Lectures on Center Vortices and Confinement

Parma, Italia
September 2005

Outline

I. What is Confinement?

II. Signals of the Confinement Phase

III. Properties of the Confining Force

IV. Confinement from Center Vortices

V. Numerical Evidence

VI. Connections to other ideas about confinement

Reviews:

- J.G., Prog. Nucl. Part. Phys. 51 (2003) 1; hep-lat/0301023

- Michael Engelhardt, plenary talk at Lattice 2004, video and preprint available at http://lqcd.fnal.gov/lattice04

Part I : What *is* Confinement?

These lectures are mainly a review of the center vortex theory of confinement:

a. Its motivation and formulation
b. What we can learn from lattice strong-coupling expansions
c. Evidence from lattice Monte-Carlo simulations
d. Relations to other ideas (involving monopoles, or the Gribov Horizon)

The numerical work is mostly for pure SU(2) Yang-Mills. Towards the end, I'll discuss what happens when matter fields are added to the system.

But to begin with: what are most people are trying to prove, when they talk about "proving" confinement? What are most of us are trying to explain, when we talk about "explaining" confinement?

A. Historical: No free particles with ± 1/3, ± 2/3 electric charge. ***(Confinement?)***

From a modern point of view, this is kind of accidental. Suppose Nature has supplied us with a scalar field in the 3 representation of color SU(3), having otherwise vacuum quantum numbers. There would be free, fractionally charged particle states in the theory, although, if the scalars were very massive, the dynamics and spectrum of the theory wouldn't otherwise be much different.

B. Color Singlets: There exist no isolated, non-color-singlet particles in Nature. ***(Confinement?)***

While true, this definition is also a little problematic, because it also holds for gauge-Higgs theories in which the gauge group is completely broken spontaneously -- it can be interpreted as a ***color screening*** criterion.

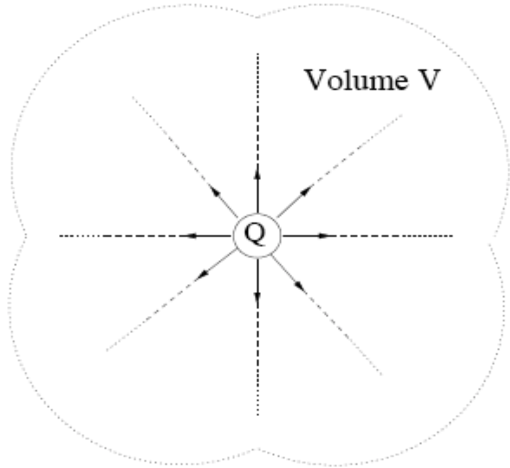
Again consider adding a scalar field in the fundamental 3 representation of color SU(3)

$$S_\phi = \quad d^4x \ (\frac{1}{2} D_\mu \phi^\dagger D_\mu \phi + \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2)$$

Two cases to consider:

- **μ^2 > 0:** unbroken SU(3). Color singlet spectrum, some additional quark-scalar bound states. External source shielded by bound state formation.

- **μ^2 ¿ 0:** spontaneous symmetry breaking. ***Still*** color singlet spectrum. External source shielded by condensate.



Bound antiquark screens
quark electric field

Or

Higgs condensate screens
quark electric field

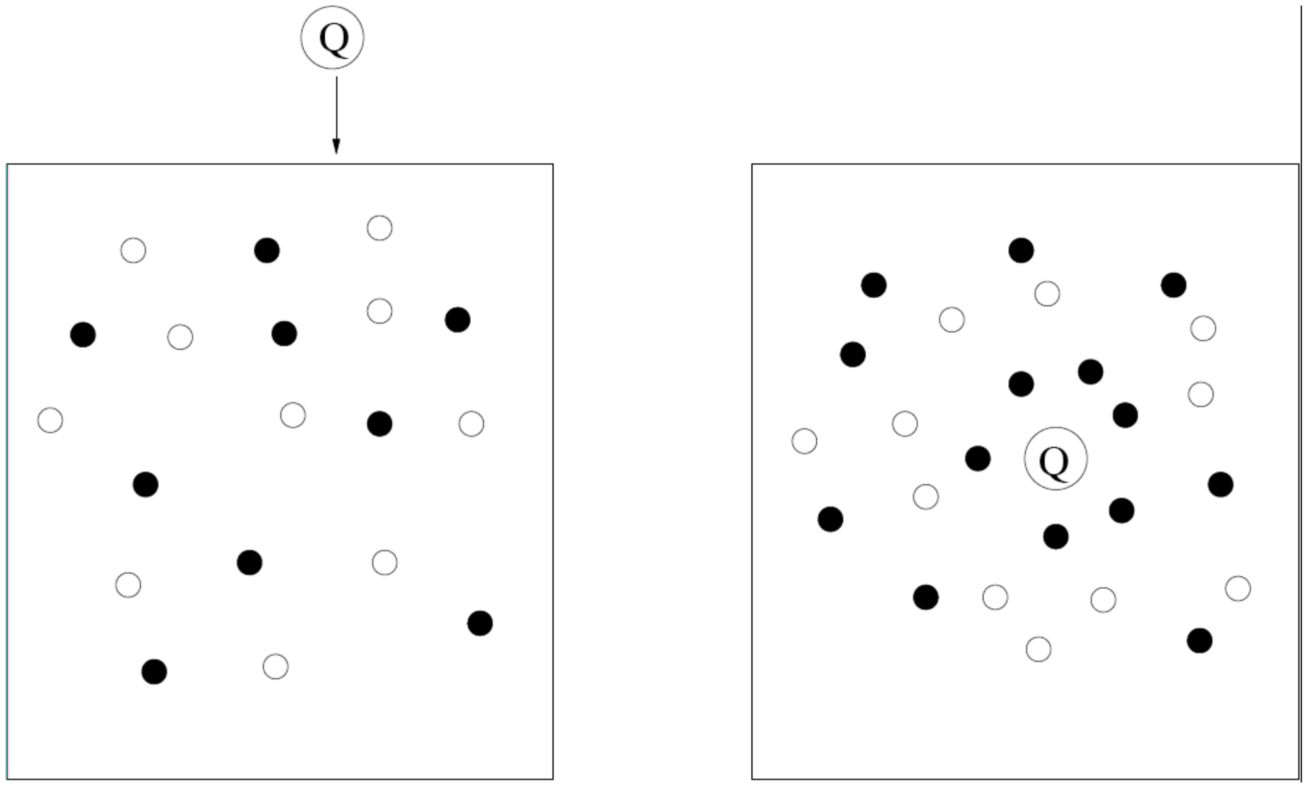
Non-abelian Gauss Law:

$$\vec{\nabla} \cdot \vec{E}^a = -c^{abc} A_k^b E_k^c - i \frac{\partial \mathcal{L}}{\partial D^0 \phi} t^a \phi$$

rhs is the 0-component of a conserved current. Absence of a color E-field outside volume V means the non-abelian charge inside is zero. In the Higgs phase, the screening charge is supplied by the condensate.

We get the same screening effect in the abelian Higgs model (an relativistic generalization of the Landau-Ginzburg superconductor), and even in an electrically charged plasma.

Inserting a charge +Q into an electrically charged plasma



Charge is screened; but this is not what we mean by "confinement".

The similarity - screening of heavy sources in both “broken” and “unbroken” gauge theories, is not a coincidence.

Consider a gauge-Higgs lattice action

$$S_{FS} = \beta_G \sum_x \sum_{\mu>\nu} \{\text{Tr}[U_\mu(x) U_\nu(x+\widehat{\mu}) U_\mu^\dagger(x+\widehat{\nu}) U_\nu^\dagger(x)] + \text{ c.c.}\}$$

$$+ \beta_H \sum_x \sum_\mu \{\phi^\dagger(x) U_\mu(x) \phi(x+\widehat{\mu}) + \text{ c.c.}\} \quad , \quad |\phi| = 1$$

Fradkin-Shenker Theorem

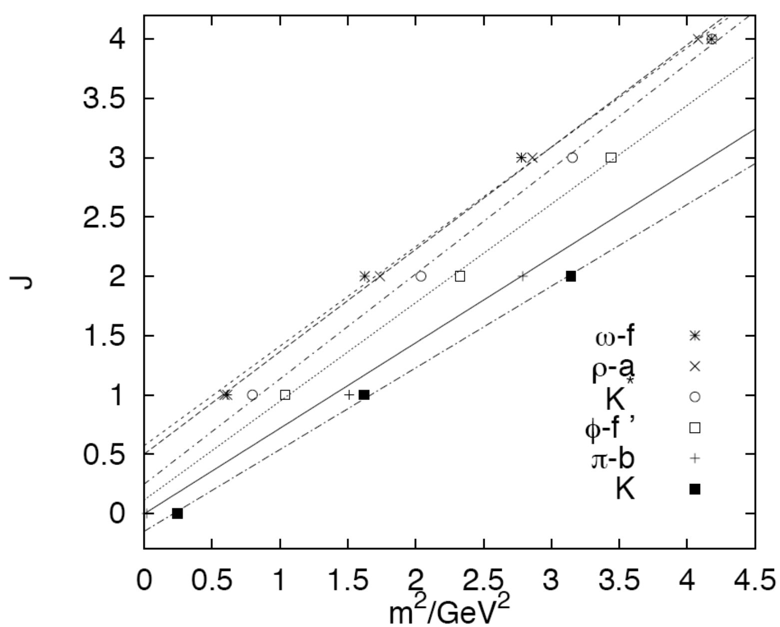
There is no thermodynamic phase boundary in the β_G - β_H phase diagram, isolating the Higgs phase from a separate, confining phase.

Consequence - Analytic free energy, no sudden qualitative change in the spectrum, e.g. from color singlets to non-singlets.

But **:** dynamics at β_H **À 1** looks a lot like, e.g., Weinberg-Salam (without the photon), and much different from the dynamics of QCD.

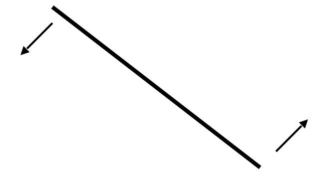
So lets focus on dynamics, rather than color singlets. What is special about the dynamics of QCD, as opposed to a Higgs theory?

C. Regge Trajectories: In QCD, in a **J** vs. $\mathbf{m^2}$ plot, mesons (and baryons) fall on linear, nearly parallel Regge trajectories.



Why??

The “spinning stick” model: View a meson as a spinning line of length L=2R, with mass/length σ, and ends moving at the speed of light.



Then, for the energy

$$
\begin{aligned}
m &= E = 2 \quad \frac{\sigma dr}{\sqrt{1 - v^2(r)}} \\
&= 2 \quad \frac{\sigma dr}{\sqrt{1 - r^2/R^2}} \\
&= \pi \sigma R
\end{aligned}
$$

and the angular momentum

$$
\begin{aligned}
J &= 2 \quad \frac{\sigma r v(r) dr}{\sqrt{1 - v^2(r)}} \\
&= \frac{2}{R} \quad \frac{\sigma r^2 dr}{\sqrt{1 - r^2/R^2}} \\
&= \frac{1}{2} \pi \sigma R^2
\end{aligned}
$$

Comparing the two expressions

$$J = \frac{1}{2\pi\sigma} m^2 = \alpha' m^2$$

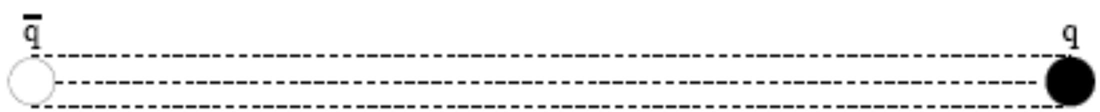
“Regge Slope”

From the data, $\alpha' = \mathbf{1/(2}\pi\sigma)$ = 0.9 Gev^{-2}, which gives a ***string tension***

$$\sigma \approx 0.18\ \text{Gev}^2 \approx 0.9\ \text{Gev/fm}$$

The model isn't perfect (data has non-zero intercept, slightly different slopes). Allow for quantum fluctuations of the stick $\longrightarrow$ ***String Theory***.

How does a string-like picture come out of QCD? ***Flux tube formation!***



$$\text{Energy} = \sigma L \quad \text{with} \quad \sigma = \quad d^2 x_\perp \, \frac{1}{2} \vec{E}^a \cdot \vec{E}^a$$

So this is the interesting dynamics:

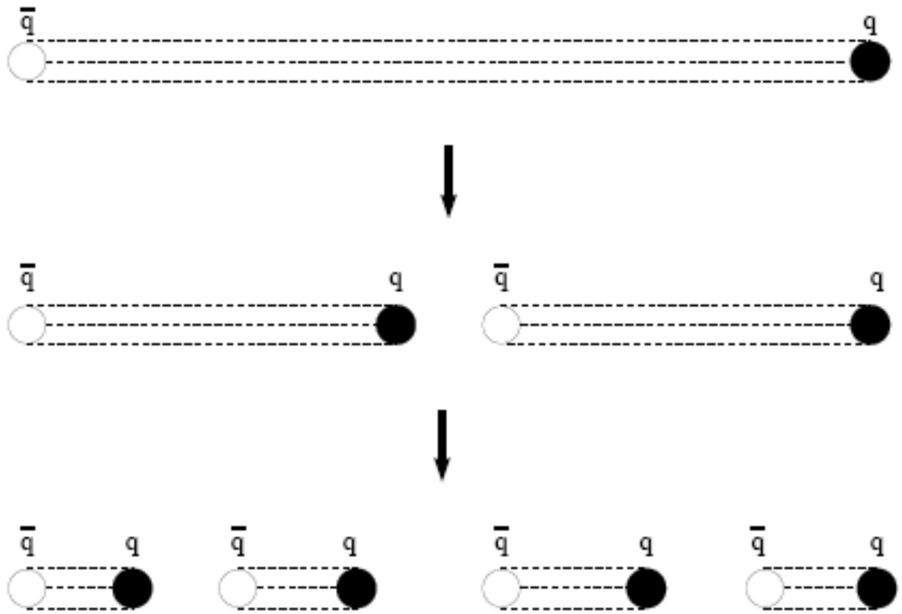
Energy = σ L

a linearly-rising potential between static sources (the “static quark potential”), and an infinite energy for infinite source separation.

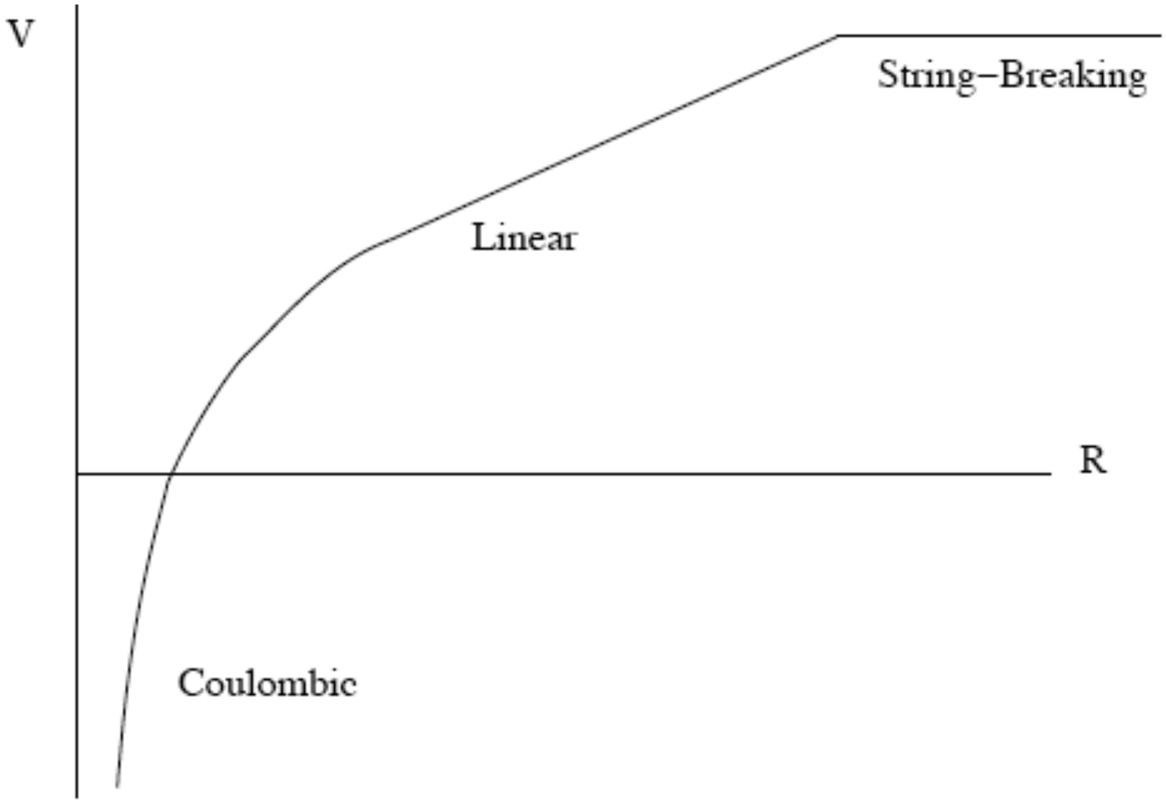
But is this what happens in real QCD?

String-Breaking and the Center of the Gauge Group

In real QCD, with light fermions, as in the Fradkin-Shenker model, the linear potential does not extend indefinitely. For $L > 2m_q$ it is energetically favorable to pair-produce quarks, and break the flux tube (or QCD "string").



Then the static quark potential looks something like this



At large distances, the color field of the static quarks are screened by the dynamical matter fields. Not so different from the Higgs physics.

If we want to explain the ***linear*** part of the potential, it is useful to work in a limit where the flux tube never breaks ("permanent confinement"), screening is suppressed, and the potential rises linearly without leveling out. In this limit, for any finite gauge group and set of matter fields, we now take note of an important fact:

<u>Permanent confinement exists only if:</u>

1. ***the unbroken gauge group has a non-trivial center subgroup, and***
2. ***all matter fields transform as singlets wrt that center subgroup.***

There are no known exceptions ***(G(2) is an example, not an exception).*** This fact provides us with an important clue about the nature of confinement.

A little group theory:

The ***center*** of a Lie Group consists of those group elements which commute with all elements of the group.

For an SU(N) gauge theory, this is the set of all group elements proportional to the identity:

$$z_n = \exp\left(\frac{2\pi i n}{N}\right) \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix} \qquad (n = 0, 1, 2, ..., N-1)$$

which form the discrete abelian subgroup Z_N of SU(N).

There are an infinite no. of representations of SU(N), but only a finite number of representations of Z_N . Every representation of SU(N) falls into one of N subsets (known as "class" or "N-ality"), depending on the representation of the Z_N subgroup.

- N-ality is given by the number of boxes in the Young tableau, mod N. Multiplication by a center element z_n , in a representation of N-ality k, corresponds to multiplication by a factor of

$$\exp(\frac{2\pi i k n}{N})$$

- N-ality = 0 representations (e.g. the adjoint representation) are special, in that all center elements map onto the identity.

- Particles of N-ality = 0 cannot bind to a particle of N-ality $\neq$ 0 to form a singlet.

- Consequence: Particles in N-ality = 0 representations can never break the string connecting two N-ality $\neq$ 0 sources.

So the limit in which the linear potential (if it exists) rises indefinitely, is the limit in which

- for QCD (or any SU(N) theory): take the masses of the quarks to infinity.

- for G(2), introduce a Higgs and break the gauge group to SU(3), taking the mass of the massive gluons to infinity.

Our goal (and the strategy of ***most*** efforts in this field) is to try to understand the linear potential in limit that it goes on forever. Once we understand confinement in this limit, we can take the masses of quarks (or broken generator gluons) to be finite, and see how the picture changes.

Part II : Signals of Confinement

A. The Wilson Loop

The Wilson loop has a dual role:

- rectangular ***timelike*** loops are related to the interaction potential of static, external sources,
- ***spacelike*** loops are probes of gauge-field fluctuations in the vacuum state, independent of external sources

Of course, spacelike and timelike loops are not intrinsically different, but related to one another by Lorentz (or, Euclidean rotation) invariance. It means that the interaction energy between static sources is related to vacuum fluctuations in the ***absence*** of external sources.

Lets start with the relation of timelike loops to the static quark potential.

Start with an SU(N) lattice gauge field, and a single quark field in color group representation r. The quark field is so massive that all quarks are static, and string-breaking effects can be ignored.

$$
\begin{aligned}
S &= \beta \sum_p \left(1 - \frac{1}{N}\text{ReTr}[U(p)]\right) \\
&+ \sum_x \left\{(m_q + 4\alpha)\overline{\psi}(x)\psi(x) - \frac{1}{2}\sum_{\mu=\pm 1}^{\pm 4} \overline{\psi}(x)(\alpha + \gamma_\mu)U_\mu^{(r)}(x)\psi(x+\widehat{\mu})\right\}
\end{aligned}
$$

Let Q be an operator which creates a color-singlet quark-antiquark state, with separation R

$$
Q(t) = \overline{\psi}(0,t)\Gamma \prod_{n=0}^{R-1} U_x^{(r)}(n\hat{i},t)\psi(R\hat{i},t)
$$

By the usual rules

$$\langle Q^\dagger(T)Q(0)\rangle = \frac{\Sigma_{nm}\langle 0|Q^\dagger|n\rangle\langle n|e^{-HT}|m\rangle\langle m|Q|0\rangle}{\langle 0|e^{-HT}|0\rangle}$$

$$= \sum_n |c_n|^2 e^{-\Delta E_n T}$$

$$= \frac{1}{Z}\int DU\,D\psi D\overline{\psi}\; Q^\dagger(T)Q(0)e^{-S}$$

For m_q very large, this is evaluated by bringing terms $\overline{\psi}(\alpha+\gamma_4)U\psi$ down from the action, and we find

$$
\begin{aligned}
\langle Q^\dagger(T)Q(0)\rangle &= C(m_q+4\alpha)^{-2T}\frac{1}{Z_U}\int DU\ \chi_r[U(R,T)]e^{-S_U} \\
&= C(m_q+4\alpha)^{-2T}W_r(R,T)
\end{aligned}
$$

Where $\chi_r(g)$ is the **group character** (trace) of group element g in representation r, U(R,T) is the **holonomy** - ordered product of link variables around the loop - and $W_r(R,T)$ is the VEV

$$
W_r(T) = \langle \chi_r[U(R,T)]\rangle
$$

“Holonomy” is just a Wilson loop, before taking the trace. In the continuum, the holonomy U(C) is

$$
U(C) = P\exp\left[i\oint_C dx^\mu\ A_\mu(x)\right]
$$

Digression on Character Expansions

A class function F[g] is a function defined on the group manifold, with the property that for any two group elements g and h

$$F[hgh^{-1}] = F[g]$$

A class function can always be expanded in terms of group characters,

$$F[g] = \sum_r c_r \chi_r[g]$$

where the sum is over representations r, and the group character is the trace of group element g in representation r.

A Fourier series expansion is an example of a character expansion; in this case the group is U(1), the sum runs over positive and negative integers, and

$$\chi_n[e^{i\theta}] = e^{in\theta}$$

So now we have

$$\sum_n |c_n|^2 e^{-\Delta E_n T} \sim C(m_q + 4\alpha)^{-2T} W_r(R,T)$$

The minimal energy is singled out in the **T ! 1** limit. The R-dependent part of the potential is obtained from

$$V_r(R) = -\lim_{T\to\infty} \log\left[\frac{W_r(R,T+1)}{W_r(R,T)}\right]$$

and this is what we refer to as the ***static quark potential*** in group representation r. The confinement problem is to show that

$$V_r(R) \sim \sigma_r R$$

at large R, for N-ality $\neq$ 0 representations, or more generally an asymptotic, area-law falloff

$$W_r(C) \sim \exp[-\sigma_r \text{ Area}(C)]$$

B. The Polyakov Line

In an SU(N) gauge theory with only N-ality=0 matter fields there is, beyond the SU(N) gauge symmetry, an additional global Z_n symmetry on a finite periodic lattice:

$$U_0(\vec{x}, t_0) \rightarrow z U_0(\vec{x}, t_0) \quad , \quad z \in Z_N \quad , \quad \text{all } \vec{x}$$

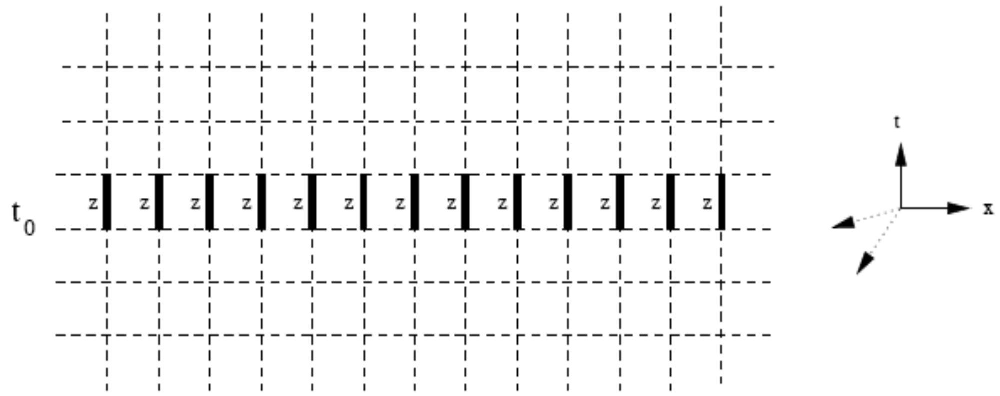


Figure 4: The global center transformation. Each of the indicated links in the t-direction, at $t = t_0$, is multiplied by a center element z. The lattice action is left unchanged by this operation.

This transformation does not change plaquettes or Wilson loops. But there are certain gauge-invariant observables which ***are*** affected.

Consider a Wilson line which winds once around the lattice in the periodic time direction

$$P(\vec{x}) = \mathrm{Tr}\left[U_0(\vec{x},1)U_0(\vec{x},2)...U_0(\vec{x},L_t)\right]$$

This is known as a ***Polyakov Loop***.

Under a center transformation $U_0(x,t_0) \longrightarrow z\, U_0(x,t_0)$, we find

$$P(\vec{x}) \rightarrow zP(\vec{x})$$

This global symmetry can be realized on the lattice in one of two ways:

$$\langle P(\vec{x})\rangle = \begin{cases} 0 & \text{unbroken } Z_N \text{ symmetry phase} \\ \text{non-zero} & \text{broken } Z_N \text{ symmetry phase} \end{cases}$$

This has a ***lot*** to do with confinement, because…

a Polyakov line can be thought of as the world-line of a massive static quark at fixed spatial position x , propagating through the periodic time direction. Then

$$\langle P(\vec{x}) \rangle = \exp[-F_q L_t]$$

where F_q is the quark free energy. In the confinement phase, the free energy is infinite, but finite in a non-confined phase. So

unbroken Z_N symmetry = confinement phase

Actually the transformation $U_0(x,t_0) \longrightarrow z\, U_0(x,t_0)$ can be generalized; it is a special example of a ***singular gauge transformation.*** Consider

$$U_0(x,t) \rightarrow g(x,t) U_0(x,t) g^\dagger(x,t+1)$$

Periodic only up to a center transformation

$$g(x, L_t + 1) = z^* g(x, 1)$$

This again transforms Polyakov lines P(x) $\longrightarrow$ z P(x), but plaquettes and ordinary Wilson loops are not affected.

In the continuum, it amounts to transforming the gauge field by the usual formula

$$A_\mu(x) \rightarrow g(x) A_\mu(x) g^\dagger(x) - \frac{i}{g_s} g(x) \partial_\mu g^\dagger(x)$$

Except, at $t=L_t$, we drop the second term (which would be a delta-function). Because of that, a “singular gauge transformation” is not a true gauge transformation.

C. The ‘t Hooft Loop

Instead of gauge transformations which are discontinuous on loops which wind around the periodic lattice, which could also consider transformations which are discontinuous on other sets of loops.

In fact there is a familiar example in classical electrodynamics: the exterior field of a solenoid is the result of a singular gauge transformation applied to A=0 !

So lets start with electrodynamics. The Wilson loop holonomy U(1) is

$$U(C) = \exp\left[ie \oint_C dx^\mu A_\mu(x)\right] = e^{i\Phi_B}$$

C is spacelike, Φ_B is the magnetic flux through the loop.

We can have Φ_B non-zero, yet B=0 along C. E.g., the vector potential exterior to a solenoid of radius R, oriented along the z-axis

$$A_\theta = \frac{\Phi_B}{2\pi e r}\hat{\theta} \qquad (r > R)$$

which is obtained from a singular gauge transformation of A=0, with

$$g(r, \theta, z, t) = \exp\left[-i\Phi_B \frac{\theta}{2\pi}\right]$$

Note the discontinuity

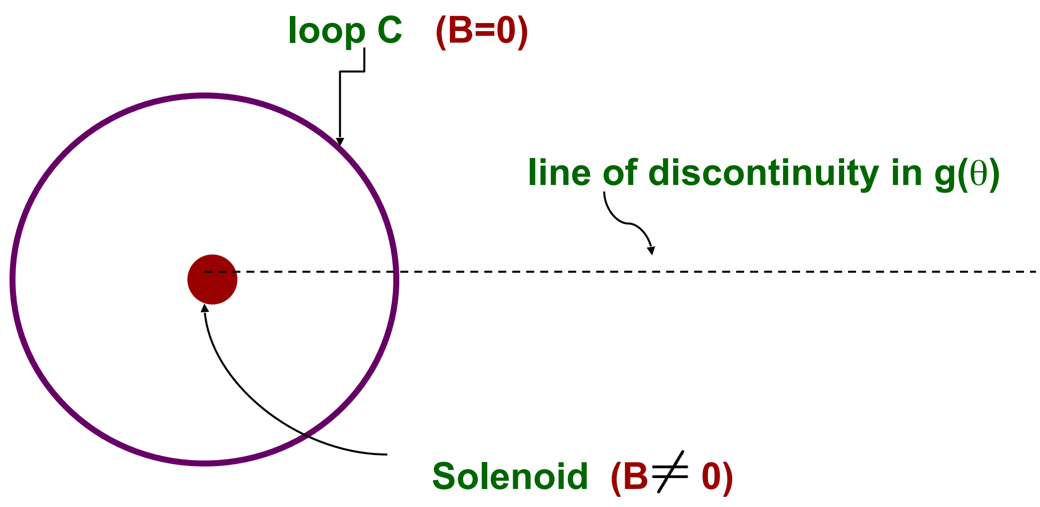
$$g(r, \theta = 2\pi, z, t) = e^{-i\Phi_B} g(r, \theta = 0, z, t)$$

We drop the delta-function that would arise in A_θ if this were a ***true*** gauge transformation.

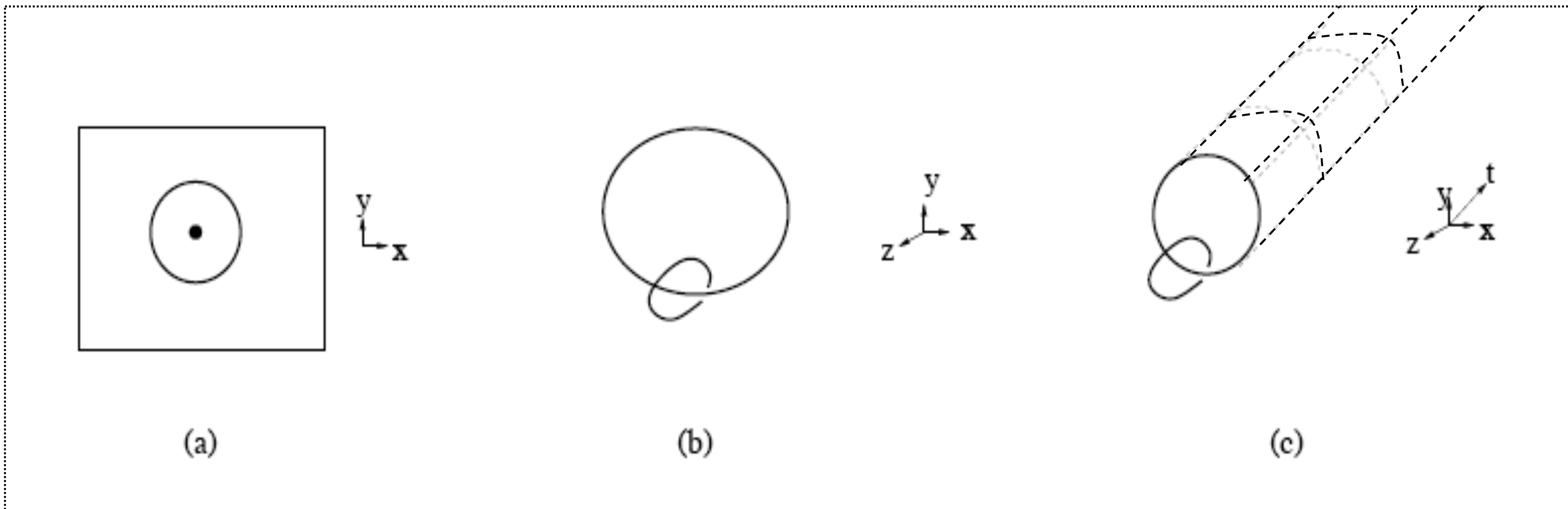
It's the exp[iΦ_B] discontinuity at θ=0,2π which is essential. Consider a loop winding n times around the z-axis: ***linking number = n.*** Any singular gauge transformation, with the same discontinuity, applied to any vector potential A, would give

$$U(C) \rightarrow e^{\pm in\Phi_B} U(C)$$

The concept of linking:



Loops link to points (D=2), other loops (D=3), surfaces (D=4).



In our example, the solenoid is a surface in the z-t plane in D=4 dimensions, and the gauge discontinuity lies on a 3-volume at y=0, x>0.

Summarize: the singular gauge transformation creates a surface of magnetic flux in the z-t plane, and any Wilson loop holonomy which is topologically linked to this surface gets multiplied by a factor of exp[iΦ_B]   U(1).

