Gravitational waves: theory and sources

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Lectures' content

Lecture 1:

– Einstein equations for weak gravitational fields

– Propagation of GWs: plane wave solution

– Interaction of GWs with free-falling particles

– local Lorentz gauge versus transverse traceless gauge

Lecture 2:

– GW energy-momentum (pseudo) tensor

– Quadrupolar wave generation in linearized Einstein theory

– Indirect detection of GWs: the Hulse-Taylor binary

Lectures' content

Lecture 3:

– **Angular distribution of GWs emitted by binaries**

– **Templates to be used in the search of GWs from compact binaries**

– **Extraction of cosmological parameters using binary black holes as standard candles**

– **How to test alternative theories of gravity using GWs**

On Thursday afternoon: Discussion/comments

Lectures' content

Lecture 4:

– GWs from black-hole's and neutron-star's ring down
– GWs from pulsars, supernovae, low-mass X-ray binaries
– Galactic binaries, supermassive black holes, extreme mass ratio binaries

Lecture 5 & 6:

– GWs from the early Universe: typical frequencies and amplitudes
– Amplification of quantum-vacuum fluctuations
– Stochastic GW background from standard inflationary models
– Examples of stochastic GW background from non-standard inflation
– GWs from first order phase transitions and cosmic strings

References

Landau-Lifshitz: *Teoria dei Campi*, **Chap. 11, 13**

B. Schutz: *A first course in general relativity*, **Chap. 8, 9**

S. Weinberg: *Gravitation and Cosmology*, **Chap. 7,10**

Misner-Thorne-Wheeler: *Gravitation*, **Chap. 8**

Course by Kip Thorne in 2002 at Caltech: Lectures 4, 5 & 6

Brief summary of Einstein equations

$$S = S_g + S_m$$

$$S_g = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R \qquad\qquad -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}$$

$\eta_{\mu\nu} = (-,+,+,+)$ **with** $\mu,\nu = 0,1,2,3$ **and** $i,j = 1,2,3$

by imposing the principle of minimal action

$$\int (G_{\mu\nu} - \frac{8\pi G}{c^4}T_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g} = 0$$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\,T_{\mu\nu}$$

Brief summary of Einstein equations [continued]

$$R^{\nu}_{\mu\rho\sigma} = \frac{\partial \Gamma^{\nu}_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\nu}_{\mu\rho}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\lambda\rho}\,\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\,\Gamma^{\lambda}_{\mu\rho}$$

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\lambda\nu}}{\partial x^{\rho}} + \frac{\partial g_{\lambda\rho}}{\partial x^{\nu}} - \frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} \right)$$

more explicitly:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu}\partial x^{\rho}} + \frac{\partial^2 g_{\nu\rho}}{\partial x^{\mu}\partial x^{\sigma}} - \frac{\partial^2 g_{\mu\rho}}{\partial x^{\nu}\partial x^{\sigma}} - \frac{\partial^2 g_{\nu\sigma}}{\partial x^{\mu}\partial x^{\rho}} \right)$$

$$+\frac{1}{2} g_{\lambda\alpha} \left(\Gamma^{\lambda}_{\nu\rho}\,\Gamma^{\alpha}_{\mu\sigma} - \Gamma^{\lambda}_{\nu\sigma}\,\Gamma^{\alpha}_{\mu\rho} \right)$$

Bianchi identity: $R^{\lambda}_{\mu\nu\rho;\sigma} + R^{\lambda}_{\mu\sigma\nu;\rho} + R^{\lambda}_{\mu\rho\sigma;\nu} = 0$

Brief summary of Einstein equations [continued]

Ricci tensor: $R_{\mu\nu} = g^{\rho\sigma}\, R_{\rho\mu\sigma\nu}$

Scalar tensor: $R = g^{\mu\nu} R_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

- **Non-linear equations with well-posed initial value structure**

- $4 \times 4 = 16$ **differential equations, but** $G_{\mu\nu}$ **and** $T_{\mu\nu}$ **are symmetric tensors** $\Rightarrow 10$ **differential equations, but because of Bianchi identity** ${G_{\mu\nu}}^{;\nu} = 0 \Rightarrow 6$ **differential equations to be solved when** $T_{\mu\nu}$ **is given**

Einstein equations for weak gravitational fields in flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho}\right) + \mathcal{O}(|h|^2)$$

$$G_{\nu\sigma} = R_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\,R =$$

$$\frac{1}{2}\left[{h_{\mu\sigma,\nu}}^{\mu} + {h_{\mu\nu,\sigma}}^{\mu} - h_{,\nu\sigma} - {h_{\nu\sigma,\mu}}^{\mu} - \eta_{\nu\sigma}{h_{\mu\alpha,}}^{\alpha\mu} + \eta_{\nu\sigma}{h_{,\alpha}}^{\alpha} + \mathcal{O}(|h|^2)\right]$$

Lorentz gauge can always be imposed ...

