau \leq 1$ and we take E constant then the lenght of the path is given by:

$$L = \int_0^1 d\tau \frac{E}{mc} \rightarrow E = mcL \tag{2.46}$$

In this gauge the action becomes:

$$S = \frac{mc}{2} \int d\tau \left[\frac{\dot{x}^2}{L} + L \right] \tag{2.47}$$

where we have gone to euclidean space ($\tau \rightarrow i\tau$). Using the previous action we can compute the probability amplitude to find the particle at the point x' for $\tau = 1$ if it was at x for $\tau = 0$ and the two points are connected by a path of lenght L. This probability amplitude is given by:

$$< x|x' >_L = \mathcal{N} \int_{x(0)=x}^{x(1)=x'} Dx^\mu e^{-\frac{mc}{2} \int d\tau \left[\frac{\dot{x}^2}{L} + L \right]} \tag{2.48}$$

The previous functional integral can be computed by expanding x^μ around a classical solution:

$$x^\mu(\tau) \equiv x^\mu_{cl} + \delta x^\mu(\tau) = x^\mu + ((x')^\mu - x^\mu)\tau + \delta x^\mu(\tau) \tag{2.49}$$

where the classsical solution satisfies the conditions $x^\mu_{cl}(0) = x^\mu$ and $x^\mu_{cl}(1) = x'$ and the fluctuation satisfies the following ones $\delta x^\mu(0) = \delta x^\mu(1) = 0$. Inserting the expansion in eq. (2.49) in eq. (2.48) we can perform the gaussian functional integral getting:

$$< x|x' >_L = \frac{\mathcal{N}}{[\det(-\frac{\partial_\tau^2}{L}]^{D/2}} e^{-\frac{mc}{2}\left[\frac{(x-x')^2}{L} + L \right]} \tag{2.50}$$

In order to compute the determinant we have to solve the eigenvalue equation:

$$-\frac{1}{L} \partial_\tau^2 \psi(\tau) = \lambda \psi(\tau) \tag{2.51}$$

with the boundary conditions $\psi(0) = \psi(1) = 0$. It is easy to get the following eigenfunctions and eigenvalues:

$$\psi_n(\tau) = c_n \sin(n\pi\tau) \quad ; \quad \lambda_n = \frac{n^2\pi^2}{L} \tag{2.52}$$

From them we can compute the determinant by using the ζ-function regularization:

$$\det(-\frac{\partial_\tau^2}{L}) = [\prod_{n=1}^{\infty} \frac{n^2\pi^2}{L}]^2 = \left(\frac{\pi^2}{L} \right)^{2\sum_{n=1}^\infty 1} e^{-4\sum_{n=1}^\infty \log n} = \left(\frac{\pi^2}{L} \right)^{2\zeta(0)} e^{-4\zeta'(0)} \tag{2.53}$$

where

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} \quad , \quad \zeta(0) = \sum_{n=1}^\infty 1 \rightarrow -\frac{1}{2} \quad , \quad \zeta'(0) = -\sum_{n=1}^\infty \log n \rightarrow -\frac{1}{2} \log(2\pi) \tag{2.54}$$

Using the previous equations we get

$$\det(-\frac{\partial_\tau^2}{L}) = 4L \tag{2.55}$$

Inserting it in eq.(2.50) we get:

$$< x|x' >_L = \mathcal{N} e^{-\frac{mc}{2}\left[\frac{(x-x')^2}{L} - L\right]} L^{-D/2} 2^{-D} \tag{2.56}$$

The normalization constant can be determined by requiring that:

$$< x|x' >_{L_1+L_2} = \int d^D x'' < x|x'' >_{L_1} < x''|x' >_{L_2} \tag{2.57}$$

Inserting eq.(2.56) in the previous equation one gets:

$$\mathcal{N} = \left(\frac{2mc}{\pi}\right)^{D/2} \tag{2.58}$$

In conclusion we get

$$< x|x' >_L = \left(\frac{mc}{2L\pi}\right)^{D/2} e^{-\frac{mc}{2}\left[\frac{(x'-x)^2}{L} + L\right]} \tag{2.59}$$

The propagator in configuration space is obtained by computing

$$\int_0^\infty \frac{dL}{2mc} < x|x' >_L = \int_0^\infty \frac{dL}{2mc} \left(\frac{mc}{2L\pi}\right)^{D/2} e^{-\frac{mc}{2}\left[\frac{(x'-x)^2}{L} + L\right]} \tag{2.60}$$

From it we can compute the propagator in momentum space given by:

$$< p|p' > = \int d^D x \int d^D x' e^{-i(p\cdot x - p'x')} \int_0^\infty \frac{dL}{2mc} < x|x' >_L = (2\pi)^D \delta^{(D)}(p-p') \frac{1}{p^2 + m^2c^2} \tag{2.61}$$

We have shown that the propagator of a spinless particle can be obtained from its particle description without needing to talk about field theory.

2.2 String action and elementary considerations

We have seen in the previous section that the basic structure used to construct an action for a spinless point particle is the infinitesimal line element dx^μ. If we consider a one dimensional extended object as a string the natural generalization of the line element is the infinitesimal area element

$$d\sigma_{\mu\nu} = dx_\mu \wedge dx_\nu \tag{2.62}$$

Choosing a system of coordinates one can write $d\sigma_{\mu\nu}$ as follows:

$$d\sigma_{\mu\nu} = \frac{\partial x_\mu}{\partial \zeta^\alpha}\frac{\partial x_\nu}{\partial \zeta^\beta} d\zeta^\alpha \wedge d\zeta^\beta = \frac{\partial x_\mu}{\partial \zeta^\alpha}\frac{\partial x_\nu}{\partial \zeta^\beta}\epsilon^{\alpha\beta} d\sigma d\tau \tag{2.63}$$

where $x_\mu(\sigma,\tau)$ are the coordinates of the world sheet of a string described by the variables σ and τ with $\zeta^0 = \tau$ and $\zeta^1 = \sigma$. $\epsilon^{\alpha\beta}$ is an antisymmetric tensor with $\epsilon^{01} = 1$.

Proceeding in analogy with the point particle we can use the area element (2.62) to write an action for a string, that is proportional to the area spanned by the string

$$S \sim \int \sqrt{-d\sigma_{\mu\nu} d\sigma^{\mu\nu}} \tag{2.64}$$

Inserting eq. (2.63) in (2.64) and fixing the proportionality constant one gets the Nambu-Goto action [16].

$$S = -cT \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \sqrt{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2} \tag{2.65}$$

where

$$\dot{x}^\mu \equiv \frac{\partial x^\mu}{\partial \tau} \qquad\qquad x'^\mu \equiv \frac{\partial x^\mu}{\partial \sigma} \tag{2.66}$$

and T is the string tension, that replaces the mass appearing in the case of a point particle. In going from eq.(2.64) to eq.(2.65) we have used the relation:

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} = \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\gamma}\eta^{\beta\delta} \tag{2.67}$$

where $\eta^{\alpha\beta}$ is the flat world sheet metric chosen to be $-\eta_{00} = \eta_{11} = 1$.

It is convenient to use the variable σ in the interval $0 \leq \sigma \leq \pi$. In the case of a closed string we must impose the periodicity condition $x^\mu(\tau, 0) = x^\mu(\tau,\pi)$ while in the case of an open string the points $\sigma = 0$ and $\sigma = \pi$ parametrize the two end points of the string.

The previous action can also be expressed in terms of the world sheet induced metric:

$$g_{\alpha\beta} = \frac{\partial x^\mu}{\partial \zeta^\alpha}\frac{\partial x_\mu}{\partial \zeta^\beta} \tag{2.68}$$

One gets

$$S = -cT \int d\tau \int d\sigma \sqrt{-\det(g_{\alpha\beta})} \tag{2.69}$$

The variables ζ^α are arbitrary variables that parametrize the world sheet of a string. We can choose another set of variables $\delta\zeta^\alpha = \epsilon^\alpha(\zeta^\alpha)$ without changing the physics of the system. This is reflected in the fact that the action (2.65) is invariant under the following transformation:

$$\delta x^\mu = \epsilon^\alpha(\zeta)\partial_\alpha x^\mu \tag{2.70}$$

where $\epsilon(\zeta)$ are arbitrary infinitesimal functions of ζ^α.

The classical equations of motion of the string can be obtained by imposing the least action principle that requires:

$$\delta S = 0 \tag{2.71}$$

for those variations such that $\delta x^\mu(\tau_i) = \delta x^\mu(\tau_f) = 0$. Starting from eq. (2.65) after a partial integration we get:

$$\delta S = \int_{\tau_i}^{\tau_f} \left[\int_0^\pi d\sigma \left(-\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial}{\partial \sigma}\frac{\partial L}{\partial x'^\mu} \right) \delta x^\mu + \frac{\partial L}{\partial x'^\mu} \delta x^\mu |_{\sigma=0}^{\sigma=\pi} \right] = 0 \tag{2.72}$$

Since δx^μ is arbitrary from eq. (2.71) the Euler-Lagrange equation follows

$$\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{x}^\mu} + \frac{\partial}{\partial \sigma}\frac{\partial L}{\partial x'^\mu} \equiv \frac{\partial}{\partial \zeta^\alpha} \left(\frac{\partial L}{\partial (\frac{\partial x^\mu}{\partial \zeta^\alpha})} \right) = 0 \tag{2.73}$$

The surface terms appearing in eq. (2.72) vanish if

$$\frac{\partial L}{\partial x'^\mu} = 0 \qquad or \qquad \delta x_\mu = 0 \qquad at \qquad \sigma = 0, \pi \tag{2.74}$$

for an open string and if

$$x^\mu(\tau, 0) = x^\mu(\tau, \pi) \tag{2.75}$$

in the case of a closed string. In the case of an open string the first kind of boundary condition in eq.(2.74) correspond to Neumann boundary condition, while the second one to Dirichlet boundary conditions. In the following we will consider only the Neumann ones because they are preserving Poincare invariance. We will come back to the Dirichlet ones when we will be considering D-branes.

From eq. (2.65) it is easy to compute:

$$\frac{\partial L}{\partial \dot{x}^\mu} \equiv P_\mu = cT \frac{\dot{x}_\mu x'^2 - x'_\mu (\dot{x} \cdot x')}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \tag{2.76}$$

and

$$\frac{\partial L}{\partial x'^\mu} = cT \frac{x'_\mu \dot{x}^2 - \dot{x}_\mu (\dot{x} \cdot x')}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \tag{2.77}$$

The relativistic invariance of the string action (2.65) implies that the four momentum:

$$p_\mu = \int_0^\pi d\sigma P_\mu(\tau, \sigma) \tag{2.78}$$

and the angular momentum

$$M_{\mu\nu} = \int_0^\pi d\sigma (x_\mu P_\nu - x_\nu P_\mu) \tag{2.79}$$

are constant of motion

$$\dot{p}_\mu = \dot{M}_{\mu\nu} = 0 \tag{2.80}$$

as a consequence of the eqs. of motion of the string.

Squaring eq. (2.77) we get the following constraint

$$\left(\frac{\partial L}{\partial x'^\mu} \right)^2 + c^2 T^2 \dot{x}^2 = 0 \tag{2.81}$$

implying, because of the boundary condition (2.74), that the end points of an open string move with the speed of light.

As in the case of a point particle we get in the Hamiltonian formalism two primary constraints between the dynamical variables, that can be obtained by squaring (2.76) and by multiplying (2.76) with x'_μ. They are given by:

$$c^2 T^2 x'^2 + P^2 = x' \cdot P = 0 \tag{2.82}$$

We could proceed to quantize the string as we have sketched in the case of a point particle using the Dirac procedure of quantization of a system with constraints. It is however for many respects more convenient to rewrite the action (2.65) in a form, that is quadratic in the string variable x^μ. This will be done in the next section.

In the last part of this section we study some simple motion of a string.

Let us choose the orthonormal gauge specified by the conditions:

$$\dot{x}^2 + x'^2 = \dot{x} \cdot x' = 0 \tag{2.83}$$

In this gauge eqs. (2.76) and (2.77) become:

$$P_\mu = cT\dot{x}_\mu \qquad\qquad \frac{\partial L}{\partial x'^\mu} = -cTx'_\mu \tag{2.84}$$

and therefore the eq. of motion in eq.(2.73) becomes:

$$\ddot{x}_\mu - x''_\mu = 0 \tag{2.85}$$

while the boundary condition in eq.(2.74) becomes:

$$x'_\mu(\sigma = 0, \pi) = 0 \tag{2.86}$$

Let us consider now some simple motion consisting of a straight open string of length 2a rigidly rotating around its center in the plane $x_1 x_2$. The coordinates of the string are given by:

$$x_1 = a\cos\sigma\cos\tau \qquad x_2 = a\cos\sigma\sin\tau$$

$$x_3 = 0 \qquad x^0 \equiv ct = a\tau \tag{2.87}$$

where $r = acos\sigma$ is the coordinate along the string that varies in the interval $-a \leq r \leq a$. The end points of the string move with the speed of light since:

$$\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 = \frac{c^2 r^2}{a^2} = c^2 \qquad if \qquad r^2 = a^2 \tag{2.88}$$

It is easy to check that (2.87) satisfy the equation of motion in eq.(2.85), the boundary conditions in eq.(2.86) and also the orthonormal gauge conditions in eq.(2.83). In conclusion the motions described by (2.87) are allowed motions of a string. The total momentum of the string is obtained from eq.(2.78). For the particular solution in eq.(2.87) one obtains:

$$p^\mu = \int_0^\pi d\sigma c T \dot{x}^\mu = cTa\pi\delta_{\mu 0} \tag{2.89}$$

Since we are in the c.o.m. frame the total three-momentum is zero, while from the energy of the string we can compute the mass

$$m \equiv \frac{p^0}{c} = Ta\pi \tag{2.90}$$

On the other hand the angular momentum of a rigidly rotating string has a non vanishing component only in the direction orthogonal to the plane in which the string rotates. It is given by:

$$J_{12} = \int_0^\pi d\sigma \, (x_1 P_2 - x_2 P_1) = \pi c T a^2/2 \tag{2.91}$$

implying the following relation between mass and angular momentum:

$$J = \alpha' \hbar (mc^2)^2 \tag{2.92}$$

with

$$T = \frac{1}{2\pi\alpha'\hbar c^3} \tag{2.93}$$

The previous two relations, that have been deduced for a particular set of motions of the string, are actually valid in general implying that the states of a string lie on linearly rising Regge trajectories. Eq. (2.93) gives the relation between the string tension and the Regge slope.