Generalize to SU(N)

We consider:

- g(x) discontinuous on a Dirac 3-volume V_3
- g(x) creates magnetic flux ***only*** on the boundary S of the 3-volume

Let C be a loop which is topologically linked to S, parametrized by

$$\vec{x}(\tau), \quad \tau \in [0,1]$$

with $$\vec{x}(1) = \vec{x}(0) \in V_3$$

What kind of discontinuity can g(x) have? Suppose g(x(1)) = h g(x(0)) where h is any SU(N) group element. In general this introduces a field strength throughout V_3 . But, as with center transformations in a periodic volume, we are looking for a discontinuity which changes the loop holonomy U[C], without changing the action in the neighborhood of C. This is accomplished by

$$g(\vec{x}(1)) = zg(\vec{x}(0)) \quad , \quad z \in Z_N$$

so that

$$U(C) \to z^* U(C)$$

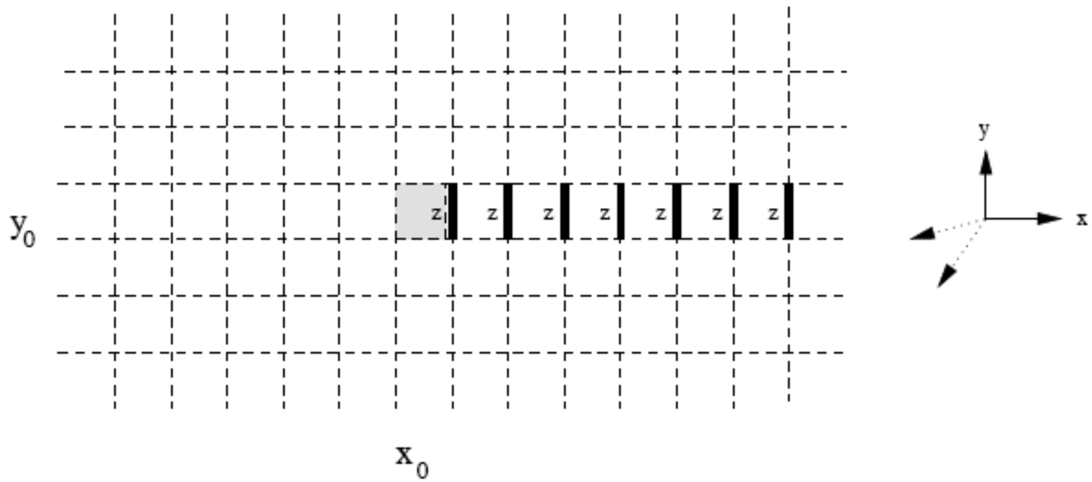
If this is true for ***any*** loop linked to S, no matter how small, then it means that the singular transformation has created a surface of infinite field strength just on S. Its called a ***thin center vortex.***

As in QED, the singular region can be smeared out in a region of finite thickness, a kind of solenoid sweeping out S. This is a ***thick center vortex.***

Creating a thin vortex on the lattice:

On every x-y plane of the lattice, set

$$U_y(x, y_0, x_\perp) \to z U_y(x, y_0, x_\perp) \quad \text{for} \quad x > x_0$$



The thin vortex is a stack of plaquettes at x_0,y_0 at all z,t. This is a surface.

Note that if the discontinuity were not a center element, then the action would be affected at all plaquettes in the Dirac volume.

Go to the Hamiltonian formulation, and let **B[C]** be an operator which creates a thin center vortex at fixed time t along curve C.

This means that B[C] performs a singular gauge transformation on gauge fields at time t.

It follow that along any loop C' linked to C in D=3 dimensions,

$$U(C')B(C) = zB(C)U(C') \quad , \qquad z \in Z_N$$

Using only this relation, 't Hooft argued that

- only area-law or perimeter-law falloff for <U[C]>, <B[C]> is possible,
- in the absence of massless excitations, it is impossible to have

$$\langle U(C)\rangle \sim e^{-aP(C)} \quad \text{and} \quad \langle B(C)\rangle \sim e^{-bP(C)}$$

So, perimeter law for B[C] implies area-law falloff for U[C] *(i.e.* ***confinement)*** .

C. The Vortex Free Energy

(Introduced by ‘t Hooft, first simulations by **Kovacs and Tomboulis**, much further work by **de Forcrand and von Smekal**)

Consider a finite 2D lattice. Usually we impose periodic boundary conditions

$$
\begin{aligned}
U_\mu(0,y) &= U_\mu(L,y) \\
U_\mu(x,0) &= U_\mu(x,L)
\end{aligned}
$$

Lets modify this by the following condition:

$$
U_y(0,y_0) = zU_y(L,y_0)
$$

This boundary condition can be absorbed into a change in the coupling **β ! z β** at the single plaquette P’ that contains $U_y(L,y_0)$, i.e

$$
\begin{aligned}
S &= \beta \sum_{p \neq P'} \mathrm{Tr}[U(p)] + c.c \\
&\quad + z\beta \mathrm{Tr}[U(P')] + c.c
\end{aligned}
$$

This is an example of ***twisted boundary conditions.*** It is impossible, with this choice, to pick the link variables such that the action vanishes.

The twisted b.c. introduces a Dirac string, starting at (L,y_0). But it has to end somewhere; at the end is a center vortex.

This is the rough argument; see ‘t Hooft for more rigor in showing that t.b.c. introduce a unit of center flux.

Generalization to higher dimensions is straighforward: A single link in D=2 becomes a line of links in D=3, and surface of links in D=4

$$U_y(0, y_0, x_\perp) = z U_y(L, y_0, x_\perp)$$

For simplicity, consider SU(2), and z = -1 for twisted boundary conditions. We define Z_+ as the lattice partition function with ordinary boundary conditions, and Z_- as the partition function with twisted boundary conditions. The magnetic free energy of a Z_2 vortex is then given by

$$e^{-F_{mg}} = \frac{Z_-}{Z_+}$$

The "electric" free energy is defined by a Z_2 Fourier transform

$$\begin{aligned} e^{-F_{el}} &= \sum_{z=\pm} z \frac{Z_z}{Z_+} \\ &= 1 - e^{-F_{mg}} \end{aligned}$$

Then let C be a rectangular loop of area A[C]. The following inequality was proved by Tomboulis and Yaffe

$$\langle \mathrm{Tr}[U(C)] \rangle \leq \{\exp[-F_{el}]\}^{\frac{A(C)}{L_x L_y}}$$

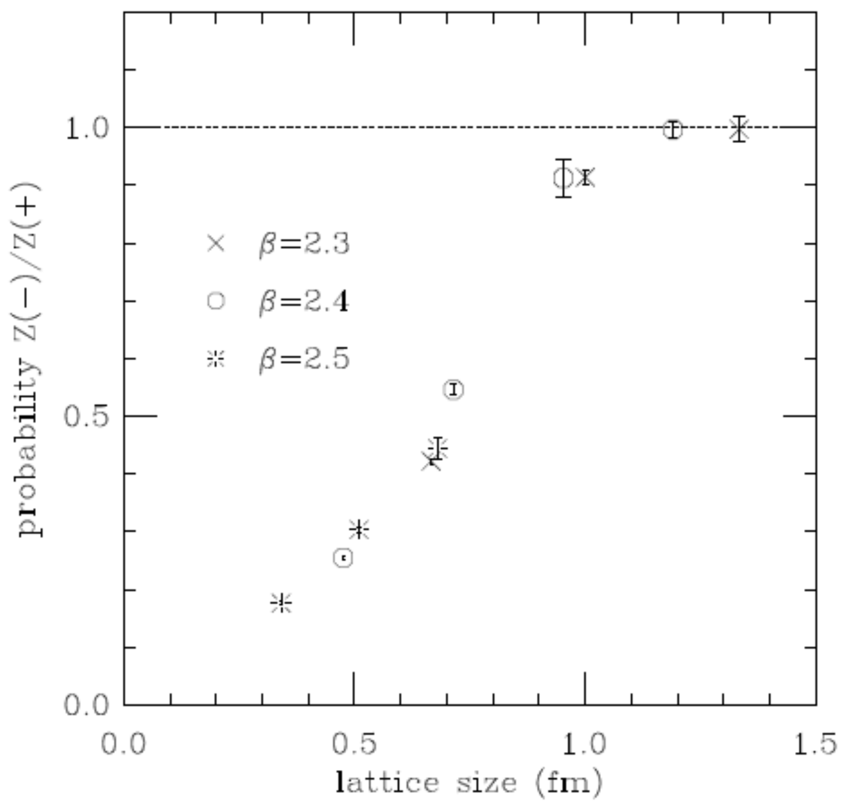
So, if the vortex free energy falls off wrt cross-sectional area $L_x L_y$ like

$$F_{mg} = c L_z L_t e^{-\rho L_x L_y}$$

then this is a sufficient condition for confinement, because it implies an area law bound for Wilson loops.

Numerical investigations of these quantities were begun by Kovacs and Tomboulis, and carried on in much more detail by de Forcrand and von Smekal.

This is the first numerical computation of Z_-/Z_+



Kovacs and Tomboulis.

Consistent with vortex free energy going rapidly to zero at large lattice volume.

To Repeat

The existence of a non-zero asymptotic string tension requires that

1. The gauge group has a non-trivial center

2. All matter fields transform in N-ality=0 representations

In that case, the action is invariant under global center transformations. All of the signals we've seen for non-vanishing asymptotic string tension:

a) area law for Wilson loops
b) vanishing Polyakov lines
c) perimeter-law 't Hooft loops
d) area-law falloff for the vortex free energy

can only be satisfied if global center symmetry exists.

This motivates the idea that vacuum fluctuations responsible for the asymptotic string tension must be, in some way, connected with center symmetry.

Part III : Properties of the Confining Force

- Asymptotic Linearity of the Static Potential
- Casimir Scaling
- N-ality dependence
- String Behavior: Roughening and the Luscher term

Asymptotic Linearity

Theorem: the force between a static quark and antiquark is everywhere attractive but cannot *increase* with distance; i.e.

$$\frac{dV}{dR} > 0 \quad \text{and} \quad \frac{d^2V}{dR^2} \leq 0$$

(Bachas)

The potential is convex, and can rise no faster than linear.

Since

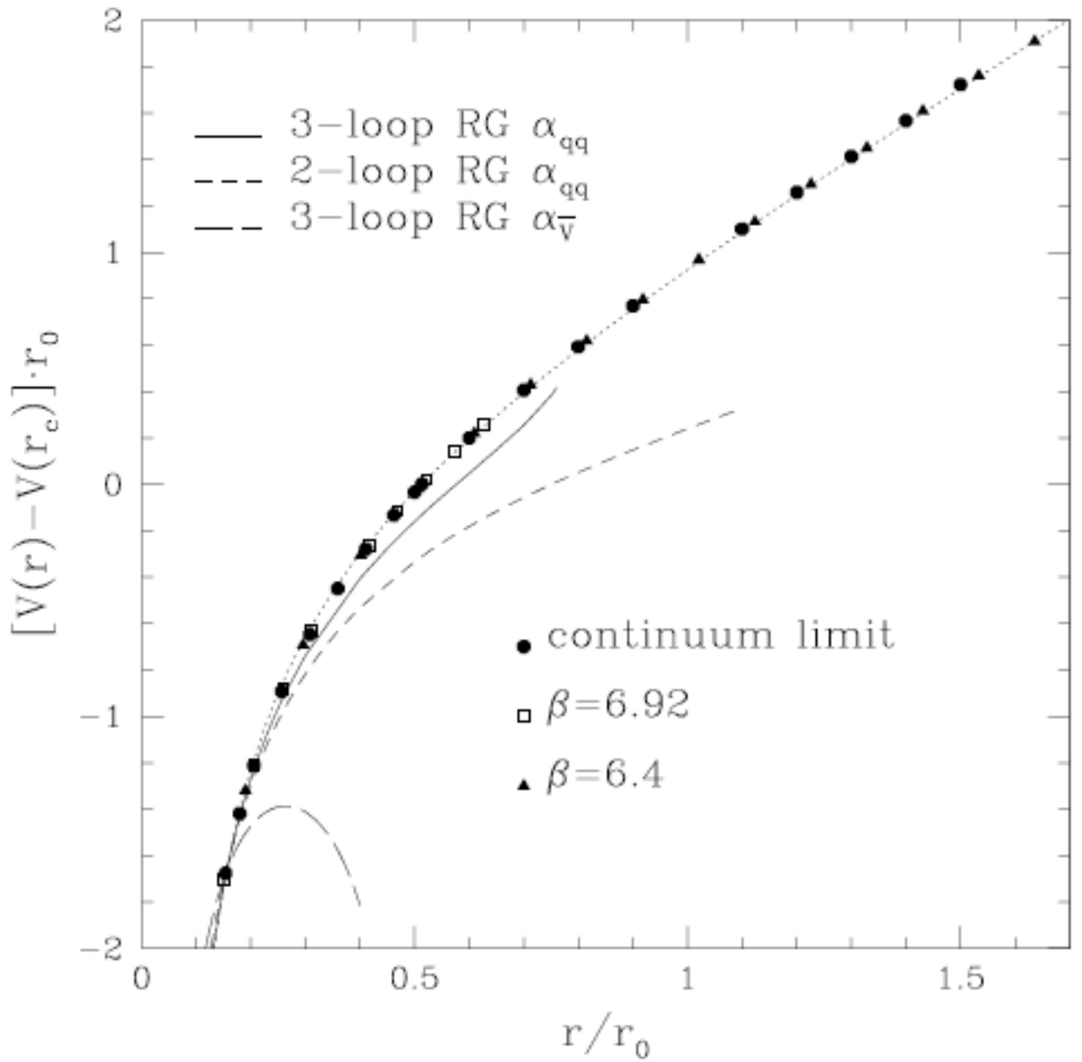
$$W(R,T) \to \exp[-V(R)T] \quad \text{as} \quad T \to \infty$$

We can compute the static potential from

$$V(R) = -\lim_{T\to\infty} \log\left[\frac{W(R,T+1)}{W(R,T)}\right]$$

There are techniques for greatly increasing convergence to the large-T limit by "smearing" the spacelike links.

Here is a typical numerical result for the static potential



Necco and Sommer

Casimir Scaling

Ambjorn & Olesen, Faber et al.

This is the idea that there is some intermediate range of distances where the string tension between a quark in group representation ***r,*** and its antiquark, is approx. proportional to the quadratic Casimir C_r of the representation.

$$\sigma_r = \frac{C_r}{C_F}\sigma_F$$

Why? - Two arguments: large-N limit, and dimensional reduction.

Large-N:
Group character $\chi_r(g)$ = trace of g in representation r has the property

$$\chi_r(g) \sim \chi_F^n(g)\chi_F^{*m}(g) + O(N^{-1})$$

n+m ¿ N is the smallest integer such that the irreducible representation r is obtained from the reduction of a product of n defining (``quark'') representations, and m conjugate (``antiquark'') representations.

Large-N also has the property of factorization: if A and B are two gauge-Invariant operators, then

$$\langle AB \rangle = \langle A \rangle \langle B \rangle$$

Put these facts together:

$$\begin{aligned} W_r(C) &= \langle \chi_r[U(C)] \rangle \\ &\overset{N\to\infty}{\sim} \langle \chi_F^n[U(C)\chi_F^{*m}[U(C)]] \rangle \\ &\sim W_F^{n+m} \end{aligned}$$

It follows that

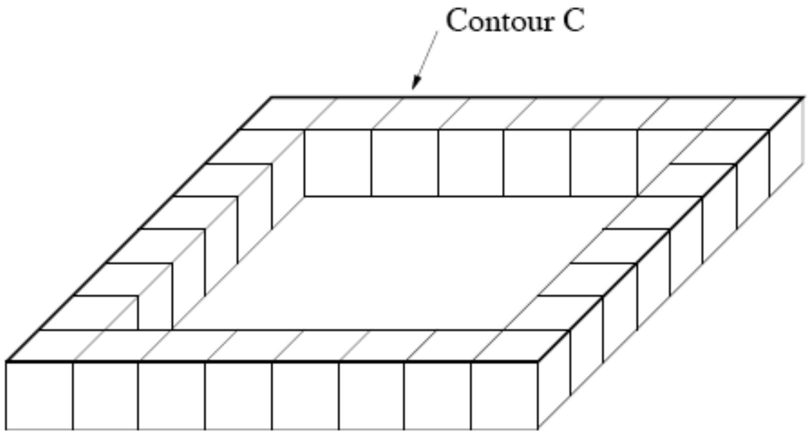
$$\sigma_r = (n+m)\sigma_F$$

which is precisely Casimir scaling in this large-N limit.

Its interesting to see how this goes in lattice strong-coupling expansions. The adjoint representation is n=m=1, and we find, for a square LxL loop in SU(N)

$$W_A[C] = N^2 e^{-2\sigma_F L^2} + e^{-16\sigma_F L}$$

where the second term comes from a “double layer” of plaquettes in the minimal area, and the second comes from the “tube” diagram



At large-N the first term / N^2 dominates initially, but eventually, for

$$L = \sqrt{\frac{1}{\sigma_F}\log(N) + 16} + 4$$

The perimeter term takes over, and the adjoint string tension drops to zero **(n-ality dependence).**

The second argument for Casimir scaling was:

Dimensional Reduction **(J.G., Olesen)**

At strong-coupling, confinement in D=4 looks the same as D=2. But in D=2, its easy to show that the confining potential at weak coupling comes from one-gluon exchange, and this leads to Casimir scaling.

So is confinement at weak-coupling in D=4 something like confinement at weak-coupling in D=2?

Argument: consider a spacelike Wilson loop

$$W_r^{D=4}(C) = \langle \Psi_0 | \chi_r[U(C)] | \Psi_0 \rangle$$

$$\Psi_0 = \exp[-R[A]]$$

In temporal gauge, R[A] is gauge-invariant, so we consider an expansion

$$R[A] = \quad d^3x \; [\alpha \text{tr}[F^2] + \beta \text{tr}[F^4] + \gamma \text{tr}[DFDF] + \ldots]$$

For small amplitude, long-wavelength configurations, perhaps only the first term dominates, in which case

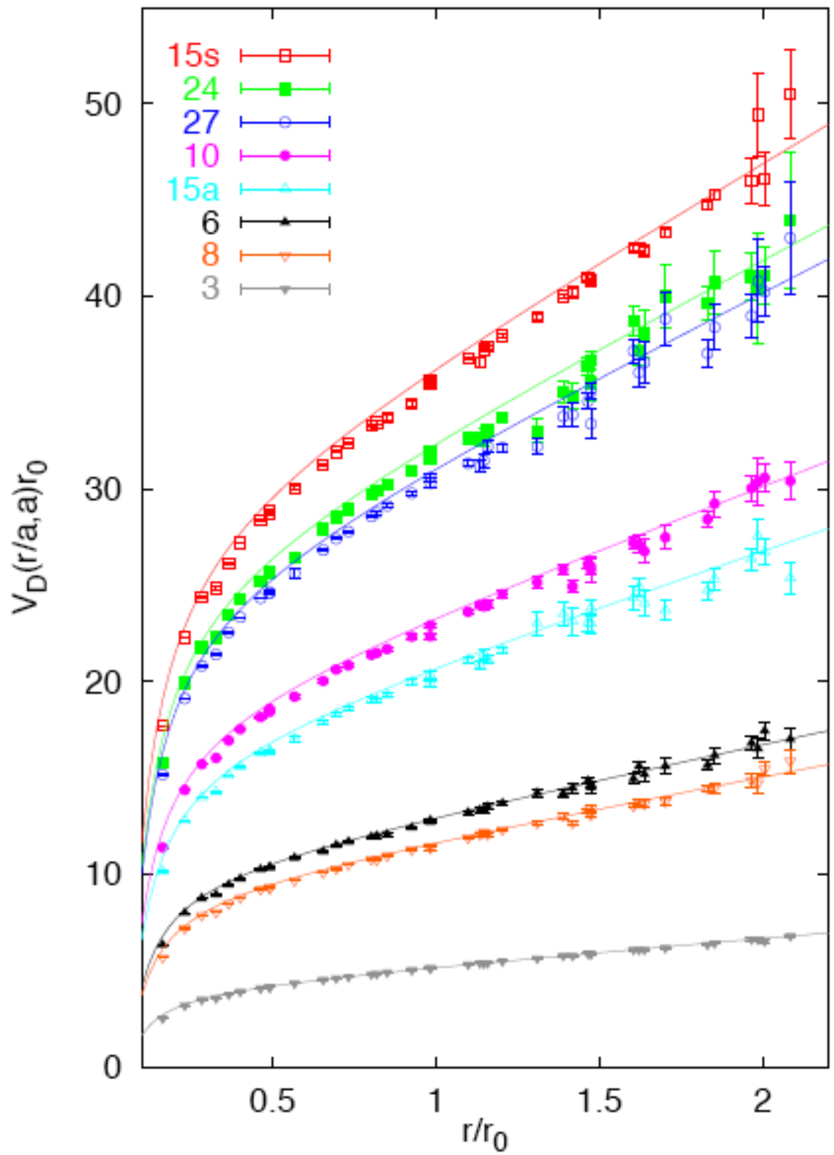
$$\begin{aligned} W_r^{D=4}(C) &= \langle \Psi_0 | \chi_r[U(C))] | \Psi_0 \rangle \\ &\sim \int DA_k \; \chi_r[U(C)] e^{-2\alpha \int d^3x \text{tr}[F^2]} \\ &\sim W_r^{D=3}(C) \end{aligned}$$

Dimensional reduce one more time, to get $W_r^{D=4}(C) \sim W_r^{D=2}(C)$

Since we have Casimir scaling in D=2, we should then get Casimir scaling in D=4.

Numerically, this works pretty well. Here is some data for SU(3):

Solid lines are a fit to the fundamental data, Multiplied by the Casimir ratio **C_r/C_F**



(Bali)

N-ality Dependence

Consider a quark-antiquark pair in representation r, with N-ality k_r. Gluons can bind to the quark and antiquark, and reduce the color charge to the lowest-dimensional representation with the same N-ality k_r . It follows that the after screening by gluons - i.e. ***asymptotically, string tension depends only on N-ality,***

$$\sigma_r = \sigma(k_r)$$

String tensions of the lowest-dimensional representation of N-ality k are often called ***“k-string tensions”.***

Two proposals for SU(N) gauge theories:

$$\sigma(k) = \begin{cases} \dfrac{k(N-k)}{N-1}\,\sigma_F & \text{“Casimir scaling” (a slight misnomer!)} \\[2em] \dfrac{\sin(\pi k/N)}{\sin(\pi/N)}\,\sigma_F & \text{Sine-law scaling} \end{cases}$$

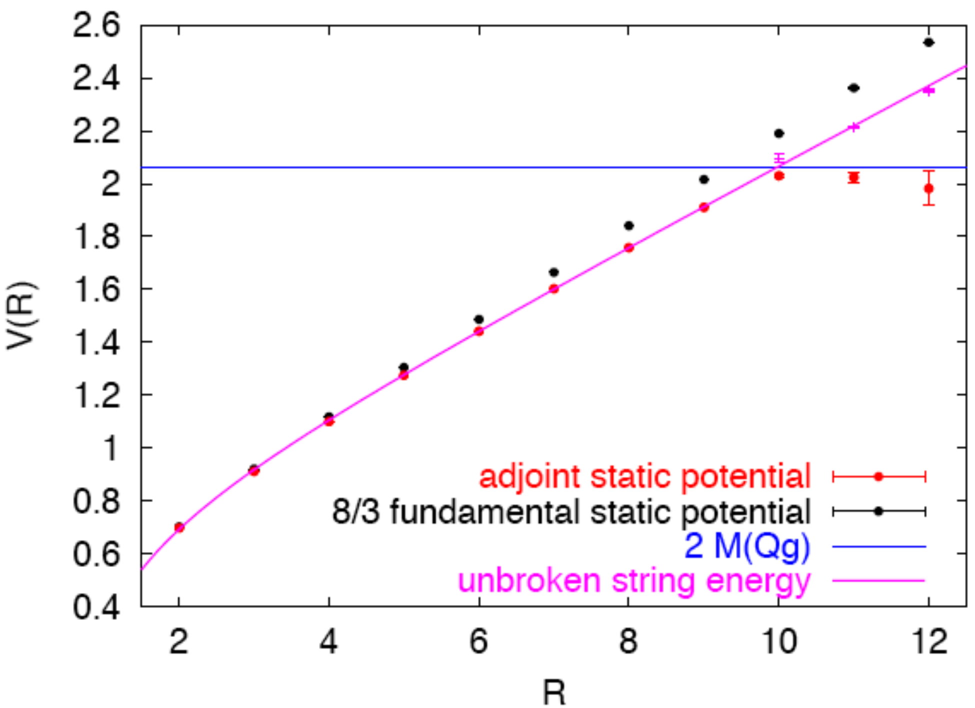
Either way, σ(k) / k for k ¿ N.

For our purposes, the crucial point is not Casimir vs Sine Law, but rather the simple fact that asymptotic string tension depends only on the N-ality of the representation.

An important example: the adjoint representation, which has k=0. We get string-breaking, and a flat static (adjoint) quark potential, when

$$\sigma_A L \approx C_A \sigma_F L > 2 m_{GL}$$

where m_{GL} is the mass of the adjoint quark-gluon bound state.



de Forcrand & Kratochvila

String Behavior (Luscher)

If the QCD flux tube resembles a Nambu string, then transverse fluctuations of the string induce a universal (coupling, scale) independent 1/R modificiation to the linear potential

the "Luscher term"

$$V(R) = \sigma R - \frac{\pi(D-2)}{24}\frac{1}{R}$$

According to the string calculation, the cross-sectional area of the flux tube should also grow logarithmically with R.

Both of these effects have been observed in numerical simulations.
(Luscher & Weisz, Kuti et al.)

Part IV: Confinement from Center Vortices

The motivations we've already seen:

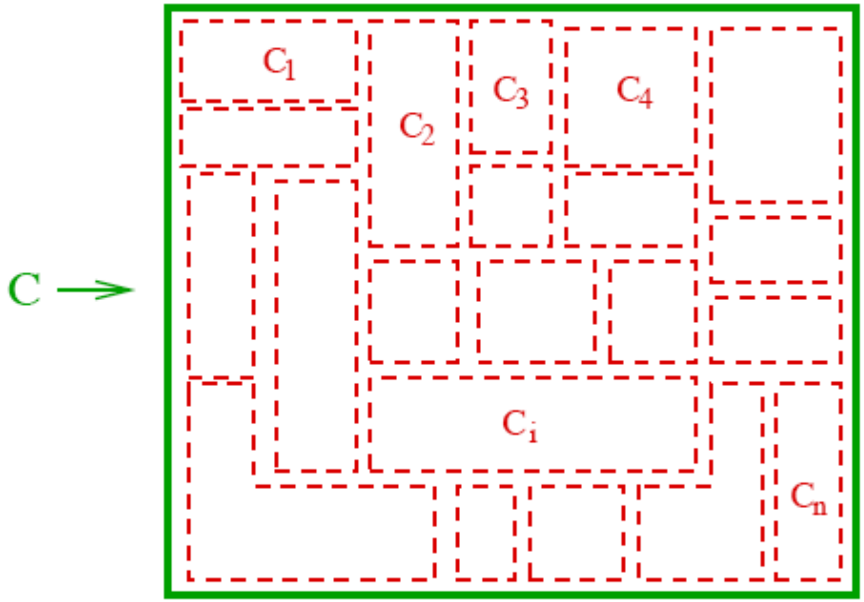
- Confinement - non-vanishing asymptotic string tension - is associated with the unbroken realization of a global center symmetry.

- The asymptotic string tension depends only on N-ality. In particular, whatever vacuum fluctuations cause σ_F **0** should not also force σ_A **0.**

- Two of the order parameters for confinement - the 't Hooft loop **B[C]** and the vortex free energy, are explicitly associated with the center vortex creation, while the non-zero value of the Polyakov line is a signal of spontaneous center symmetry breaking.

What can we learn from strong-coupling expansions, where confinement can be derived analytically?

Lets consider SU(2) lattice gauge theory at strong couplings, and U[C] is an SU(2) holonomy around loop C. Suppose C is very large, and we subdivide the minimal area into a set of sub-areas bounded by $\{U(C_i)\}$ as shown



Do the individual holonomies C_i fluctuate independently, if all loops are very large, or are they correlated somehow?

In general - observables a,b,c… are uncorrelated iff

<abc…> = <a><b><c>….

So lets find out if

$$< \prod_i F[U(C_i)] > \stackrel{?}{=} \prod_i < F[U(C_i)] >$$

where F[g] is any class (gauge inv) function with s dg F[g] =0. Such functions can be expanded in group characters

$$F[g] = \sum_{j \neq 0} f_j \chi_j[g]$$

If we evaluate the VEVs in D=2 dimensions, the equality is satisfied (and dominated by the lowest dimensional group representation)

$$e^{-\sigma A(C)} \prod_i f_{1/2} = \prod_i f_{1/2} e^{-\sigma A(C_i)}$$

This works because $A(C) = \sum_i A(C_i)$.

But for D>2, the N-ality=0 group characters have an asymptotic perimeter law falloff, and the lowest dimensional of these (j=1) dominates the VEVs. At strong coupling, we find

$$e^{-4\sigma P(C)} \prod_i \frac{1}{3} f_1 \gg \prod_i f_1 e^{-4\sigma P(C_i)}$$

Because $$\sum_i P(C_i) \gg P(C)$$

CONCLUSION: At D>2 the $U(C_i)$ do **not** fluctuate independently. But how does the area law arise, for j=half-integer Wilson loops?

Extract a center element from the holonomies

$$z[U(C)] = \mathrm{sign}\,\mathrm{Tr}[U(C)] \in Z_2$$

And ask if ***these*** fluctuate independently.

i.e., does

$$< \prod_i z[U(C_i)] > \stackrel{?}{=} \prod_i < z[U(C_i)] >$$

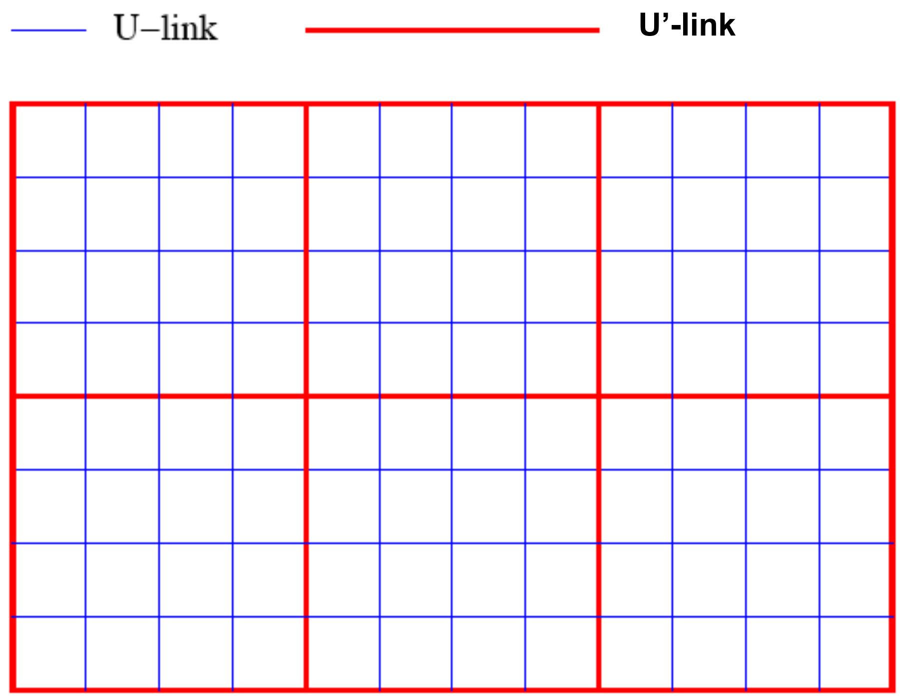
It does!

$$e^{-\sigma A(C)} \prod_i \frac{3}{4\pi} = \prod_i \frac{3}{4\pi} e^{-\sigma A(C_i)}$$

Confining disorder (for D>2) is ***center*** disorder, at least at strong couplings. It is natural to suspect that the source of this disorder is the center vortices, which affect loop holonomies only by a factor of a center element.

Its therefore interesting that thin center vortices are ***saddlepoints*** of the strong-coupling effective action.

Start with the strong-coupling Wilson action on a "fine" lattice of lattice spacing ***a*** and do a "blocking" transformation to arrive at an effective action on a lattice with spacing ***a'=La***



$$U'_\mu(x) = U_\mu(x)U_\mu(x+\hat{\mu}]U_\mu(x+2\hat{\mu})\ldots U_\mu(x+L\hat{\mu})$$

The blocking transformation

$$e^{-S'[U']} = \int DU \prod_{l'} \delta[U'_{l'} - (UUU...U)_{l'}] e^{-S[U]}$$

can be carried out analytically, and the result is **(Faber et al.)**

$$\begin{aligned} -S'[U'] \;&=\; \sum_{p'} \Bigg\{ 2N \left(\frac{1}{g^2 N}\right)^{L^2} \mathrm{ReTr}_F[U'(p')] \\ &\qquad\qquad +4(D-2)\left(\frac{1}{g^2 N}\right)^{4(4L-4)} \mathrm{Tr}_A[U'(p']\Bigg\} \\ &\qquad +\text{larger adjoint loops with } e^{-P} \text{ coefficients} \end{aligned}$$

For simplicity, truncate to the leading one-plaquette terms…

$$-S'_{trunc}[U'] = \sum_{p'} \{c_0 \text{ReTr}_F[U'(p')] + c_1 \text{Tr}_A[U'(p')]\}$$

with

$$c_0 \sim \exp[-\sigma \ \text{Area}(p')] \quad , \quad c_1 \sim \exp[-4\sigma \ \text{Perimeter}(p')]$$

Question: Does this action have a local minimum, other than vacuum (U=I)?

Answer: Yes, for c_1 À c_0 any center configuration, gauge-equivalent to

$$U_\mu(x) \ = \ Z_\mu(x) I_N$$

is a saddlepoint (local minimum) of the effective action, where

$$Z_\mu(x) = \exp\left(\frac{2\pi i n_\mu(x)}{N}\right), \quad n_\mu(x) = 1, 2, \ldots, N-1.$$

is a center element.

Proof: Consider small fluctuations of link variables around center elements

$$U_\mu(x) = Z_\mu(x)V_\mu(x), \qquad V_\mu(x) = e^{iA_\mu(x)}$$

And denote the product of V links around a plaquette as V_P

$$V_P = e^{iF_P} = I_N + iF_P - \frac{1}{2}F_P^2 + \mathcal{O}(F_P^3),$$

Then the effective action, to $O(F_P^2)$ is

$$S'_{trunc} = \text{const} + \frac{1}{2}\sum_{p'}\left\{c_0 \cos\left(\frac{2\pi n_{p'}}{N}\right)\text{Tr}_F[F_{p'}^2] + c_1 \text{Tr}_A[F_{p'}^2]\right\}$$

We see by inspection that for c_1 À c_0, the action has a local minimum at F_P=0. QED.