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\, h$$

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \eta^{\mu\rho}\,\xi^{\nu}{}_{,\rho} - \eta^{\lambda\nu}\,\xi^{\mu}{}_{,\lambda} + \eta^{\mu\nu}\xi^{\rho}{}_{,\rho}$$

$$\bar{h}'^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \eta^{\lambda\nu}\,\xi^{\mu}{}_{,\lambda\nu}$$

$$\eta^{\lambda\nu}\,\xi^{\mu}{}_{,\lambda\nu} = \bar{h}^{\mu\nu}{}_{,\nu}$$

Imposing transverse-traceless gauge

We choose $\xi_\mu = B_\mu \, e^{ik_\alpha x^\alpha}$ with $k_\alpha k^\alpha = 0$ $(\eta^{\rho\nu}\partial_\rho\partial_\nu\xi_\mu = 0)$

$$\mathcal{A}'_{\mu\nu} = \mathcal{A}_{\mu\nu} - iB_\mu k_\nu - iB_\nu k_\mu + i\eta_{\mu\nu}B^\rho k_\rho$$

We impose:

1. $\mathcal{A}'_{\mu\nu}k^\nu = 0$
2. $\mathcal{A}'_{\mu\nu}\eta^{\mu\nu} = 0$
3. If U^ν is a constant timelike unit vector $(U_\nu U^\nu = -1)$ we impose $\mathcal{A}'_{\mu\nu}U^\mu = 0$

This set of equations determine B_μ

Linearly and circularly polarized gravitational waves

- **Linearly polarized GW:**

$$\mathbf{e}_+ = \mathbf{e}_x \wedge \mathbf{e}_x - \mathbf{e}_y \wedge \mathbf{e}_y \qquad \text{and} \qquad \mathbf{e}_\times = \mathbf{e}_x \wedge \mathbf{e}_y + \mathbf{e}_y \wedge \mathbf{e}_x$$

$$(\mathbf{u} \wedge \mathbf{v})(\lambda, \mathbf{q}) = (\lambda \cdot \mathbf{u})(\mathbf{q} \cdot \mathbf{v})$$

- **Circularly polarized GW:**

$$\mathbf{e}_R = \mathbf{e}_+ + i\mathbf{e}_\times \qquad \text{and} \qquad \mathbf{e}_L = \mathbf{e}_+ - i\mathbf{e}_\times$$

Equation of geodesic deviation

Pair of nearby freely-falling particles traveling on trajectories $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu$

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau}$$

$$0 = \frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x + \xi) \frac{d(x^\mu + \xi^\mu)}{d\tau} \frac{d(x^\lambda + \xi^\lambda)}{d\tau}$$

Taking the difference and limiting to first order in ξ

$$\nabla_U \nabla_U \xi^\lambda = R^\lambda_{\nu\mu\rho} \xi^\mu U^\nu U^\rho \qquad U^\alpha = \frac{dx^\alpha}{d\tau}$$

From TT gauge to local Lorentz gauge: non-static tidal potential

If $\bar{x}^\mu$ and $\bar{g}^{\mu\nu}$ refer to TT gauge $(h_{xy}^{\rm TT} = 0, h_{xx}^{\rm TT} \equiv h^{\rm TT} \neq 0)$:

$$\bar{g}^{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{\rm TT} & 0 & 0 \\ 0 & 0 & -h^{\rm TT} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The local Lorentz metric $g_{\mu\nu}$ should reduce to Minkowski metric at the origin and all its first derivatives must vanish

$$\bar{t} = t - \dot{h}^{\rm TT}\,(x^2 - y^2)/4$$
$$\bar{x} = x - h^{\rm TT}\,x/2$$
$$\bar{y} = y + h^{\rm TT}\,y/2$$
$$\bar{z} = z + \dot{h}^{\rm TT}\,(x^2 - y^2)/4$$

$$g^{\mu\nu} = \eta_{\mu\nu} - 2 \begin{pmatrix} \Phi(t) & 0 & 0 & \Phi(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi(t) & 0 & 0 & \Phi(t) \end{pmatrix}$$

$$\Phi(t) = -\tfrac{1}{4}\ddot{h}^{\rm TT}\,(x^2 - y^2) \Rightarrow \frac{d^2\xi^j}{dt^2} = -\frac{\partial^2\Phi}{\partial x^j \partial x^k}\,\xi^k$$