Another interesting feature of a string is that, if we put a charge g at one end point and we compute the gyromagnetic ratio G we find the result that $G = 2$. The string has therefore no anomalous magnetic moment. This property will be now checked for a rigidly rotating string, that generates a current given by:

$$j = g\frac{c}{2\pi a} \tag{2.94}$$

corresponding to a dipole magnetic moment

$$\mu = \frac{j}{c} A \tag{2.95}$$

where $A = \pi a^2$ is the area spanned by the string. Inserting (2.94) in (2.95) one obtains

$$\mu = \frac{g}{2mc} G J = g a / 2 \tag{2.96}$$

Using the expressions for the mass and the angular momentum in eqs.(2.90) and (2.91) for J and m we get the final result:

$$G = 2 \tag{2.97}$$

This result, explicitly shown for some particular motion, holds in fact for an arbitrary motion of the string.

Finally in the last part of this section we want to show the following relation between the slopes of open and closed string:

$$\alpha'_{cl} = \frac{1}{2} \alpha'_{op} \tag{2.98}$$

Using the argument already used for an open string it is easy to show that an allowed motion for a closed string is the one consisting of two straight open strings attached at the end points and rotating together around this common center.

Since the energy density for such closed string is twice the one of an open string, its squared mass will be four times the one of an open string:

$$m^2_{closed} = 4 m^2_{open} \tag{2.99}$$

On the other hand its angular momentum is only twice of that of an open string:

$$J_{closed} = 2 J_{open} \tag{2.100}$$

Combining eqs. (2.99) (2.100) with eq. (2.92) we get eq. (2.98).

2.3 Classical theory in the conformal gauge

In many considerations instead of the Nambu-Goto action (2.65) it is more convenient to use an action for the string that is quadratic in the coordinate $x^\mu(\sigma, \tau)$. This alternative action that is the generalization to the string of eq. (2.43) for a point particle is given by [17]:

$$S(x^\mu, g_{\alpha\beta}) = -\frac{T}{2} \int d\tau \int_0^\pi d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \tag{2.101}$$

where $\eta^{\mu\nu} = (1, 1...1, -1)$ is the D-dimensional metric $[\mu, \nu = 1, 2, ...D]$, $T = 1/(2\pi\alpha')^2$, $g = \det(g_{\alpha\beta})$, and $g_{\alpha\beta}$ is the world sheet metric tensor.

Viewed as a two dimensional field theory the action in eq. (2.101) describes the interaction of a set of D massless fields with an external gravitational field. In this case the D-dimensional Lorentz index plays the role of a flavour index.

Tha action in eq. (2.101) is invariant under arbitrary reparametrizations of the coordinates of the world sheet of the string. They act in the following way on the variables x^μ and $g_{\alpha\beta}$:

$$x^\mu(\xi) \to x^\mu(\xi') \quad , \quad g_{\alpha\beta)}(\xi) \to \frac{\partial(\xi')^\gamma}{\partial\xi^\alpha}\frac{\partial(\xi')^\delta}{\partial\xi^\beta} g_{\gamma\delta}(\xi') \tag{2.102}$$

that reduce to

$$\delta x^\mu = \epsilon^\alpha \partial_\alpha x^\mu \qquad \delta g_{\alpha\beta} = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_a \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma} \tag{2.103}$$

for infinitesimal transformations ($\xi' = \xi + \epsilon$). The equivalence of (2.101) with the Nambu-Goto action can be immediately seen by writing down the algebraic equations of motion for $g_{\alpha\beta}$. They imply the vanishing of the two dimensional energy-momentum tensor

$$\theta_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0 \tag{2.104}$$

that is a consequence of the fact that, because of the reparametrization invariance, there is no physical degree of freedom in the two dimensional space of the world sheet of the string. In deriving eq.(2.104) we have used the relation:

$$\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}} = -\frac{1}{2} g_{\alpha\beta}\sqrt{-g} \tag{2.105}$$

From eq. (2.104) it follows

$$\det(\partial_\alpha x \cdot \partial_\beta x) = \frac{g}{4}\left[g^{\gamma\delta}\partial_\gamma x \cdot \partial_\delta x\right]^2 \tag{2.106}$$

that, when inserted in eq. (2.101), reproduces the Nambu-Goto action (2.65).

In conclusion the two classical actions (2.65) and (2.101) are completely equivalent. The action (2.101) has the big advantage of being quadratic in the "matter field" x^μ and therefore the functional integration over x^μ in the quantum theory can be easily performed.

It must be stressed here, that although the action (2.101) describes a string moving in a D-dimensional Minkowski space, it can also be viewed as a two dimensional general invariant theory and therefore all the machinery of two dimensional field theories can be used in the string theories.

[2]In the following we will use units where $\hbar = c = 1$

In particular, because of reparametrization invariance, it is convenient to choose the conformal gauge where

$$g_{\alpha\beta} = \rho(\zeta)\eta_{\alpha\beta} \qquad \eta_{11} = -\eta_{00} = 1 \tag{2.107}$$

In this gauge the vanishing of the two dimensional energy-momentum tensor (2.104) implies the conditions:

$$\dot{x} \cdot x' = \dot{x}^2 + x'^2 = 0 \tag{2.108}$$

that correspond to the choice of an orthonormal system of coordinates in the world sheet of the string. For this reason the conformal gauge has been also called the orthonormal gauge.

In this gauge the Lagrangian in eq. (2.101) becomes a "conformal invariant theory"[3]:

$$L = -\frac{T}{2}\partial_\alpha x \cdot \partial^\alpha x \tag{2.109}$$

and the equation of motion (2.73) becomes

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) x^\mu(\sigma,\tau) = 0 \tag{2.110}$$

The boundary conditions (2.74) reduce to

$$\frac{\partial}{\partial\sigma} x^\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0 \tag{2.111}$$

for an open string, while for a closed string we must impose the periodicity condition

$$x^\mu(\tau,0) = x^\mu(\tau,\pi) \tag{2.112}$$

The most general solution of the eq. of motion and of the boundary conditions can be written as follows:

$$x^\mu(\tau,\sigma) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}[a_n^\mu e^{-in\tau} - a_n^{+\mu} e^{in\tau}]\frac{cos n\sigma}{\sqrt{n}} \tag{2.113}$$

for an open string and

$$x^\mu(\tau,\sigma) = q^\mu + 2\alpha' p^\mu \tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}[\tilde{a}_n^\mu e^{-2in(\tau+\sigma)} - \tilde{a}_n^{+\mu} e^{2in(\tau+\sigma)}]\frac{1}{\sqrt{n}} +$$

$$+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}[a_n^\mu e^{-2in(\tau-\sigma)} - a_n^{+\mu} e^{2in(\tau-\sigma)}]\frac{1}{\sqrt{n}} \tag{2.114}$$

[3]In the following, when not explicitly mentioned, we will be using units where $\alpha' = 1/2$ and $T = 1/\pi$.

for a closed string.

The choice of the conformal gauge does not fix uniquely the gauge; we can still perform gauge transformations that leave in the conformal gauge. They are the conformal transformations characterized by a parameter $\epsilon^\alpha(\sigma,\tau)$ in eq. (2.103) satisfying the conditions

$$\partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha - \eta_{\alpha\beta}\partial^\gamma \epsilon_\gamma = 0 \tag{2.115}$$

They are more transparent if we introduce light-cone coordinates:

$$\zeta^\pm = \zeta^0 \pm \zeta^1 \quad , \quad \epsilon^\pm = \epsilon^0 \pm \epsilon^1 \quad , \quad \frac{\partial}{\partial \zeta^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial \zeta^0} \pm \frac{\partial}{\partial \zeta^1}\right) \tag{2.116}$$

In terms of those variables the conditions (2.115) reduce to

$$\frac{\partial}{\partial \zeta^-}\epsilon^+ = \frac{\partial}{\partial \zeta^+}\epsilon^- = 0 \tag{2.117}$$

In conclusion the transformations that leave in the conformal gauge are characterized by two arbitrary functions $\epsilon^+(\zeta^+)$ and $\epsilon^-(\zeta^-)$, that transform the variables $\zeta^\pm$ as follows:

$$\delta\zeta^+ = \epsilon^+(\zeta^+) \qquad \qquad \delta\zeta^- = \epsilon^-(\zeta^-) \tag{2.118}$$

In the case of an open string we must impose additional restrictions on these functions. In fact in this case we have parametrized the end points of the string with the values $\sigma = 0, \pi$ and it is convenient to require that this parametrization is not changed by a reparametrization. From eq. (2.118) it follows that:

$$\delta\sigma = \frac{1}{2}[\epsilon^+(\tau+\sigma) - \epsilon^-(\tau-\sigma)] \tag{2.119}$$

and we require that $\delta\sigma|_{\sigma=0,\pi} = 0$. This implies that the two functions ϵ^+ and ϵ^- are restricted by

$$\epsilon^+(\tau) = \epsilon^-(\tau) \equiv \epsilon(\tau) \qquad \qquad \epsilon(\tau-\pi) = \epsilon(\tau+\pi) \tag{2.120}$$

The generators of the conformal transformations that leave unchanged the parametrization of the end points of the string can be written in terms of the two independent components of $\theta_{\alpha\beta}$:

$$L_\epsilon = \frac{1}{8\alpha'\pi}\int_0^\pi d\sigma \left[(\dot{x}+x')^2(\tau+\sigma)\epsilon(\tau+\sigma) + (\dot{x}-x')^2(\tau-\sigma)\epsilon(\tau-\sigma)\right] \tag{2.121}$$

$(\dot{x} \pm x')^2$ are only functions of $\tau \pm \sigma$ respectively as follows from the eq. of motion (2.110), that implies:

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)(\dot{x}-x')^2 = \left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right)(\dot{x}+x')^2 = 0 \tag{2.122}$$

They express the conservation of the two dimensional energy-momentum tensor, and they imply that L_ϵ is independent of τ . As it will be shown in Section (2.5) the previous equations hold in any two dimensional conformal invariant theory.

Using the explicit solution (2.113) it is easy to see that:

$$(\dot{x} + x')^2(\tau, \sigma) = (\dot{x} - x')^2(\tau, -\sigma) \tag{2.123}$$

We can therefore rewrite (2.121) in more compact form:

$$L_\epsilon = \frac{1}{8\alpha'\pi} \int_{-\pi}^{\pi} d\sigma (\dot{x} + x')^2(\tau + \sigma)\epsilon(\tau + \sigma) \tag{2.124}$$

Finally, because of the symmetry between τ and σ, we can integrate over τ instead of σ and put $\sigma = 0$. In so doing we get:

$$L_\epsilon = \frac{1}{8\alpha'\pi} \int_{-\pi}^{\pi} d\tau \dot{x}^2(\tau)\epsilon(\tau) \tag{2.125}$$

where we have used the boundary conditions (2.111).

Using the explicit expression for $\dot{x}^\mu$ in terms of the oscillators

$$\dot{x}^\mu(\tau, 0) = 2\alpha' p^\mu + \sqrt{2\alpha'} \sum_{m=1}^{\infty} \sqrt{m}[a^\mu_m e^{-im\tau} + a^{+\mu}_m e^{im\tau}] \tag{2.126}$$

and choosing $\epsilon(\tau) = e^{in\tau}$ we get from eq. (2.125):

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m \tag{2.127}$$

where

$$\alpha^\mu_n = \begin{cases} \sqrt{n} a^\mu_n & if \;\; n > 0 \\ \sqrt{2\alpha'} p^\mu & if \;\; n = 0 \\ \sqrt{n} a^{+\mu}_n & if \;\; n < 0 \end{cases} \tag{2.128}$$

Finally introducing the variable $z = e^{i\tau}$ we can rewrite eq. (2.125) in the final form:

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} \left[-\frac{1}{4\alpha'} \left(\frac{\partial x^\mu}{\partial z} \right)^2 \right] \tag{2.129}$$

In the string theory the conformal invariance is not a classification symmetry as isospin or the Poincarè group. It is instead a residual gauge invariance corresponding to the reparametrizations that leave in the conformal gauge. It is a situation rather similar to a gauge field theory when one chooses the temporal or the Lorentz gauge. In this cases in fact the gauge is not completely fixed; one can still perform gauge transformations staying in those gauges and the generators of this residual invariance are identically vanishing.

The same is true in the case of the string in the conformal gauge, where in fact, because of the conditions (2.108), the generators of the conformal transformations (2.121) are identically vanishing implying that

$$L_n = 0 \tag{2.130}$$

for any integer n.