Now we move away from strong coupling, and suppose that vacuum configurations in SU(N) gauge theory can be decomposed into a relatively smooth confining background, and high-frequency fluctuations around that background

$$A_\mu(x) = \underbrace{\mathcal{A}_\mu(x)}_{\text{Confining background}} + \underbrace{a_\mu(x)}_{\text{fluctuations}}$$

An important hint about $\mathcal{A}_\mu(x)$ is N-ality dependence. An N-ality = 0 Wilson loop should have no area law falloff; i.e. should be not depend much on $\mathcal{A}_\mu(x)$. Suppose its not affected ***at all***. Then, writing holonomies

$$U(C) = P\exp\left[i\oint_C dx^\mu \; (\mathcal{A}_\mu(x) + a_\mu(x))\right]$$

$$u(C) = P\exp\left[i\oint_C dx^\mu \; a_\mu(x)\right]$$

$$\mathcal{U}(C) = P\exp\left[i\oint_C dx^\mu \; \mathcal{A}_\mu(x)\right]$$

it means that for N-ality=0

$$\chi_r[U(C)] = \chi_r[u(C)]$$

But this can only be true if (up to a gauge transformation)

$$U(C) = Z(C)u(C)$$

$$\Rightarrow \quad \mathcal{U}_\mu(C) = Z(C)$$

Link variables which give center element holonomies are gauge-equivalent
To the link variables of a Z_N lattice gauge theory

$$\mathcal{U}_\mu(x) = z_\mu(x)g(x)g^{-1}(x+\widehat{\mu}) \quad , \quad z_\mu(x) \in Z_N$$

whose excitations consist only of center vortices.

The Center Vortex Confinement Mechanism (finally!)

We assume that variables can be expressed

$$
\begin{aligned}
U_\mu(x) &= g(x) z_\mu(x) u_\mu(x) g^{-1}(x) \\
U(C) &= Z(C) g u(C) g^{-1}
\end{aligned}
$$

such that

1. large holonomies u[C] and Z[C] are only weakly correlated

$$\langle U(C) \rangle \approx \langle Z(C) \rangle \langle u(C) \rangle$$

2. for any two large non-overlapping loops

$$\langle Z(C_1) Z(C_2) \rangle \approx \langle Z(C_1) \rangle \langle Z(C_2) \rangle$$

as we found to be true in strong-coupling lattice gauge theory.

This is enough to give us confinement!

Here's how: Consider for simplicity a large rectangular Wilson loop C of area A, in group representation r of N-ality k. We have, by assumption

$$W_r(C) = \langle \chi_r[U(C)] \rangle = \langle Z^k(C) \rangle \langle \chi_r[u(C)] \rangle$$

Now subdivide the area A into square LxL subareas bounded by loops $\{C_i\}$.

$$\begin{aligned} \langle Z^k(C)] \rangle &= \langle \prod_{i=1}^{A/L^2} Z^{k_r}(C_i) \rangle \\ &\approx \prod_{i=1}^{A/L^2} \langle Z^{k_r}(C_i) \rangle \\ &= \exp[-\sigma(k_r)A] \end{aligned}$$

where

$$\sigma(k) = -\frac{\log[\langle Z^k(C_i) \rangle]}{L^2}$$

An area law which depends only on N-ality

How big should we make these LxL subareas?

(It doesn’t matter, as long as the $Z(C_k)$ are uncorrelated.)

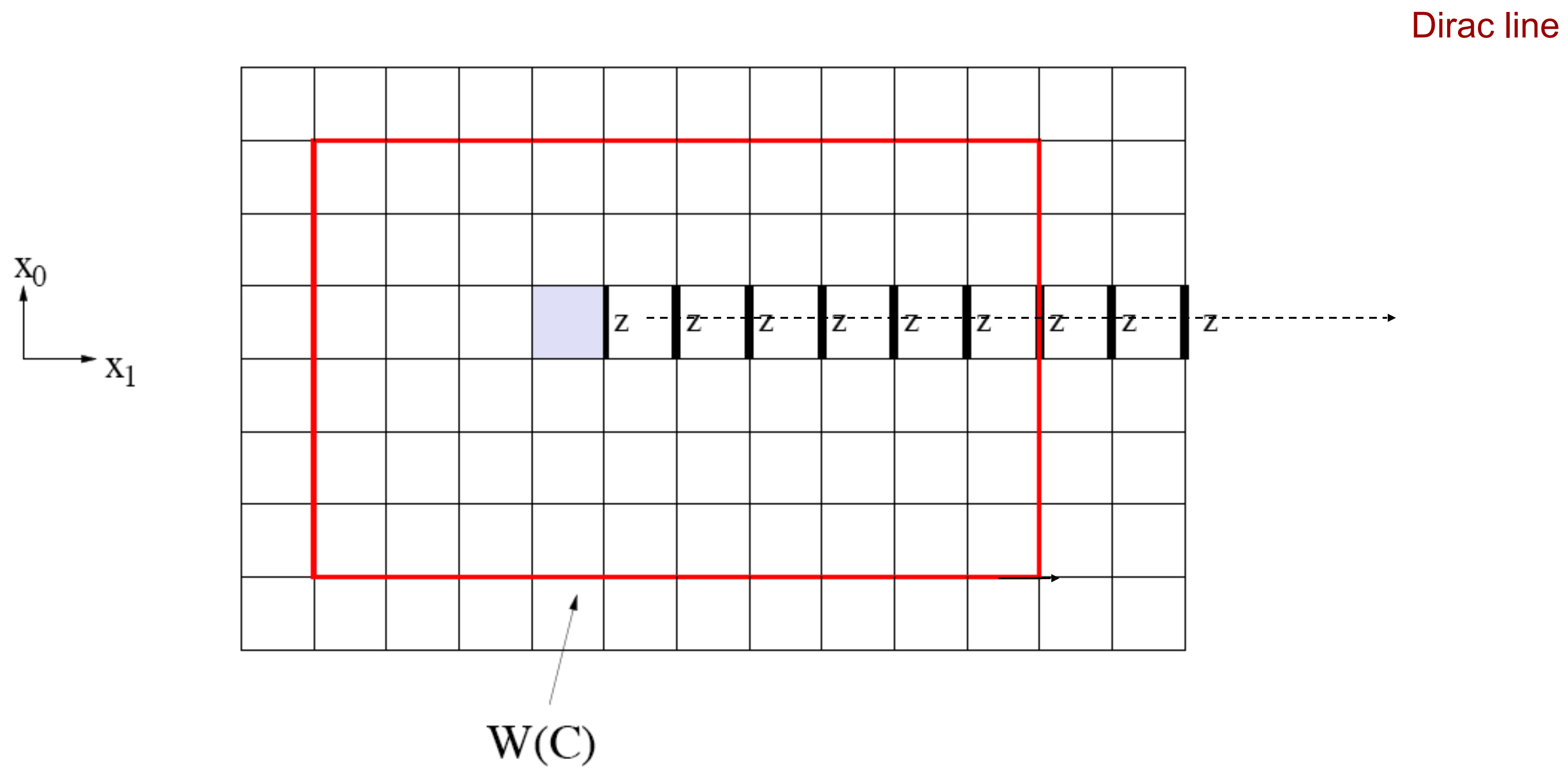
Very simplest case: LxL is 1x1.

This means that the probability **f** for a given plaquette p to have z(p) = -1 is uncorrelated with the values of z(p’) on other plaquettes p’ in the same plane.

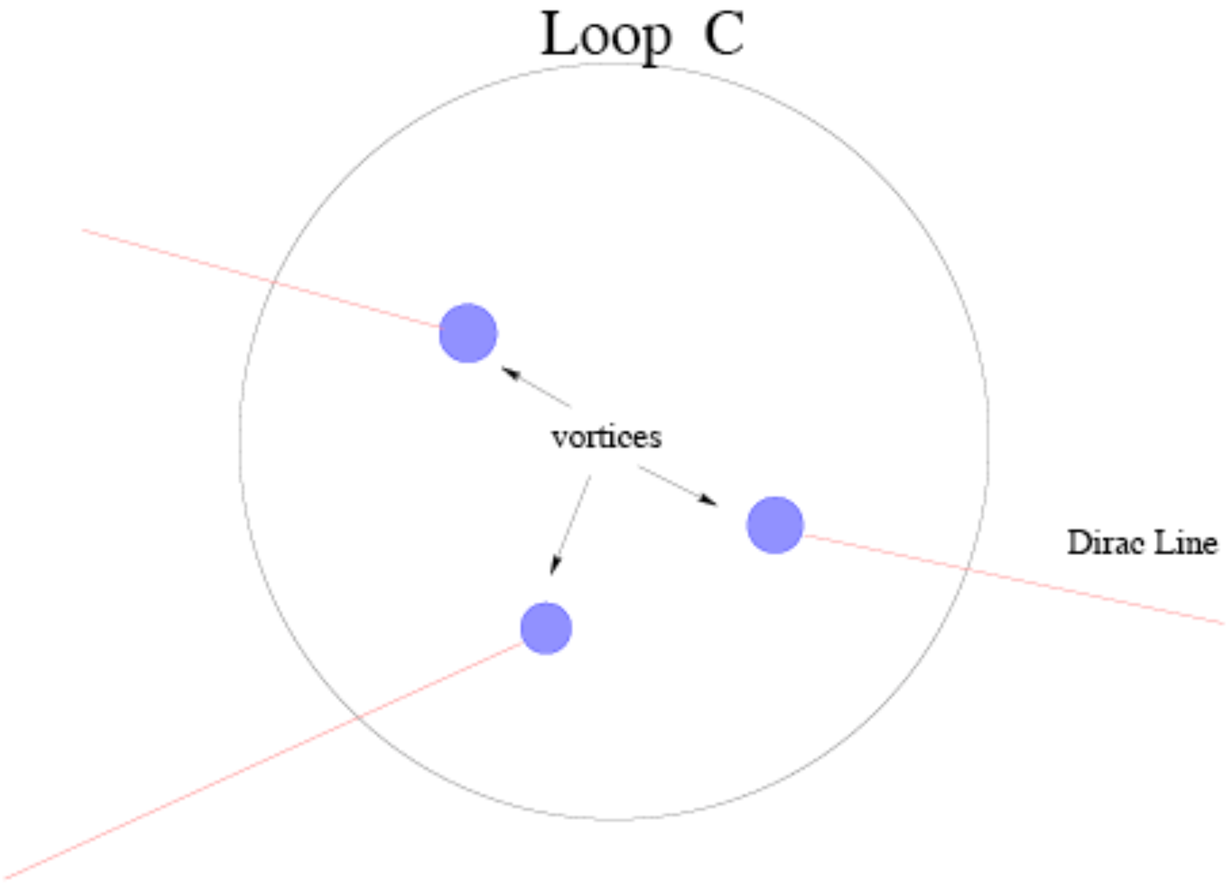
Lets run through the argument again. The decomposition

$$U_\mu(x) = g(x) z_\mu(x) u_\mu(x) g^{-1}(x)$$

gives us a confining background of thin center vortices (the $z_\mu(x)$),
With non-confining fluctuations gug^{-1} around that background.



Wilson loop W[C] is multiplied by a factor of z for each vortex piercing the minimal area.



Then

$$Z(C) = \prod_{p \in Area(C)} z(p)$$

and

$$z(p) = \begin{cases} -1 & \text{vortex pierces the plane at } p \\ +1 & \text{otherwise} \end{cases}$$

Again define

f = prob. that a vortex pierces any given plaquette
= prob. that z(p) = -1

and assume that

1. Piercings are uncorrelated;
2. Fluctuations u[C] are uncorrelated with Z[C] for large loops

Denote $$W_j^0(C) = \langle \chi_j[u(C)] \rangle$$

Then

$$
\begin{aligned}
W_j(C) &= \langle \prod_{p\in A(C)} z^{2j}(p)\rangle W_j^0(C) \qquad \left(j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right) \\
&= \prod_{p\in A(C)} [(1-f)(+1) + f(-1)^{2j}] W_j^0(C) \\
&= \exp[\ln(1 - f + (-1)^{2j} f) A(C)] W_j^0(C)
\end{aligned}
$$

String tensions are

$$
\sigma_j = \begin{cases} -\ln(1-2f) & j = \text{half-integer} \quad \text{(N-ality=1)} \\ 0 & j = \text{integer} \qquad\qquad \text{(N-ality=0)} \end{cases}
$$

and depend only on N-ality.

Part V: Numerical Evidence

The vortex mechanism is probably the simplest route to confinement, and well motivated by local gauge-invariant order parameters for confinement (‘t Hooft loop, Polyakov loop, vortex free energy), and by the known facts about N-ality dependence.

But is it right?

To find out, we turn to lattice Monte Carlo simulations. The first problem is to figure out how to spot thick center vortices in a list of what looks like random numbers - i.e. the lattice link variables.

Finding Center Vortices in Thermalized Lattices

An "adjoint gauge" is a gauge which completely fixes link variables U_A in the adjoint representation, leaving a residual Z_N gauge symmetry. An example is the adjoint Landau gauge, which maximizes

$$\begin{aligned} R &= \sum_x \sum_\mu \mathrm{Tr}[U_{A\mu}(x)] \\ &= \sum_x \sum_\mu \mathrm{Tr}[U_\mu(x)]\mathrm{Tr}[U_\mu^\dagger(x)] \; + \; \text{const.} \end{aligned}$$

This gauge is also known as ***direct maximal center gauge***.

In an adjoint gauge, we factor each link variable into a center and coset part

$$U_\mu(x) = Z_\mu(x) V_\mu(x)$$

where $Z_\mu(x)$ is the center element closest to $U_\mu(x)$ on the Lie group manifold. E.g., for SU(2):

$$Z_\mu(x) = \mathrm{signTr}[U_\mu(x)] \; .$$

Center Projection is the replacement of link variables U_μ in an adjoint gauge by their closest center elements, i.e.

$$U_\mu(x) \rightarrow Z_\mu(x)$$

which maps the SU(N) configuration into a Z_N configuration. Plaquettes Z(p) 1 on the projected lattice are known as **P-plaquettes,** and together they form thin center vortices known as **P-vortices**.

The claim is that this procedure locates center vortices on the unprojected lattice.

The idea is that P-vortices lie somewhere in the middle of the “thick” center vortices on the uprojected lattice.

Motivation

Suppose we have some lattice configuration U, and insert "by hand" a thin center vortex, via a singular gauge transformation, somewhere on the lattice. Will this center vortex be among the set of vortices identified by the center projection procedure?

The answer is yes. Let **U'** be the lattice with the inserted vortex. In the adjoint representation, the corresponding $\mathbf{U_A}$ and $\mathbf{U'_A}$ are gauge-equivalent. Then in adjoint center gauge, $\mathbf{U_A}$ and $\mathbf{U'_A}$ transform into the same configuration, call it $\mathbf{U^{ag}_A}$.

It follows that in the fundamental representaion, **U** and **U'** can only differ by center elements, i.e.

$$
\begin{aligned}
U'^{ag}_\mu(x) &= Z'_\mu(x) U^{ag}_\mu(x) \\
&= Z'_\mu(x) Z_\mu(x) V_\mu(x)
\end{aligned}
$$

Inserting a thin vortex, into either an SU(N) or Z_N lattice configuration, is gauge equivalent to multiplication by a set of center elements $Z'_\mu(x)$. It follows that the inserted vortex is among the vortices identified by center projection

$$U'^{ag}_\mu(x) \rightarrow Z'_\mu(x) Z_\mu(x)$$

So in principle, center projection in ***any*** adjoint gauge has the "vortex-finding property" for thin vortices.

Weaknesses

- Vortices in SU(N) are not thin. They have finite thickness.
- Gribov copies.

For these reasons, while all adjoint gauges are in principle equal, "some are more equal than others". The only real justifications are empirical.

Useful Adjoint Gauges

1. **Direct Maximal Center (DMC) gauge** **(Faber et al)**
   Again, this is the lattice Landau gauge in adjoint representation. The prescription is to maximize

$$R = \sum_x \sum_\mu \mathrm{Tr}_A[U_\mu(x)]$$

This is the most "intuitive" choice, because it is equivalent to making the best fit of a given lattice configuration $U_\mu(x)$ by the thin vortex configuration

$$\mathcal{U}_\mu(x) = g(x) Z_\mu(x) g^\dagger(x+\mu)$$

Why? - Since the adjoint representation is blind to $Z_\mu(x)$, start by making a best fit of $U_{A\mu}(x)$ to a pure gauge, by minimizing

$$\begin{aligned} d_A^2 &= \sum_{x,\mu} \mathrm{Tr}_A \left[\left(U_\mu(x) - g(x) g^\dagger(x+\hat{\mu}) \right) \times (\mathrm{h.c.}) \right] \\ &= \sum_{x,\mu} 2\mathrm{Tr}_A \left[I - g^\dagger(x) U_\mu(x) g(x+\hat{\mu}) \right] \end{aligned}$$

But this is completely equivalent to maximizing

$$R = \sum_x \sum_\mu \mathrm{Tr}_A[U_\mu(x)]$$

which is the DMC gauge. Next, choose $Z_\mu(x)$ so as to minimize the distance function in the fundamental representation

$$d^2 = \sum_{x,\mu} \mathrm{Tr}\left[\left(U_\mu(x) - g(x)Z_\mu(x)g^\dagger(x+\hat{\mu})\right) \times (\mathrm{h.c.})\right]$$

which is achieved for SU(2) by

$$Z_\mu(x) = \mathrm{sign}\,\mathrm{Tr}[{}^g U_\mu(x)]$$

Or, in SU(N), setting $Z_\mu(x)$ to be the closest center element to ${}^g U_\mu(x)$. This is the center-projection prescription.

2. Indirect Maximal Center (IMC) gauge (Faber et al.)

This gauge is useful in exploring connections between abelian monopoles and vortices.

First go to maximal abelian gauge, which minimizes the off-diagonal elements of the link variables, leaving a residual $U(1)^{N-1}$ gauge invariance. For SU(2), maximize

$$R = \sum_{x,\mu} \mathrm{Tr}[U_\mu(x)\sigma_3 U_\mu^\dagger(x)\sigma_3]$$

The link variables are decomposed as

$$U_\mu(x) = C_\mu(x) D_\mu(x)$$

Where D is the diagonal part of the link variable, rescaled to restore unitarity

$$\begin{aligned} D_\mu &= \frac{1}{\sqrt{|[U_\mu]_{11}|^2 + |[U_\mu]_{22}|^2}} \begin{bmatrix} [U_\mu]_{11} & 0 \\ 0 & [U_\mu]_{22} \end{bmatrix} \\ &= \begin{bmatrix} e^{i\theta_\mu} & 0 \\ 0 & e^{-i\theta_\mu} \end{bmatrix} \end{aligned}$$

2. (cont) Then we use the residual U(1) gauge symmetry to maximize

$$\widetilde{R} = \sum_{x,\mu} |\mathrm{Tr}[D_\mu(x)]|^2 = \sum_{x,\mu} 4\cos^2(\theta_\mu(x))$$

leaving a residual U(1) symmetry.

Both the DMC and IMC gauges have Gribov copies: for any U, there are a huge number of local maxima of R (much like spin glasses), and no known technique for finding the global maximum.

There are two strategies for dealing with the Gribov problem:

1. Make an effort to come as close as possible to a global (rather than local) maximum of R, using e.g. a simulated annealing algorithm. **(Bornyakov et al.)**

2. Give up on the global maximum, and average over all copies (MC simulations $\longrightarrow$ pick a copy at random).

Strategy 2 makes sense if most copies arrive at the same vortex content, with relatively small variations in vortex location. This can (and has been) be checked for DMC gauge. The vortices in randomly selected Gribov copies are closely correlated.

Another possibility is to use a Laplacian gauge, which avoids the Gribov copy problem…

3. Laplacian Center (LC) Gauge de Forcrand and Pepe

Consider Yang-Mills theory with two Higgs fields ϕ_1^c, ϕ_2^c in the adjoint representation. The unitary gauge

$$\begin{aligned} \phi_1^c(x) &= \rho(x)\delta^{c3} \\ \phi_2^2(x) &= 0 \quad , \quad \phi_2^1(x) > 0 \end{aligned}$$

then leaves only a residual Z_2 symmetry. In the LC gauge, the two "Higgs" fields are taken to be the two lowest eigenmodes

$$\sum_y \Delta_{ij}(x,y) f_j^\alpha(y) = \lambda_\alpha f_i^\alpha(x)$$

of the lattice Laplacian operator in adjoint representation

$$\begin{aligned} \Delta_{ij}(x,y) &= -\sum_\mu \Big([U_{A\mu}(x)]_{ij}\delta_{y,x+\hat{\mu}} \\ &\qquad + [U_{A\mu}(x-\widehat{\mu})]_{ji}\delta_{y,x-\hat{\mu}} \ - 2\delta_{xy}\delta_{ij} \Big) \end{aligned}$$

4. Direct Laplacian Center (DLC) Gauge (Faber et al)

A hybrid. Uses three lowest eigenmodes to select a particular Gribov copy of DMC gauge.

Most of the numerical results I'll show were obtained from DLC gauge; but these are almost identical to what is obtained by simply picking DMC Gribov copies at random.

The numerical results I will now present fall into several categories:

I. **Center Dominance** - what string tension is obtained from P-vortices?

II. **Vortex-Limited Wilson loops** - what is the correlation between center-projected loops and Wilson loops on the unprojected lattice? Do P-vortices locate thick vortices on the original lattice?

III. **Vortex Removal** - what is the effect of removing vortices, identified by center projection, from the lattice configuration?

IV. **Scaling** - does the density of vortices scale according to the asymptotic freedom prediction?

V. **Finite Temperature**

VI. **Chiral Condensate/Topological Charge**

VII. **Casimir Scaling** - and vortex thickness.

VIII. **Vortices and Matter Fields** - what happens to the vortex picture if we break center symmetry by introducing matter fields, as in real QCD?

Finally, I want to show the close connection of the vortex picture and two other proposed confinement mechanisms:

IX. **Monopoles and vortices** - monopole worldlines lie on vortex sheets.

X. **Vortices and the Gribov Horizon** - there are very close connections between the vortex mechanism, and a confinement mechanism suggested by Gribov and Zwanziger in Coulomb gauge.

Center Dominance

The very first question is whether, under the factorization in maximal center gauge

$$U_\mu(x) = Z_\mu(x) V_\mu(x)$$

The variables $Z_\mu(x)$ carry the confining disorder.

Let Z(I,J) represent a Wilson loop in the projected lattice, on a rectangular IxJ contour. The corresponding center-projected Creutz ratios are

$$\chi_{cp}[I,J] \equiv -\log\left\{\frac{Z[I,J]Z[I-1,J-1]}{Z[I-1,J]Z[I,J-1]}\right\}$$

At large I,J, does this quantity equal the asymptotic string tension?

A little digression on Creutz ratios

Rectangular Wilson loops W[R,T] typically fall off this way:

$$W(R,T) = \exp[-(\sigma RT + \mu(R+T) - a(T/R + R/T) + b]$$

The term μ(R+T) is a self-energy term, and is divergent in the continuum limit. Creutz noticed that one could form a ratio of rectangular loops such that the self-energy terms would cancel out

$$\chi[R,T] \equiv -\log\left\{\frac{W[R,T]W[R-1,T-1]}{W[R-1,T]W[R,T-1]}\right\}$$

and one can check that in the limit of large loop area

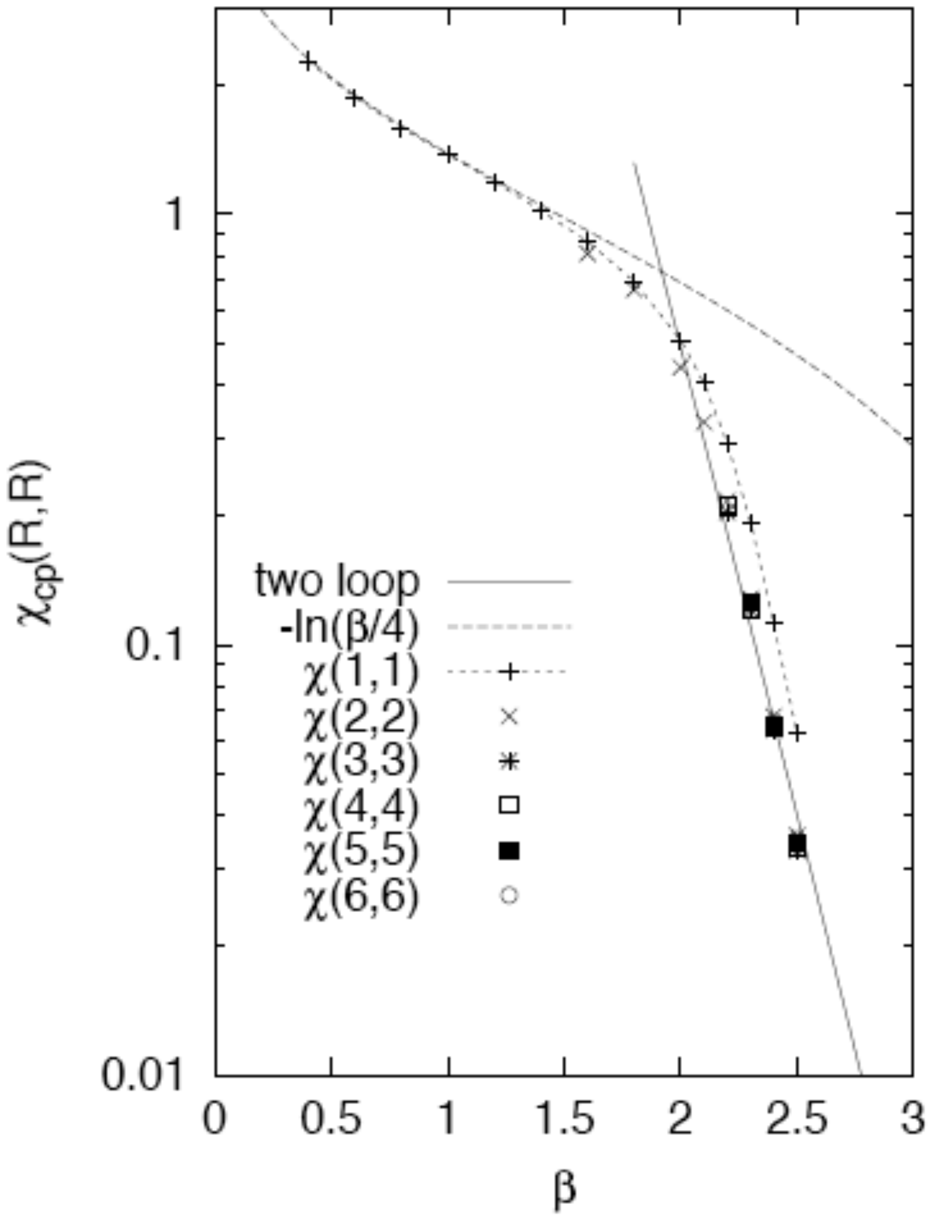
$$\chi[R,T] \to \sigma$$

Here is a first look at the data for χ_{cp}**(R,R)**. Note that:

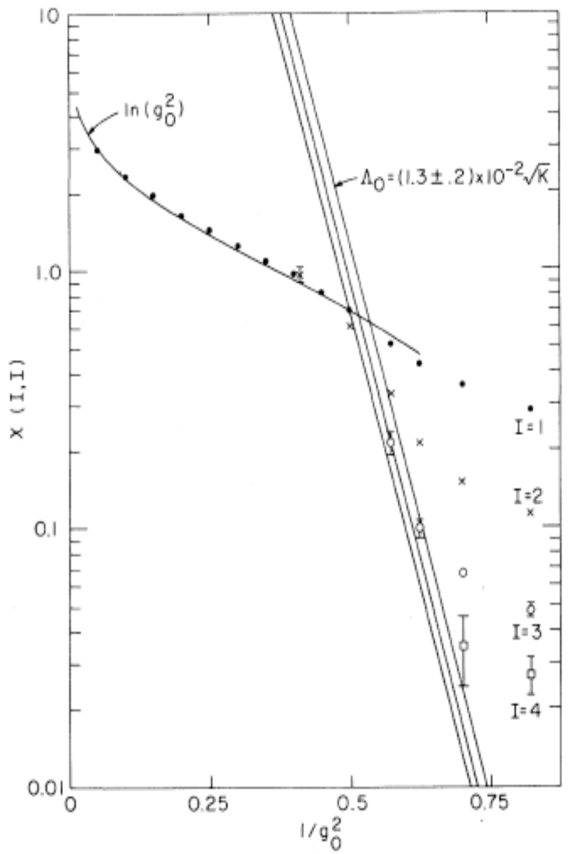
a) At each β, the different χ_{cp}**(R,R)** are almost identical, for R > 1.

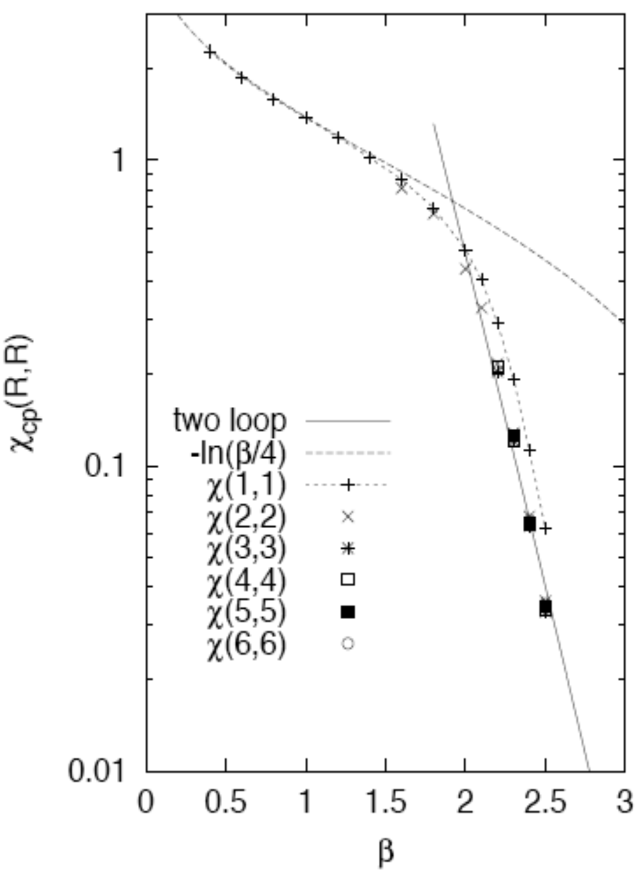
a) There is excellent agreement with asymptotic freedom.

b) Even the R=1 data points seem to scale.



It is worth comparing the center-projected data with the original Creutz plot from 1980.

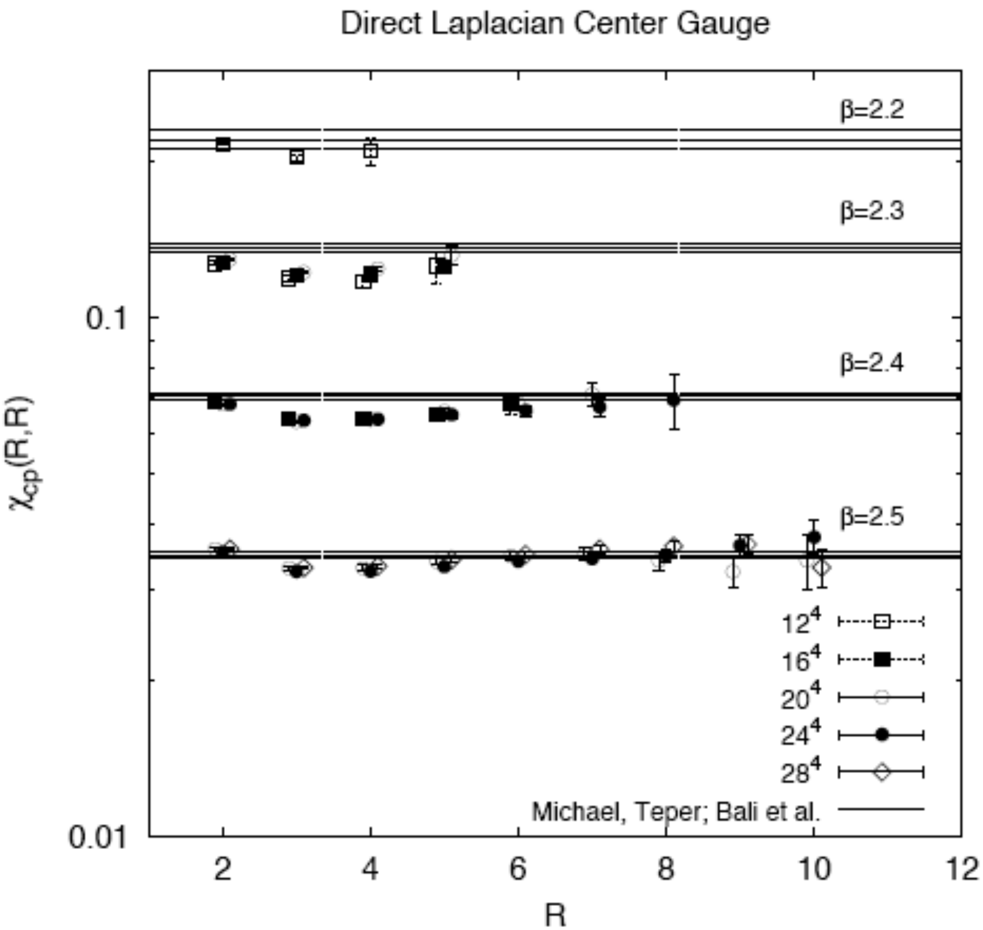




Creutz

Center projection

Here is a closer look. The solid line is the accepted asymptotic string tension at the given β value.



The fact that χ_{cp}**(R,R)** is nearly R-independent means that the center-projected potential is linear starting from R=2; there is no Coulomb piece. This feature is known as Precocious Linearity.

Why precocious linearity?

Center vortices on the unprojected lattice are thick objects, and the full effect on a Wilson loop - multiplication by a center element - only occurs for large loops. Center projection “shrinks” the thickness of the vortex to one lattice spacing; the full effect of linking to a vortex appears for even the smallest center-projected Wilson loops.