GWs interacting with free-falling particles

- If the two particles are originally separated along x

$$\frac{\partial^2 \xi^x}{\partial^2 t} = \frac{L}{2} \frac{\partial^2 h_{xx}^{\rm TT}}{\partial^2 t}$$

$$\frac{\partial^2 \xi^y}{\partial^2 t} = \frac{L}{2} \frac{\partial^2 h_{xy}^{\rm TT}}{\partial^2 t}$$

- If the two particles are originally separated along y

$$\frac{\partial^2 \xi^x}{\partial^2 t} = \frac{L}{2} \frac{\partial^2 h_{xy}^{\rm TT}}{\partial^2 t}$$

$$\frac{\partial^2 \xi^y}{\partial^2 t} = -\frac{L}{2} \frac{\partial^2 h_{xx}^{\rm TT}}{\partial^2 t}$$

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Content of Lecture 2

– **Energy-momentum pseudo tensor**

– **Quadrupolar wave generation in linearized Einstein theory**

– **Indirect detection of GWs: the Hulse-Taylor binary**

Energy-momentum pseudotensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| << 1$$

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, R^{(1)} = \frac{8\pi G}{c^4}\,(T_{\mu\nu} + \tau_{\mu\nu})$$

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\left(\frac{\partial^2 {h_\lambda}^\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 {h^\lambda}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 {h^\lambda}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right)$$

$$\tau_{\mu\nu} = \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, R - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\, R^{(1)}\right)$$

$$t^{\nu\lambda} = \eta^{\nu\mu}\,\eta^{\lambda\alpha}\,(T_{\mu\alpha} + \tau_{\mu\alpha})$$

$t^{\mu\nu}$ is locally conserved $\Rightarrow {t^{\mu\nu}}_{,\nu} = 0$

[matter EMT satisfies ${T^{\mu\nu}}_{;\nu} = 0$]

Quadrupole nature of GW emission [naive way]

EM theory: Luminosity $\propto \ddot{\mathbf{d}}^2$ $\mathbf{d} = e\,\mathbf{x} \Rightarrow$ electric dipole moment

- GW theory: electric dipole moment $\Rightarrow$ mass dipole moment

$\mathbf{d} = \sum_i m_i\,\mathbf{x}_i \Rightarrow \dot{\mathbf{d}} = \sum_i m_i\,\dot{\mathbf{x}}_i = \mathbf{P}$

Conservation of momentum $\Rightarrow$ no mass dipole radiation exists in GR

- GW theory: magnetic dipole moment $\Rightarrow$ current dipole moment

$\mu = \sum_i m_i\,\mathbf{x}_i \times \dot{\mathbf{x}}_i = \mathbf{J}$

Conservation of angular momentum $\Rightarrow$ no current dipole radiation exists in GR

Derivation of quadrupole formula

Eq. (1): $$\frac{\partial T_{0i}}{\partial x^i} - \frac{\partial T_{00}}{\partial x^0} = 0$$

Eq. (2): $$\frac{\partial T_{ji}}{\partial x^i} - \frac{\partial T_{j0}}{\partial x^0} = 0$$

multiplying Eq. (2) by x^k and integrating on all space

$$\int x^k \frac{\partial T_{ji}}{\partial x^i} dV = \int x^k \frac{\partial T_{j0}}{\partial x^0} dV = \frac{\partial}{\partial x^0} \int x^k \, T_{j0} dV$$

integrating by parts the LHS and assuming that the source decays sufficiently fast at ∞

$$- \int T_{ji} \, \delta^k_i \, dV = \frac{\partial}{\partial x^0} \int x^k \, T_{j0} dV$$

symmetrizing $\Rightarrow \int T_{kj} \, dV = -\frac{1}{2} \frac{\partial}{\partial x^0} \int (x_k \, T_{j0} + x_j \, T_{k0}) dV$

Derivation of quadrupole formula [continued]