The previous considerations can be extended to the closed string, where the generators of the conformal transformations are characterized by two independent functions $\epsilon^+(\tau+\sigma) \equiv \epsilon(\tau+\sigma)$ and $\epsilon^-(\tau-\sigma) \equiv \tilde{\epsilon}(\tau-\sigma)$ and they are given by

$$\tilde{L}_\epsilon = \frac{1}{16\alpha'\pi} \int_0^\pi d\sigma (\dot{x} + x')^2 (\tau+\sigma) \tilde{\epsilon}(\tau+\sigma) \tag{2.131}$$

and

$$L_\epsilon = \frac{1}{16\alpha'\pi} \int_0^\pi d\sigma (\dot{x} - x')^2 (\tau-\sigma) \epsilon(\tau-\sigma) \tag{2.132}$$

Introducing the two variables

$$z = e^{2i\zeta^+} \qquad \bar{z} = e^{2i\zeta^-} \tag{2.133}$$

and using the relations

$$(\dot{x} + x')^2 = -16 z^2 \left(\frac{\partial x}{\partial z} \right)^2 \qquad (\dot{x} - x')^2 = -16 \bar{z}^2 \left(\frac{\partial x}{\partial \bar{z}} \right)^2 \tag{2.134}$$

we can rewrite eqs. (2.131) and (2.132) in the following way:

$$\tilde{L}_n = \frac{1}{2\pi i} \oint dz z^{n+1} \left[-\frac{1}{\alpha'} \left(\frac{\partial x^\mu}{\partial z} \right)^2 \right] \tag{2.135}$$

$$L_n = \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \left[-\frac{1}{\alpha'} \left(\frac{\partial x^\mu}{\partial \bar{z}} \right)^2 \right] \tag{2.136}$$

where we have chosen $\epsilon(\tau+\sigma) = z^n$ and $\tilde{\epsilon}(\tau-\sigma) = \bar{z}^n$. In terms of the harmonic oscillators introduced in eq. (2.114) we get

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m \cdot \alpha_{n-m} \quad ; \quad \tilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} \tag{2.137}$$

where for the non zero modes we have used the convention in (2.128), while the zero mode is given by:

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{2\alpha'} \frac{p^\mu}{2} \tag{2.138}$$

Notice the different overall factors in the brackets (2.129) and (2.135) and (2.136) related to a different normalization used for the coordinate x^μ for the open and closed string. We would have found the same normalization if we had chosen for the closed string the variable σ to vary in the interval $0 \leq \sigma \leq 2\pi$. This is a more natural choice since a closed string can always be thought as two open strings attached at the end points.

Finally also for a closed string the conformal generators are vanishing quantities:

$$L_n = \tilde{L}_n = 0 \tag{2.139}$$

for any integer n.

2.4 Quantization in the light-cone gauge

In the previous section we have seen that the choice of the conformal gauge does not fix uniquely the gauge. We can in fact still perform conformal transformations and stay in the conformal gauge.

One way of quantizing the theory is to first fix completely the gauge in the classical theory and then quantize only the independent left over physical degrees of freedom.

A very convenient way of fixing completely the gauge is by choosing the light-cone gauge characterized by the condition:

$$x^+ = 2\alpha' p^+ \tau \tag{2.140}$$

where

$$x^\pm = \frac{x^D \pm x^{D-1}}{\sqrt{2}} \qquad\qquad x_\pm = \frac{x_D \pm x_{D-1}}{\sqrt{2}} \tag{2.141}$$

The lightcone coordinates with upper and lower indices are related through the relations: $x_+ = -x^-, x_- = -x^+$. In terms of the lightcone coordinates the scalar product of two vectors can be written as:

$$A_\mu B^\mu = A_i B^i - A^+ B^- - A^- B^+ \tag{2.142}$$

Although very similar to the gauge choice in the fourth eq.(2.87), (2.140) has the advantage of allowing also the elimination of the degrees of freedom corresponding to x^- as we will see later on. This is a possible gauge choice inside the conformal gauge as we show in the case of a closed string.This follows from the fact that the most general solution of the equation of motion (2.110) and of the boundary condition (2.112) is given by:

$$x^\mu = \tilde{\phi}^\mu(\tau+\sigma) + \phi^\mu(\tau-\sigma) + 2\alpha' p^\mu \tau \tag{2.143}$$

and that under a conformal transformation x^μ transforms as follows:

$$\delta x^\mu = \epsilon^+(\tau+\sigma)\frac{\partial}{\partial\zeta^+}x^\mu + \epsilon^-(\tau-\sigma)\frac{\partial}{\partial\zeta^-}x^\mu \tag{2.144}$$

where $\tilde{\phi}^\mu(\tau+\sigma), \phi^\mu(\tau-\sigma)$ and $\epsilon^\pm(\tau\pm\sigma)$ are all periodic function with period equal to 2π.

Therefore by performing a suitable conformal transformation with periodic functions $\epsilon^\pm(\tau\pm\sigma)$ we can bring one component of $x^\mu(\tau,\sigma)$ [say the component x^+] in a form where $\phi^\pm = 0$, that is in fact the form proposed in eq. (2.140).

In the case of an open string the two functions appearing in (2.143) and (2.144) are not independent. They are both periodic with period equal to 2π and they must be identified: $\phi(\tau) = \bar{\phi}(\tau)$ and $\epsilon^+(\tau) = \epsilon^-(\tau)$. Therefore the gauge (2.140) can also be chosen for an open string.

In the light-cone gauge the only independent degrees of freedom are the transverse ones. The longitudinal and scalar ones can be in fact expressed in terms of the transverse ones.

This follows from the constraints in eq.(2.108), that with the choice (2.140) allow one to fix x^- as a function of the transverse components $x^i[i=1,...D-2]$, that are orthogonal to both $\pm$ directions. We get

$$\dot{x}^- = \frac{1}{4\alpha' p^+}(\dot{x}_i^2 + x'^2_i) \qquad x'^- = \frac{1}{2\alpha' p^+}\dot{x}_i \cdot x'_i \tag{2.145}$$

that up to a constant of integration determine completely x^- as a function of x^i.

In the following we want to obtain the Hamiltonian of the string in the lightcone gauge and from it determine its spectrum. The Hamiltonian can be determined in various ways. The first one is the following.

Since the Hamiltonian of a system is the conjugate variable to the evolution parameter, in the light-cone gauge, where the evolution parameter τ is proportional to x^+, the Hamiltonian density will be proportional to P^-. More precisely we get:

$$\mathcal{H} = -i\frac{\partial}{\partial\tau} = (-i)2\alpha' p^+ \frac{\partial}{\partial x^+} = 2\alpha' p^+ P^- \tag{2.146}$$

From (2.109) we get:

$$P^- = \frac{\dot{x}^-}{2\alpha'\pi} \tag{2.147}$$

and therefore, using eqs.(2.146) and (2.147), the Hamiltonian of the string in the light-cone gauge is given by:

$$H \equiv \int_0^\pi d\sigma \mathcal{H} = 2\alpha' p^+ \int_0^\pi d\sigma P^- = 2\alpha' p^+ p^- = \frac{1}{4\alpha'\pi}\int_0^\pi d\sigma [\dot{x}_i^2 + x'^2_i] \tag{2.148}$$

where we have used the first equation in (2.145).

A second way to get the string Hamiltonian is to start from the following action containing only the transverse degrees of freedom, namely:

$$S = -\frac{T}{2}\int d\tau \int_0^\pi d\sigma \eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i \tag{2.149}$$

and then from it compute the corresponding Hamiltonian. In this way one gets the same expression as in eq.(2.148).

Using the gauge choice (2.140) in the most general solution (2.113) and (2.114) and remembering (2.128) we get that

$$\alpha_n^+ = 0 \qquad\qquad n \neq 0 \tag{2.150}$$

for an open string and

$$\alpha_n^+ = \tilde{\alpha}_n^+ = 0 \qquad\qquad n \neq 0 \tag{2.151}$$

in the case of a closed string.

On the other hand eqs. (2.145) determine the oscillators α_n^- in terms of the transverse ones. We get

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_{m=-\infty}^{\infty}\alpha_{n-m}^i\alpha_m^i \qquad n \neq 0 \tag{2.152}$$

in the case of an open string and

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_{m=-\infty}^{\infty}\alpha_{n-m}^i\alpha_m^i$$

$$\sqrt{2\alpha'}\tilde{\alpha}_n^- = \frac{1}{2p^+}\sum_{m=-\infty}^{\infty}\tilde{\alpha}_{n-m}^i\tilde{\alpha}_m^i \tag{2.153}$$

for a closed string. The previous expressions can also equivalently be obtained by inserting eq.(2.152) in eq.s (2.127) and (2.130) for the open string and by inserting eq. (2.153) in eq.s (2.137) and (2.139) for the closed string.

In conclusion in the light-cone gauge the only independent degrees of freedom are the transverse oscillators supplemented by the zero modes p^μ and q^μ.

The open string can therefore be quantized by imposing the following commutation relations:

$$[\alpha_n^i, \alpha_m^j] = n\delta^{ij}\delta_{n+m,0} \qquad [q^\mu, p^\nu] = ig^{\mu\nu} \tag{2.154}$$

In the case of a closed string we must add the commutation relations for the oscillators $\bar{\alpha}_n^i$:

$$[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0} \tag{2.155}$$

The previous commutation relations for the transverse degrees of freedom can be obtained starting from the action (2.149) and applying the standard quantization procedure. This will be done explicitly when we will discuss the covariant quantization in Section 2.6) and therefore is not repeated here.

The spectrum of the open string can be computed from the Hamiltonian (2.148), that in terms of the oscillators is given by

$$2\alpha' p^+ p^- = \alpha' p^i p^i + \sum_{n=1}^{\infty} n a_n^{+i} a_n^i + c \tag{2.156}$$

where we have added an arbitrary constant to take into account of the arbitrariness in the ordering of the harmonic oscillators.

In the usual harmonic oscillator the constant c is given by the zero point energy. Here in the string theory the zero point energy is formally infinite being equal to

$$c = \frac{D-2}{2} \sum_{n=1}^{\infty} n \tag{2.157}$$

and it must be regularized.

Brink and Nielsen [19] have shown how to obtain a finite expression from (2.157). Here we use another regularization scheme proposed by Gliozzi [20] called ζ-function regularization, that amounts to replace (2.157) with

$$c = \frac{D-2}{2} \lim_{s \to -1} \sum_{n=1}^{\infty} n^{-s} = \frac{D-2}{2} \lim_{s \to -1} \zeta_R(s) \tag{2.158}$$

where $\zeta_R(s)$ is the Riemann ζ-function, that is an analytic function for $s = -1$ and its value is given by $\zeta_R(-1) = -1/12$. Inserting this value in eq. (2.158) we get the following expression for the zero point energy:

$$c = -\frac{D-2}{24} \tag{2.159}$$

and therefore we can rewrite eq. (2.156) in the following form:

$$\alpha(M^2) = \sum_{n=1}^{\infty} n a_n^{+i} a_n^i \tag{2.160}$$

where

$$\alpha(M^2) = \alpha_0 + \alpha' M^2 \tag{2.161}$$

with

$$\alpha_0 \equiv = -c = \frac{D-2}{24} \qquad\qquad M^2 = -p^2 \tag{2.162}$$

Another way to obtain (2.160) is to use in eq. (2.130) for $n = 0$ the gauge choice (2.150) written in terms of the oscillators allowing for a constant c as done in (2.156). We get

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} n a_n^{+i} a_n^i = -c \tag{2.163}$$

that coincides with eq. (2.156).

We have quantized the open string in the light-cone gauge loosing the manifest Lorentz invariance of the original action (2.101).

We must therefore check that the quantization procedure preserves the Lorentz invariance of the original theory by constructing the Lorentz generators and show that they satisfy the Lorentz algebra.

A natural expression for them is given by:

$$J^{ij} = \ell^{ij} - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n)$$

$$J^{+-} = \ell^{+-} \qquad\qquad J^{i+} = \ell^{i+}$$

$$J^{i-} = \ell^{i-} - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^i_{-n} \alpha^-_n - \alpha^-_{-n} \alpha^i_n) \tag{2.164}$$

where

$$\ell^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu \tag{2.165}$$

and α^-_n is given by eq. (2.152):

It can be shown that the operators in (2.164) satisfy the Lorentz algebra only if:

$$c = -1 \qquad\qquad D = 26 \tag{2.166}$$

Therefore only for those values of c and D we have succeeded in preserving the Lorentz invariance in the quantum theory.

Inserting (2.166) in (2.160) we get the final expression for the masses of the string states

$$\alpha' M^2 = \sum_{n=1}^{\infty} n a^{+i}_n a^i_n - 1 = N - 1 \tag{2.167}$$

The lowest state is given by the vacuum $|0>$ that corresponds to a tachyon with $M^2 = -\frac{1}{\alpha'}$. The next level, corresponding $M^2 = 0$, is given by the state $a^+_{1;i}|0>$, that describes the transverse components of a massless spin 1 particle ("photon"). At the level $N = 2$ we find the two states:

$$a^+_{1;i} a^+_{1;j} |0> \qquad\qquad a^+_{2;i}|0> \tag{2.168}$$

that describe a massive spin 2 particle with $M^2 = \frac{1}{\alpha'}$.

Because of Lorentz invariance in the center of mass frame the states of various levels must be classified according to the representations of $SO(D-1)$.

The number of states appearing at the level $N = 2$ can be obtained from eq. (2.168) and it is given by

$$\frac{(D-2)(D-1)}{2} + D - 2 = \frac{(D-2)(D+1)}{2} \tag{2.169}$$

It is nice to see that it coincides with the number of components of a spin 2 in $SO(D-1)$ given by $\frac{D(D-1)}{2} - 1$.