Therefore, if P-vortex plaquettes are completely uncorrelated in a plane, then we must see a linear potential from the smallest distances.
If no precocious linearity, then either

a) the vortex surface is very rough, bending in and out of the plane, or
b) there are very small vortices.

In either case there are correlations between nearby P-plaquettes, and a delay in the onset of the linear projected potential.

Vortex-Limited Wilson Loops

Even if P-vortices get the asymptotic string tension roughly right, what tells us that they are really correlated with fat center vortices on the unprojected lattice?

What we need to do is to test the correlation of P-vortices with gauge-invariant observables, such as Wilson loops.

A **"vortex-limited Wilson loop"** $\mathbf{W_n[C]}$ is the VEV of a Wilson loop on the unprojected lattice, evaluated in the subensemble of configurations in which the minimal area of the loop is pierced by precisely n P-vortices (i.e. there are n P-plaquettes in the minimal area).

Here the center projection is used only to select the data set; the loop itself is evaluated using unprojected link variables.

If P-vortices on the projected lattice locate center vortices on the unprojected lattice, then for SU(2) we would expect, asymptotically, that

$$\frac{W_n(C)}{W_0(C)} \to (-1)^n$$

Reason

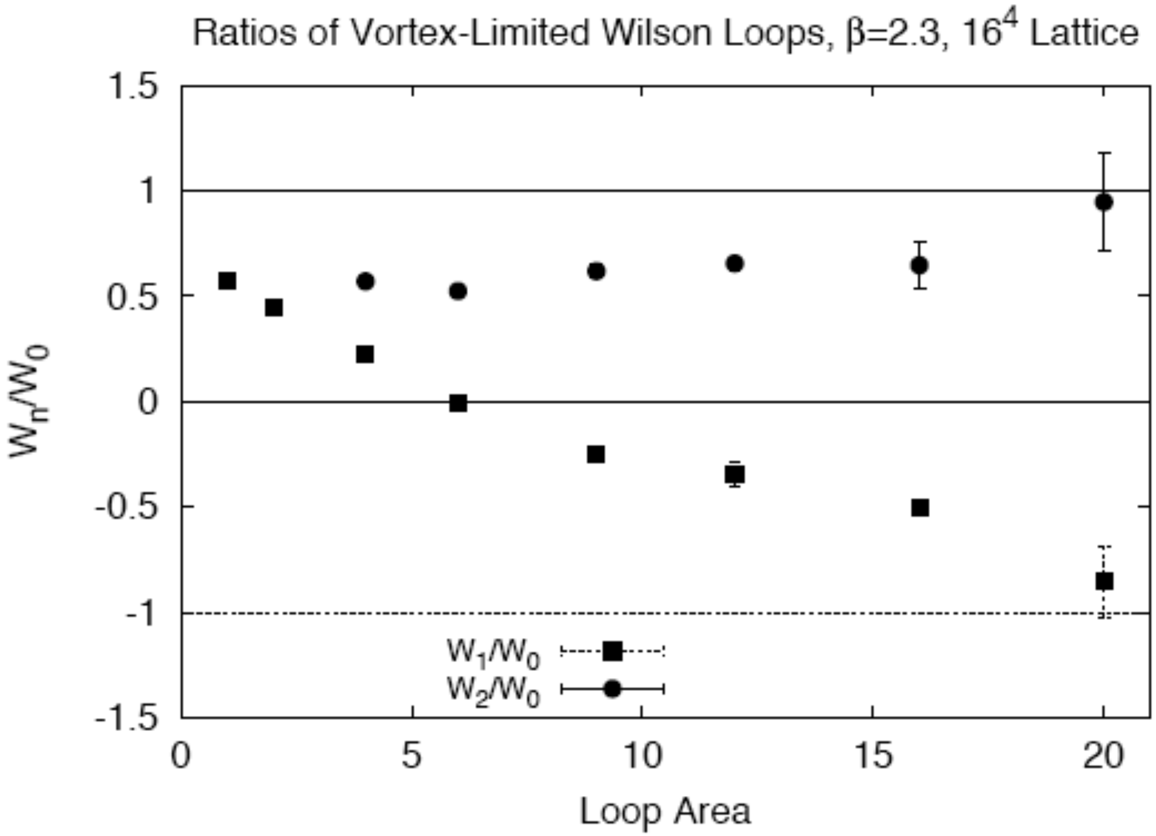
$$\begin{aligned} W_n(C) &= \langle Z(C)\,\mathrm{Tr}[V(C)]\rangle_{n_p(C)=n} \\ &= (-1)^n \langle \mathrm{Tr}[V(C)]\rangle_{n_p(C)=n} \end{aligned}$$

If we assume that $\mathbf{V_\mu(x)}$ has only short range correlations, then on a large loop this variable is insensitive to the presence or absence of vortices deep in the interior of the loop, i.e.

$$\langle \mathrm{Tr}[V(C)]\rangle_{n_p(C)=n} \approx \langle \mathrm{Tr}[V(C)]\rangle_{n_p(C)=0}$$ for large loops

The ratio $\mathbf{W_n/W_0}$ **!** $\mathbf{(-1)^n}$ follows immediately.

Here are the numerical results, which are consistent with this reasoning.



One can also looks at loops with, e.g. even or odd numbers of P-vortices piercing the loop. We find, for SU(2)

- $$W_{odd}(C)/W_{even}(C) \rightarrow -1$$
- $$\chi_{even}(R,R) \rightarrow 0$$
- $$\chi_0(R,R) \rightarrow 0$$

From the fact that **W_n/W_0 ! $(-1)^n$**, we conclude that P-vortices are correlated with the sign of the Wilson loop, in just the way expected if these P-vortices are correlated with center vortices.

From center dominance, we conclude that it is the sign fluctuations in **Z[C]** , rather than in **TrV[C]** , that is responsible for the string tension.

Vortex Removal (De Forcrand and D'Elia)

A powerful consistency test: Suppose we "remove" center vortices from the unprojected configuration, by replacing

$$U_\mu(x) \rightarrow U'_\mu(x) = Z_\mu(x) U_\mu(x) = V_\mu(x)$$

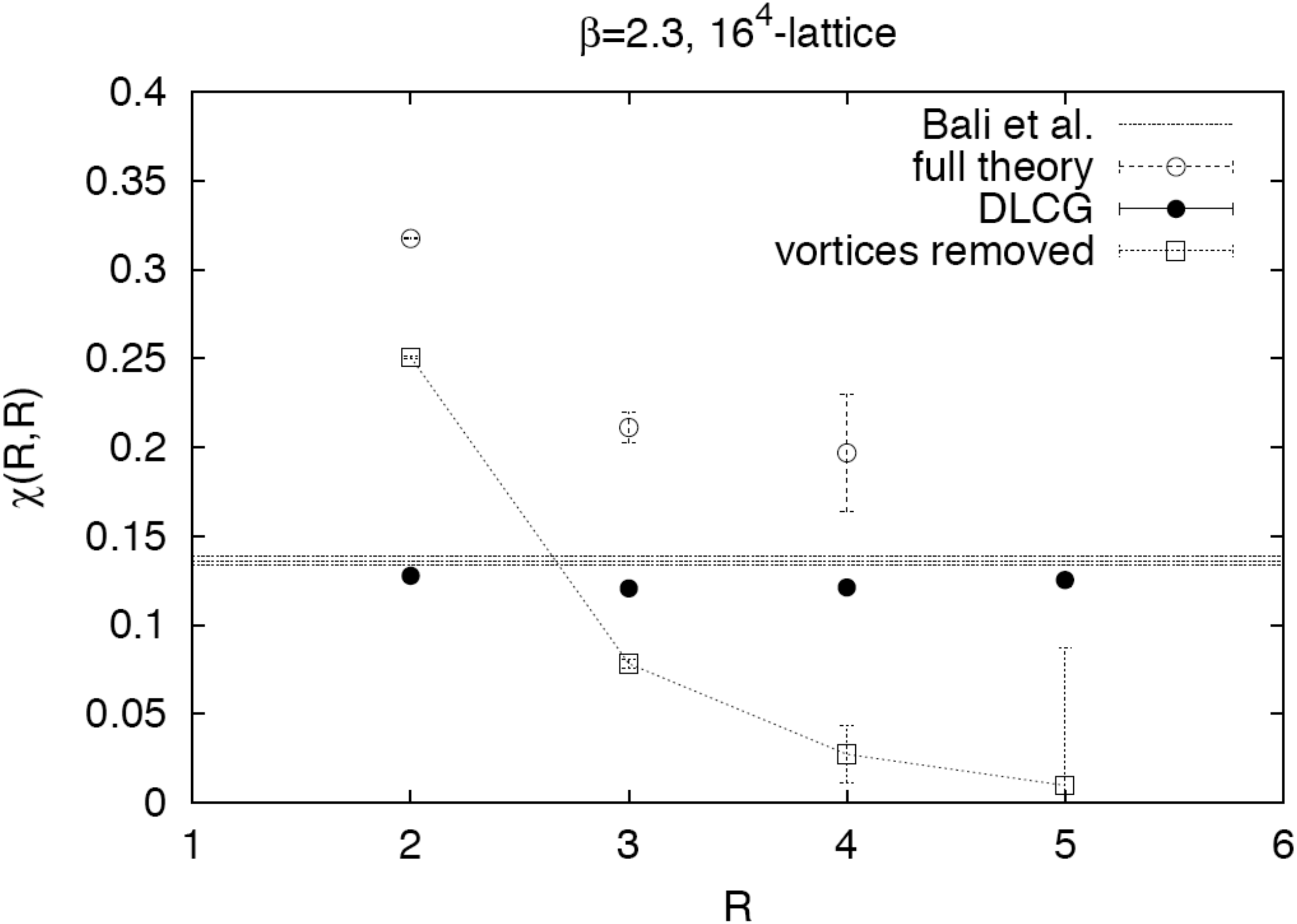
This inserts a thin vortex in the middle of a thick vortex. The asymptotic fields of the thin and thick vortex would cancel out, removing the vortex disordering effect on large loops.

Thus, if

a) P-vortices locate thick vortices (evidence is vortex-limited loops), and

b) Vortex disorder is confining disorder (evidence is center dominance), then

removing vortices in this way should also remove the asymptotic string tension.



β=2.3, 16^4-lattice
Bali et al.
full theory
DLCG
vortices removed
χ(R,R)
R
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6

Scaling of the P-Vortex Density (Tuebingen group)

If center vortices are physical objects, it makes sense that their density in the vacuum (vortex area/volume) is lattice-spacing independent in the continuum limit. If P-vortices lie in the middle of center vortices, it would follow (there are some caveats) that P-vortex density is lattice-spacing independent. Let

N_{vor} = total no. of P-plaquettes = total P-vortex area (lattice units)
N_T = total no. of plaquettes = total lattice volume X 6

Then the density of P-plaquettes **p** is related to the vortex density in physical units, ρ, via

$$
\begin{aligned}
p &= \frac{N_{vor}}{N_T} = \frac{N_{vor}a^2}{N_T a^4} a^2 \\
&= \frac{\text{Total Vortex Area}}{6 \times \text{Total Lattice Volume}} a^2 \\
&= \frac{1}{6} \rho a^2
\end{aligned}
$$

If ρ is a physical quantity (i.e. β-independent), then we can substitute the asymptotic freedom expression for lattice spacing **$a(\beta,\Lambda)$** to get

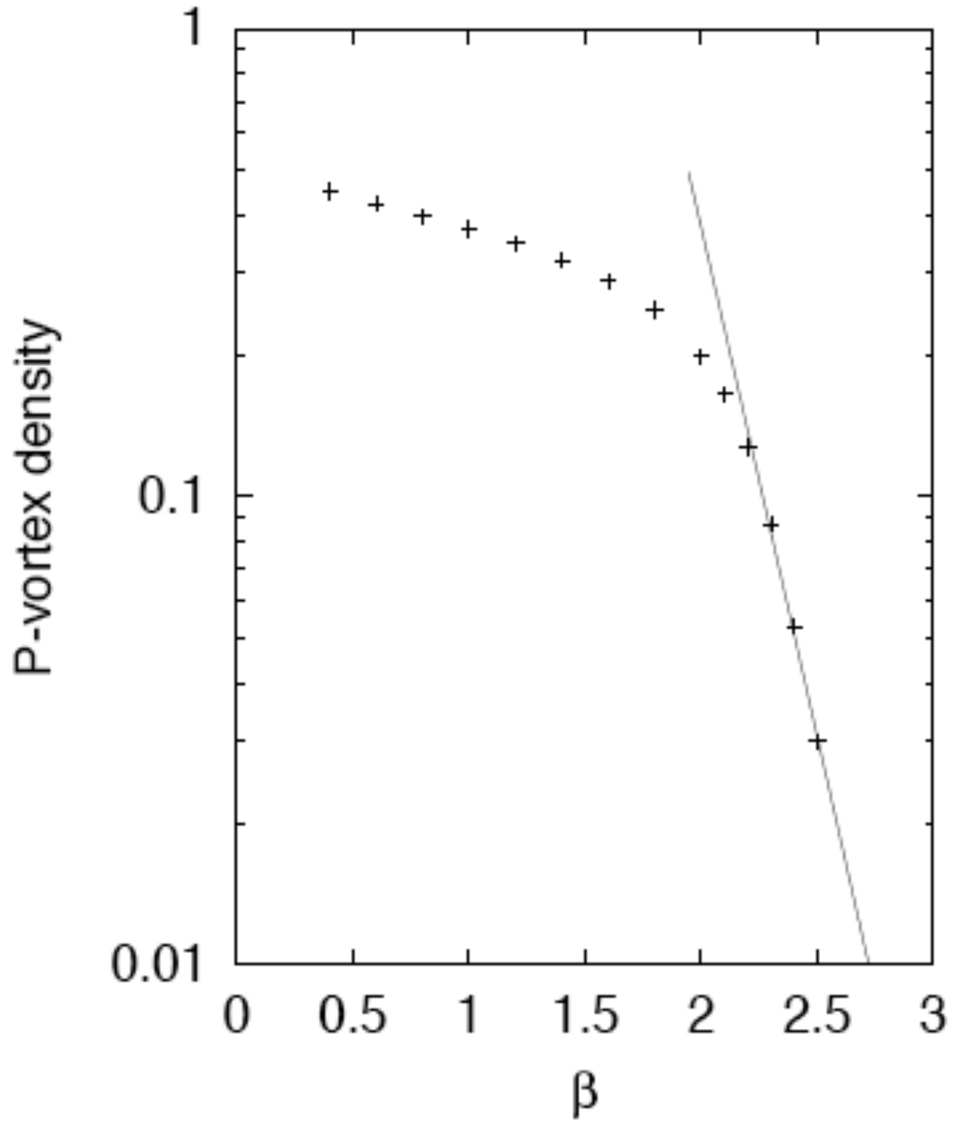
$$p = \frac{\rho}{6\Lambda^2}\left(\frac{6\pi^2}{11}\beta\right)^{102/121} \exp\left[-\frac{6\pi^2}{11}\beta\right]$$

The average value of **p** is obtained from center-projected plaquettes, because

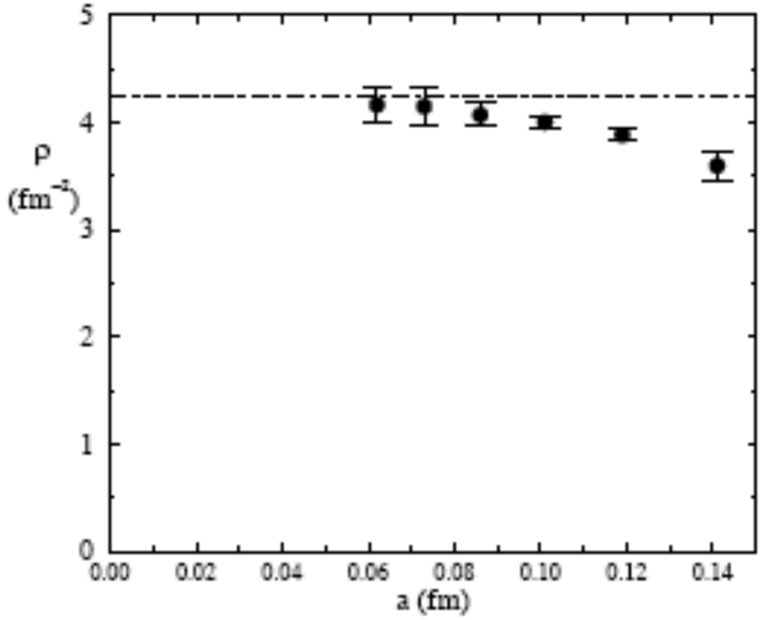
$$\langle Z(1,1)\rangle = (1-p) + p \times (-1) = 1 - 2p$$

The solid line is the asymptotic freedom prediction, with

$$\sqrt{\rho/6\Lambda^2} = 50$$



Here are related results of **Gubarev et al.**, in the IMC gauge



Finite Temperature (Tuebingen group)

At a temperature T_c=220 Mev, Yang-Mills theory goes through a "deconfinement" transition, where hadrons dissolve into their constitutents. On the lattice, finite temperature is represented by time-asymmetric lattices, with temperature proportional to $1/L_t$.

One can show numerically that at T>T_c, the quark free energy measured by the Polyakov line becomes finite.

Yet, the vacuum of the "deconfined" phase in not exactly non-confining. It has also been shown that spacelike Wilson loops (a measure of vacuum fluctuations) retain an area law and asymptotic string tension beyond the phase transition, even though the static quark potential measured by Polyakov line correlators goes flat.

A theory of confinement must be consistent with ***both*** features at T>T_c

- non-zero Polyakov lines
- non-zero string tension for spacelike Wilson loops

How do center vortices fit in? At zero temperature:

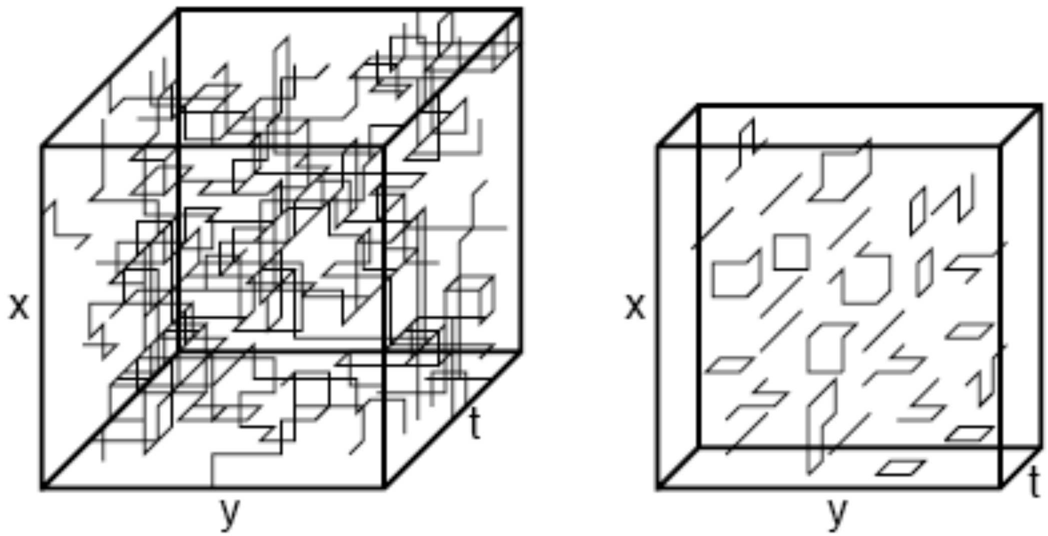
Vortex conf. mechanism → uncorrelated piercings of minimal surface area

→ extension of vortex is order of lattice size (else piercings of a large loop are paired)

→ vortices ***percolate*** through the lattice

Above T_c , the picture must be that vortices percolate in a time-slice (fixed t), so that spacelike Wilson loops have an area law, but cease to percolate in a space-slice (e.g. fixed z), so that Polyakov line correlators do not fall off exponentially with distance.

Schematically, here is what we expect on a space-slice (constant z).
Projection of a surface in 4D becomes closed lines in 3D.



confined phase

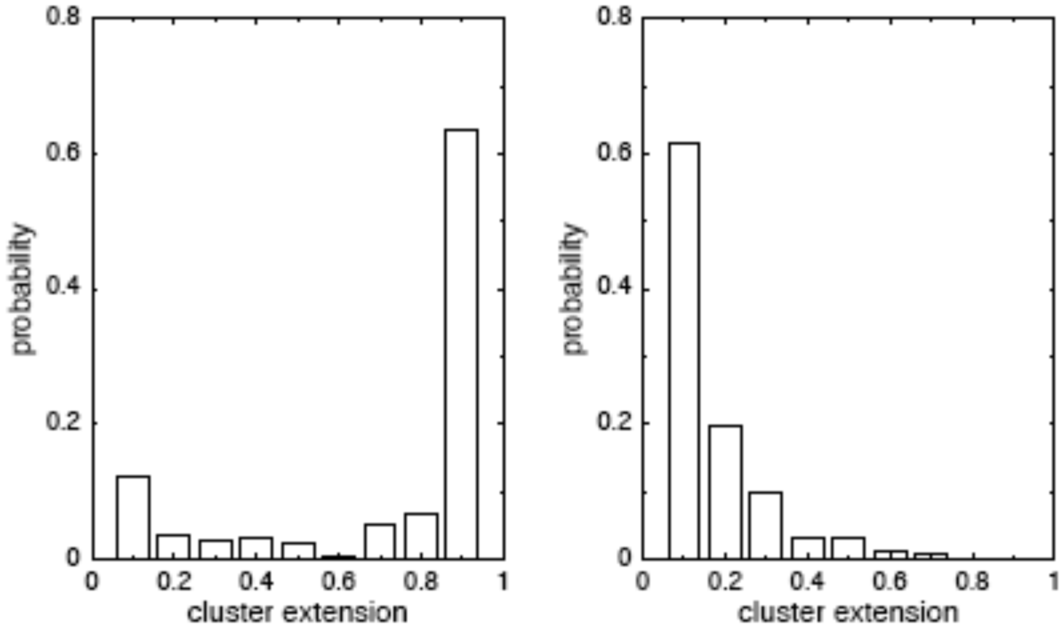
deconfined phase

percolation

no percolation

Note loops closed through periodicity in the (small) time direction.

So here is some actual data for a space-slice at finite temperature. The x-axis is in units of the maximal extension in the $L^2 L_t$ 3-volume. The y-axis is the percentage of P-vortex plaquettes belonging to a loop of a given extension.

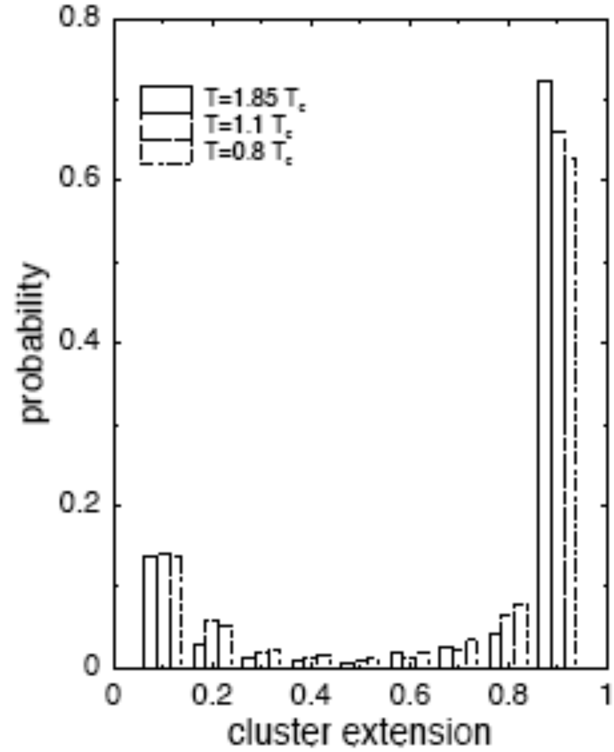


(Tuebingen)

Figure 22: Histograms of vortex extension in a space-slice at finite temperature, both below (left figure, $T = 0.7T_c$) and above (right figure, $T = 1.85T_c$) the deconfinement phase transition. The data is for $\beta = 2.4$ on a $12^3 \times 7$ (confined) and $12^3 \times 3$ (deconfined) lattices; center projection in DMC gauge. From Engelhardt et al., ref. [87]

No percolation at high T, no confinement.

Now for the same data on a time-slice, above and below T_c.



Percolation at all temperatures, spacelike string tension in both the confined and “deconfined” phase!

Chiral Condensates

Chiral symmetry breaking is associated with a non-zero VEV (chiral condensate)

$$\langle \overline{\psi}\psi \rangle \neq 0$$

which, for unbroken chiral symmetry, would necessarily vanish. According to the celebrated Banks-Casher formula, the finite value of the chiral condensate is directly related to a finite density of near-zero eigenvalues of the Dirac operator

$$\langle \overline{\psi}\psi \rangle = \pi \frac{\rho(0)}{V}$$

$\rho(\lambda)$ = density of eigenvalues of the Dirac operator

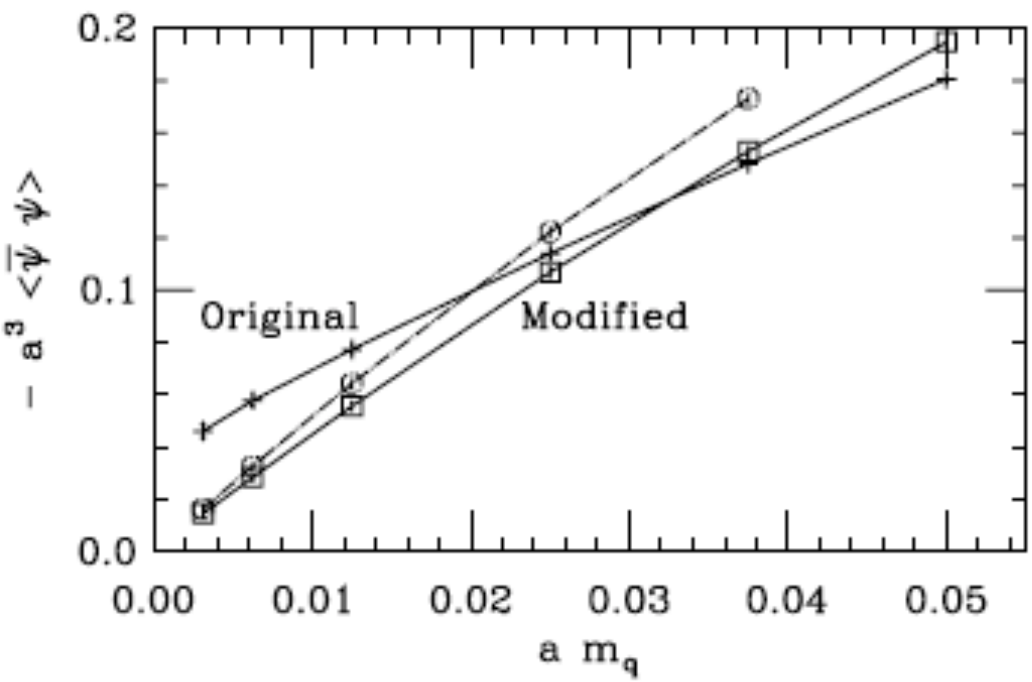
What happens if vortices are removed? It was found by de Forcrand and D'Elia that

a) chiral symmetry goes away;

b) The total topological charge of the configuration is reset to zero.

Here is the chiral condensate data:

(de Forcrand and D'Elia)

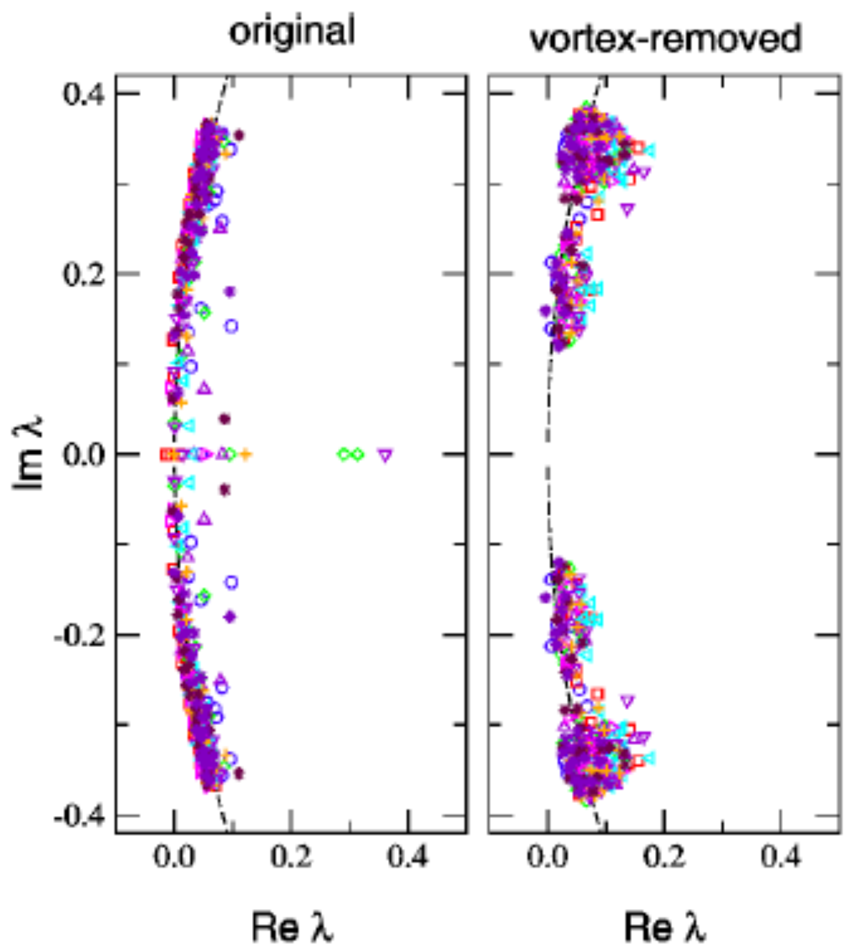


"Modified" is the vortex-removed data.

Banks-Casher formula:

$$\langle \overline{\psi}\psi \rangle = \pi \frac{\rho(0)}{V}$$

where V is the lattice volume, and $\rho(\lambda)$ is the density of eigenvalues of the Dirac operator. **Gattnar et al.** have calculated the low-lying eigenvalues of a certain “chirally improved” Dirac operator, and what they find is that removing vortices send $\rho(0)$! 0.



The fact that topological charge is generally non-zero for full configurations, U=ZV, but vanishes when vortices are removed, U=V, suggests that the thin vortex degrees of freedom Z are crucial in some way. Topological charge is defined in the continuum as

$$Q = \frac{1}{32\pi^2} \int d^4x \; \epsilon_{\mu\nu\alpha\beta} \mathrm{Tr}[F_{\mu\nu} F_{\alpha\beta}]$$

On a thin vortex surface, topological charge density can arise at sites on the surface where the tangent vectors to the surface are in all four space-time directions. These sites are of two sorts

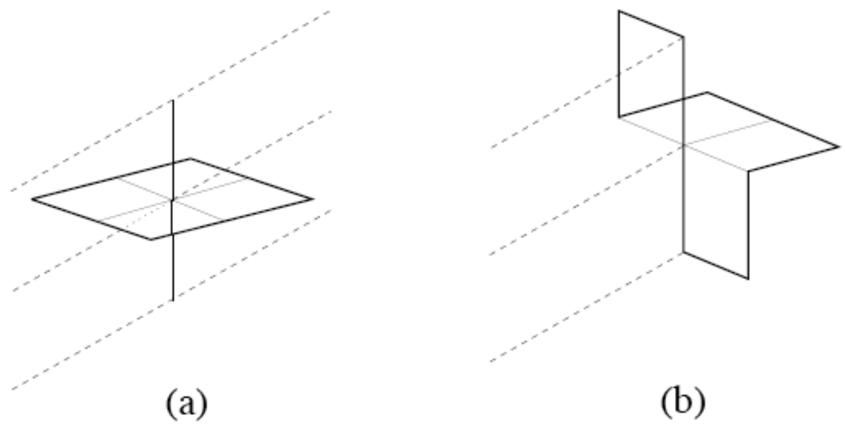
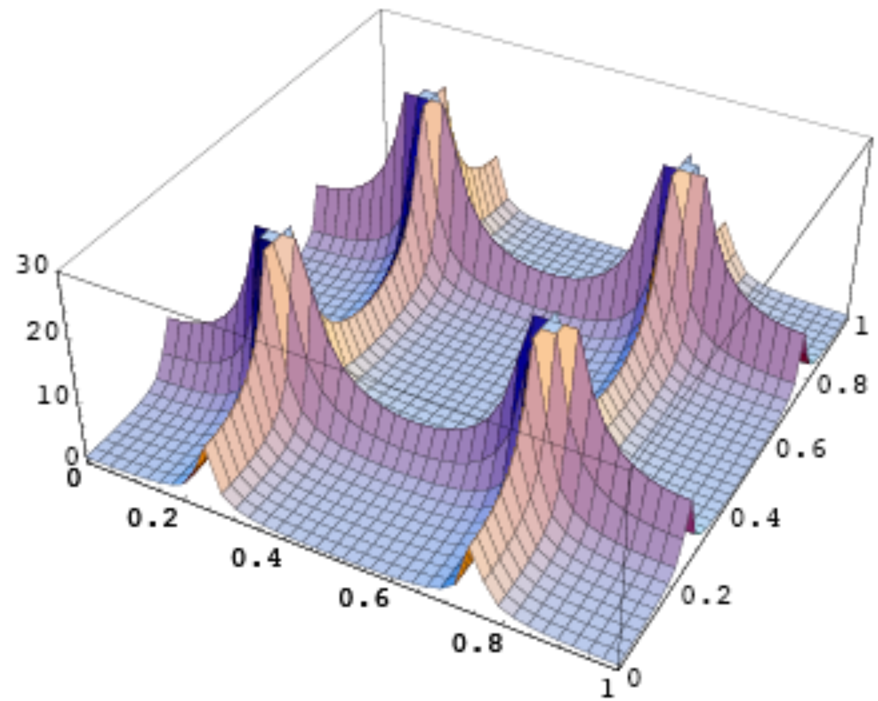


Figure 25: Intersection points (a) and writhing points (b) which contribute to the topological charge of a P-vortex surface. From Reinhardt, ref. [93].

One can also show that zero modes of the Dirac operator tend to peak at these intersection and writhing points.

(Teubingen)



This plot shows the modulus of Dirac zero modes in a background of intersecting vortex sheets.