Eq. (1): $$\frac{\partial T_{0i}}{\partial x^i} - \frac{\partial T_{00}}{\partial x^0} = 0$$

Eq. (2): $$\frac{\partial T_{ji}}{\partial x^i} - \frac{\partial T_{j0}}{\partial x^0} = 0$$

multiplying Eq. (1) by $x_k\, x_j$ and integrating on all space

$$\frac{\partial}{\partial x^0} \int T_{00}\, x_k\, x_j dV = \int \frac{\partial T_{0i}}{\partial x^i} x_k\, x_j\, dV$$

integrating by parts the RHS and assuming that the source decays sufficiently fast at ∞

$$\frac{\partial}{\partial x^0} \int T_{00}\, x_k\, x_j dV = - \int (x_k\, T_{j0} + x_j\, T_{k0}) dV$$

combining $\Rightarrow \int T_{kj} dV = \frac{1}{2} \frac{\partial^2}{\partial x^{0\,2}} \int T_{00}\, x_k\, x_j\, dV$

Derivation of quadrupole formula [continued]

Eq. (1): $$\frac{\partial T_{0i}}{\partial x^i} - \frac{\partial T_{00}}{\partial x^0} = 0$$

Eq. (2): $$\frac{\partial T_{ji}}{\partial x^i} - \frac{\partial T_{j0}}{\partial x^0} = 0$$

multiplying Eq. (1) by x_k and integrating on all space

$$\frac{\partial}{\partial x^0} \int T_{00}\, x_k\, dV = \int \frac{\partial T_{0i}}{\partial x^i}\, x_k\, dV$$

integrating by parts the RHS and assuming that the source decays sufficiently fast at ∞

$$\int T_{0k} dV = -\frac{\partial}{\partial x^0} \int T_{00}\, x_k dV$$

Total power radiated in GWs ...

$$
\begin{aligned}
2\,\overline{\epsilon_{ij}\,\epsilon_{kl}} &= 2\,\frac{1}{4}\,\{n_i\,n_j\,n_k\,n_l + n_i\,n_j\,\delta_{kl} + n_k\,n_l\,\delta_{ij} - \\
&\quad (n_i\,n_k\,\delta_{jl} + n_j\,n_k\,\delta_{il} + n_i\,n_l\,\delta_{jk} + n_j\,n_l\,\delta_{ik}) \\
&\quad -\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{jk}\,\delta_{il}\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{n_i\,n_j} &= \frac{1}{3}\delta_{ij} \\
\overline{n_i\,n_j\,n_k\,n_l} &= \frac{1}{5}(\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})
\end{aligned}
$$

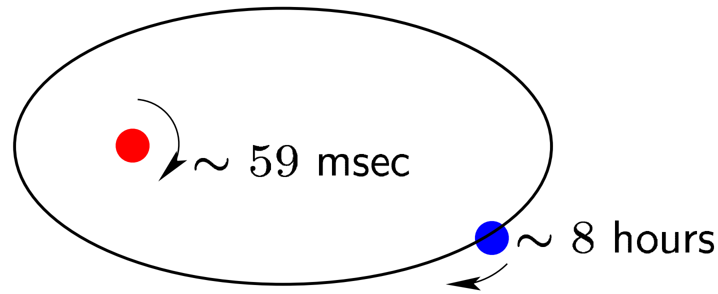
GW radiated in non-precessing binaries in elliptical orbits

$$-\frac{dE}{dt} = \frac{32}{5}\,\frac{G^4}{c^5}\,\frac{m_1^2\,m_2^2\,M}{a^5\,(1-e^2)^{7/2}}\left(1+\frac{73}{24}\,e^2+\frac{37}{96}\,e^4\right) \qquad a=\frac{r_1+r_2}{2}$$

Indirect observation of gravitational wav

Neutron Binary System: PSR 1913 +16 - Timing Pulsars

Hulse & Taylor discovery (1974)



Separated by $\sim 10^6$ Km, $m_1 = 1.4 M_\odot$, $m_2 = 1.36 M_\odot$, $\epsilon = 0.617$

- **Prediction from GR: rate of change of orbital period**
- **Emission of gravitational waves:**
  - **due to loss of orbital energy**
  - **orbital decay in agreement with GR at the level of 0.5%**

Hulse-Taylor binary: cumulative shift of periastron time

To show agreement with GR, they compared the *observed* **phase of the orbit with a theoretical template phase as function of time**