The degeneracy of states at an arbitrary level N can be obtained from the partition function:

$$G(w) = \frac{1}{w} [f(w)]^{2-D} \qquad\qquad f(w) = \prod_{n=1}^{\infty} (1 - w^n) \tag{2.170}$$

that is obtained from:

$$G(w) = Tr\left(w^{(\sum_{n=1}^{\infty} n a_n^{+i} a_n^i - 1)}\right) \tag{2.171}$$

where the term -1 comes again from the zero point energy.

From (2.171) it follows that, if we fix the level N, the degeneracy $T_d(N)$ of states at the level is the coefficient of the power w^{N-1} in the expansion of (2.170) in power series around $w = 0$:

$$G(w) = \frac{1}{w} \sum_{N=0}^{\infty} T_d(N) w^N \tag{2.172}$$

where $d = D - 2$.

The function $G(w)$ is well known to the mathematicians, who called it "partitio numerorum".

The degeneracy of states at the level N can be obtained from (2.172) and it is given by:

$$T_d(N) = \frac{1}{2\pi i} \oint dw w^{-N-1} [f(w)]^d \tag{2.173}$$

Since $T_d(N)$ has the following asymptotic behaviour for $N \to \infty$

$$T_d(N) \sim \frac{1}{\sqrt{2}} \left(\frac{d}{24}\right)^{d+1/2} N^{-(\frac{d+3}{4})} e^{2\pi\sqrt{\frac{dN}{6}}} \tag{2.174}$$

one is led to a density of states per unit of mass given by:

$$N(M) \sim A M^{-B} e^{\beta_0 M} \tag{2.175}$$

where

$$B = \frac{d+1}{2} \qquad \beta_0 = 2\pi\sqrt{\frac{d\alpha'}{6}} \tag{2.176}$$

and we have used $N \sim \alpha' M^2$.

The exponential increase of the density of states in eq. (2.175) implies the existence of a limiting temperature if we consider the partition function of an ensemble of resonances:

$$Z(T) = \int_0^{\infty} dM N(M) e^{-\frac{M}{T}} \tag{2.177}$$

It is a well defined quantity only if the temperature is smaller than a limiting temperature $T_0 = 1/\beta_0$.

Since the string describes the hadronic phase, T_0 corresponds to the temperature in which the constituents of the hadrons get liberated.

In the quantum theory of a closed string we can proceed analogously as in the case of an open string and get the spectrum from the conditions (2.139) for $n = 0$, that imply

$$L_0 - 1 = \tilde{L}_0 - 1 = 0 \tag{2.178}$$

where the arbitrary constant c and the space-time dimension D have been chosen as for the open string in order to have a Lorentz invariant theory.

Summing the expressions in (2.178) after having written them in terms of the oscillators we get:

$$2 + \frac{\alpha'}{2} M^2 = \sum_{n=1}^{\infty} n[a_n^{+i} a_n^i + \tilde{a}_n^{+i} \tilde{a}_n^i] \tag{2.179}$$

while if we subtract them we get:

$$N = \sum_{n=1}^{\infty} n a_n^{+i} a_n^i = \tilde{N} = \sum_{n=1}^{\infty} n \tilde{a}_n^{+i} \tilde{a}_n^i \tag{2.180}$$

Taking into account that $\alpha_c{}' \equiv \alpha'/2$ we can rewrite the equations (2.179) and (2.180) characterizing the spectrum of a closed string in the following final form

$$\alpha_c{}' M^2 = N + \tilde{N} - 2 \tag{2.181}$$

and

$$N = \tilde{N} \tag{2.182}$$

The lowest state of the spectrum is a tachyon with mass $M^2 = -2/\alpha_c{}'$ described by the vacuum $|0>$.

The first excited level containing massless states is described by the states:

$$a_{1;i}^{+} \tilde{a}_{1;j}^{+} |0> \tag{2.183}$$

The symmetric and traceless state corresponds to the graviton, the trace of (2.183) corresponds to a dilaton and finally the antisymmetric state describes an antisymmetric tensor.

In general the degeneracy at an arbitrary level can be obtained from the "partition function":

$$F(\rho) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{|z|^2} \prod_{n=1}^{\infty} \left[\frac{1}{|1 - z^n|^2} \right]^{D-2} \tag{2.184}$$

where $z = \rho e^{i\theta}$.

Notice that the integration over θ allows one to take into account of the condition (2.182).

2.5 Conformal invariant theories

In Section (2.3) we have seen that a string in the conformal gauge is described by a two-dimensional "free massless bosonic theory", that is conformal invariant.

In this section we want to give a general treatment of a conformal invariant theory in two dimensions, that will be very useful for the covariant quantization of the bosonic string described in Section (2.6) and that is also so general that it would also be applied to more complicated string theories.

A two dimensional conformal invariant quantum field theory is characterized by the existence of a conserved symmetric and traceless energy-momentum tensor:

$$\partial_\alpha \theta^{\alpha\beta} = \theta^\alpha_\alpha = 0 \qquad \theta^{\alpha\beta} = \theta^{\beta\alpha} \tag{2.185}$$

Since it is symmetric and traceless, it has only two independent components. If we use light-cone coordinates:

$$A^\pm = A^0 \pm A^1 \qquad\qquad \zeta^\pm = \zeta^0 \pm \zeta^1 \tag{2.186}$$

the two independent components are

$$\theta^{++} = \theta^{0+1,0+1} = 2(\theta^{01} + \theta^{00}) \quad , \quad \theta^{--} = \theta^{0-1,0-1} = 2(\theta^{00} - \theta^{01}) \tag{2.187}$$

while

$$\theta^{+-} = \theta^{-+} = 0 \tag{2.188}$$

The conservation equation (2.185) implies the two equations for θ^{++} and θ^{--}:

$$\frac{\partial}{\partial \zeta^+}\theta^{++} = \frac{\partial}{\partial \zeta^-}\theta^{--} = 0 \tag{2.189}$$

As a consequence $\theta^{++}(\zeta^-)[\theta^{--}(\zeta^+)]$ is only a function of $\zeta^-[\zeta^+]$. Equations (2.185) imply that

$$\frac{\partial}{\partial \zeta^\beta}[\epsilon_\alpha \theta_{\alpha\beta}] = 0 \tag{2.190}$$

if ϵ_α satisfies the condition (2.115) characterizing a conformal transformation. Because of (2.190) we can construct the following constants of motion:

$$L_\epsilon = \int d\sigma \epsilon^\alpha \theta_{\alpha 0} \tag{2.191}$$

depending on a function ϵ^α satisfying eq. (2.115). In terms of the light-cone variables (2.191) becomes:

$$L_\epsilon = \frac{1}{4}\int d\sigma[\epsilon^+(\zeta^+)\theta^{--}(\zeta^+) + \epsilon^-(\zeta^-)\theta^{++}(\zeta^-)] \tag{2.192}$$

where we have chosen σ to vary in the interval $0 \leq \sigma \leq \pi$ and therefore the various functions appearing in (2.192) are periodic functions of period π. It is convenient in many cases as for instance in string theories to introduce the new variables:

$$z = e^{2i\zeta^-} \qquad \bar{z} = e^{2i\zeta^+} \qquad \zeta^\pm = \tau \pm \sigma \tag{2.193}$$

related to the original ones by a conformal transformation. In euclidean space where $\tau \to i\tau$ $\bar{z}$ becomes the complex conjugate of z.

A conformal or primary field $\Phi(z,\bar{z})$ in the notation of Ref. [21] is defined as an object that transforms in the following way under a finite conformal transformation:

$$\Phi(z,\bar{z}) \to \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} \Phi(w,\bar{w}) \tag{2.194}$$

where $w = w(z)$ and $\bar{w} = \bar{w}(\bar{z})$. (2.194) implies the infinitesimal transformations:

$$\delta\Phi(z,\bar{z}) = [\epsilon(z)\frac{\partial}{\partial z} + \Delta\epsilon'(z)]\Phi(z,\bar{z}) + [\bar{\epsilon}(\bar{z})\frac{\partial}{\partial \bar{z}} + \bar{\Delta}\bar{\epsilon}'(\bar{z})]\Phi(z,\bar{z}) \tag{2.195}$$

where $w(z) = z+\epsilon(z)$ and $\bar{w}(\bar{z}) = \bar{z}+\bar{\epsilon}(\bar{z})$, with $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ infinitesimal quantities.

The parameters Δ and $\bar{\Delta}$ are the left and right conformal dimensions of $\Phi(z,\bar{z})$.

Since there is a complete symmetry between left and right variable and transformations, for the sake of simplicity we will omit in the following the dependence on the variable $\bar{z}$.

It is very useful to rewrite the transformation (2.195) in terms of the operator product expansion (OPE) of the energy momentum tensor with $\Phi(z,\bar{z})$. First of all it is easy to rewrite the first term of (2.192) in terms of the variable (2.193):

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \tag{2.196}$$

where $\epsilon^{+}(\tau+\sigma) = z^{n+1}$, $\theta^{--}(\tau+\sigma) \equiv \frac{2}{\pi} z^2 T(z)$ and the integral is performed around the origin.

Let us then introduce the notion of radially ordered OPE between two fields in euclidean space:

$$R\left(A(z)B(\zeta)\right) = \begin{cases} A(z)B(\zeta) & if\ \ |z| > |\zeta| \\ \pm B(\zeta)A(z) & if\ \ |z| < |\zeta| \end{cases} \tag{2.197}$$

where the minus sign must be chosen only if both fields are fermions.

The OPE between a primary field and the energy-momentum tensor $T(z)$ is given by:

$$R(T(z)\Phi(\zeta)) = \frac{\partial/\partial\zeta\Phi(\zeta)}{z-\zeta} + \Delta\frac{\Phi(\zeta)}{(z-\zeta)^2} + \cdots \tag{2.198}$$

It is in fact easy to prove that it implies the transformation (2.195). Since

$$\delta\Phi \equiv [L_{\epsilon}, \Phi(\zeta)] \tag{2.199}$$

we can rewrite L_{ϵ} as in (2.196) with $\epsilon(z) = z^{n+1}$ and get:

$$\delta\Phi = \frac{1}{2\pi i}\left(\oint_{|z|>|\zeta|} dz\epsilon(z)T(z)\Phi(\zeta) - \oint_{|z|<|\zeta|} dz\epsilon(z)\Phi(\zeta)T(z)\right) =$$

$$= \frac{1}{2\pi i} \oint_{\zeta} dz \epsilon(z) R(T(z)\Phi(\zeta)) = \epsilon(\zeta) \frac{\partial}{\partial \zeta} \Phi(\zeta) + \Delta \epsilon'(\zeta) \Phi(\zeta) \tag{2.200}$$

where the integral has been performed in the complex z plane around the point ζ and we have used eq. (2.198).

In conclusion we have shown that the singular terms in the OPE (2.198) between $T(z)$ and a primary field $\Phi(\zeta)$ are completely fixed by the conformal invariance of the theory.

The energy-momentum tensor is also a primary field with conformal dimension $\Delta = 2$ implying the following OPE:

$$R(T(z)T(\zeta)) = \frac{\partial/\partial\zeta T(\zeta)}{z-\zeta} + 2\frac{T(\zeta)}{(z-\zeta)^2} + \frac{c/2}{(z-\zeta)^4} + \cdots \tag{2.201}$$

The last term in (2.201) containing an arbitrary parameter c is allowed for a primary field with conformal dimension $\Delta = 2$ being consistent with closure of the conformal algebra.

From eqs. (2.200) and (2.201) we get:

$$\delta T(\zeta) \equiv [L_\epsilon, T(\zeta)] = \oint_\zeta \frac{dz}{2\pi i} \epsilon(z) \left[\frac{\partial/\partial\zeta T(\zeta)}{z-\zeta} + 2\frac{T(\zeta)}{(z-\zeta)^2} + \frac{c/2}{(z-\zeta)^4} \right] \tag{2.202}$$

Performing the integral we get

$$[L_\epsilon, T(\zeta)] \equiv \delta T(\zeta) = [\epsilon(\zeta)\frac{\partial}{\partial\zeta} + 2\epsilon'(\zeta)]T(\zeta) + \frac{c}{12}\epsilon'''(\zeta) \tag{2.203}$$

that implies the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m;0} \tag{2.204}$$

containing an arbitrary parameter c that depends on the particular conformal theory that we consider.

As an explicit application of the previous considerations let us consider a free massless bosonic theory described by the following action:

$$S = -\frac{1}{2\pi} \int d\tau \int_0^\pi d\sigma \partial_\alpha \Phi \partial^\alpha \Phi \tag{2.205}$$

The two independent components of the energy-momentum tensor are:

$$\theta^{++} = \frac{1}{2\pi} : (\dot{\Phi} - \Phi')^2 : \qquad \theta^{--} = \frac{1}{2\pi} : (\dot{\Phi} + \Phi')^2 : \tag{2.206}$$

and therefore

$$T(z) = -\frac{1}{2} : \left(\frac{\partial \Phi}{\partial z} \right)^2 : \tag{2.207}$$

The propagator of the field Φ can be easily computed and it is given by:

$$< \Phi(z,\bar{z})\Phi(\zeta,\bar{\zeta}) >= -\log[(z-\zeta)(\bar{z}-\bar{\zeta})] \tag{2.208}$$

Neglecting the dependence on the variables $\bar{z}$ and $\bar{\zeta}$ and using the contractions:

$$< \Phi(z)\Phi(\zeta) >= -\log(z-\zeta) \qquad < \Phi(z)\frac{\partial}{\partial\zeta}\Phi(\zeta) >= \frac{1}{z-\zeta} \tag{2.209}$$

it is easy to check that $\partial\Phi(z)$ is a primary field with dimension $\Delta = 1$ by computing its OPE with (2.207) and the the c-number in the Virasoro algebra has the value $c = 1$.

2.6 Old covariant quantization

In this section we want to quantize the string in the conformal gauge obtaining a quantum theory that is manifestly Lorentz covariant.