Casimir Scaling and Vortex Thickness (Faber et al)

Although the asymptotic string tension only depends on N-ality, so that for SU(2)

$$\sigma_j = \sigma_{1/2} \qquad \text{j=half-integer}$$
$$\sigma_j = 0 \qquad \text{j=integer}$$

there is still an intermediate range of distances where Casmir scaling applies (at least approximately), i.e. for SU(2)

$$\sigma_j / (1/2)\, j(j+1)$$

How do vortices fit in, since they are motivated by (and ***seem*** to only give rise to) N-ality dependence?

In fact, j=integer Wilson loops are only unaffected by ***thin*** center vortices, as already noted for Precocious Linearity.

Thick center vortices can affect j=integer Wilson loops if the vortex "core" overlaps the loop.

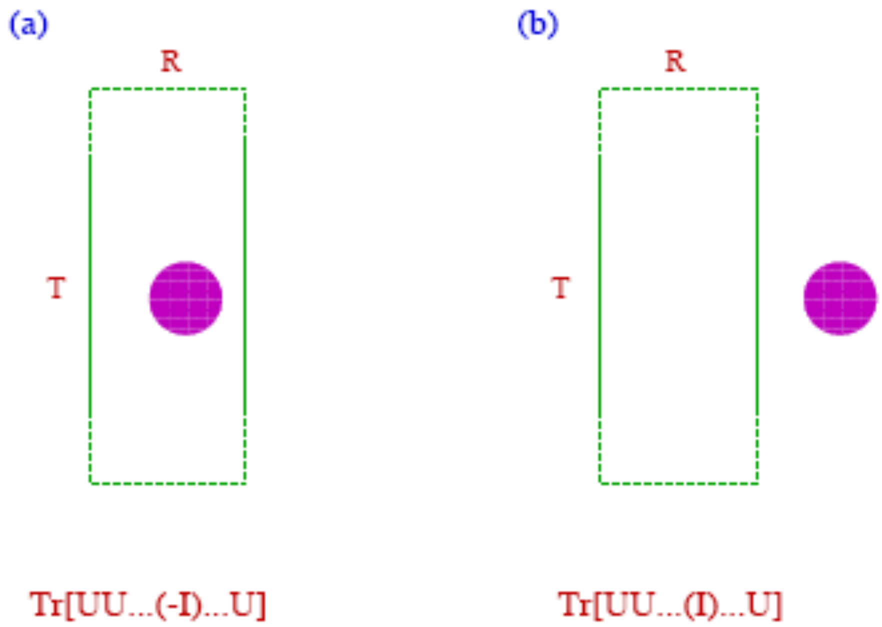
How thick are center vortices? From three different arguments:

1. Adjoint string-breaking at around 1.25 fm
2. $W_1/W_0 \longrightarrow -1$ for LxL loops, at L around 1 fm
3. Vortex free energy $\rightarrow$ 0 for lattices of extension beyond 1 fm

We can estimate the thickness of center vortices to be around one fm.

Not so small! Does this make a difference, for Wilson loops of extension less than one fm?

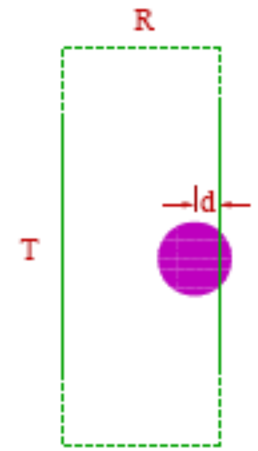
If the core doesn't overlap the loop, the effect is multiplication by a center element.



What happens if the core ***does*** overlap the loop?

An (over)simple model:

If the vortex core overlaps the loop perimeter, we represent its effect as multiplication by a group element G, as in an abelian theory, which smoothly interpolates from -I to I.



$$G = S \exp[i\alpha_C(x)\sigma_3/2]S^\dagger \qquad 0 \le \alpha_C(x) \le 2\pi$$

In our simple model, we assume S to be a ***random*** SU(2) group element.

If we then consider RxT Wilson loops, T À R, in group representation j, the model predicts

$$
\begin{aligned}
V_j(R) &= -\sum_{n=-\infty}^{\infty} \ln\{(1-f) + f\mathcal{G}_j[\alpha_R(x_n)]\} \\
\mathcal{G}_j[\alpha] &= \frac{1}{2j+1}\sum_{m=-j}^{j} \cos(\alpha m)
\end{aligned}
$$

where **f** is the probability for the middle of a vortex to pierce any given plaquette, and $\mathbf{x_n=n+1/2}$ is the coordinate in the R-direction. For large loops, where most piercings don't overlap the loop, we get the prediction

$\sigma_k = -\ln(1-2f)$ k=half-integer

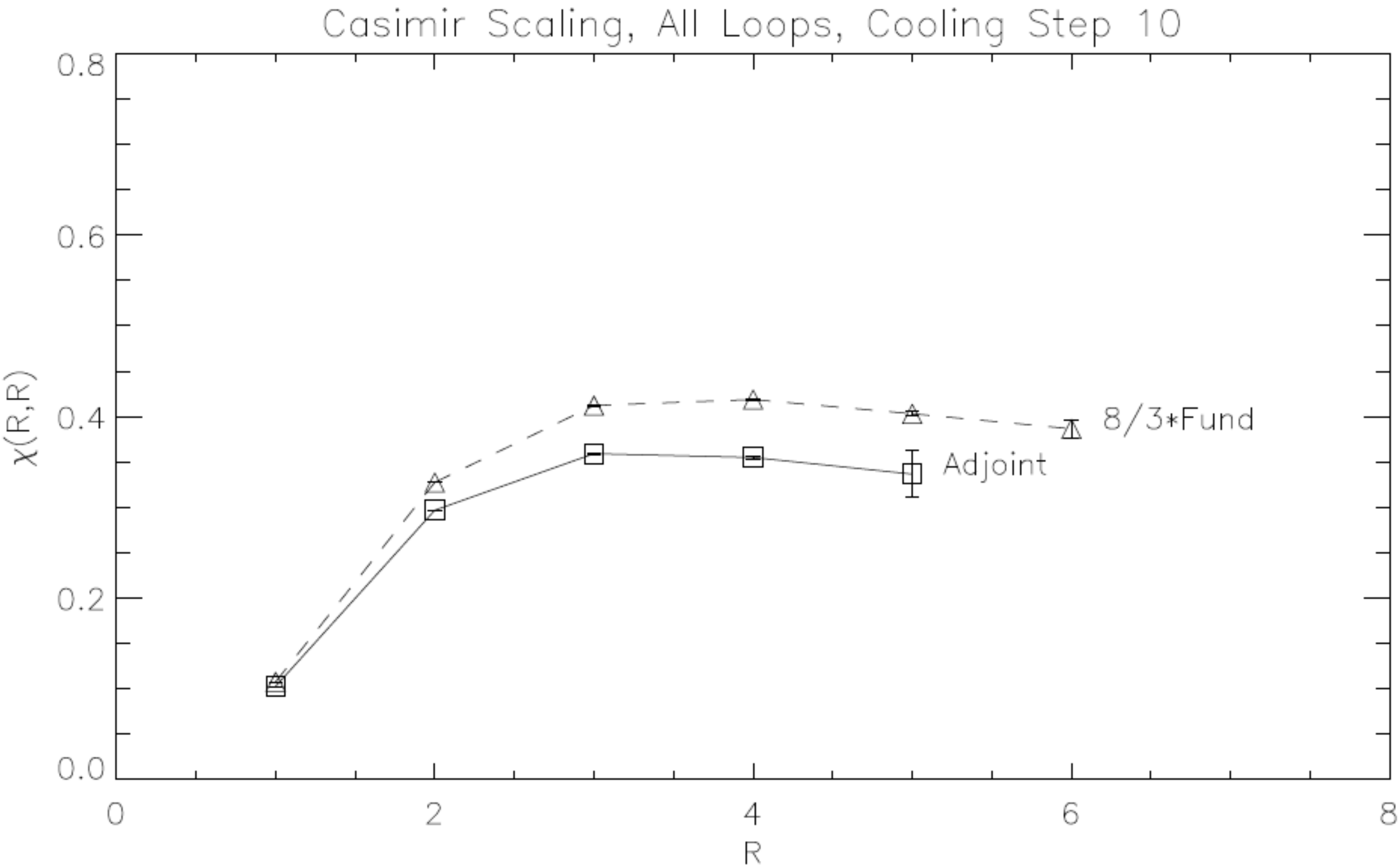
$\sigma_k = 0$ k=integer

However, for R very small compared to vortex thickness, so that
$\alpha_R(x)$ ¿ 2π , we find instead

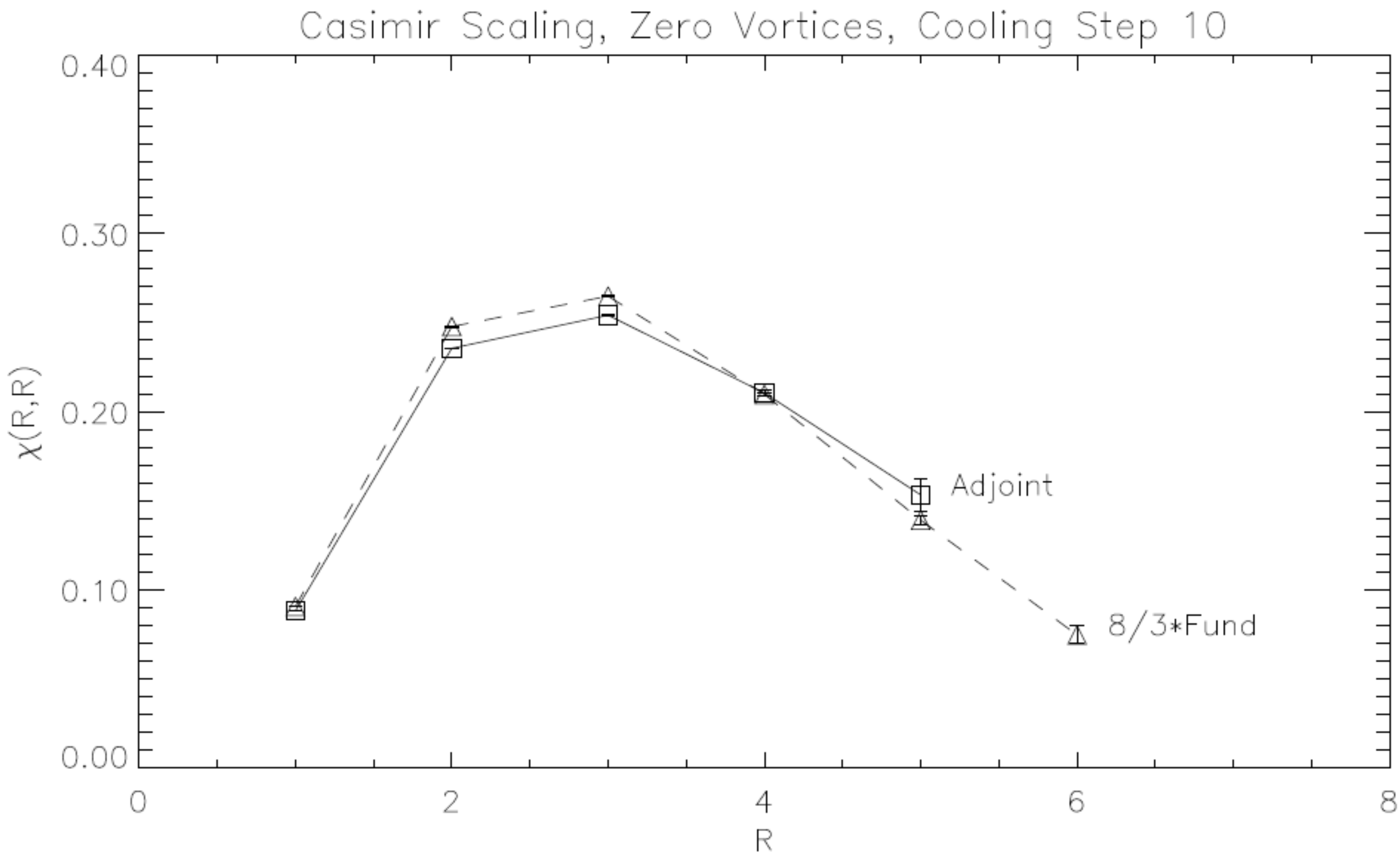
$$V_j(R) = \left\{\frac{f}{6}\sum_{n=-\infty}^{\infty} \alpha_R^2(x_n)\right\} j(j+1)$$

which is proportional to the quadratic Casimir. Whether this potential Is also linear depends on assumptions which are made about the precise x-dependence of $\alpha_R(x)$. Most simple choices give approximate linearity, and approximate Casimir scaling, over some intermediate range of distances.

Numerical evidence: Casimir scaling, no vortices removed



And with vortices removed…



Vortices and Matter Fields

The principal motivation for the center vortex confinement mechanism is the fact that the existence of an asymptotic string tension is ***always*** associated with a global center symmetry.

But in real QCD, the global center symmetry is broken by quark fields in the fundamental representation of the gauge group. So what happens to center vortices?

Possibilities:

- Vortices don't exist, or are irrelevant to the static potential, for even the tiniest explicit breaking of center symmetry (e.g. very large but not infinite quark mass).

- Vortices exist but cease to percolate. They break into clusters of fixed average extension, independent of lattice size.

- Vortices continue to percolate (perhaps as branched polymers on large scales), and are crucial to the linear potential up to the "string-breaking" scale.

Instead of quark fields, its easier to introduce a Higgs field in the fundamental representation of SU(2)

$$S = S_W + \sum_x \{\phi^\dagger(x)\phi(x) + \lambda[\phi^\dagger(x)\phi(x) - 1]^2\}$$

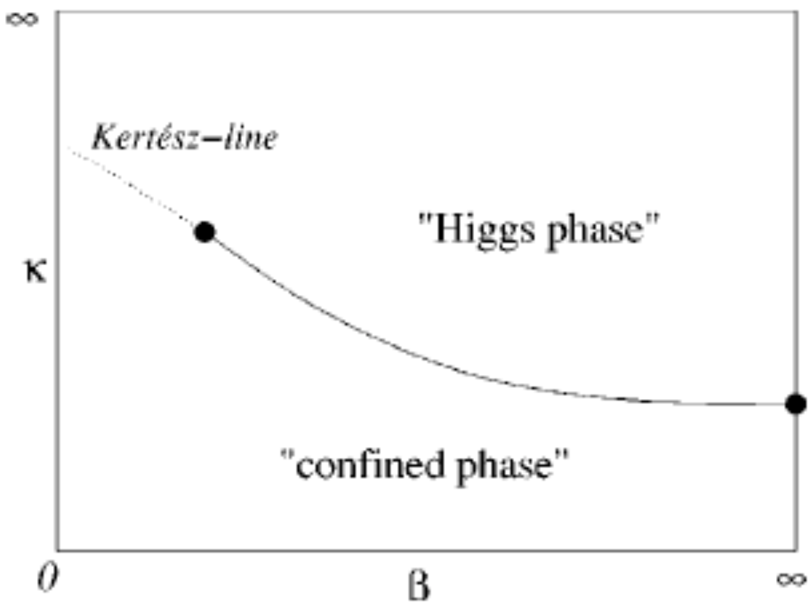
$$-\kappa \sum_{x,\mu} [\phi^\dagger(x) U_\mu(x) \phi(x+\mu) + \text{c.c}]$$

where S_W is the usual Wilson one-plaquette action.

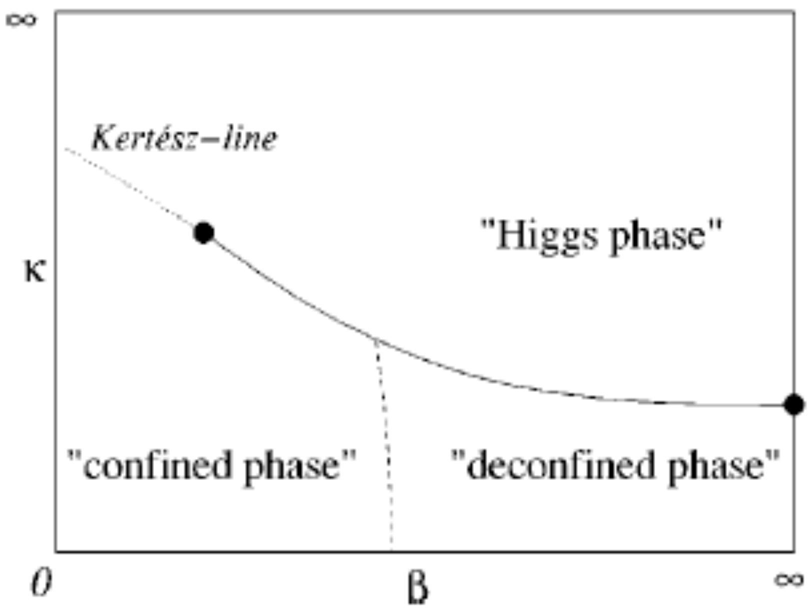
This theory has a "confinement-like" region, where there is a linear potential up to some string-breaking distance, and a Higgs-like region, where there is no linear potential at all.

Fradkin-Shenker Theorem: There is no thermodynamic phase separation between the confinement-like and Higgs-like regions.

Schematic Phase Diagrams at fixed λ



Zero temperature



Finite temperature

We have worked at β**=2.2** in the λ**=1** limit, where **|ϕ|=1**, and also at λ**=0.5**.

At λ**=1** , β**=2.2** , the first order transition line is at κ**=0.22** . We will work in the “confinement-like” region, just before the transition, at κ=0.20 .

Once again, fix to DMC gauge, center project, and make the usual tests. We find the usual effects:

1. Center Dominance σ_{cp} ¼ σ

2. W_1/W_0 ! -1

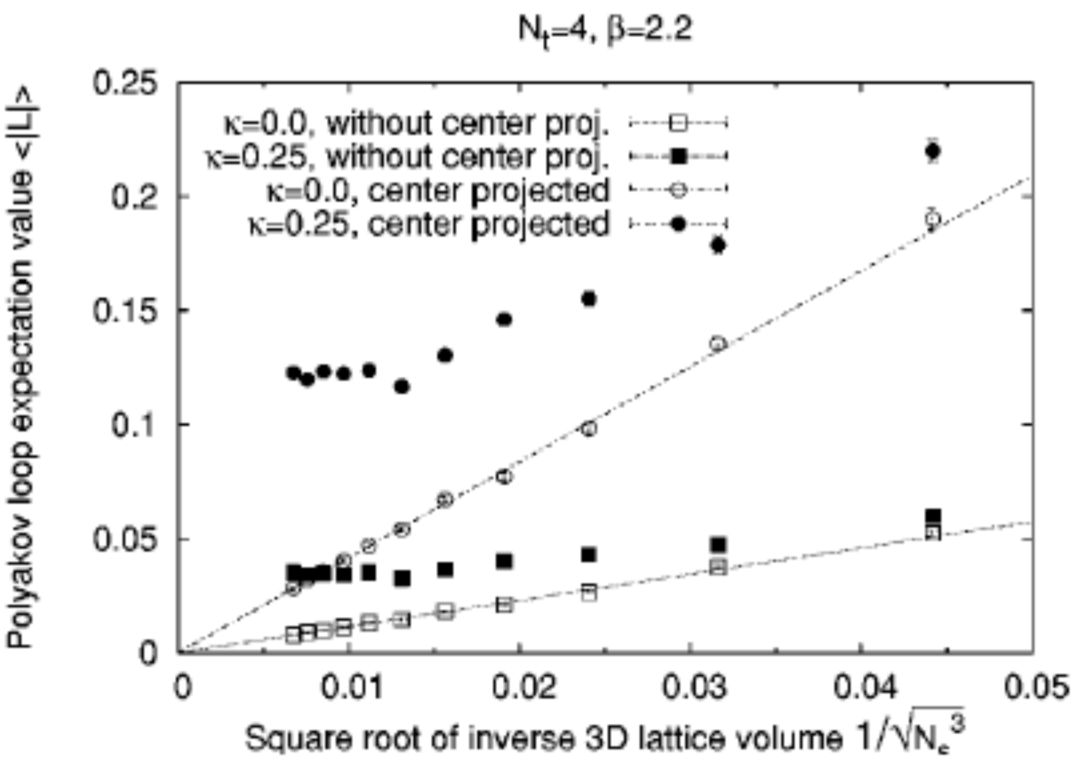
3. σ ! 0 for vortex-removed loops

In the Higgs region, above κ=0.22, we find that $\sigma_{\mathbf{cp}}$ **¼** σ **¼ 0**.

But what about screening/string-breaking due to matter fields in the confinement-like region? Do the vortices see that too?

It not easy to spot string-breaking with Wilson loops, even center-projected Wilson loops. Instead, we look at Polyakov lines in the finite T theory, below the high-temperature “deconfinement” transition.

This calculation was done at λ=0.5 .



For κ 0, Polyakov   0 for projected and unprojected loops.

This means that the vortex ensemble “sees” string breaking by the matter field.

Since vortices get so many features right for the gauge-matter system, we would like to know what happens to the vortex ensemble as we go from the confinement-like region to the Higgs-like region without crossing the 1st-order transition line.

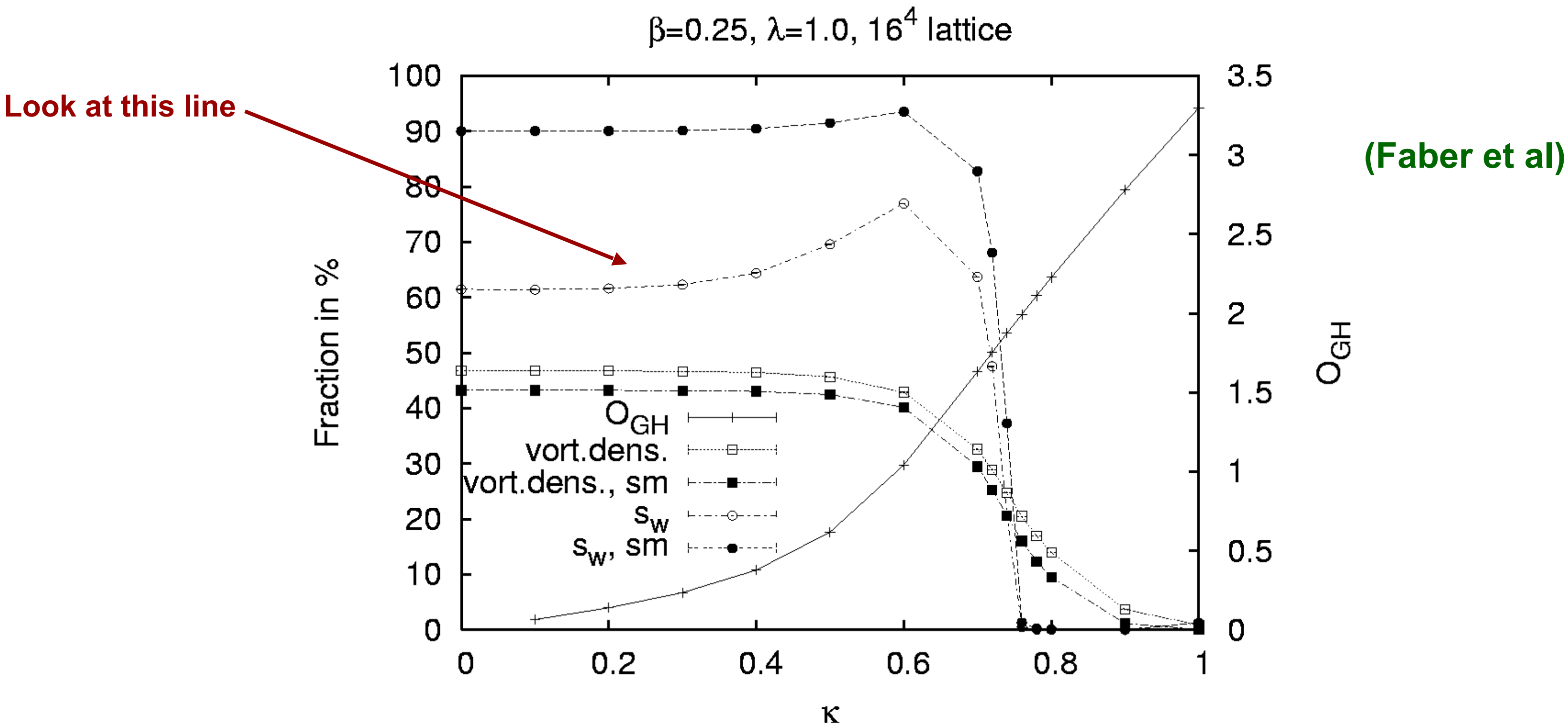
Let **f[p]** be the fraction of the total number of P-plaquettes, carried by the vortex containing P-plaquette p.

We define $\mathbf{s_w}$ as **f[p]** averaged over all P-plaquettes. This is the fraction of the total number of P-plaquettes contained in the “average” P-vortex.

$s_w = 1$ if there is only one percolating cluster

$s_w = 0$ if there is no percolation (infinite volume limit)

If $\mathbf{s_w}$ is non-zero, it means that the size of the average vortex grows with lattice size, typical of percolation.



The calculation was carried out for a variable-modulus Higgs field with quartic self-coupling **λ=1**, and κ is the gauge-Higgs coupling. The sudden drop in s_w indicates the transition to the non-percolating phase.

Conclusion

The confinement-Higgs transition for the SU(2) Higgs model can be understood as a vortex depercolation transition.

The operator s_w is highly non-local. There is no contradiction to the Fradkin-Shenker theorem.

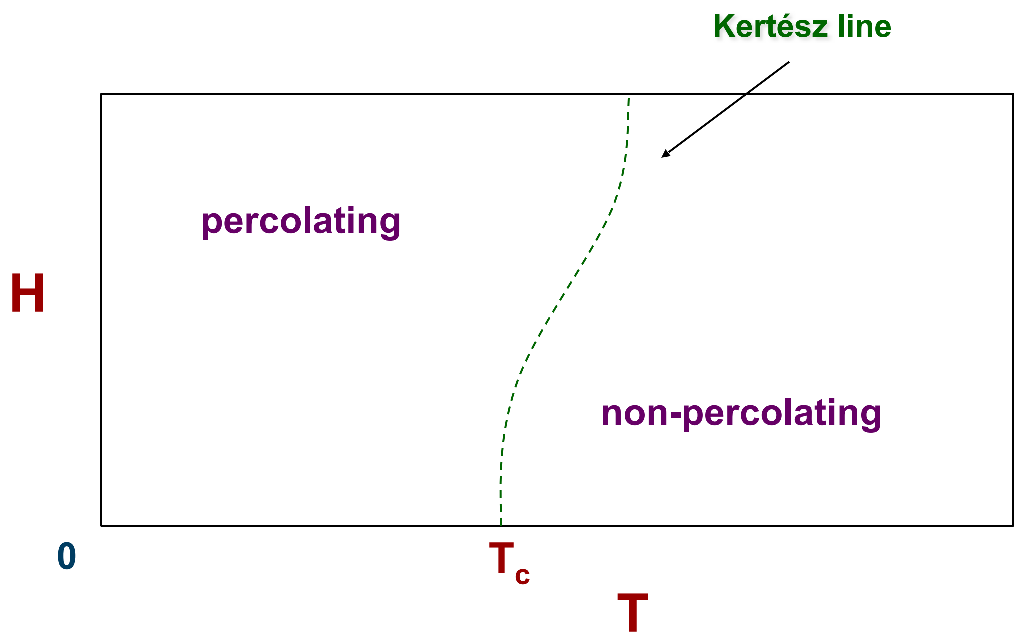
The depercolation transition line, which is not necessarily a line of thermodynamic transitions, is known in stat mech as a **Kertesz line**.

The Kertész line

How can there be ***any*** change of phase, in the gauge-fundamental Higgs theory, in the absence of any non-analyticity in the free energy?

This question has come up before, in the context of the Ising model.

For external magnetic field H>0, the free energy is analytic. But the Ising model can be reformulated in terms of clusters of connected sites which may or may not percolate. There exists a sharp line of ***percolation*** transitions – known as the **Kertész line** – separating the high and low temperature phases.



Weak Points

- Gribov copy problem (average over all copies? Pick a “best” copy?)
- Origin of the Luscher term?
- SU(3)

1. $W_1/W_0 \rightarrow \exp[i2\pi/3]$ **Good!**
2. Vortex removal: σ **! 0** **Good!**
3. **σ_{cp} ¼ (2/3) σ** **Not so good….**

Part VI: Connections to other ideas

- Monopole confinement:

't Hooft's abelian-projection scenario

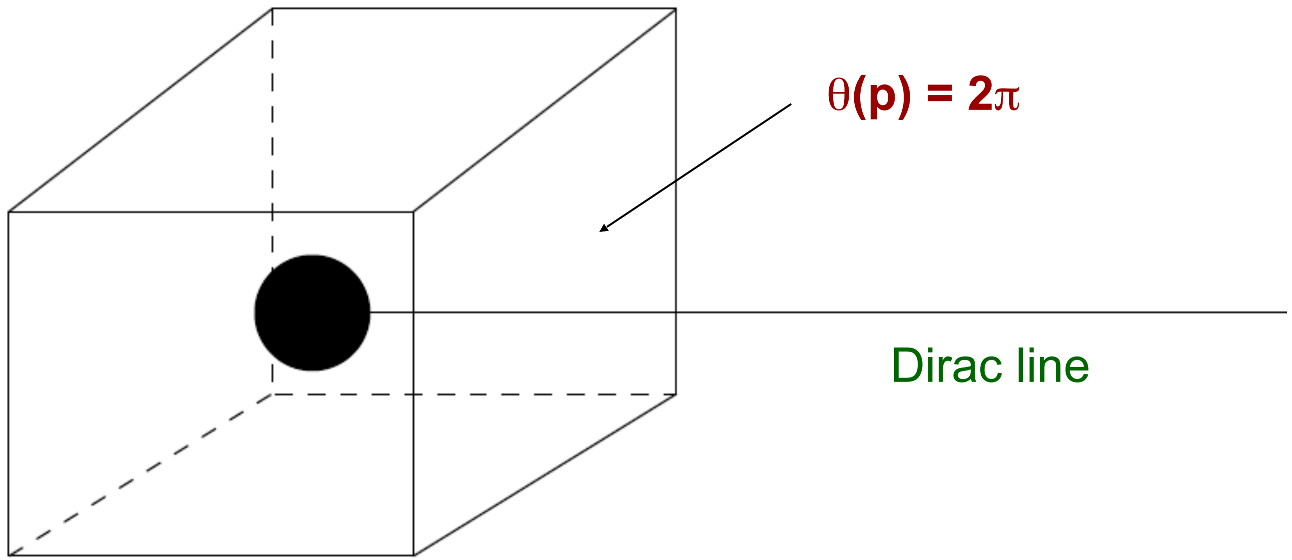
- the Gribov-Zwanziger scenario:

confinement by one-gluon exchange in Coulomb gauge

Confinement in Compact QED_3

Compact QED has monopole as well as photon excitations

$$U(p) = \exp[i\theta(p)]$$



Details: monopole currents are identified by the DeGrand-Toussaint criterion:

$$k_\mu(x) = \frac{1}{4\pi}\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha \overline{f}_{\beta\gamma}(x)$$

where

$$\begin{aligned} f_{\mu\nu}(x) &= \partial_\mu\theta_\nu(x) - \partial_\nu\theta_\mu(x) \\ &= \overline{f}_{\mu\nu}(x) + 2\pi n_{\mu\nu}(x) \end{aligned}$$

Then one constructs “monopole dominance” link variables

$$\begin{aligned} u_\mu^{mon}(x) &= \exp[i\theta_\mu^{mon}(x)] \\ \theta_\mu^{mon}(x) &= -\sum_y D(x-y)\partial'_\nu n_{\mu\nu}(y) \end{aligned}$$

neglects the photon contributions

Where D(x-y) is the lattice Coulomb propagator.

Polyakov was able to show that in D=3 dimensions, compact QED_3 is equivalent to the partition function of a monopole plasma. Its possible to change variables from links to integer-valued monopole variables m[r], living at sites on the “dual” lattice

$$Z_{mon} = \sum_{m(r)=-\infty}^{\infty} \exp[-\frac{2\pi^2}{g^2 a} \sum_{r,r'} m(r')G(r-r')m(r)]$$

where G(r) » 1/r . Then one adds a Wilson loop source exp[is dr$^{\mu}$ A$_{\mu}$] to the partition function, where

$$\oint_C dr_\mu A_\mu(r) = \int_{S(C)} dS_\mu(r)\ H_\mu(r) = \int d^3 r\ \eta_{S(C)}(r)\ m(r),$$

where

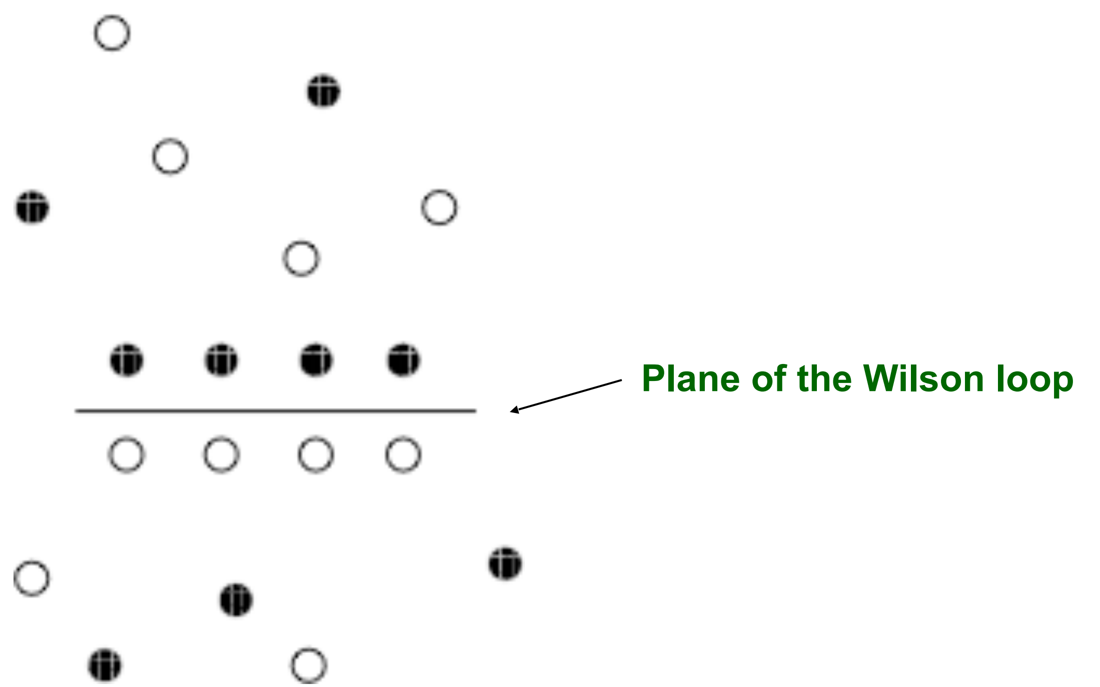
$$\eta_{S(C)}(r) = -\frac{1}{2}\frac{\partial}{\partial r_\mu}\int_{S(C)} dS_\mu(r')\ \frac{1}{|r-r'|}.$$

Everything can be calculated explicitly in D=3 dimensions, and the result is that for a Wilson loop of charge n

$$\langle U_n(C)\rangle \approx \exp[-n\sigma \ \mathrm{area}(C)]$$

Where σ is a function of coupling g and monopole mass (» 1/a).