The starting point is the Lagrangian (2.109), from which we can compute the four-momentum density:

$$P_\mu(\tau,\sigma) = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2\pi\alpha'}\dot{x}_\mu \tag{2.210}$$

The theory is then quantized by requiring the canonical commutation relations:

$$[x^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = ig^{\mu\nu}\delta(\sigma-\sigma') \tag{2.211}$$

Inserting the expansion (2.113) in (2.211) we see that (2.211) is satisfied if the oscillators and the center of mass variables satisfy the following commutation relations:

$$[a_{n,\mu}, a^+_{m,\nu}] = g^{\mu\nu}\delta_{nm} \tag{2.212}$$

$$[q^\mu, p^\nu] = ig^{\mu\nu} \tag{2.213}$$

The connection between (2.211) and (2.212) and (2.213) can be easily established by using the following definition of the δ-function.

$$\sum_{n=-\infty}^{\infty} \cos n\sigma \cos n\sigma' = \pi\delta(\sigma-\sigma') \tag{2.214}$$

valid for functions expandable in a Fourier series of $\cos n\sigma$.

The generators L_n of the conformal transformations given in the classical theory by introducing a normal ordered expression:

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z) \tag{2.215}$$

where

$$T(z) = -\frac{1}{4\alpha'} : \left(\frac{\partial x}{\partial z} \right)^2 : \tag{2.216}$$

as already done in the free massless bosonic theory at the end of last section.

Using the results of the last section with the contraction given by

$$< x^\mu(z) x^\nu(\zeta) >= -2\alpha' \log(z - \zeta) \tag{2.217}$$

it is easy to show that the Virasoro generators satisfy the Virasoro algebra with $c = D$ [22], where D is the dimension of the space-time. Notice that the contraction in eq. (2.217) can be obtained using the commutation relations in eq.s (2.212) and (2.213) by computing the following correlator:

$$< q = 0 | x^\mu(z) x^\nu(\zeta) | p = 0 > \tag{2.218}$$

where the bra (ket) is an eigenstate of the operator q (p) with vanishing eigenvalue and is annihilated by the creation (annihilation) operators. The Virasoro generators have the following expression in terms of the harmonic oscillators:

$$L_n = \sqrt{2\alpha'} p \cdot a_n \sqrt{n} + \sum_{m=1}^{\infty} \sqrt{m(m+n)} a_{n+m} \cdot a_m^+ + \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a_m \cdot a_{n-m} \quad n > 0$$

$$L_{-n} = L_n^+ \qquad\qquad L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} n a_n^+ \cdot a_n \tag{2.219}$$

Notice that, because of the Lorentz metric, the commutation relations are realized in a linear space with indefinite metric exactly as in QED when one quantizes the theory à la Gupta-Bleuler.

However the physical states span a subspace of the entire space, that is characterized by the conditions [23]

$$L_n | Phys >= (L_0 - 1) | Phys >= 0 \tag{2.220}$$

In fact the conditions (2.130) cannot be imposed in the quantum theory as operator identities. They can only be imposed as conditions between physical states:

$$< Phys' | L_n | Phys >=< Phys' | (L_0 - 1) | Phys >= 0 \tag{2.221}$$

where $n \neq 0$. Eqs. (2.221) are satisfied if (2.220) are required.

As in the quantization in the light-cone gauge an arbitrary constant can be added to L_0, that in the covariant gauge must be chosen to be equal to 1 if we require the same spectrum as in the light-cone gauge. The value 1 can be obtained in the conformal gauge only if we add the coordinate of the ghost as we will show in Chapter 4. This is the first sign of the incompleteness of the old covariant quantization.

The mass shell condition $(L_0 - 1)|Phys >= 0$ determines the spectrum of the theory, that is given by:

$$[1 + \alpha' M^2 - \sum_{n=1}^{\infty} n a_n^{+} \cdot a_n]|Phys >= 0 \qquad (2.222)$$

Notice that all Lorentz components of the oscillators are present in (2.222), while in (2.156) we had only the transverse oscillators.

In order to eliminate the negative norm states present among the solutions of (2.222) we need to impose the additional constraints:

$$L_n|Phys >= 0 \qquad\qquad n > 0 \qquad (2.223)$$

The state with lowest mass is the vacuum state $|0>$ that satisfies (2.223) for any positive n and (2.222) if $M^2 = -1/\alpha'$. Therefore $|0>$ corresponds to a tachyon.

The next level is spanned by the states $\epsilon^\mu a_{1\mu}^{+}|0>$ corresponding to a massless gauge particle. The only nontrivial condition that we get from eq. (2.223) on a combination of photon states come from L_1 and reduces to:

$$L_1 \epsilon^\mu a_{1\mu}^{+}|0>= (p \cdot a_1)\epsilon^\mu a_{1\mu}^{+}|0> \qquad (2.224)$$

where ϵ^μ are arbitrary coefficients and p^μ is the four-momentum of the photon. (2.224) is the Lorentz condition imposed on the physical states in the Gupta-Bleuler quantization of QED. It requires a restriction of the parameters ϵ^μ:

$$p \cdot \epsilon = 0 \qquad (2.225)$$

If we choose a frame of reference where the momentum of the photon is given by $p^\mu \equiv (0, 0....0, p, p)$, (2.225) implies that the only physical states are:

$$\epsilon^i a_{1i}^{+}|0> + \epsilon(a_{1;D-1}^{+} - a_{1;D})|0> \qquad (2.226)$$

where ϵ^i and ϵ are arbitrary parameters.

(2.226) is the most general state of the level $N = 1$ satisfying the conditions (2.223). The first state in eq. (2.226) has positive norm, while the second one has zero norm contains states with positive norm and is orthogonal to all other physical states since it can be written as follows:

$$(a_{1;D-1}^{+} - a_{1;D}^{+})|0>= L_1^{+}|0> \qquad (2.227)$$

in the frame of reference where $p^\mu \equiv (0, ...0, p, p)$.

Because of the previous property it is decoupled from the physical states together with its conjugate:

$$(a_{1,D-1}^{+} + a_{1,D}^{+})|0> \qquad (2.228)$$

In conclusion we are left with only transverse physical states given by the first term in (2.226), that are exactly those found at the level $N = 1$ in the light-cone gauge.

Finally let us analyze the level $N = 2$, where the most general state with $M^2 = 1/\alpha'$ is given by:

$$[\alpha^{\mu\nu} a^+_{1,\nu} + \beta^\mu a^+_{2,\mu}]|0> \tag{2.229}$$

where $\alpha^{\mu\nu}$ and b^μ are arbitrary parameters.

In the center of mass frame where $p^\mu = (\vec{0}, M)$ the most general physical state satisfying the conditions (2.223) is given by:

$$|Phys>= \alpha^{ij}[a^+_{1,i}a^+_{1,j} - \frac{1}{(D-1)}\delta_{ij}\sum_{k=1}^{D-1} a^+_{1,k}a^+_{1,k}]|0> +$$

$$+\beta^i[a^+_{2,i} + a^+_{1,D}a^+_{1,i}]|0> +$$

$$+\sum_{i=1}^{D-1}\alpha^{ii}\left[\sum_{i=1}^{D-1} a^+_{1,i}a^+_{1,i} + \frac{D-1}{5}(a^{+2}_{1,D} - 2a^+_{2,D})\right]|0> \tag{2.230}$$

where the indices i, j run over the $D-1$ space components.

The first term in (2.230) correspond to a spin 2 in $(D-1)$ dimensional space and has a positive norm being made with space indices. The second term has zero norm and is orthogonal to the other physical state since it can be written in the form $L^+_1 a^+_{1,i}|0>$. It must be therefore eliminated from the physical spectrum together with its conjugate, as already explained in the case of the state (2.227) at the level $N = 1$. Finally the last state in (2.230) is spinless and has a norm given by:

$$2(D-1)(26-D) \tag{2.231}$$

If $D < 26$ it corresponds to physical spin zero particle with positive norm. If $D > 26$ it is a ghost. Finally if $D = 26$ it has a zero norm and is also orthogonal to the other physical states since it can be written in the form:

$$(2L^+_2 + 3L^{+2}_1)|0> \tag{2.232}$$

It does not belong therefore to the physical spectrum.

In conclusion if $D = 26$ we find at the level $N = 2$ the same number of physical states as in the light-cone gauge. If instead $D < 26$ the transverse oscillators are not sufficient to reproduce the full degeneracy; we need to add the so-called Brower's states [24, 25].

We conclude this section saying that also the closed string can be quantized in the conformal gauge following a procedure, that is completely analogous to the one used for the open string. We get in this case two sets of harmonic oscillators and of conformal generators. The Virasoro generators are defined by (2.135) and (2.136) with normal ordered expressions.

The on shell physical states are characterized by the following conditions:

$$L_n|Phys>= \tilde{L}_n|Phys>= 0$$

$$(L_0 + \tilde{L}_0 - 2)|Phys>= (L_0 - \tilde{L}_0)|Phys>= 0 \tag{2.233}$$

and for $D = 26$ one gets the same number of physical states as in the light-cone gauge keeping Lorentz invariance. On the other hand the covariant procedure developed in this section seems to work also for $D < 26$ and one needs also non-transverse oscillators (Brower's states) in order to describe the full spectrum. This apparent disagreement between the covariant and light-cone quantization is due to the fact that the procedure following in this section is not quite correct, because, fixing the conformal gauge, we have neglected the contribution of the ghost, that is an important ingredient anytime we quantize a gauge theory covariantly. In Chapter 4 we will show that the inclusion of the ghost will eliminate the contradictions between covariant and light-cone quantization.

3 Interacting bosonic string: tree diagrams

3.1 Vertex operators for open string

The simplest way to introduce the interaction in the string model is by adding to the free action, that has been discussed in the previous chapter, a term that describes the interaction of a string with an external field [26]:

$$S_{INT} = \int d^D y \Phi_L(y) J_L(y) \tag{3.234}$$

where $\Phi_L(y)$ is the external field and J_L is the current generated by the string. The index L stands for possible Lorentz indices that are saturated in order to have a Lorentz invariant action.

In the case of a point particle such an interaction term will not give in general any information on the self-interaction of a particle.

In the case of a string instead S_{INT} will describe the interaction among strings because the only external fields that can consistently interact with a string are exactly those that correspond to the various states of the string, as it will become clear later.

This is a consequence of the fact that, for the sake of consistency, we must require the following restriction on S_{INT}:

i It must be a well defined operator in the space spanned by the string oscillators.

ii It must preserve the invariances of the free string theory. In particular in the "conformal gauge" it must be conformal invariant. In general reparametrization invariant actions are written by using covariant derivatives, that also in the conformal gauge, defined by (2.107), contain the extra degree of freedom $\rho(\zeta)$ as it will be shown in the next chapter. For the sake of simplicity we consider only the case of critical dimension $D = 26$, where $\rho(\zeta)$ can be consistently set equal to 1. For a general treatment see for instance Ref. [27].

iii In the case of an open string the interaction occurs at the end point of a string (say at $\sigma = 0$). This follows from the fact that two open strings interact attaching to each other at the end points.

Let us concentrate on the open string and start with some examples.

The simplest scalar current generated by the motion of a string can be written as follows

$$J(y) = \int d\tau \int d\sigma \delta(\sigma) \delta^{(D)}[y^\mu - x^\mu(\tau, \sigma)] \tag{3.235}$$

where $\delta(\sigma)$ has been introduced because the interaction occurs at the end of the string. For the sake of simplicity we omit to write a coupling constant g in (3.235) and in the following.

Inserting (3.235) in (3.234) and using for $\Phi(y) = e^{ik\cdot y}$ a plane wave we get the following interaction:

$$S_{INT} = \int d\tau : e^{ik\cdot x(\tau,0)} : \tag{3.236}$$

where the normal ordering has been introduced in order to have a well defined operator according to i).

The invariance of (3.236) under a conformal transformation $\tau \to w(\tau)$ requires the following identity:

$$S_{INT} = \int d\tau : e^{ik\cdot x(\tau,0)} : = \int dw : e^{ik\cdot x(w,0)} : \tag{3.237}$$

or in other words that

$$: e^{ik\cdot x(\tau,0)} : \longrightarrow w'(\tau) : e^{ik\cdot x(w,0)} : \tag{3.238}$$

Since in general a conformal or primary field $\Phi(\tau)$ transforms as follows under a conformal transformation:

$$\Phi(\tau) \to [w'(\tau)]^\Delta \Phi(w) \tag{3.239}$$

as it has been explained in Section(2.5), the requirement in eq. (3.238) implies that the vertex operator $: e^{ik\cdot x(\tau,0)} :$ must transform as a conformal field with $\Delta = 1$.

It is convenient to consider the vertex operator as a function of $z = e^{i\tau}$ instead of τ. In the following we will omit to write explicitly the dependence on σ: $x_\mu(\tau, 0) \equiv x_\mu(\tau)$.

The transformation properties of $e^{ik\cdot x(z)}$ under as conformal transformation can be determined by computing its OPE with the energy-momentum tensor, that in our case is given in eq. (2.216). Using the contraction in eq. (2.217) it is easy to obtain:

$$T(z) : e^{ik\cdot x(\zeta)} := \frac{\partial/\partial\zeta : e^{ik\cdot x(\zeta)} :}{z - \zeta} + \alpha' k^2 \frac{: e^{ik\cdot x(\zeta)} :}{(z-\zeta)^2} + \cdots \tag{3.240}$$

that implies that the vertex operator is a conformal field with $\Delta = \alpha' k^2$. In conclusion S_{INT} in (3.236) is conformal invariant only if the external field is on shell with $\alpha' k^2 = 1$, corresponding to the tachyonic lowest state of the bosonic string.