A very rough image of whats going on: monopoles and antimonopoles line up along the minimal area, and screen out the magnetic field that would be generated by the Wilson (current) loop source



The Abelian Projection

Just as the center of a group is the set of which commutes with all group elements, one can also identify a **Cartan subalgebra**, formed by the maximum number of commuting group generators, and these generate the **Cartan subgroup**.

For example, the generators of SU(2) are the three Pauli matrices. The U(1) subgroup generated by any one (or linear combination of) Pauli matrices can be taken as the Cartan subgroup. For SU(3), one could choose, e.g., the third component of isospin I_3, and hypercharge Y, forming the subgroup U(1)xU(1).

In general, for any SU(N) group, the Cartan subgroup is $U(1)^{N-1}$.

‘t Hooft’s idea - choose a gauge which leaves the Cartan subalgebra unbroken. For example, one could pick a gauge where F_{12} is a diagonal matrix. Then in this gauge the theory can be thought of as an abelian $U(1)^{N-1}$ theory (photons and monopoles), interacting with charged matter (the other gluons). Then confinement is due to monopole plasma/condensation, just as in compact QED_3.

How can we test if abelian monopoles, identified in an abelian projection gauge, carries the information about confinement?

Does any abelian projection gauge work?

Not every abelian projection gauge works. One which works pretty well is the **maximal abelian gauge**, which requires that link variables are as diagonal as possible. In SU(2), the condition is that

$$R = \sum_{x,\mu} \mathrm{Tr}[U_\mu(x)\sigma_3 U_\mu^\dagger(x)\sigma_3] \qquad \text{is a maximum}$$

which allows the residual U(1) gauge symmetry

$$U_\mu(x) \rightarrow e^{i\phi(x)\sigma_3} U_\mu(x) e^{-i\phi(x+\widehat{\mu})\sigma_3}$$

Let $u_\mu(x)$ be the diagonal part of $U_\mu(x)$, rescaled to restore unitarity

$$U_\mu(x) = a_0 I + i\vec{a}\cdot\vec{\sigma}$$

$$\begin{aligned} u_\mu(x) &= \frac{1}{\sqrt{a_0^2 + a_3^2}}[a_0 I + i a_3 \sigma_3] \\ &= \begin{bmatrix} e^{i\theta_\mu(x)} & \\ & e^{-i\theta_\mu(x)} \end{bmatrix} \end{aligned}$$

Decompose

$$\begin{aligned} U_\mu(x) &= C_\mu u_\mu(x) \\ &= \begin{bmatrix} (1-|c_\mu(x)|^2)^{1/2} & c_\mu(x) \\ -c_\mu^*(x) & (1-|c_\mu(x)|^2)^{1/2} \end{bmatrix} \begin{bmatrix} e^{i\theta_\mu(x)} & \\ & e^{-i\theta_\mu(x)} \end{bmatrix} \end{aligned}$$

What is interesting is that under the remnant U(1) gauge symmetry, u_μ transforms like a gauge field, and C_μ transforms like a matter field

$$\begin{aligned} u_\mu(x) &\to e^{i\phi(x)\sigma_3} u_\mu(x) e^{-i\phi(x+\widehat{\mu})\sigma_3} \\ c_\mu(x) &\to e^{2i\phi(x)} c_\mu(x) \end{aligned}$$

Now the steps are as follows:

1. Identify monopole currents $k_\mu(x)$ from the abelian links

2. Find the gauge fields (link variables denoted u_μ^{mon}) due to those monopole current sources.

3. Compute Wilson loops from the abelian gauge fields u_μ^{mon}) derived from the monopole currents alone.

This works out pretty well, in the sense of getting string tensions about right for single charged (n=1) Wilson loops.

$$W_n^{ab}(C) = \langle [\mathrm{Tr}\, u^{mon}(C)]^n \rangle$$

But there is a big problem for n>1. It should be that, because of screening by the "charged" (off-diagonal) gluons

$$\sigma_n = \begin{cases} \sigma_1 & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

This has been checked for Polyakov loops.

Instead, the monopole-dominance approximation just gives the QED result

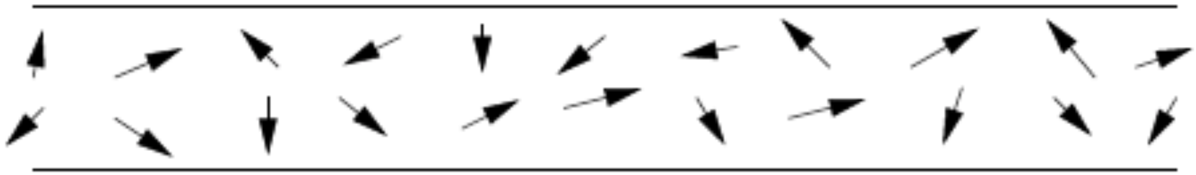
$$\sigma_n = n\sigma_1$$

Even if the confining disorder is dominated (in some gauge) by abelian configurations, the distribution of abelian flux ***cannot*** be that of a monopole Coulomb gas.

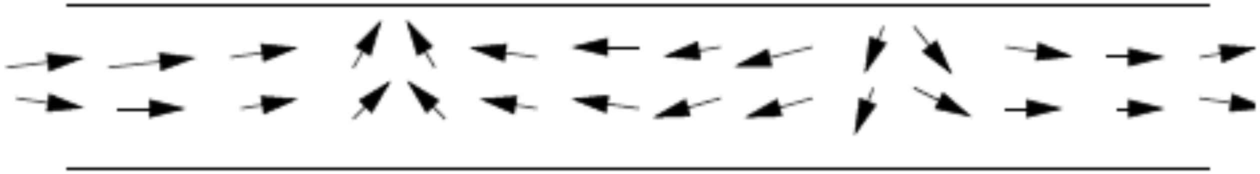
Still, the monopole projection does get some things right.

To see whats going on, lets think about what vortices would look like in maximal abelian gauge, at some fixed time t.

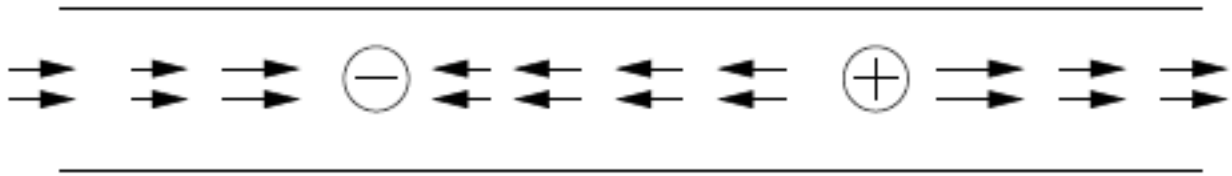
In the absence of gauge-fixing, the vortex field strength $F_{\mu\nu}^{a}$
Points in random directions in the Lie algebra



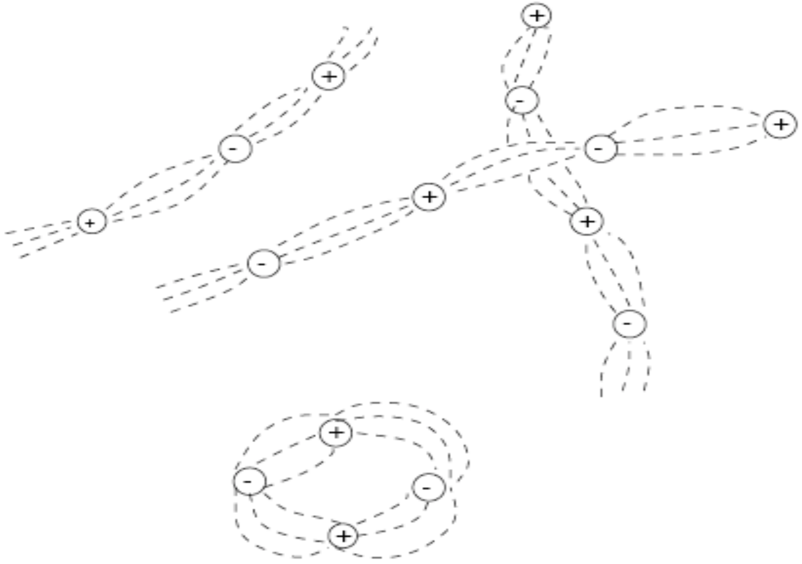
Fixing to maximal abelian gauge, the field tends to line up in the +/- σ_3 direction. But there will still be regions where the field strength rotates in group space, from $+\sigma_3$ to $-\sigma_3$



Now, if we keep only the abelian part of the link variables (“abelian projection”), we get a ***monopole-antimonopole chain***, with π flux running Between a monopole and neighboring antimonopole (total monopole flux is +/- 2π, as it should be.



Then a typical vacuum configuration at a fixed time, after abelian projection, looks something like this:



These configurations will ensure that

$$\sigma_n = \begin{cases} \sigma_1 & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

as it should.

Numerical Tests

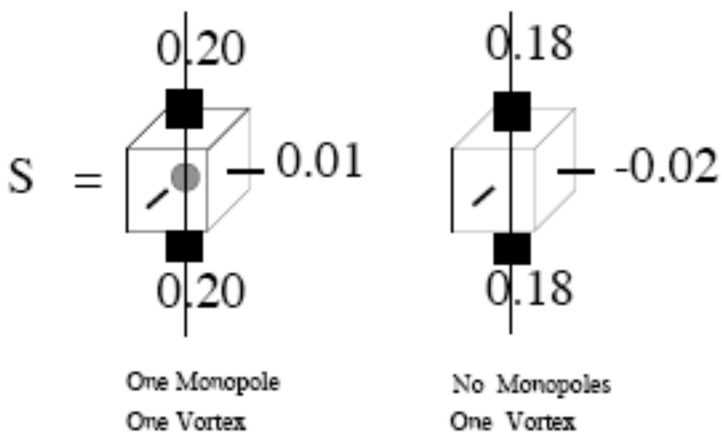
We work in IMC gauge, which uses maximal abelian gauge as an intermediate step. We identify the locations of both monopoles (by abelian projection) and vortices (by center projection). We also measure the excess action (above the average plaquette value S_0), on plaquettes Belonging to monopole "cubes", and on plaquettes pierced by vortex lines.

Results:

- Almost all monopoles and antimonopoles (97%) lie on vortex sheets.

- At fixed time, the monopoles and antimonopoles alternate on the vortex lines, in a chain.

- Excess (gauge-invariant) plaquetted action is highly directional, and lies mainly on plaquettes pierced by vortex lines. The presence or absence of a monopole is not so important to excess action.



Very similar results are obtained for 2- and 3-cubes surrounding monopoles.

Vortices and the Gribov Horizon

The Gribov-Zwanziger idea – confinement in Coulomb gauge is due to one-gluon exchange, with 0-0 propagator

$$\langle A_0^a(x) A_0^b(y) \rangle = D(\vec{x} - \vec{y}) \delta^{ab} \delta(x_0 - y_0) + \text{non-instantaneous}$$

where

$$D(\vec{x} - \vec{y}) \delta^{ab} = \left\langle \left[\frac{1}{\nabla \cdot D[A]} (-\nabla^2) \frac{1}{\nabla \cdot D[A]} \right]_{x,y}^{a,b} \right\rangle$$

$D_k[A]$ is a covariant derivative.

This quantity is directly related to Coulomb potential in Coulomb gauge.

We recall the classical Coulomb-gauge Hamiltonian $H = H_{glue} + H_{coul}$
where

$$
\begin{aligned}
H_{glue} &= \frac{1}{2}\int d^3x \; (E^{tr2} + B^2) \\
H_{coul} &= \frac{1}{2}\int d^3x d^3y \; \rho^a(x) K^{ab}(x,y) \rho^b(y) \\
K^{ab}(x,y;A) &= \left[\frac{1}{\nabla \cdot D(A)}(-\nabla^2)\frac{1}{\nabla \cdot D(A)}\right]^{ab}_{xy} \\
\rho^a &= \rho^a_m - g f^{abc} A^b_k E^c_k
\end{aligned}
$$

Note that **hKi** is the instantaneous piece of the **hA$_0$A$_0$i** gluon propagator.

Gribov and Zwanziger argue that **hKi** is enhanced by configurations on the Gribov Horizon, where **r ¢ D(A)** has zero eigenvalues, such that

$$\langle K(x,y;A)\rangle \sim \sigma |\vec{x} - \vec{y}|$$

Confinement from one-gluon exchange

Non-Perturbative Coulomb Potential

Let $$|\Psi_{qq}\rangle = \overline{q}(0)q(R)|\Psi_0\rangle$$

be a physical state with two static charges in Coulomb gauge. Then

$$\begin{aligned} \mathcal{E} &= \langle\Psi_{qq}|H|\Psi_{qq}\rangle - \langle\Psi_0|H|\Psi_0\rangle \\ &= V_{coul}(R) + E_{se} \end{aligned}$$

where the Coulomb potential $\mathbf{V_{coul}}$ comes from the non-local $\rho_{\mathbf{m}}\mathbf{K}\rho_{\mathbf{m}}$ term in the Hamiltonian.

Questions

- Is $\mathbf{V_{coul}(R)}$ confining?
- If confining, is it asymptotically linear?
- If linear, does $\sigma_{\mathbf{coul}} = \sigma$?
- What about center vortices? What happens to the Coulomb potential if vortices are removed?

To compute the Coulomb potential numerically, define the correlator, in Coulomb gauge, of two timelike Wilson lines (*not* Polyakov lines)

$$
\begin{aligned}
G(R,T) &= \langle \mathrm{Tr}[L^\dagger(0,T)L(R,T)]\rangle \\
&= \langle \Psi_{qq}|e^{-(H-\mathcal{E}_0)T}|\Psi_{qq}\rangle
\end{aligned}
$$

where

$$
L(\vec{x},T) = \exp\left[i\int_0^T dt\, A(\vec{x},t)\right]
$$

The existence of a transfer matrix implies

$$
G(R,T) = \sum_n \left|\langle \Psi_n|\Psi_{qq}\rangle\right|^2 e^{-E_n T}
$$

Denote

$$
V(R,T) = -\frac{d}{dT}\log[G(R,T)]
$$

Then its not hard to see that

$$\begin{aligned} \mathcal{E} &= V_{coul}(R) + E_{se} \\ &= V(R,0) \end{aligned}$$

while

$$\begin{aligned} \mathcal{E}_{min} &= V(R) + E'_{se} \\ &= \lim_{T\to\infty} V(R,T) \end{aligned}$$

where $\mathcal{E}_{min}$ is the minimum energy of the $q\overline{q}$ system, and V(R) is the (confining) static quark potential.

With lattice regularization, E'_{se} and E_{se} are negligible at large R, compared to V(R).

Then, since $\mathcal{E}_{min} \le \mathcal{E}$ it follows that

$$V(R) \le V_{coul}(R)$$ **(Zwanziger)**

So V_{coul}(R) confines if V(R) confines.

If confinement exists, it exists already at the level of one-gluon exchange.

Latticize

$$L(x,T) = U_0(x,a)U_0(x,2a)\cdots U_0(x,T)$$

$$V(R,T) = \frac{1}{a}\log\left[\frac{G(R,T)}{G(R,T+a)}\right]$$

then

$$\lim_{\beta\to\infty} V(R,0) \;=\; V_{coul}(R) \;+\; \text{const.}$$

$$\lim_{T\to\infty} V(R,T) \;=\; V(R) \;+\; \text{const.}$$

where

$$V(R,0) = -\frac{1}{a}\log[G(R,1)]$$

So now we can get an estimate (exact in the continuum) of $V_{coul}(R)$ from V(R,0), and compare to the static potential V(R).

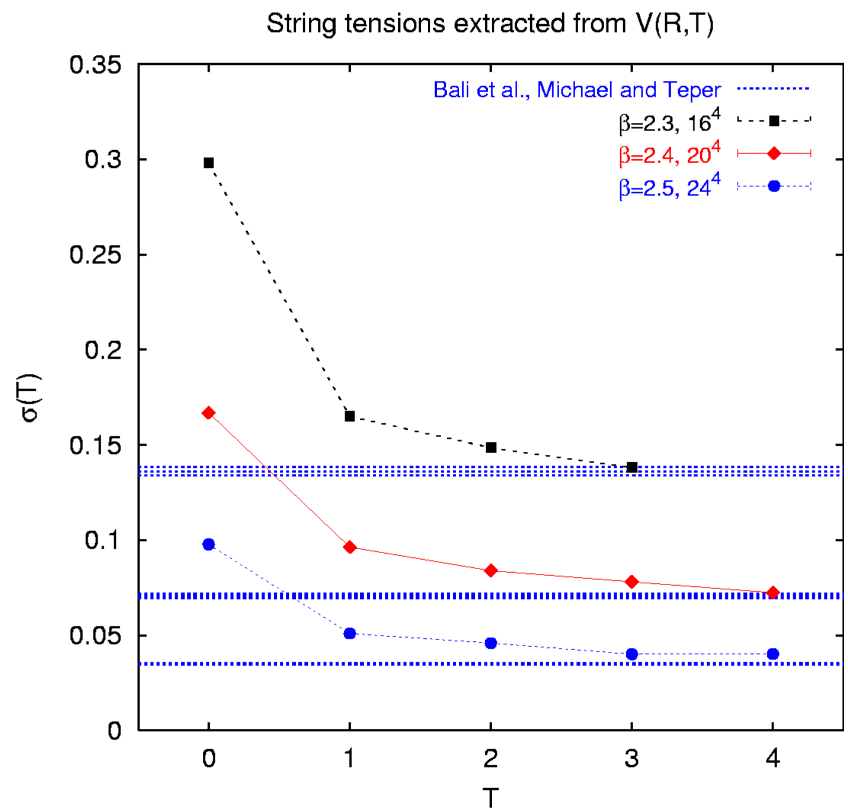
A Check

Define σ(T) from a fit of V(R,T) to

$$V(R,T) = c(T) - \frac{\pi}{12R} + \sigma(T)R$$

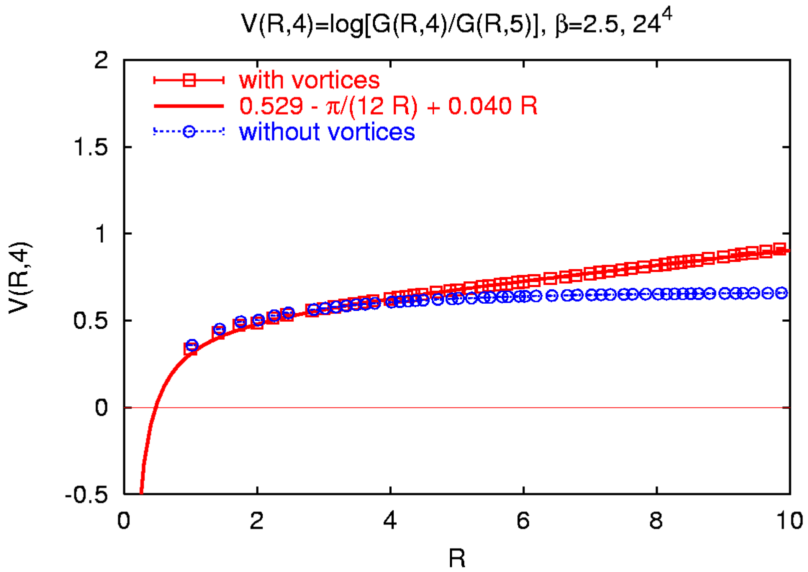
and check to see if

σ(T) ! σ as T ! 1



This seems to work out pretty well.

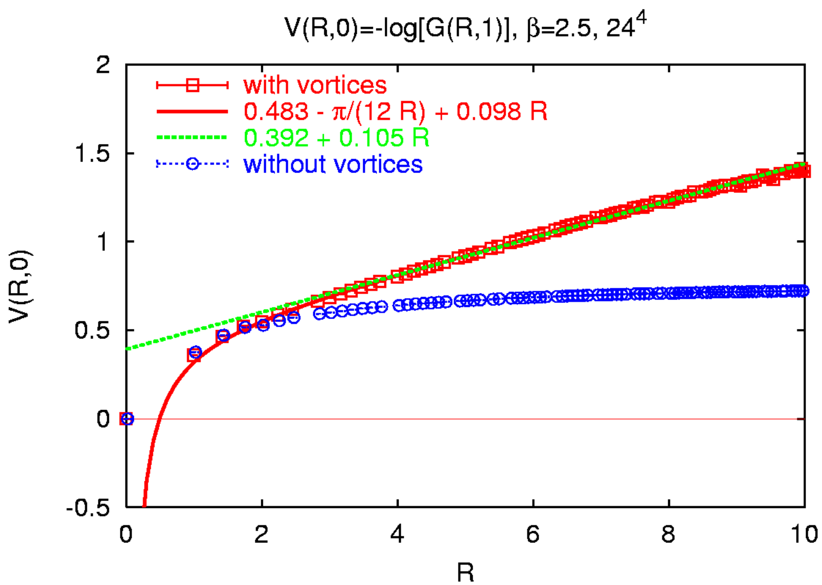
Results at β=2.5 **(Olejnik & JG, 2003)**



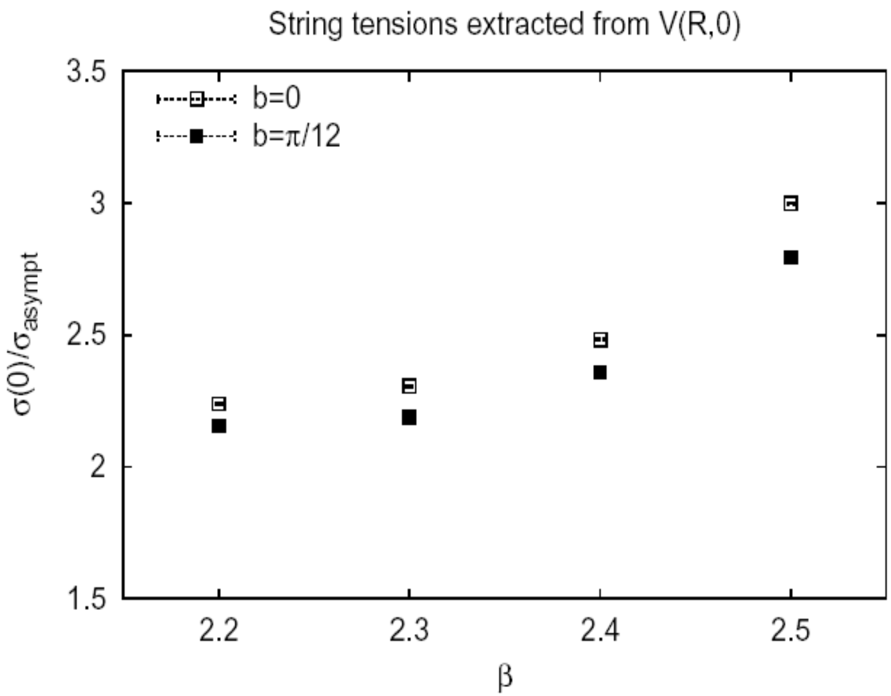
Notice the difference in slopes $\sigma(T)$ between V(R,0) (Coulomb) and V(R,4) (**red** data points).

In fact we find that

$$\sigma_{coul} \approx 3\sigma$$



When vortices are removed (**blue** data points), both σ_{coul} and σ vanish.



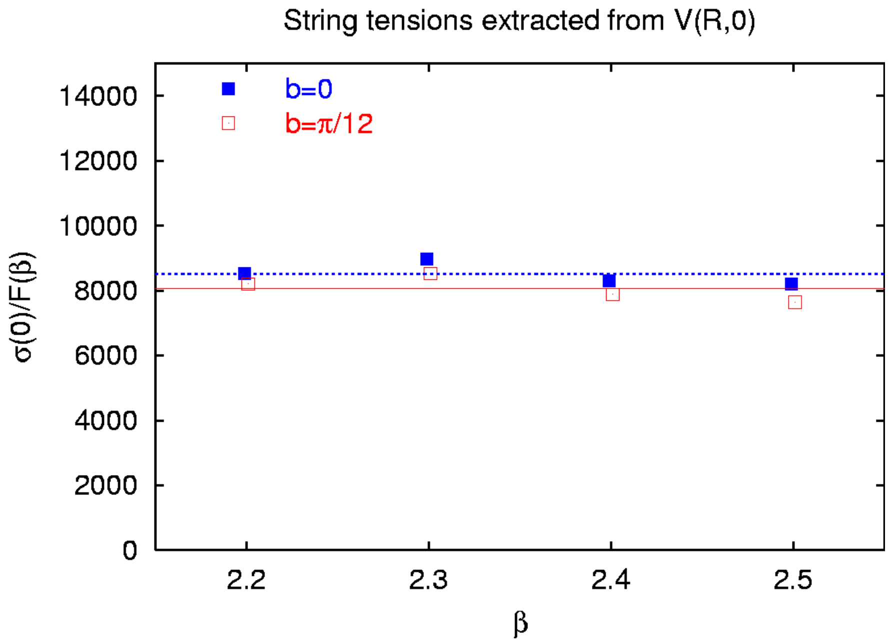
V(R,0) - essentially the Coulomb potential – is *linear*, in agreement with previous results of Zwanziger and Cucchieri.

However, $\sigma(0)$ (! σ_{coul} in the continuum) is substantially larger than the asymptotic string tension, at least in this coupling range.

The evidence is that the Coulomb potential *overconfines.*

According to asymptotic freedom, the quantity $\sigma_{coul}/F(\beta)$ should go to a constant at large β, where

$$F(\beta) = \left(\frac{6\pi^2}{11}\beta\right)^{\frac{102}{121}} \exp\left(-\frac{6\pi^2}{11}\beta\right)$$



σ_{coul} scales better than σ itself, in this coupling range.

Coulomb Propagator & Coulomb Potential

V(R,0) from one-gluon exchange:

$$V(R,0) = \frac{3}{4}[-D(R) + D(0)]$$

Its natural to associate $\mathbf{V_{coul}(R) = -\ (3/4)\ D(R)}$. This is wrong, however, because **D(0) = 1** in an infinite volume. (why? – because **D(0)** is proportional to the energy of an isolated, single quark state, which is *infrared* infinite if Q=0).

Then, since **V(R,0)** is finite, it follows that **D(R)** has an infrared infinity at *any* **R**.

These infrared infinities cancel in color singlet states, but lead to infinite energies in color non-singlet states, e.g. a quark-quark state.

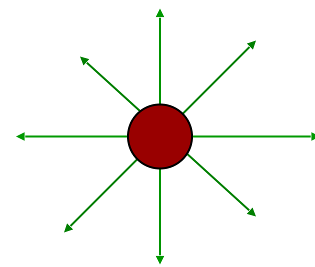
If **V_{coul}(R)** is defined so as to be finite in both the infinite volume and continuum limits, we must introduce a subtraction at some **$R=R_0$**, i.e.

$$V_{coul}(R) = -\frac{3}{4}[D(R) - D(R_0)]$$

In any case the *total* energy **V(R,0)** is finite at a>0 (for a color singlet), and unambiguous.

Coulomb Energy in Other Phases

1. <u>Massless Phase</u>
   compact QED_4
   SU(N) in D=5



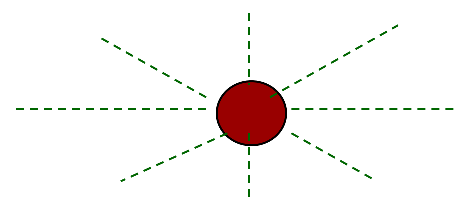
2. <u>Confinement Phase</u>
   pure SU(N)
   SU(N) + adjoint matter

(Z_N center symmetric)

3. <u>Screened Phases</u>

   SU(N) + fund matter
   G_2 gauge theory

   SU(N) + adj, higgs phase
   High T deconfined



trivial center symmetry

Spontaneously broken center symmetry

A New Order Parameter

Coulomb gauge leaves a remnant gauge symmetry

$$\begin{aligned} U_k(x,t) &\rightarrow g(t)U_k(x,t)g^\dagger(t) \\ U_0(x,t) &\rightarrow g(t)U_0(x,t)g^\dagger(t+1) \end{aligned}$$

On any time slice, this is a *global* symmetry, which can be spontaneously broken. If broken, then

$$\langle U_0(x,t)\rangle \neq 0 \qquad \text{at fixed t}$$

in an infinite volume, and as a result

$$\begin{aligned} \lim_{R\rightarrow\infty} G(R,1) &> 0 \\ \lim_{R\rightarrow\infty} V(R,0) &= \text{finite const.} \\ \sigma_{coul} &= 0 \end{aligned}$$

So Coulomb confinement or non-confinement can be understood as the symmetric or broken realizations of a remnant gauge symmetry. Not a new idea! (e.g. Marinari et al, 1993)

Order Parameter

Let ***Q*** be the modulus of the spatial average of timelike links

$$U_0^{av}(t) = \frac{1}{L^3}\sum_{\vec{x}} U_0(\vec{x},t)$$

$$Q = \left\langle \sqrt{\text{Tr}[U_0^{av}(t)U_0^{av\dagger}(t)]} \right\rangle$$

On general grounds

$$Q = c + \frac{b}{L^{3/2}}$$

with $c = 0$ in the symmetric phase

$c \neq 0$ in the broken phase

Q>0 implies $\mathbf{V_{coul}(R)}$ is non-confining, and since $\mathbf{V_{coul}(R)}$ is an upper bound on **V(R),** this means that

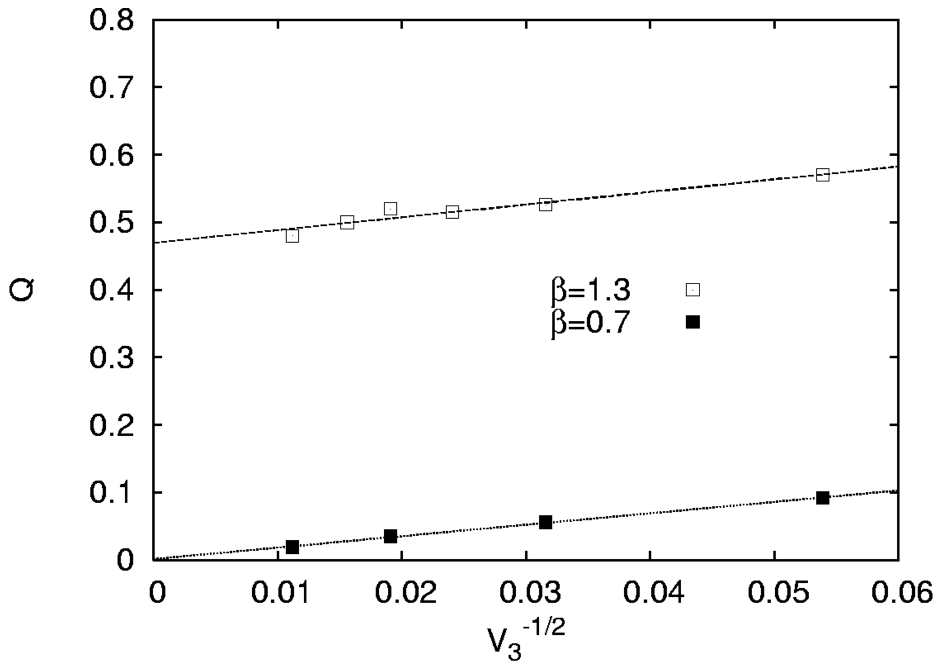
Q=0 is a necessary condition for confinement

Its interesting to try this out in QED_4, where we know there is a transition from confinement to a spin-wave phase at $\beta=1$. In particular:

β = 0.7 is in the confining phase

β = 1.3 is in the massless phase

In compact QED_4, the symmetry-breaking order parameter ***Q*** seems to nicely distinguish between the two phases.

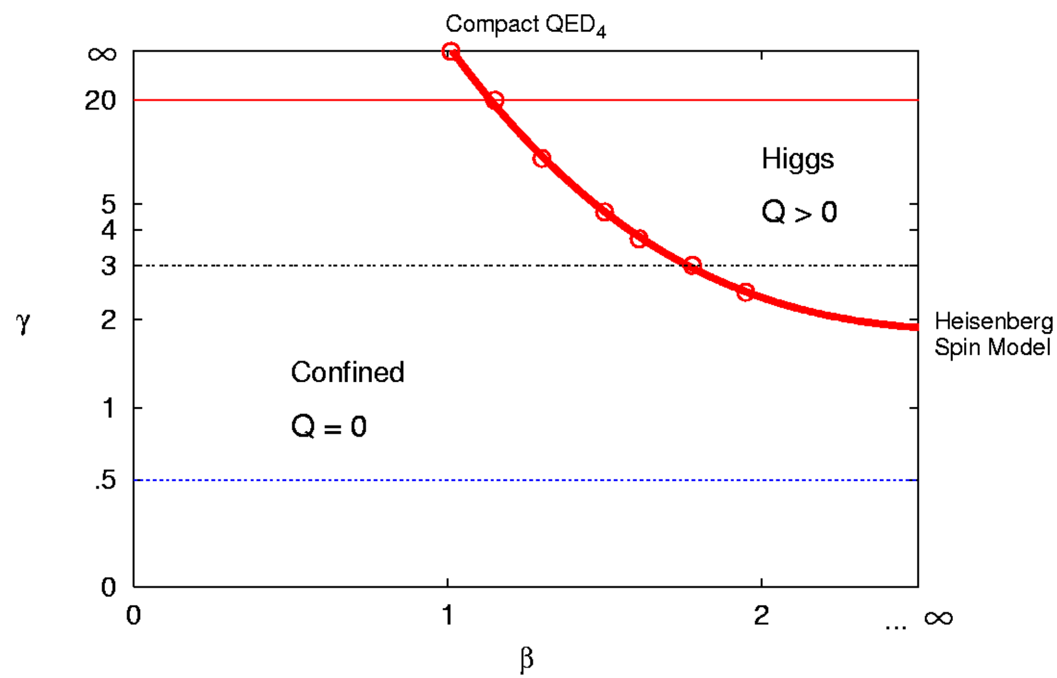


SU(2) Gauge-Adjoint Higgs Action

We add a "radially frozen" **|φ|=1** scalar field, in the adjoint representation, to the SU(2) Wilson action

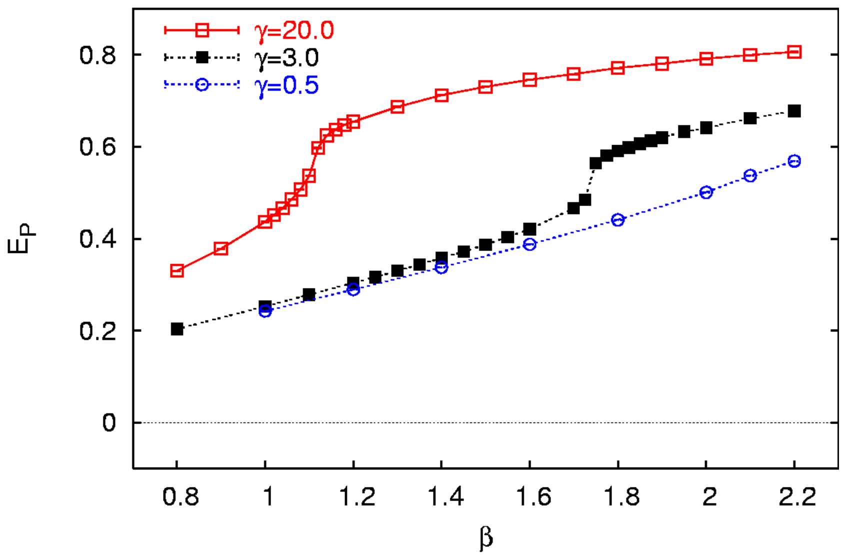
$$S = \beta \sum_{plaq} \frac{1}{2}\mathrm{Tr}[UUU^\dagger U^\dagger] + \gamma \sum_{x,\mu} \phi^a(x)\phi^b(x+\widehat{\mu})\frac{1}{2}\mathrm{Tr}[\sigma^a U_\mu(x)\sigma^b U_\mu^\dagger(x)]$$

The phase structure in coupling space looks like this: (Brower et al., 1982)

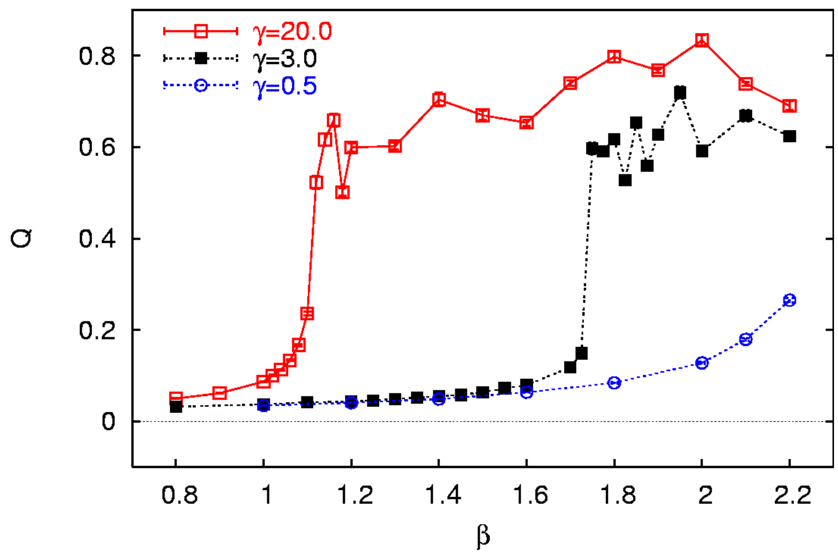


We find that the transition from **Q=0** to **Q>0** simply follows the transition from confinement to the Higgs (non-confined) phase.

SU(2) with adjoint Higgs, 12^4 lattice



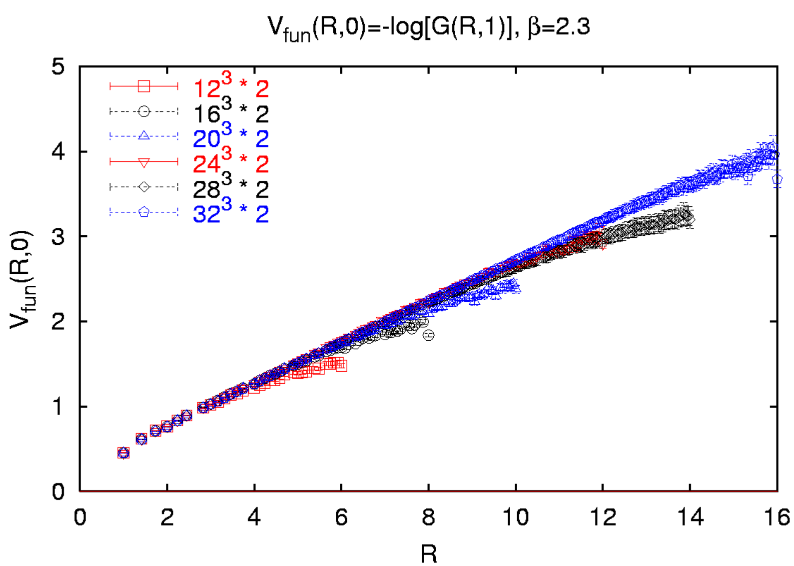
SU(2) with adjoint Higgs, 12^4 lattice

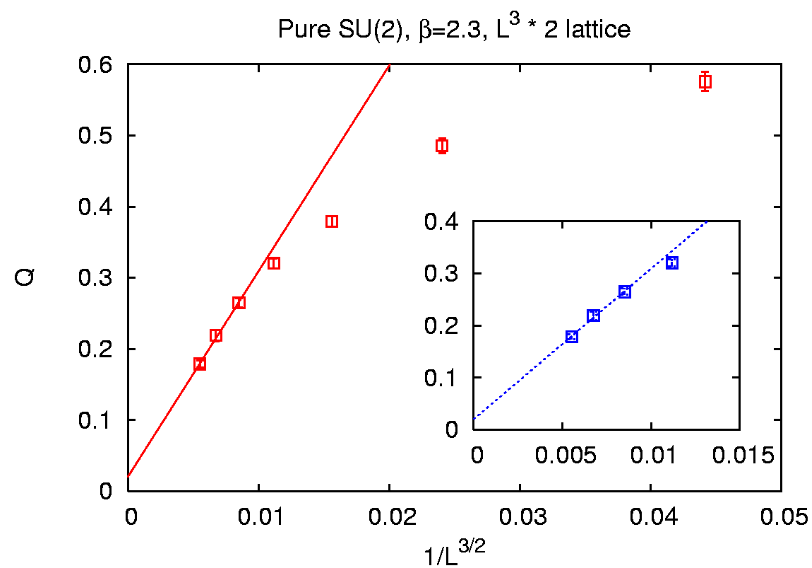


A Surprise (?) at High Temperature

We have calculated $\mathbf{V_{coul}(R)}$ in the high temperature deconfined phase, expecting to see σ_{coul}=0.

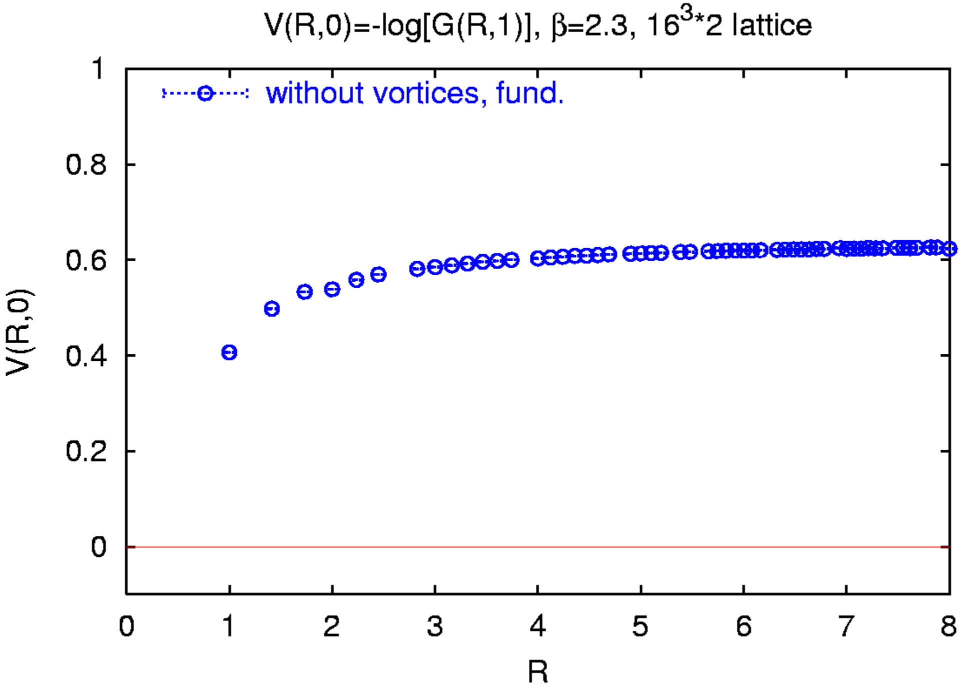
In fact, the opposite result was obtained (L_t=2, β=2.3).





Are center vortices somehow important to this high-temperature result?

Here is the Coulomb potential in the deconfined phase **(β=2.3, L_t=2)** in configurations with vortices removed…



We already knew that vortices explain the string tension of ***spatial*** Wilson loops in the deconfined phase. (Reinhardt et al)

Is it a surprise that removing vortices also removes the Coulomb string tension in that phase?

Maybe we shouldn't be too surprised.

- $V_{coul}(R)$ depends only on the space components of the vector potential on a timeslice, recall

$$D(\vec{x}-\vec{y})\delta^{ab} = \left\langle \left[\frac{1}{\nabla\cdot D[A]}(-\nabla^2)\frac{1}{\nabla\cdot D[A]} \right]^{a,b}_{x,y} \right\rangle$$

  But spacelike links still form a 3D confining ensemble, even past the deconfinement phase transition.

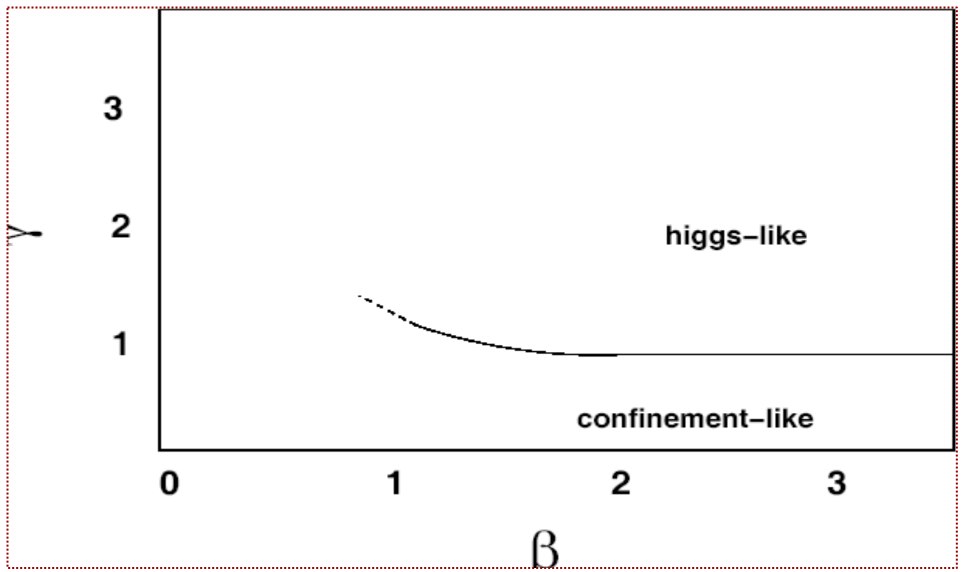
- Thin center vortices lie on the Gribov horizon, defined as the region of configuration space where **r ¢ D(A)** has a zero eigenvalue. Configurations on the Gribov horizon are essential to the Gribov-Zwanziger conjecture about confining one-gluon exchange.

SU(2) Gauge-Higgs Theory

We add a “radially frozen” scalar field, in the fundamental representation, to the SU(2) Wilson action

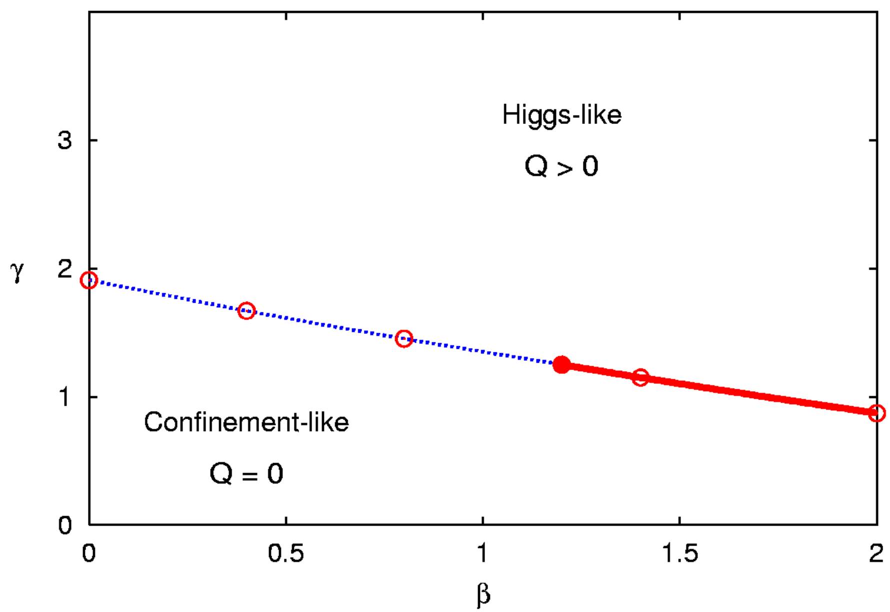
$$\begin{aligned} S &= \beta \sum_{plaq} \frac{1}{2} \mathrm{Tr}[UUU^\dagger U^\dagger] \\ &+ \gamma \sum_{x,\mu} \frac{1}{2} \mathrm{Tr}[\phi^\dagger(x) U_\mu(x) \phi(x+\widehat{\mu})] \end{aligned}$$

where ϕ is SU(2) group-valued; i.e. $\phi\, \phi^y$ = **I** . The phase structure in coupling space looks like this: (Lang et al, 1981)



Note that the Higgs-like and confinement-like phases are connected (Fradkin & Shenker), and no local order parameter can distinguish them. There is only one *screened* phase.

But as far as Q is concerned, it looks like there are *two* phases…

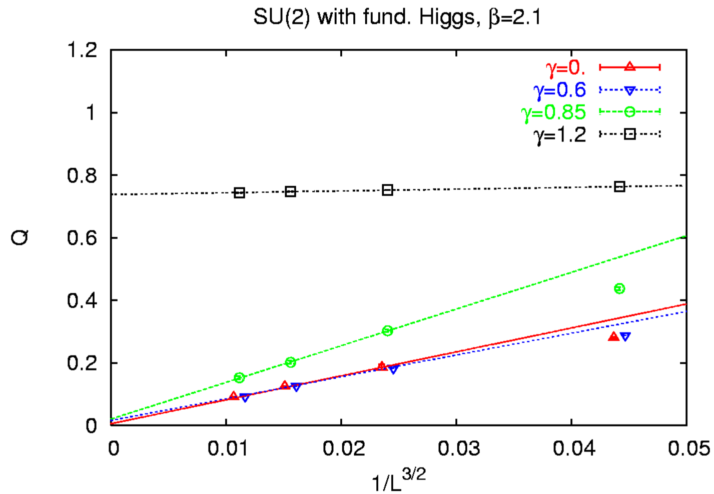


Q is discontinuous along the solid line

Q increases from 0 at the dashed line

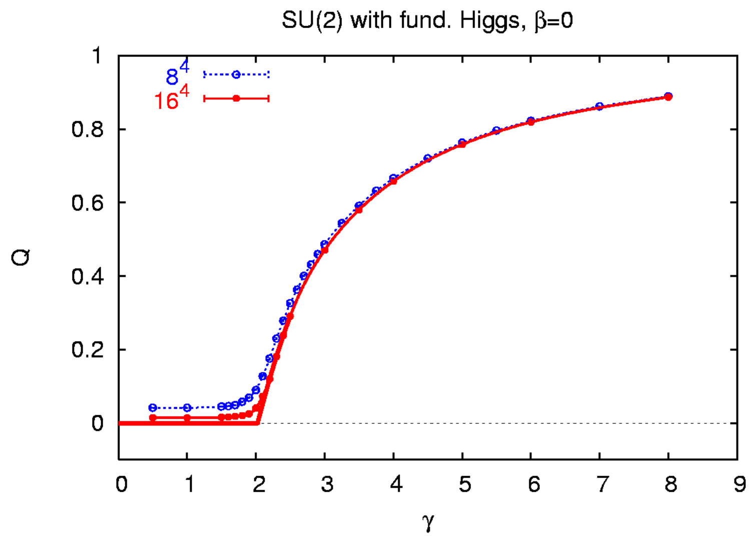
The remnant symmetry breaking transition does *not* correspond to any non-analyticity of the free energy. It does, however, correspond to what we have seen before - the vortex depercolation transition across a Kertesz line!

SU(2) Gauge-Higgs System



Q at several γ's at $\beta = 2.1$

Evidently a discontinuity in **Q** around γ = **0.9**



Q vs γ at $\beta = 0$

Transition from **Q=0 to Q>0** at γ ¼ **2.**

Yet the g.inv free energy at β=0

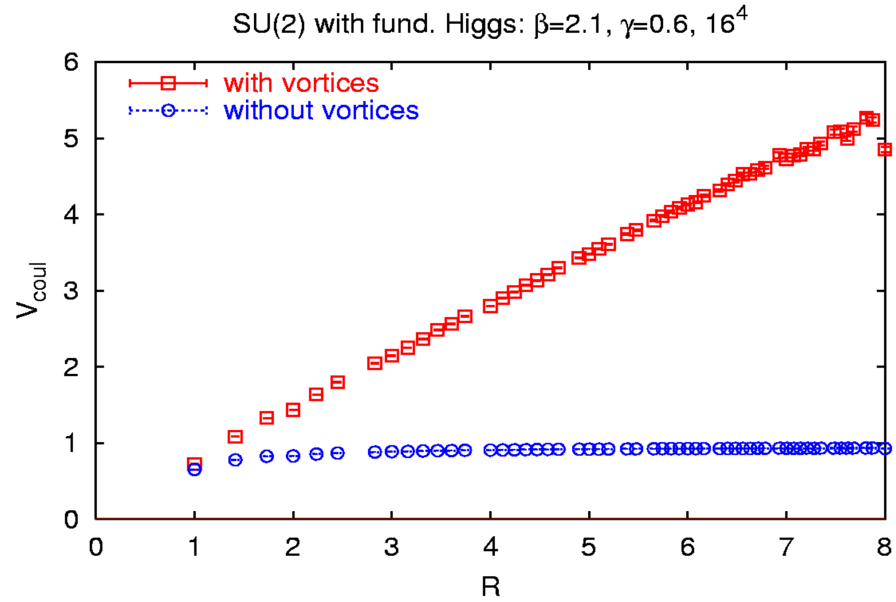
$$F(\gamma) = 4V \log\left[\frac{2I_1(\gamma)}{\gamma}\right]$$

is analytic at γ**=2.**

Effect of Vortex Removal in the gauge-Higgs theory

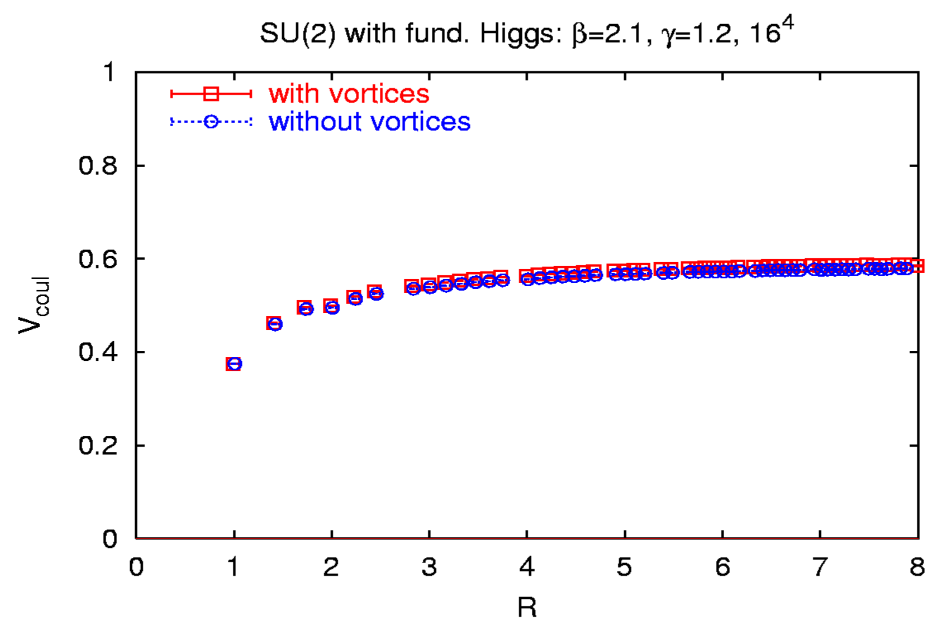
Confinement-like region:

β=2.1, γ=0.6



Higgs-like region:

β=2.1, γ=1.2



Our findings are relevant to this question:

In what sense does real QCD, or any theory with matter in the fundamental representation, "confine" color?

There are no transitions in the free energy, or any local order parameter, which isolate the Higgs from the confinement-like regions of the phase diagram. ***Fradkin and Shenker (1979)***

There are, nonetheless, two physically distinct phases, separated by a sharp percolation transition. The "confinement-like" phase is distinguished from the Higgs phase by having

- **a symmetric realization of the remnant symmetry**
- **a confining gluon propagator, and $\sigma_{coul} > 0$**
- **percolating center vortices**

Conclusions

- Coulomb energy rises linearly with quark separation.
- Coulomb energy *overconfines,* σ_{coul} ¼ 3 σ . Overconfinement is essential to the gluon chain scenario.
- Center symmetry breaking (σ **= 0**) does not necessarily imply remnant symmetry breaking (σ_{coul}**=0**). In particular

  1. σ_{coul} **> 0** in the high-T deconfined phase.
  2. σ_{coul} **> 0** in the confinement-like phase of gauge-higgs theory.

  The transition to the higgs phase is a remnant-symmetry breaking, vortex depercolation transition.

- In every case, center vortex removal also removes the Coulomb string tension, which strongly suggests a connection between the center vortex and Gribov horizon scenarios for confinement….

Center Vortices and the Gribov Horizon

In a confining theory, the energy of a color-nonsinglet state is infinite. In Coulomb gauge, such a state is, e.g.

$$\Psi^a[A;x] = \psi^a(x)\Psi_0[A]$$

In an abelian theory, the same state in temporal gauge has the well-known form

$$\Psi[A;x] = \exp\left[ie\int d^3z\ A(z)\cdot\nabla\frac{1}{4\pi|x-z|}\right]\psi(x)\Psi_0[A]$$

This conversion of the charged Coulomb gauge state to temporal gauge can be extended to non-abelian theories. **Lavelle and McMullan**

Lets warm up by computing the energy of a charged state in an abelian theory, in Coulomb gauge, in a way which will generalize to the non-abelian theory.

Coulomb Self Energy - QED

A familiar calculation: the Coulomb self-energy of a static charge, in a box of extension ***L***, with an ultraviolet cutoff $k_{max}=1/a$. Start with

$$H = \int d^3x \, (E^2 + B^2) \; + \; H_{coul}$$

$$H_{coul} = \int d^3x d^3y \; \rho(x) K(x,y) \rho(y)$$

$$K(x,y) = M^{-1}(-\nabla^2) M^{-1}$$ where $M = -r^2$

$$\mathcal{E} = e^2 K(x,x)$$

where $\mathcal{E}$ is the self-energy of a static charge at point ***x***. Without a UV cutoff, $K(x,y) \sim 1/|x-y|$, so $K(x,x)=1$.

M* = -*r*²** is the Faddeev-Popov operator for the abelian theory, obtained by variation of the gauge-fixing functional ***r ¢ A wrt an infinitesimal gauge transformation.

$$\begin{aligned} M_{xy} &= \frac{\delta}{\delta\theta(x)} \nabla \cdot [A(y) - \nabla\theta] \\ &= -\nabla^2 \delta(x-y) \end{aligned}$$

The eigenstates

$$M\phi^{(n)} = \lambda_n \phi^{(n)}$$

are of course just the plane wave states, with $\lambda_n = k_n^2$. On the lattice these states are discrete, and we can write the Green's function

$$G_{xy} = \left[M^{-1}\right]_{xy} = \sum_n \frac{\phi_x^{(n)} \phi_y^{(n)*}}{\lambda_n}$$

some simple manipulations...

$$
\begin{aligned}
\mathcal{E} &= e^2 (M^{-1}(-\nabla^2)M^{-1})_{xx} \\
&= \frac{e^2}{L^3} \sum_x \sum_{y_1 y_2} G_{x y_1} (-\nabla^2)_{y_1 y_2} G_{y_2 x} \\
&= \frac{e^2}{L^3} \sum_x \sum_{y_1 y_2} \sum_m \sum_n \frac{\phi_x^{(m)} \phi_{y_1}^{(m)*}}{\lambda_m} (-\nabla^2)_{y_1 y_2} \frac{\phi_{y_2}^{(n)} \phi_x^{(n)*}}{\lambda_n} \\
&= \frac{e^2}{L^3} \sum_{y_1 y_2} \sum_n \frac{\phi_{y_1}^{(n)*} (-\nabla^2)_{y_1 y_2} \phi_{y_2}^{(n)}}{\lambda_n^2}
\end{aligned}
$$

Then

$$\mathcal{E} = \frac{e^2}{L^3} \sum_n \frac{F(\lambda_n)}{\lambda_n^2}$$

where

$$F(\lambda_n) = (\phi^{(n)}|(-\nabla^2)|\phi^{(n)})$$

Let $\rho(\lambda)$ denote the density of eigenvalues, scaled so that $s\ d\lambda\ \rho(\lambda) = 1$.
Then at large volumes we can approximate the sum over eigenstates by an integral, and

$$\mathcal{E} = e^2 \int d\lambda\ \frac{\rho(\lambda)F(\lambda)}{\lambda^2}$$

In QED, its easy to show that

$$\rho(\lambda) = \frac{\sqrt{\lambda}}{4\pi^2} \quad , \quad F(\lambda) = \lambda$$

and also λ_{min} ***~ 1/L²*** **,** λ_{max} ***~ 1/a²*** , so that putting it all together

$$\begin{aligned} \mathcal{E} &= e^2 \int d\lambda \, \frac{\rho(\lambda) F(\lambda)}{\lambda^2} \\ &\sim e^2 \left(\frac{1}{a} - \frac{1}{L} \right) \end{aligned}$$

which is finite, at finite UV cutoff ***a***, as ***L ! 1*** . But IR finiteness clearly depends on the small λ behavior of ρ***(***λ***) F(***λ***)*** . If instead

$$\lim_{\lambda \to 0} \frac{\rho(\lambda) F(\lambda)}{\lambda} > 0$$

then the Coulomb energy would be IR infinite.

Yang-Mills: The Gribov Horizon

Coulomb gauge-fixing on the lattice involves minimizing

$$R[U] = -\sum_x \sum_{k=1}^{3} \mathrm{Tr}[U_k(x)]$$

Denote gauge transformed links

$${}^{\theta}U_\mu(x) = e^{i\theta^a(x)L_a} U_k(x) e^{-i\theta^a(x+\hat{k})L_a}$$

then

$$\left(\frac{\delta R[{}^{\theta}U]}{\delta \theta^a(x)} \right)_{\theta=0} = \nabla \cdot A^a(x) = 0$$

Coulomb gauge condition

$$\left(\frac{\delta R[{}^{\theta}U]}{\delta \theta^a(x) \theta^b(y)} \right)_{\theta=0} = M^{ab}(x,y)$$

Faddeev-Popov operator

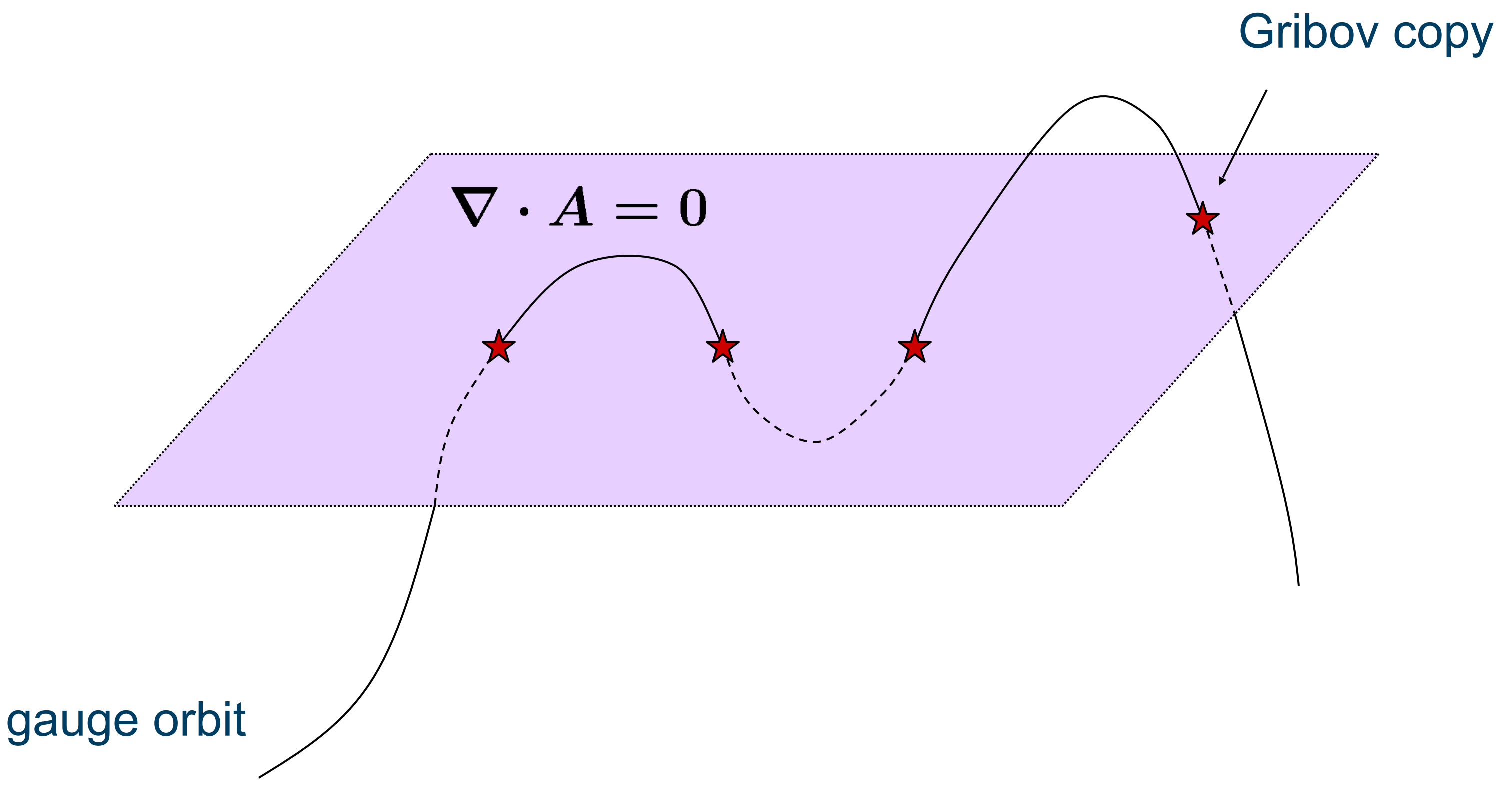
- In non-abelian theories, more than one point on the gauge orbit satisfies that Coulomb gauge condition. These are known as ***Gribov copies.***

- Gribov copies with only positive λ_n are said to lie inside the ***Gribov Region***, where Gribov copies are local minima of ***R[U]*** .