The tachyonic state can be obtained from the vertex operator

$$\lim_{z\to 0} : e^{ik\cdot x(z,0)} : |0> = |0,k> \equiv e^{ik\cdot q}|0, p=0> \tag{3.241}$$

by using the following explicit formula in terms of the harmonic oscillators:

$$: e^{ik\cdot x(z,0)} := e^{k\cdot\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}}a_n^{+}} e^{ik\cdot q} e^{k\cdot p\log z} e^{-k\cdot\sum_{n=1}^{\infty}\frac{z^{-n}}{\sqrt{n}}a_n} \tag{3.242}$$

where $|0>$ is the vacuum of the oscillators and of the zero mode p. Notice that the normal ordering means that the creation operators are put on the left with respect to the annihilation ones and also that the operator p is put before q. It is easy to check that the vertex operator in eq. (3.242) satisfies the following hermiticity property:

$$[: e^{-ik\cdot x(1/z)} :]^{\dagger} = z^2 : e^{ik\cdot x(z)} : \tag{3.243}$$

that allows to obtain the tachyonic state as a bra in the limit of $z \to \infty$. In fact one gets:

$$\lim_{z\to 0} < 0| : \left(e^{ik\cdot x(z)}\right)^{\dagger} := \lim_{z\to\infty} < 0|z^2 : e^{-ik\cdot x(\frac{1}{z})} :=< 0|e^{-ik\cdot xq} \tag{3.244}$$

Another simple current generated by the string is given by:

$$J_{\mu}(y) = \int d\tau \int d\sigma \delta(\sigma) \dot{x}_{\mu}(\tau,\sigma) \delta^{(D)}(y - x(\tau,\sigma)) \tag{3.245}$$

Inserting (3.245) in (3.234) we get

$$S_{INT} = \int d\tau \dot{x}_{\mu}(\tau, 0)\epsilon^{\mu} e^{ik\cdot x(\tau,0)} \tag{3.246}$$

if we use a plane wave for $\Phi_{\mu}(y) = \epsilon_{\mu} e^{ik\cdot y}$. Proceeding as in the case of the tachyon vertex operator in order to check the conformal invariance of (3.246) we must compute the following OPE:

$$T(z)V(\zeta;k) = \frac{\partial/\partial\zeta V(\zeta;k)}{z-\zeta} + (1+\alpha' k^2)\frac{V(\zeta;k)}{(z-\zeta)^2} + \sqrt{2\alpha'}\frac{(\epsilon\cdot k) : e^{ik\cdot x(\zeta)}}{(z-\zeta)^3} + \cdots \tag{3.247}$$

where

$$V(\zeta;k) = \epsilon^{\mu} x'_{\mu}(\zeta) e^{ik\cdot x(\zeta)} \tag{3.248}$$

and

$$x'_{\mu}(z) = \frac{dx_{\mu}}{dz} = -\frac{i}{z}\sqrt{2\alpha'}\left[\sqrt{2\alpha'}p_{\mu} + \sum_{n=1}^{\infty}\sqrt{n}(a_n z^{-n} + a_n^{+} z^n)\right] \tag{3.249}$$

According to the OPE in eq.(3.247) the vertex operator in eq. (3.246) is conformal invariant only if

$$k^2 = \epsilon \cdot k = 0 \tag{3.250}$$

and therefore the external vector must be the massless photon state of the string.

As in the case of the tachyon such a photon state can be obtained from the vertex operator in the following way:

$$\lim_{z\to 0}\frac{dx^{\mu}}{dz}\epsilon_{\mu}e^{ik\cdot x(z)}|0>= -i(\epsilon\cdot a_1^{+})|0> \tag{3.251}$$

The photon vertex operator in eq. (3.248) satisfies in addition the following hermiticity property:

$$[V(\frac{1}{z};-k)]^{+} = -z^2 V(z;k) \tag{3.252}$$

In conclusion in two particular examples we have seen that an open string can interact with an external field in a consistent way only if it corresponds to an on shell state of the string.

We are now going to show that the previous result is actually valid for an arbitrary external field. The most general vertex operator will be a combination of terms of the type:

$$:\left(\frac{\partial x_{\mu_1^{(1)}}}{\partial z}\cdots\frac{\partial x_{\mu_{n_1}^{(1)}}}{\partial z}\right)\left(\frac{\partial^2 x_{\mu_1^{(2)}}}{\partial z^2}\cdots\frac{\partial^2 x_{\mu_{n_2}^{(2)}}}{\partial z^2}\right)\cdots e^{ik\cdot x}: \tag{3.253}$$

with the same amount of Lorentz indices and with the restriction $\sum in_i = N$ in order to describe states at the same level. The additional terms in eq. (3.253) involve terms with higher z derivatives. Finally the normal ordering has been inserted in order to have a well defined operator.

Since the various terms of the combination must have the same amount of Lorentz indices, in general the generic term will have some indices saturated within themselves and others with the invariant tensors $g^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$ and with the momentum k_{μ}.

The requirement of conformal invariance implies the following OPE for the combination of terms that we call $V_{\alpha}(z;k)$:

$$T(z)V_{\alpha}(z;k) = \frac{\partial/\partial z V_{\alpha}(z;k)}{z-\zeta} + \frac{V_{\alpha}(z;k)}{(z-\zeta)^2} + \cdots \tag{3.254}$$

without higher singularities.

In general for a term as in eq.(3.253) higher singularities will be present, but they will be cancelled by taking suitable combinations of terms of the type (3.253). The coefficient of the term $(z-\zeta)^{-1}$ will be always correct, while that of the term $(z-\zeta)^{-2}$ will be the same for each term and given by

$$\alpha' k^2 + \sum_{i=1}^{\infty} in_i = \alpha' k^2 + N \tag{3.255}$$

Conformal invariance then implies that

$$\alpha' k^2 + N = 1 \tag{3.256}$$

that is the mass shell condition for an arbitrary state of the string.

Eq. (3.254) implies that the vertex operator satisfies the following commutation relations with the Virasoro generators as follows from eq.s (2.199) and (2.200):

$$[L_n, V_\alpha(z;k)] = \frac{d}{dz}[z^{n+1}V_\alpha(z;k)] \tag{3.257}$$

and therefore the state:

$$|\alpha >= \lim_{z\to 0} V_\alpha(z;k)|0> \tag{3.258}$$

satisfies the conditions (2.220) for an on shell physical state of the string.

Finally it is easy to check that the generic term (3.253) and therefore also a combination of terms with $\sum i n_i = N$ satisfies the following hermiticity property:

$$V_\alpha(1/z; -k)^+ = (-1)^N z^2 V_\alpha(z;k) \tag{3.259}$$

In conclusion we have shown that the requirement i), ii) and iii) imply that the external field must correspond to one of the on shell physical states of the string, the interaction of which with the string is described by a vertex $V_\alpha(z;k)$ satisfying the following conditions:

$$\begin{aligned} |\alpha >&= \lim_{z\to 0} V_\alpha(z;k)|0> \\ V_\alpha(1/z; -k)^+ &= (-1)^N z^2 V_\alpha(z;k) \\ [L_n, V_\alpha(z;k)] &= \frac{d}{dz}[z^{n+1}V_\alpha(z;k)] \end{aligned} \tag{3.260}$$

that were derived in Ref. [28] and that characterize the on shell physical states.

At the end of this section let us consider some explicit examples that will clarify the previous general considerations.

The states on the leading trajectory are described by the following vertex operator:

$$V_N(z;k) = \epsilon^{\mu_1\cdots\mu_N} : x'_{\mu_1}(z)\cdots x'_{\mu_N}(z)e^{ik\cdot x(z)} \tag{3.261}$$

that satisfies the OPE (3.254) if the polarization tensor satisfies the conditions:

$$\epsilon^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_N} g_{\mu_i\mu_j} = k_{\mu_i}\epsilon^{\mu_1\cdots\mu_i\cdots\mu_N} = 0 \tag{3.262}$$

for any choice of the indices μ_i and μ_j. In addition the mass shell condition $\alpha' k^2 + N = 1$ must be also satisfied.

At the level $N = 2$ the most general vertex operator is given by the following combination:

$$: [\alpha^{\mu\nu} x'_\mu(z) x'_\nu(z) + \beta^\mu x''_\mu(z)]e^{ik\cdot x(z)} \tag{3.263}$$

It is easy to check that the coefficients of the terms $(z-\zeta)^{-1}$ and $(z-\zeta)^{-2}$ are as in (3.254) if $\alpha' k^2 = -1$. In general however the OPE of $T(z)$ with the vertex operator in eq. (3.263) will contain also higher singularities, that are eliminated if we impose the following conditions:

$$\beta^\mu = i k_\nu \alpha^{\mu\nu} \qquad (g^{\mu\nu} - 2k^\mu k^\nu)\alpha_{\mu\nu} = 0 \tag{3.264}$$

In the center of mass frame where $k_\mu \equiv (M, \vec{0})$ with $\alpha' M^2 = 1$ we get the following primary fields:

$$: [x'_i x'_j - \frac{1}{D-1}\delta_{ij} \sum_{k=1}^{D-1} x'_k x'_k] e^{ik\cdot x} : \tag{3.265}$$

$$: [x'_i x'_D + i\sqrt{\alpha'} x''_i] e^{ik\dot{x}} : \tag{3.266}$$

and

$$: [-\frac{1}{D-1} \sum_{k=1}^{D-1} x'^2_i + \frac{1}{5}\left(x'_D x'_D + 2i\sqrt{\alpha'} x''_D\right)] e^{ik\cdot x} : \tag{3.267}$$

It is easy to prove that they correspond to the states already found at the level $N = 2$ in eq. (2.230) and that are obtained from eq.s (3.265), (3.266) and (3.267) as in eq. (3.258).

3.2 Scattering amplitude for open strings

In this section we want to compute the probability amplitude for the emission of $(N-2)$ external fields from a string.

Starting from the total action $S = S_0 + S_{INT}$ the S-matrix for the emission of the external field from the string is given in perturbation theory by:

$$S = \lim_{\tau_i \to -\infty; \tau_f \to \infty} T[e^{iS_{INT}}] = \lim_{\tau_i \to -\infty; \tau_f \to \infty} T[e^{i\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma L_{INT}}] \tag{3.268}$$

where

$$\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma L_{INT} = \int_{\tau_i}^{\tau_f} d\tau V_\alpha(e^{i\tau}; k) \tag{3.269}$$

V_α is the vertex operator corresponding to a certain external field.

The amplitude for the emission of $(N-2)$ external fields is given by a sum of $(N-2)!$ terms that correspond to the different terms of the T-ordered product in (3.268). A single term is given by:

$$A(\alpha_1, k_1; \cdots \alpha_N, k_N) = \int_0^\infty \prod_{i=3}^{N-1} d\tau_i \theta(\tau_{i+1} - \tau_i) < \alpha_1, k_1| \prod_{i=2}^{N-1} V_{\alpha_i}(e^{i\tau_i}; k_i) |\alpha_N, k_N > \tag{3.270}$$

From (3.268) and (3.269) we would actually obtain the same vertex operator in the product in (3.270). We have extended the result to include different external fields at different "times" τ_i.

The variable τ_2 has been chosen to be equal to 0 because of the translational invariance of the matrix element in (3.270). The integral over τ_i is performed along the positive real axis. But since the vertex operator depends on $e^{i\tau_i}$, the integral in (3.270) is not well defined. It can be made convergent by means of a Wick rotation

$\tau \to i\tau$. After the introduction of the Koba-Nielsen variables $z_i = e^{-\tau_i}$ we can rewrite (3.270) as follows

$$A(\alpha_1, k_1; \cdots \alpha_N, k_N) = \int_0^1 \prod_{i=3}^{N-2} [dz_i \theta(z_i - z_{i+1}) < \alpha_1, k_1| \prod_{i=2}^{N-1} V_{\alpha_i}(z_i; k_i) |\alpha_N; k_N > \tag{3.271}$$

where

$$izV_\alpha(z;k) = V_\alpha(e^{i\tau}; k) \tag{3.272}$$

After the Wick rotation the integral in (3.271) is perfectly well defined.

Finally an internal symmetry can be introduced by multiplying the amplitude (3.271) with a Chan-Paton factor:

$$Tr(\lambda_1 \lambda_2 \cdots \lambda_N) \tag{3.273}$$

where the λ's are the generators of some internal symmetry group.

The scattering amplitude can be written in a more symmetric way by introducing also the Koba-Nielsen variable z_1, z_2 and z_N. Then (3.271) becomes:

$$A(\alpha_1, k_1; \cdots \alpha_N, k_N) = \int_{-\infty}^{\infty} \frac{\prod_{i=1}^{N} [dz_i \theta(z_i - z_{i+1})]}{dV_{abc}} < 0| \prod_{i=1}^{N} V_{\alpha_i}(z_i; k_i)|0 > \tag{3.274}$$

where

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_a - z_b)(z_a - z_c)(z_b - z_c)} \tag{3.275}$$

The three variables z_a, z_b, z_c can be fixed arbitrarily because the expression under the integral in (3.274) is invariant under the projective transformations:

$$z_i \to z_i' = \frac{Az_i + B}{Cz_i + D} \qquad ; \qquad AD - BC = 1 \tag{3.276}$$

This follows from the fact that dV_{abc} is left invariant under the projective transformations (3.276) and that the vertex operator is a primary field with $\Delta = 1$ as shown in section (3.1):

$$V_{\alpha_i}(z_i, k_i) \to (Cz_i + D)^{-2} V_{\alpha_i}(z_i'; k_i) \tag{3.277}$$

In addition from eq. (3.276) we get also

$$dz_i{}' = \frac{dz_i}{(Cz_i + D)^2} \tag{3.278}$$

implying that $V_\alpha(z;k)dz$ is projective invariant. Fixing $z_a = z_1 = \infty$, $z_b = z_2 = 1$ and $z_c = z_N = 0$ we get the starting expression (3.271).