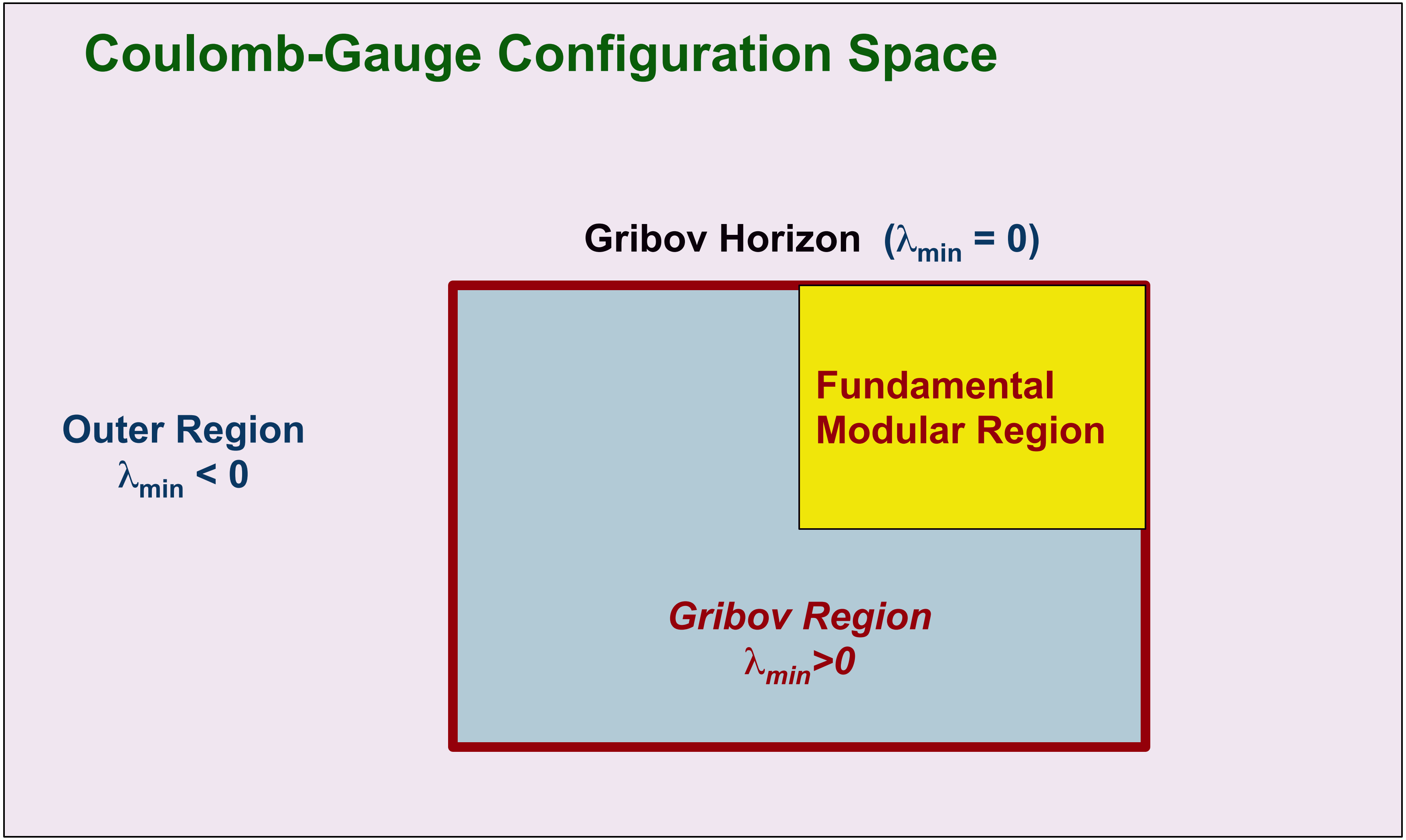
- Global minima of ***R[U]*** lie inside the ***Fundamental Modular Region***, which is a subspace of the Gribov Region.

- The ***Gribov Horizon*** is the boundary of the Gribov Region, where $M^{ab}(x,y)$ has a zero eigenvalue $\lambda_{min}=0$.

<u>Full Configuration Space</u>



Shaded region is the Coulomb-gauge configuration space



Outer Region: $\sum$ **Tr[U]** stationary (many gauge copies)

Gribov Region: $\sum$ **Tr[U]** a ***local*** maximum (many gauge copies)

Fund. Mod. Region: $\sum$ **Tr[U]** a ***global*** maximum (unique)

Typical configurations in the Gribov region are expected to approach the Gribov horizon in the infinite-volume limit. This is true even at the perturbative level, where λ_{min} » $1/L^2$.

But what counts for confinement is the density of eigenvalues $\rho(\lambda)$ near $\lambda=0$, and the lack of "smoothness" of these near-zero eigenvalues, as measured by $F(\lambda)$. This is what determines whether the Coulomb confinement criterion

$$\lim_{\lambda\rightarrow 0} \frac{\rho(\lambda) F(\lambda)}{\lambda} > 0$$

is fulfilled in non-abelian gauge theories.

Coulomb Self Energy – Yang-Mills

In Yang-Mills theory the Faddeev-Popov operator depends on the gauge field

$$M[A] = \nabla \cdot \mathcal{D}[A]$$

The self-energy of an isolated static charge in color group rep. r, Casimir C_r, is $\mathcal{E}_r = g^2 C_r \mathcal{E}$, where

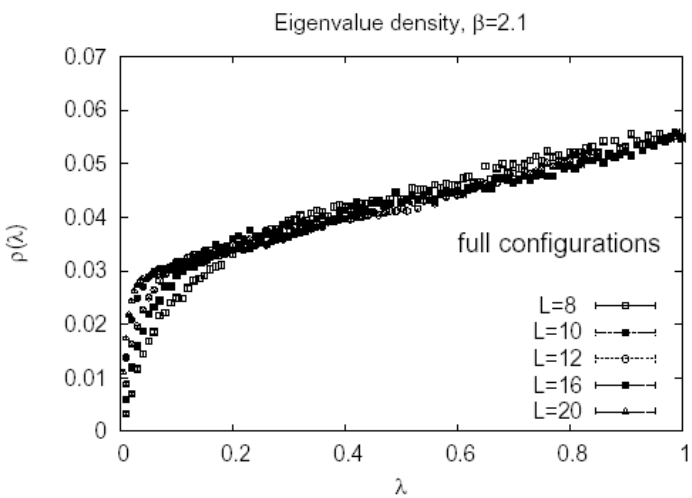
$$\mathcal{E} = \int d\lambda \, \frac{\rho(\lambda) F(\lambda)}{\lambda^2}$$

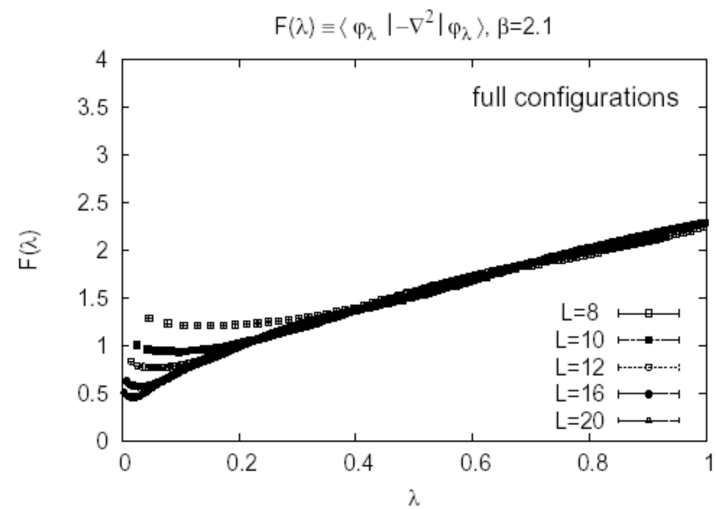
$$F(\lambda_n) = (\phi^{(n)}|(-\nabla^2)|\phi^{(n)})$$

We calculate $\rho(\lambda)$, $F(\lambda)$ numerically, on finite-size lattices, and extrapolate to infinite volume.

Procedure

- gauge fields from lattice Monte Carlo, fix to Coulomb gauge
- find the first 200 eigenstates of the lattice Faddeev-Popov operator on each time-slice of each lattice configuration (Arnoldi algorithm)
- calculate **hρ(λ)i, hF(λ) i**. Results, L=8 – 20:





From scaling of the distributions at small λ with **L**, we estimate at **L! 1**

$$\rho(\lambda) \sim \lambda^{0.25} \quad , \quad F(\lambda) \sim \lambda^{0.4}$$

which implies

$$\mathcal{E} = \int d\lambda \, \frac{\rho(\lambda) F(\lambda)}{\lambda^2} \to \infty$$

in the infrared

(i.e. confinement)

Scaling of the Eigenvalue Distribution

In certain ***N* £ *N*** matrix models, the density of near-zero eigenvalues in the ***N* !** ∞ limit

$$\rho(\lambda) = \kappa\lambda^{\alpha}$$

can be deduced if the eigenvalues display a universal scaling behavior with ***N***, where ***$N=3V_3$*** for the F-P operator. "Universal" means that under the scaling

$$z = \lambda V_3^{\frac{1}{1+\alpha}}$$

the density of eigenvalues, the average spacing between low-lying eigenvalues, and the probability distribution **$P(z_n)$** for the value of the n-th low-lying eigenvalue, agree for every lattice 3-volume ***$V_3=L^3$***.

The argument is simple. The number of eigenvalues **N[λ, Δ λ]** in the range

$[\lambda - \Delta\lambda/2, \lambda + \Delta\lambda/2]$ is

$$N[\lambda, \Delta\lambda] = 3V_3\rho(\lambda)\Delta\lambda$$

If we rescale eigenvalues by some power ***p*** of the lattice volume, $z = \lambda V_3^p$, then

$$N[\lambda, \Delta\lambda] = 3\kappa V_3^{1-p(1+\alpha)} z^\alpha \Delta z$$

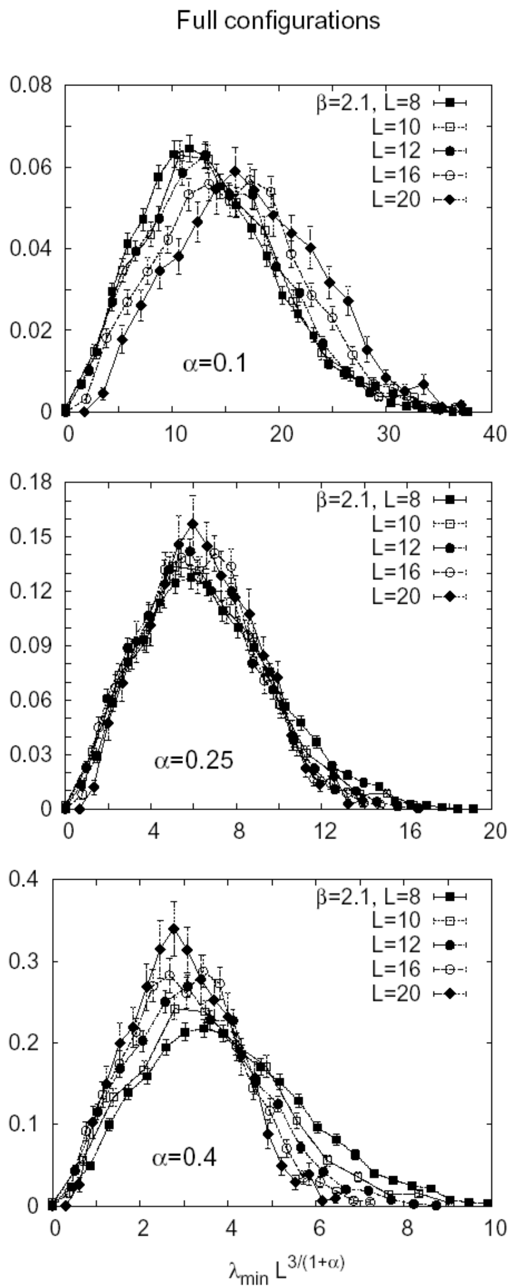
and this depends only on the rescaled quantities ***z*** and Δ ***z*** if we choose ***p=1/(1+α)*** , so that

$$z = \lambda V_3^{\frac{1}{1+\alpha}}$$

The strategy is to compute the frequency distribution $P(z_{min})$ of the lowest non-trivial eigenvalue z_{min} at various volumes, and see if there is some value of α where the data sets fall on top of each other. If so, this implies universality, and determines α in $\rho(\lambda)=\kappa\,\lambda^\alpha$.

This is the frequency distribution ***P(z_{min})*** for the values of the lowest eigenvalue ***z_{min}*** at various lattice sizes L=8-20, at three different values of α.

Notice that at α ***= 0.25***, the curves more-or-less fall on top of each other.



Now for ***F*(λ)** . We have fit our data for ***F*(λ$_{min}$)** to the form

$$F(\lambda_{min}) = a + \frac{b}{L^p}$$

and find

$$F(\lambda_{min}) = \frac{10}{L}$$

From scaling

$$\lambda_n = z_n L^{-\frac{3}{1+\alpha}}$$

and in particular, if α = 0.25

$$\lambda_{min} \sim \frac{1}{L^{2.4}}$$

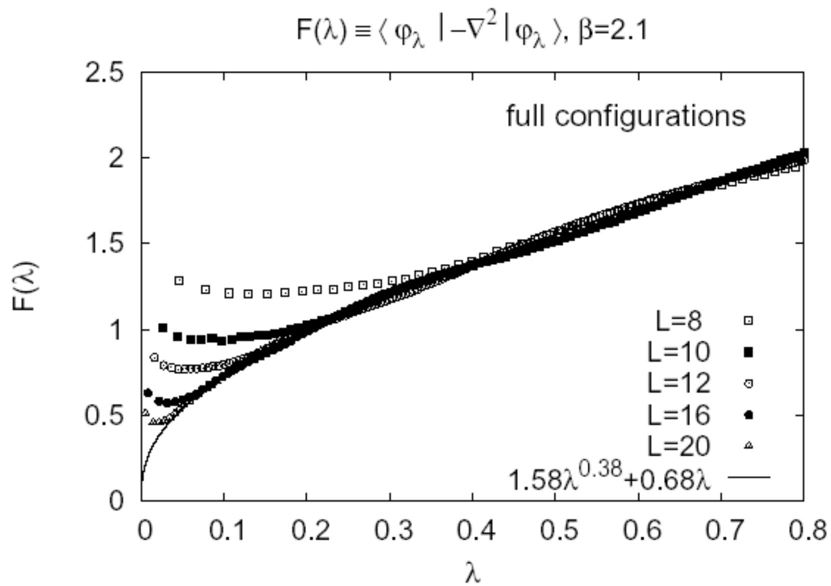
Together, these facts suggest that at small λ

$$F(\lambda) \sim \lambda^{0.42}$$

(perturbative at high λ)

So this motivates a fit of $F(\lambda)$ to

$$F(\lambda) = a\lambda^p + b\lambda$$



The best fit gives **p=0.38**, which is not far off our guess of **p=0.42**.

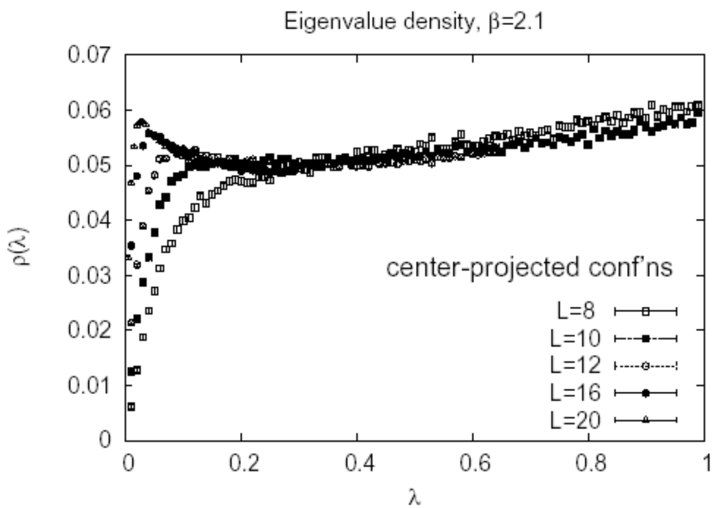
This is how we have arrived at our estimates

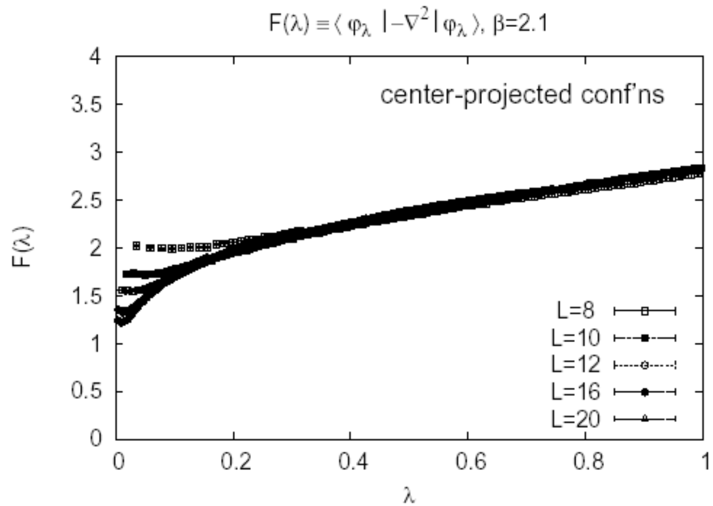
$$\rho(\lambda) \sim \lambda^{0.25} \quad , \quad F(\lambda) \sim \lambda^{0.4}$$

which leads to an infrared-divergent Coulomb energy for color-charged states.

Using standard methods, we can decompose any lattice configuration into vortex-only (Z_μ) and vortex-removed ($U'_\mu = Z_\mu U_\mu$) configurations, which we transform to Coulomb gauge.

Here is the result for the vortex-only configurations





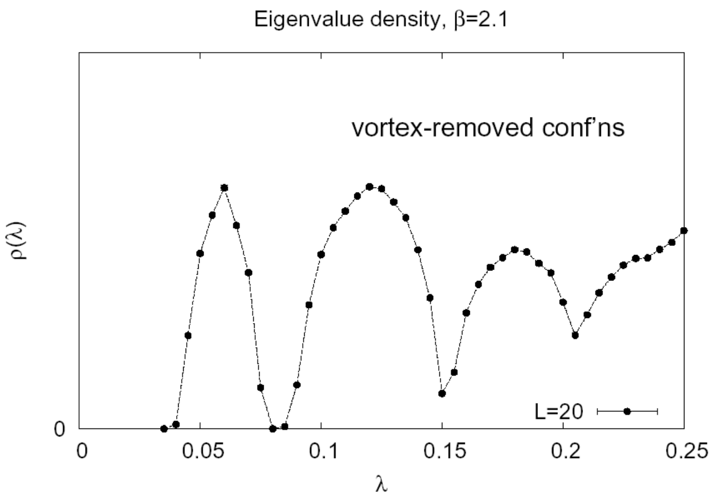
As before, we use eigenvalue-scaling to estimate

$$\rho(\lambda) \sim \lambda^{0\pm 0.05} \quad , \quad F(\lambda) \approx 1$$

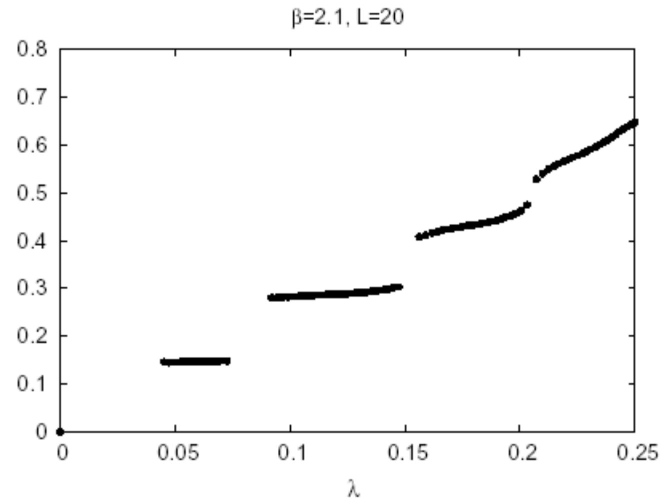
and again

$$\mathcal{E} = \int d\lambda \, \frac{\rho(\lambda) F(\lambda)}{\lambda^2} \to \infty \qquad \textbf{(confinement)}$$

Here is the result for the no-vortex configurations



“peaks”



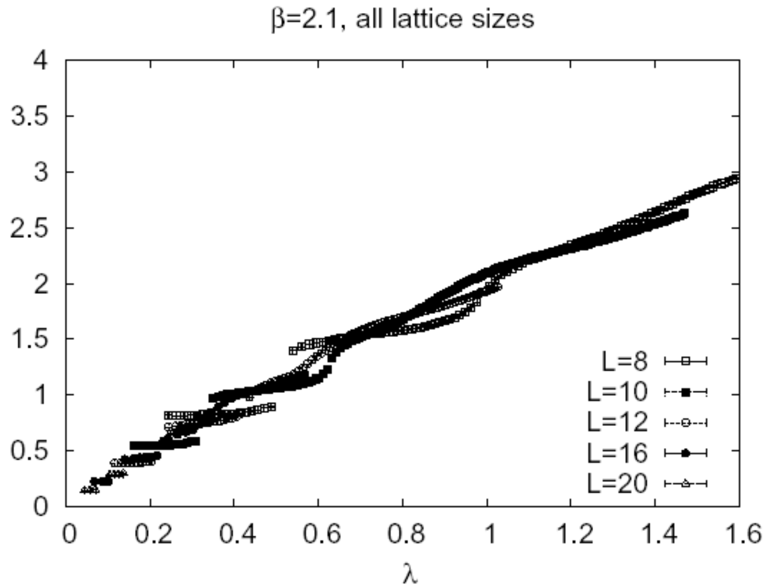
“bands”

The number of eigenvalues in each “peak” of $\rho(\lambda)$, and each “band” of $F(\lambda)$, matches the degeneracy in the first few eigenvalues of $(-r^2)$, the zeroth-order Faddeev-Popov operator.

$\rho(\lambda)$ for the $(-r^2)$ operator is just a series of δ-function peaks. In the vortex-only configurations, these peaks broaden to finite width, but the qualitative features of $\rho(\lambda)$ $F(\lambda)$ at zeroth order - no confinement - remain.

Further evidence: the low-lying eigenvalues scale with ***L*** as $\lambda_n \sim \frac{1}{L^2}$

just like in the abelian theory, and looking at ***F(λ)*** at all lattice volumes



it seems that ***F(λ) »*** λ , again as in the abelian theory.

Gauge-Higgs Theory

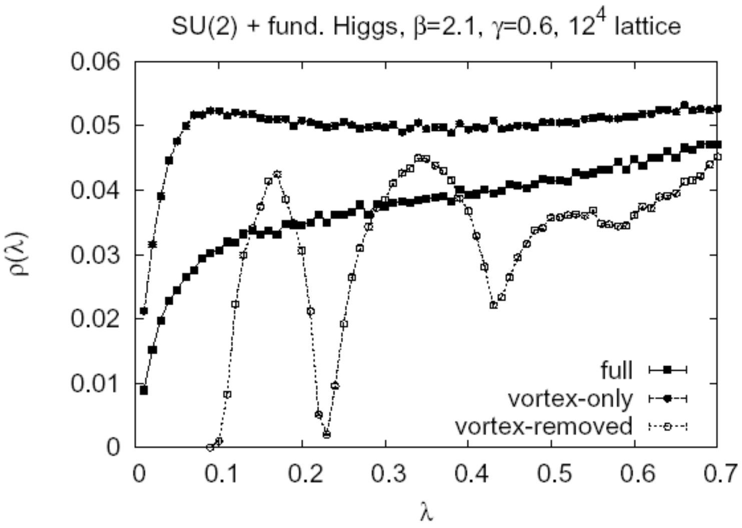
Next we add a fixed-modulus scalar field in the fundamental representation. In SU(2) this can be expressed as

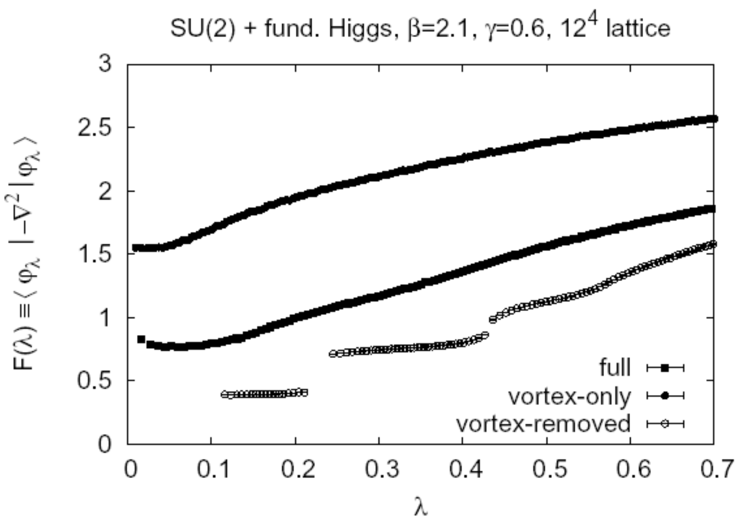
$$\begin{aligned} S &= \beta \sum_{plaq} \frac{1}{2} \mathrm{Tr}[UUU^\dagger U^\dagger] \\ &+ \gamma \sum_{x,\mu} \frac{1}{2} \mathrm{Tr}[\phi^\dagger(x) U_\mu(x) \phi(x+\widehat{\mu})] \end{aligned}$$

We have seen that, while there is no ***thermodynamic*** transition from the Higgs phase to a confinement phase (***Osterwalder & Seiler, Fradkin & Shenker)*** there are, nonetheless two distinct phases in this theory, separated by a sharp transition.

Olejnik, Zwanziger & JG,
Bertle, Faber, Olejnik & JG
Langfeld

Here are our results in the confinement-like phase (β=2.1, γ=0.6) on a 12^4 lattice





for $\rho(\lambda)$ and $F(\lambda)$

It looks just like in the pure-gauge theory (γ = ***0***).

But things change drastically in the Higgs phase β=2.1, γ=1.2

These graphs are for the ***FULL UNMODIFIED*** configurations in the Higgs phase.

They look almost identical to results in the ***VORTEX-REMOVED*** configurations of the pure (γ=0) gauge theory!

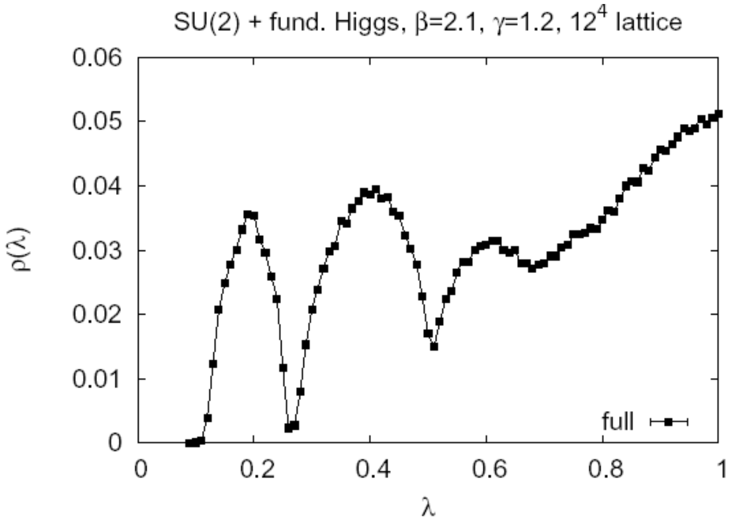


FIG. 11: The vortex density in the Higgs (broken remnant symmetry) phase, for the full configurations.

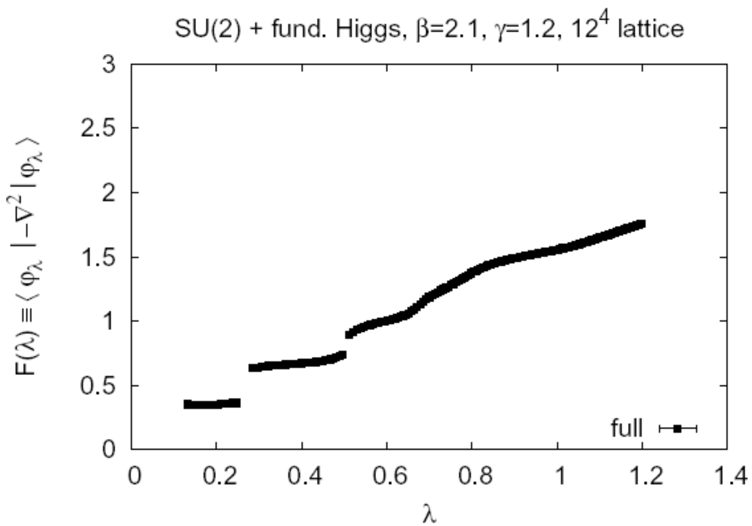


FIG. 12: $F(\lambda)$ in the Higgs phase, for full configurations.

Thin Vortices and the Eigenvalue Density

Infrared divergent Coulomb energy is due to an enhancement of $\rho(\lambda)$ near $\lambda=0$, which we have attributed to percolating center vortices.

It is interesting to start with the trivial, zero-field configuration, add thin vortices by hand, and watch what happens to $\rho(\lambda)$.

A configuration containing a single thin vortex (two planes in the 4D lattice), closed by lattice periodicity, is created by setting $U_2 = -1$ at sites

$$1 \le x \le \frac{L}{2}, \quad y = 1, \quad \text{all } z, t.$$

with all other $U_\mu = +1$. This creates two vortex sheets parallel to the zt-plane.

We can similarly create any number of vortices parallel to any lattice plane. Let ***(N,P)*** denote N vortices created in each P orientations.

P=1 means: N vortices created parallel to the zt plane

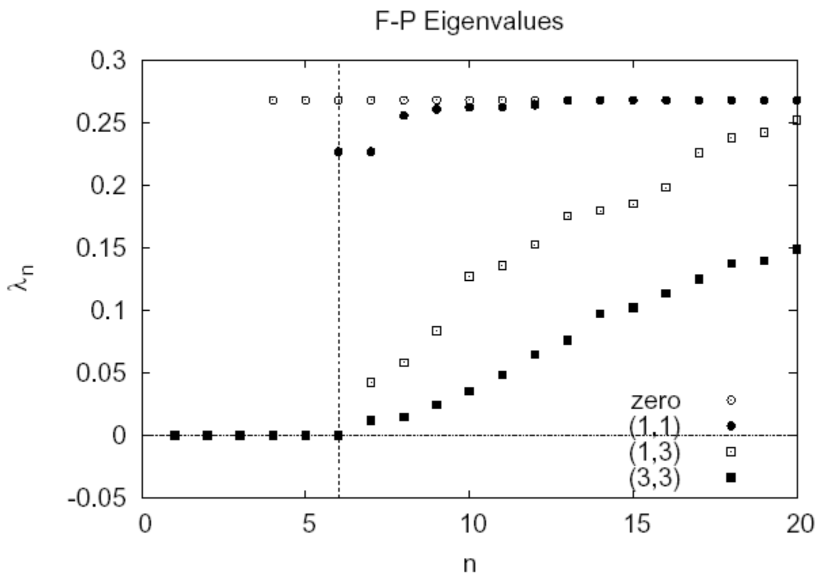
P=3 means: N vortices created parallel to the xt, yt, zt planes (3N total)

Then we just calculate the first 20 eigenvalues $\{\lambda_n\}$ on a 12^4 lattice in these configurations, and here is the result:

Note the

1. breaking of degeneracy
2. **the drastic drop in eigenvalue magnitude**

as vortex number increases.



Some Analytical Results

Facts about vortices and the Gribov horizon, stated here without proof:

- Vortex-only configurations have non-trivial Faddeev-Popov zero modes, and therefore lie precisely on the Gribov horizon.

- The Gribov horizon is a convex manifold in the space of gauge fields, both in the continuum and on the lattice. The Gribov region, bounded by that manifold, is compact.

- Vortex-only configurations are conical singularities on the Gribov horizon.

So thin vortices appear to have a special geometrical status in Coulomb gauge. The physical implications of this fact are not yet understood.

Conclusions

- The Coulomb self-energy of a color non-singlet state is infrared divergent, due to the enhanced density $\rho(\lambda)$ of Faddeev-Popov eigenvalues near $\lambda = 0$.

This supports the Gribov-Zwanziger picture of confinement.

- The confining property of the F-P eigenvalue density can be entirely attributed to center vortices:

    1. Enhancement of $\rho(\lambda)$ is found in vortex-only configurations.
    2. The confining properties of $\rho(\lambda)$, $F(\lambda)$ disappear whenever vortices are either removed from lattice configurations, or cease to percolate.

These results establish a connection between the center vortex and Gribov horizon scenarios for confinement.

This is an important point, and worth restating:

The excitations of Z_N lattice gauge theory are equivalent to a set of thin center vortices, and vice versa.

Exercise

a) Convince yourself of this fact for Z_2 lattice gauge theory in D=2 dimensions.

b) What is the analog, in Z_2 lattice gauge theory, of the Bianchi identity in electrodynamics

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0$$

Figure from a paper by Phillipe de Forcrand, who uses the z-t plane instead of the x-y plane for the negative plaquettes.

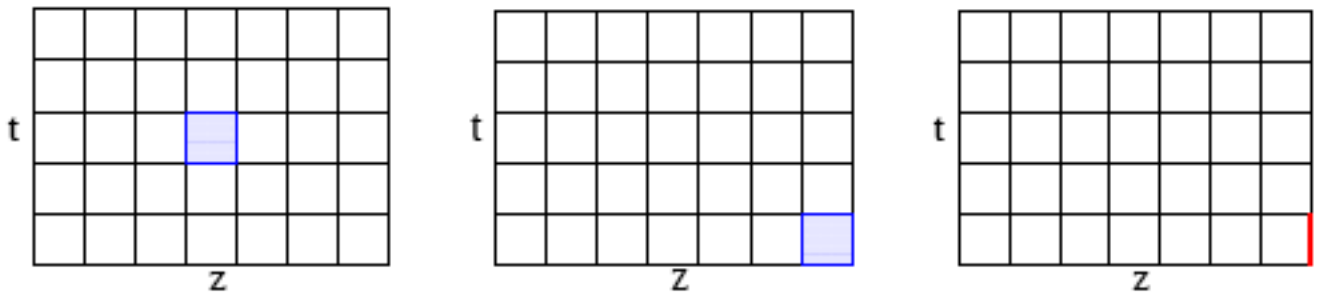


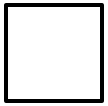
Figure 1: How to create a 't Hooft loop $\partial\tilde{\Sigma}$ in the (x, y) plane: *(left)* in each (z, t) plane intersecting $\tilde{\Sigma}$, multiply by a center element one plaquette with coordinates (z_0, t_0); *(middle)* equivalently, choose (z_0, t_0) in the corner; *(right)* equivalently, multiply by a center element the link U_t at the boundary. Thus, a 't Hooft loop of maximal size ($\tilde{\Sigma}$ intersects all (z, t) planes) is equivalent to twisted boundary conditions for the Polyakov loop.

Exercise

a) Quickly verify (from the previous equation) that, even ***without*** the adjoint plaquette term, there exist center vortex saddlepoints of the ordinary SU(N) Wilson action, providing N>5.

b) Consider adding a "rectangle" 2-plaquette term to the SU(N) Wilson action, i.e.

$$S_I = c_0 \sum_P \left(N - \mathrm{ReTr}[U(P)]\right) + c_1 \sum_R \left(N - \mathrm{ReTr}[U(R)]\right)$$



Find the inequality that c_0 , c_1 must satisfy, such that zero field strength configurations are global minima of this action. Then prove that if this condition is satisfied, and if N>5, center vortex saddlepoints of the Wilson action are also local mimima of this extended action.

(This action, with various choices of c_0, c_1, appears in the Iwaskai, tadpole-improved, Symanzik, and DBW2 extensions of the Wilson action.)