Notice that $V_\alpha(z;k)dz$ is invariant under an arbitrary conformal transformation; but the volume element dV_{abc} is not.

Notice that the integrand in eq.(3.274) is invariant under cyclic permutations of the external legs. This can be shown by using the following reordering property of the vertex operators :

$$V(z;k)V(\zeta;h) = V(\zeta;h)V(z;k)e^{2\pi i\alpha' k\cdot h\epsilon(z-\zeta)} \tag{3.279}$$

If we commute the last operator V_{α_N} in (3.274) with the other operators bringing it in front of the other terms of the product, we will get, because of (3.279), a factor $e^{2i\pi\alpha' k_N\cdot\sum_{i=1}^{N-1}k_i}$, that is equal to 1 as a consequence of the conservation of momentum and of the mass shell condition $\alpha' k_N^2 = integer$. This shows that (3.274) is invariant under a cyclic permutation of the external legs: $A(1,2,\cdots,N) = A(N,1,\cdots,N-1)$.

In conclusion the scattering amplitude in eq. (3.274) is invariant under cyclic permutations. In order to get a crossing symmetric amplitude we need to sum only over the $(N-1)!$ permutations that are not cyclic:

$$A = \sum Tr(\lambda_1\lambda_2\ldots\lambda_N)A(\alpha_{i_1},k_1;\cdots,\alpha_{i_N},k_N) \tag{3.280}$$

and not only over the $(N-2)!$ terms of the T-ordered product in (3.268) that make the scattering amplitude symmetric only under any exchanges of the $N-2$ particles related to the external field. This asymmetry between the first and the last particle and the other $N-2$ particles is due to the fact that we have treated them differently. In fact the first and the last particles in (3.270) have been treated as states of the string, while the others as external fields. In order to restaure the symmetry we should sum over non cyclic permutations as in (3.280). We conclude this section by computing the scattering amplitude involving N tachyons. It is given by:

$$A(k_1,k_2,\cdots k_N) = \int\frac{\prod_{i=1}^N[dz_i\theta(z_i-z_{i+1})}{dV_{abc}} < 0|\prod_{i=1}^N : e^{ik_i\cdot x(z_i)} : |0> \tag{3.281}$$

If we use the explicit expression (3.242) of $: e^{ik\cdot x(z)} :$ in terms of the harmonic oscillators and the Baker-Hausdorf formula

$$e^A e^B = e^B e^A e^{[A,B]} \tag{3.282}$$

it is easy to get the following expression:

$$: e^{ik\cdot x(z)} :: e^{ih\cdot x(\zeta)} :=: e^{ik\cdot x(z)}e^{ih\cdot x(\zeta)} : (z-\zeta)^{2\alpha' k\cdot h} \tag{3.283}$$

that allows to compute the vacuum expectation value in (3.281) obtaining:

$$A(k_1,k_2,\cdots k_N) = (2\pi)^D\delta^{(D)}(\sum_{i=1}^N p_i)\int\frac{\prod_{i=1}^N[dz_i\theta(z_i-z_{i+1})}{dV_{abc}}\prod_{i<j}(z_i-z_j)^{2\alpha' k_i\cdot k_j} \tag{3.284}$$

The factor $(z-\zeta)^{2\alpha' k\cdot h}$ in (3.283) can also be directly obtained using the contraction (2.217).

After having fixed $z_1 = \infty, z_2 = 1, z_3 = z$ and $z_4 = 0$ we get the following 4-point tachyon amplitude:

$$A(k_1, k_2, k_3, k_4) = \int_0^1 dz z^{2\alpha' k_3 \cdot k_4} (1-z)^{2\alpha' k_2 \cdot k_3} \tag{3.285}$$

In terms of the Regge trajectories in the s and t-channel

$$\alpha_s = 1 - \alpha'(k_3 + k_4)^2 \qquad \alpha_t = 1 - \alpha'(k_3 + k_2)^2 \tag{3.286}$$

we get for (3.285) the famous Veneziano formula:

$$A(k_1, k_2, k_3, k_4) = \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s - \alpha_t)} \tag{3.287}$$

3.3 Vertex operators for closed strings

For a closed string we can proceed as in the case of an open string, but we shall drop the property **iii)** of the beginning of this chapter. The simplest scalar current generated by the string is given by:

$$J(y) = \int d\tau \int d\sigma \delta^{(D)}(y^\mu - x^\mu(\tau, \sigma)) \tag{3.288}$$

Inserting (3.288) in (3.234) and taking a plane wave for the external field we get the following vertex operator:

$$S_{INT} = \int d\tau \int d\sigma : e^{ik \cdot x(\tau, \sigma)} : \tag{3.289}$$

with the normal ordering prescription in order to have a well defined operator.

According to (2.114) we can write $x_\mu(\tau, \sigma)$ as follows:

$$x_\mu(\tau, \sigma) = x_\mu(\tau + \sigma) + \bar{x}_\mu(\tau - \sigma) \tag{3.290}$$

where

$$x^\mu(\zeta^+) = \frac{1}{2}\left[q^\mu + 2\alpha' p^\mu \zeta^+ + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-2in\zeta^+}\right] \tag{3.291}$$

and

$$\bar{x}^\mu(\zeta^-) = \frac{1}{2}\left[q^\mu + 2\alpha' p^\mu \zeta^- + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-2in\zeta^-}\right] \tag{3.292}$$

with $\zeta^\pm = \tau \pm \sigma$.

They contain the same zero modes, but completely independent non zero modes.

Using the decomposition in eq.(3.290) we can rewrite eq. (3.289) as follows apart from an overall factor:

$$S_{INT} = \int d\zeta^+ : e^{ik \cdot x(\zeta^+)} : \int d\zeta^- : e^{ik \cdot \bar{x}(\zeta^-)} : \tag{3.293}$$

where the normal ordering for the zero mode is as in eq.(3.1.12). Conformal invariance requires that the vertex operator transforms as a primary field with both right and left dimensions $\Delta = \bar{\Delta} = 1$. Both these conditions imply that

$$\frac{\alpha'}{2} k^2 = 2 \tag{3.294}$$

that implies

$$(\alpha_0)_{cl} = (2\alpha_0)_{op} \qquad\qquad \alpha'_{c'} = \frac{\alpha'}{2} \tag{3.295}$$

In conclusion the external scalar field must have the same mass as the tachyonic state of the closed string.

In terms of the variables z and $\bar{z}$ defined in eq. (2.133) the tachyon of the closed string is described by the following vertex operator:

$$: e^{ik\cdot x(z)} e^{ik\cdot \bar{x}(\bar{z})} : \tag{3.296}$$

where the normal ordering for the zero modes is defined with the q_μ-operator on the left of the p_μ as in (3.242). Therefore the vertex of a tachyon of the closed string is the product of two vertices of the tachyon of the open string that are functions of the variables z and $\bar{z}$ respectively.

The next possibility is to consider the following current generated by the string:

$$J_{\mu\nu}(y) = \int d\sigma \int d\tau \frac{\partial x_\mu}{\partial \zeta^+} \frac{\partial x_\nu}{\partial \zeta^-} \delta(y_\mu - x_\mu(\tau, \sigma) \tag{3.297}$$

the symmetric part of which is equal to the D-dimensional energy-momentum tensor of the string.

From (3.297) using a plane wave $\Phi_{\mu\nu} = \epsilon_{\mu\nu} e^{ik\cdot y}$ for the external field we get the following interaction

$$S_{INT} = \int d\zeta^+ \int d\zeta^- \epsilon^{\mu\nu} \frac{\partial x_\mu}{\partial \zeta^+} \frac{\partial x_\nu}{\partial \zeta^-} e^{ik\cdot x(\zeta^+, \zeta^-)} \tag{3.298}$$

where ϵ is the polarization tensor of the external field.

Using the decomposition (3.290) we can rewrite (3.298) as follows:

$$S_{INT} = \epsilon^{\mu\nu} \int d\zeta^+ \frac{\partial x_\mu}{\partial \zeta^+} e^{ik\cdot x(\zeta^+)} \int d\zeta^- \frac{\partial x_\nu}{\partial \zeta^-} e^{ik\cdot x(\zeta^-)} \tag{3.299}$$

(3.299) is of course conformal invariant if $k^2 = 0$ with the external field corresponding to a state of the massless level of a closed string. In particular if we choose $\epsilon^{\mu\nu}$ symmetric and traceless we get the interaction of a string with an external gravitational field, while if we choose it antisymmetric we get the interaction with an external antisymmetric tensor field. In both cases the conformal invariance of (3.299) requires that $k^\mu \epsilon_{\mu\nu} = k^\nu \epsilon_{\mu\nu} = 0$.

Finally if we take

$$\epsilon^{\mu\nu} = \frac{1}{D-2}\left[\eta^{\mu\nu} - k_\mu \ell_\nu - k_\nu \ell \mu\right] \tag{3.300}$$

with $\ell^2 = 0$ and $\ell \cdot k = 1$, we get the vertex operator corresponding to a dilaton.

In terms of the variables z and $\bar{z}$ the vertex for a massless state for a closed string can be written as follows:

$$: x'_\mu(z) e^{i(k/2)\cdot x(z)} \bar{x}'_\nu(\bar{z}) e^{i(k/2)\cdot \bar{x}(\bar{z})} : \epsilon^{\mu\nu} \tag{3.301}$$

where $x'(z)$ means the derivative with respect to the argument.

It is easy to convince oneself that the most general vertex operator for a closed string is in general a product of two vertex operators of an open string as follows:

$$V_{\alpha\beta}(z, \bar{z}; k) =: V_\alpha(z; k) V_\beta(\bar{z}; k) : \tag{3.302}$$

where :: means that the q-operator appears always on the left of the p-operator.

Moreover, since we know that the states of a closed string must satisfy the condition (2.182), the two open string states α and β must be chosen to belong to the same level.

3.4 Scattering amplitude for closed strings

The amplitudes for the emission of N external fields from a closed string can be again easily computed by using the general formula (3.2.1). We get:

$$A(\alpha_1, \beta_1, k_1; \alpha_2, \beta_2, k_2; \ldots \alpha_N, \beta_N, k_N) = \int_0^\infty \prod_{i=3}^{N-1} d\tau_i \int_0^\pi d\sigma_i$$

$$< \alpha_1, \beta_1, k_1 | T\left(\prod_{i=2}^{N-1} V_{\alpha_i\beta_i}(e^{2i\zeta_i^+}, e^{2i\zeta_i^-}; k_i)\right) |\alpha_N, \beta_N, k_N > \tag{3.303}$$

where the variable τ_2 and σ_2 have been taken equal to zero because of the translational invariance of the matrix element.

By making a Wick rotation $\tau \to i\tau$, as in the case of an open string, the two variables z and $\bar{z}$ become one the complex conjugate of the other

$$z = e^{-2\tau} e^{2i\sigma} \qquad \bar{z} = e^{-2\tau} e^{-2i\sigma} \tag{3.304}$$

In terms of the variables z and $\bar{z}$ (3.4.15) becomes

$$A(\alpha_1, \beta_1, k_1; \alpha_2, \beta_2, k_2; \ldots \alpha_N, \beta_N, k_N) =$$

$$= \int \prod_{i=3}^{\infty} d^2 z_i < \alpha_1, \beta_1, k_1 | R\left(\prod_{i=2}^{N-1} V_{\alpha_i\beta_i}(z_i, \bar{z}_i; k_i)\right) |\alpha_N, \beta_N, k_N > \tag{3.305}$$

where the T-ordering becomes now an ordering on the modulus of z_i and the integrals are performed over the entire complex plane of the variables z_i.

Notice that the jacobian for the change of variables $\sigma, \tau \to z, \bar{z}$ cancels with the factor coming out from a different definition in the normal ordering of the zero mode of the vertex operator as a function of (τ, σ) as in (3.293) or as a function of $(z, \bar{z})$ as in (3.296).

If we restrict ourselves to the scattering of N tachyons we can use the vertex in eq. (3.296) in eq. (3.305). The matrix element in eq. (3.305) will give the same expression for each term of the R-ordered product, that can be immediately obtained from the analogous calculation already performed in Sect. 3.2 for an open string.

For an open string the matric element in (3.281) is given by $\prod_{i<j;i=2}^{N-1}(z_i - z_j)^{2\alpha' k_i \cdot k_j}$. In the case of a closed string, where the vertex is a product of two vertices of an open string the matric element in (3.305) is given by

$$\prod_{i<j;i=2}^{N-1} (z_i - z_j)^{2\alpha' k_i \cdot k_j/4} (\bar{z}_i - \bar{z}_j)^{2\alpha' k_i \cdot k_j/4} \tag{3.306}$$

The factor 4 in the denominator of the two exponents follows from the factor 1/2 appearing in front of the brackets in (3.291) and (3.292).

Since the various terms in (3.290) will all give the same result, the terms with the product of θ-functions will sum up to 1.

In conclusion the scattering amplitude for N tachyons is given by

$$A(k_1, k_2, \ldots, k_N) = \int \prod_{i=3}^{N-1} d^2 z_i \prod_{i<j;i=2}^{N-2} |z_i - z_j|^{\alpha' k_i \cdot k_j} \tag{3.307}$$

It can be rewritten in a more symmetric form as in the case of an open string, by introducing also the variables z_1 and z_N . Then (3.5.5) becomes:

$$A(k_1, k_2, \ldots, k_N) = \int \frac{\prod_{i=1}^{N} d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j} \tag{3.308}$$

where

$$dV_{abc} = \frac{d^2 z_a d^2 z_b d^2 z_c}{|z_a - z_b|^2 |z_a - z_c|^2 |z_b - z_c|^2} \tag{3.309}$$

In the case of the scattering of 4 tachyons the amplitude (3.307) becomes:

$$A(k_1, k_2, k_3, k_4) = \int d^2 z |z|^{\alpha' k_3 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_3} \tag{3.310}$$

Using the following equation:

$$\int d^2 z |z|^{\alpha} |1 - z|^{\beta} = \pi \frac{\Gamma(1 + \frac{\alpha}{2}) \Gamma(1 + \frac{\beta}{2}) \Gamma(-1 - \frac{\alpha+\beta}{2})}{\Gamma(-\frac{\alpha}{2}) \Gamma(-\frac{\beta}{2}) \Gamma(2 + \frac{\alpha+\beta}{2})} \tag{3.311}$$

we get finally for eq. (3.310) the scattering amplitude proposed by Virasoro:

$$A = \pi \frac{\Gamma(-\frac{\alpha(u)}{2}) - \Gamma(\frac{\alpha(s)}{2})\Gamma(-\frac{\alpha(t)}{2})}{\Gamma(1+\frac{\alpha(u)}{2})\Gamma(1+\frac{\alpha(s)}{2})\Gamma(1+\frac{\alpha(t)}{2})} \tag{3.312}$$

where

$$\alpha(s) = 2 - (k_1+k_2)^2 \qquad \alpha(t) = 2-(k_3+k_2)^2 \qquad \alpha(u) = 2-(k_1+k_4)^2 \tag{3.313}$$

3.5 Normalization of scattering amplitudes

In this section we give the rules for obtaining amplitudes for open and closed strings with the correct normalization factors. We will have two kinds of factors. The first one is a factor that correctly normalizes the vertex operator associated to each string state, while the second one refers to the topology of the string diagram. Let us start from the open string tree amplitudes where, for the sake of generality, we consider open strings living on Dp-branes, i.e. strings having Neumann boundary conditions on $(p+1)$ directions and Dirichlet ones along the remaining $(D-p-1)$. Let us start from the massless vector vertex operator in the bosonic string that we write as follows:

$$V_\epsilon(z,k) = N_o\, \lambda^a\, (i\epsilon \cdot \partial_z \hat{x}(z)) \mathrm{e}^{i\sqrt{2\alpha'}k\cdot\hat{x}} \quad , \quad k^2 = \epsilon\cdot k = 0 \tag{3.314}$$

where we have introduced the dimensionless coordinate:

$$\hat{x}(z) = \frac{x(z)}{\sqrt{2\alpha'}} = \hat{q} - i\hat{p}\log z + i\sum_{n\neq 0}\frac{\alpha_n}{n}z^{-n} \tag{3.315}$$

with

$$\hat{q} = \frac{q}{\sqrt{2\alpha'}} \quad , \quad \hat{p} = \sqrt{2\alpha'}\,p \tag{3.316}$$

Since

$$\partial_z\hat{x}(z) = (-i)\left(\frac{\hat{p}}{z} + \sum_{n\neq 0}\alpha_n z^{-n-1}\right) = -i\sum_n \alpha_n z^{-n-1} \tag{3.317}$$

we see that the factor i in eq.(3.314) is needed in order to cancel the factor $(-i)$ appearing in eq.(3.317). The normalization factor N_o is given by:

$$N_o^{(p)} = (2r)^{1/2} g_o (2\alpha')^{(d-4)/4}(2\pi\alpha')^{(p+1-d)/2} = g_{p+1}(2\alpha')^{1/2} \tag{3.318}$$

where g_{p+1} is the gauge coupling constant given by

$$g_{p+1} = (2r)^{1/2} g_o (2\pi\sqrt{\alpha'})^{(p-3)/2}(2\pi^2)^{(4-d)/4} \tag{3.319}$$

that appears in the Yang-Mills action:

$$L_{YM} = -\frac{1}{4} F^a_{\alpha\beta} F^{\alpha\beta}_a \quad , \quad F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + g_{p+1} f^{abc} A^b_\alpha A^c_\beta \tag{3.320}$$

Finally the Chan-Paton factors λ^a are normalized as:

$$Tr(\lambda^a \lambda^b) = \frac{\delta^{ab}}{r} \tag{3.321}$$

The factor $N_o^{(p)}$ that correctly normalizes the vector vertex operator is actually the same for all other vertex operators provided that one always uses the variable $\hat{x}$. In particular the tachyon vertex operator is given by:

$$V(z,k) = N_o^{(p)} \ \mathrm{e}^{i\sqrt{2\alpha'} k\cdot\hat{x}} \quad , \quad \alpha' k^2 = 1 \tag{3.322}$$

Having correctly normalized the vertex operators we must also add a normalization factor that depends on the topology of the diagram. In the case of open string tree diagrams corresponding to the topology of the disk we need to include the factor:

$$C_0^{(p)} = \frac{2r}{g^2_{p+1}(2\alpha')^2} \tag{3.323}$$

satisfying the following relation:

$$C_0^{(p)} (N_o^{(p)})^2 \alpha' = r \tag{3.324}$$

It insures the correct factorization properties of the tree diagrams.

Let us use the previous normalization factors for computing the three-gluon amplitude. It is given by the sum of two planar diagrams. The first one corresponding to the cyclic ordering 123 is equal to:

$$C_0 N_o^3 Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) < 0|V_1(z_1)V_2(z_2)V_3(z_3)|0 > (z_1-z_2)(z_2-z_3)(z_1-z_3) \tag{3.325}$$

that, after some calculation, can be written as follows:

$$C_0 N_o^3 Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\sqrt{2\alpha'}\,[(\epsilon_1\cdot\epsilon_2)(k_1\cdot\epsilon_3) + (\epsilon_1\cdot\epsilon_3)(k_3\cdot\epsilon_2) + (\epsilon_2\cdot\epsilon_3)(k_2\cdot\epsilon_1)] \tag{3.326}$$

while the second one corresponding to the ordering 132 can be obtained from the previous one by the substitution

$$Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) \rightarrow -Tr(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}) \tag{3.327}$$

Notice that in order to obtain eq.(3.326) we have used the conservation of momentum and the mass-shell conditions: $k_i^2 = 0, i = 1,2,3$ to cancel the dependence on the Koba-Nielsen variables. Summing the two contributions one gets

$$C_0 N_o^3 Tr(\lambda^{a_1}[\lambda^{a_2},\lambda^{a_3}])\sqrt{2\alpha'}\,[(\epsilon_1\cdot\epsilon_2)(k_1\cdot\epsilon_3) + (\epsilon_1\cdot\epsilon_3)(k_3\cdot\epsilon_2) + (\epsilon_2\cdot\epsilon_3)(k_2\cdot\epsilon_1)] \tag{3.328}$$

By using the commutation relations:

$$[\lambda^a, \lambda^b] = i f^{abc} \lambda^c \tag{3.329}$$

the normalization given in eq.(3.321), the relation in eq.(3.324) and eq.(3.318) we get for the three-gluon amplitude:

$$i g_{p+1} f^{a_1 a_2 a_3} \left[(\epsilon_1 \cdot \epsilon_2)((k_1 - k_2) \cdot \epsilon_3) + (\epsilon_1 \cdot \epsilon_3)((k_2 - k_3) \cdot \epsilon_2) + (\epsilon_2 \cdot \epsilon_3)((k_2 - k_3) \cdot \epsilon_1) \right] \tag{3.330}$$

that is equal to the 3-gluon vertex that one obtains from the Yang-Mills action in eq.(3.320).

The factor given in eq.(3.323) generalizes to a surface with h boundaries as follows

$$C_h = g_o^{2h-2} N^h (2\pi)^{-(p+1)h} (2\alpha')^{-(p+1)/2} \tag{3.331}$$

It reduces of course to the one given in eq.(3.323) for $h = 0$.

Let us consider now the normalization factors in the case of a closed string. In this case it is convenient to write the string coordinate as follows:

$$x(z, \bar{z}) = \frac{1}{2} \left(x(z) + \tilde{x}(\bar{z}) \right) \equiv \frac{\sqrt{2\alpha'}}{2} \hat{x}(z, \bar{z}) \tag{3.332}$$

where

$$x(z) = q - i\alpha' p \log z + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \tag{3.333}$$

and

$$\tilde{x}(z) = q - i\alpha' p \log \bar{z} + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\tilde{\alpha}_n}{n} \bar{z}^{-n} \tag{3.334}$$

In eq.(3.332) we have again introduced the dimensionless quantity that we will use in the vertex operator. We write the vertex for the massless states as follows:

$$(i\epsilon \cdot \partial_z \hat{x}(z, \bar{z}))(i\tilde{\epsilon} \partial_{\bar{z}} \hat{x}^\nu(z, \bar{z}))\, N_c\, e^{ik \cdot x(z, \bar{z})} \tag{3.335}$$

where the polarization tensor has been written as $\epsilon_{\mu\nu} = \epsilon_\mu \tilde{\epsilon}_\nu$. The normalization factor for a closed string vertex is given by:

$$N_c = (4\pi)^{1/2} g_c (\alpha')^{(d-2)/4} = \frac{\kappa_d}{\pi} \tag{3.336}$$

and the d-dimensional Newton constant is given by:

$$\kappa_d = 2\pi^{3/2} g_c (\alpha')^{(d-2)/4} \tag{3.337}$$

If we compute tree diagrams for closed strings corresponding to the topology of a sphere we must add the following factor:

$$\hat{C}_0 = g_c^{-2} (\alpha')^{-d/2} \tag{3.338}$$

that satisfies the following relation:

$$\hat{C}_0 N_c^2 \alpha' = 4\pi \tag{3.339}$$

that insures the correct factorization properties of the tree diagrams.

From the previous normalizations we can compute the three-graviton amplitude that is given by:

$$\hat{C}_0 N_c^3 < 0|V_1(z_1,\bar{z}_1)V_2(z_2,\bar{z}_2)V_3(z_3,\bar{z}_3)|0 > |z_1 - z_2|^2 |z_2 - z_3|^2 |z_1 - z_3|^2 \tag{3.340}$$

Using eqs.(3.339) and (3.336) one gets

$$\begin{aligned} 2\kappa_D \left[(\epsilon_1 \cdot \epsilon_3)(k_3 \cdot \epsilon_2) + (\epsilon_1 \cdot \epsilon_3)(k_1 \cdot \epsilon_3) + (\epsilon_2 \cdot \epsilon_3)(k_2 \cdot \epsilon_1)\right] \cdot \\ \cdot \left[(\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_3)(k_3 \cdot \tilde{\epsilon}_2) + (\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_3)(k_1 \cdot \tilde{\epsilon}_3) + (\tilde{\epsilon}_2 \cdot \tilde{\epsilon}_3)(k_2 \cdot \tilde{\epsilon}_1)\right] \end{aligned} \tag{3.341}$$

that agrees with the three-graviton amplitude that one gets from the Einstein's action.

The factor for the sphere given in eq.(3.338) generalizes in the case of a Riemann surface with genus g to:

$$\hat{C}_g = g_c^{2g-2}(2\pi)^{-dg}(\alpha')^{-d/2} \tag{3.342}$$

We have introduced a string coupling constant for the open and a different one for the closed string. It turns out that the open string one is related to the closed string one by a numerical relation. It can be derived by observing that the Yang-Mills coupling constant g_{p+1} can also be obtained by expanding the Born-Infeld action:

$$S_{BI} = -\frac{T_p}{\kappa_D}\int d^{p+1}\xi\sqrt{-\det\left(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)} \tag{3.343}$$

In this way one obtains the following expression for the Yang-Mills coupling constant:

$$g_{p+1}^2 = \frac{r\kappa_d}{T_p(2\pi\alpha')^2} \;,\; T_p = \frac{\sqrt{\pi}}{2^{(d-10)/4}}(2\pi\sqrt{\alpha'})^{\frac{d}{2}-p-2} \;,\; \tau_p \equiv \frac{T_p}{\kappa_d} = \frac{(2\pi)^{d/2-3}}{2^{(d-10)/4}}\frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha' g_c} \tag{3.344}$$

where T_p can be computed from the boundary state. Finally comparing the first equation in (3.344) with eqs. (3.319) and (3.337) we get the following relation between the open and closed string coupling constants:

$$\frac{g_o^2}{g_c} = 2\pi^{d/2}2^{(d-10)/4} \tag{3.345}$$

The previous normalizations have been determined for the case of the bosonic string where actually $d = 26$. It turns out, however, that they also provide the correct normalization for the superstring amplitudes if we put $d = 10$. In the case of superstring we have also the fermionic coordinate that we use in the vertex operators. Also in this case we have to use a dimensionless quantity corresponding to the one given for the bosonic coordinate in eq.(3.315). It is given by:

$$\hat{\psi} = -i \sum_t \psi_r z^{-t-1/2} \tag{3.346}$$

where the sum is over integers [half-integers] values of r for the Ramond [Neveu-Schwarz] sector. The factor $(-i)$ appearing in eq.(3.346) is a consequence of the same factor appearing in eq.(3.317). We have the same normalization also in the case of the closed string.

The closed string coupling constant g_c is related to the most common used g_s in superstring theory through the relation:

$$g_s = \frac{g_c}{(2\pi)^2} \tag{3.347}$$

that implies

$$\frac{g_o^2}{g_s} = 8\pi^7 \tag{3.348}$$

In terms of this new string coupling constant the duality relation between the coupling constants of two dual theories is simply given by $g_s \to 1/g_s$.

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