If f_b **varies slowly with time, then to first order in a Taylor expansion**

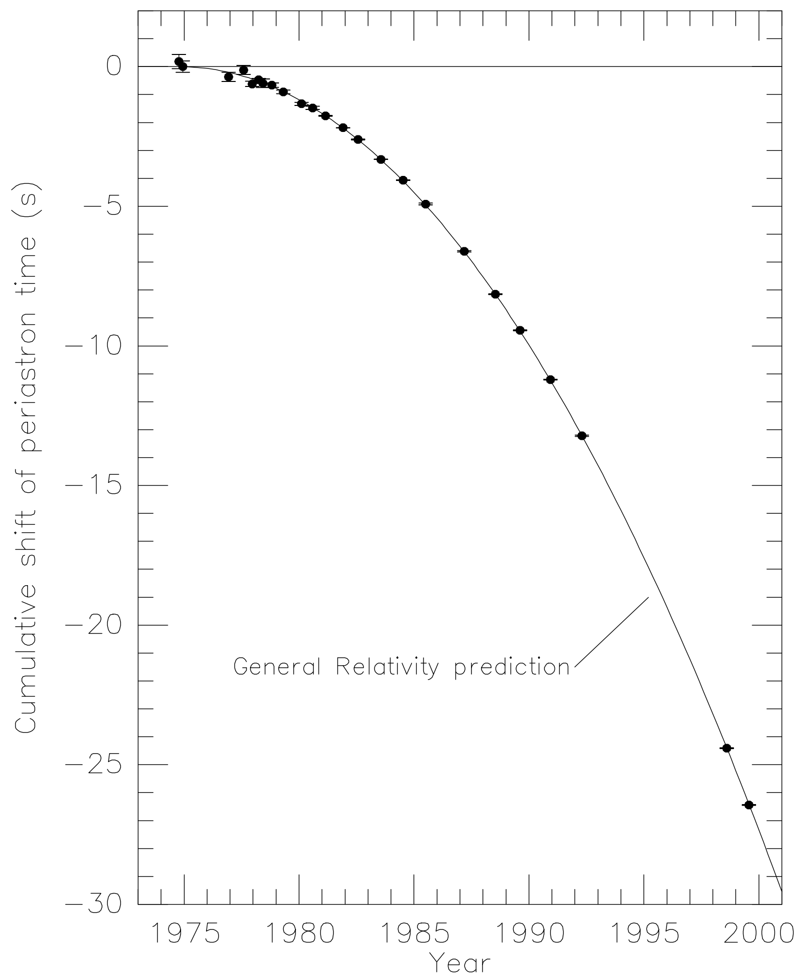
$$\Phi_b(t) = 2\pi\, f_b\, t + \pi \dot{f}_b\, t^2$$

Assuming that t_P **is the periastron passage time defined as**

$$\Phi(t_p) = 2\pi N \qquad N \text{ being an integer}$$

$$2\pi N = 2\pi f_b\, t_p + \pi \dot{f}_b\, t_p^2 \quad \Rightarrow \quad t_p - N/f_b = -\tfrac{1}{2}\dot{f}_b/f_b\, t^2$$

Hulse-Taylor binary: cumulative shift of periastron time



[from Taylor & Weisberg 2000]

Known binaries

PSR B1534+12 (Wolszczan 1991): NS-NS

rot. period 38 ms

orb. period 10 h

$e = 0.27$

$\Delta\phi = 4.2^o/yr$ $\qquad \dot{P} = -2.4 \times 10^{-12}$

distance $\sim 1kpc$

PSR J1141-6545 (discovery 99/timing Bailes et al. 2003): NS-WD

rot. period 400 ms

orb. period 4 h and 45 min

$e = 0.17$

$\Delta\phi = 1.8^o/yr$ $\qquad \dot{P} = -1.4 \times 10^{-13}$

distance $> 4kpc$

Known double pulsar binaries

PSR J0737-3039 (timing Burgay et al. 2003/double pulsar Lyne et al. 2004): NS-NS

rot. period A 23 ms

rot. period B 2.8 ms

orb. period 2 h and 27 min (will merge in 85 Myr!)

$e = 0.088$

$\Delta\phi = 17^o/yr$ (large!) $\dot{P}$ is not known yet

distance $\sim 0.6 kpc$ (close!)

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Content of Lecture 3

– **Angular distribution of GWs emitted by binaries**

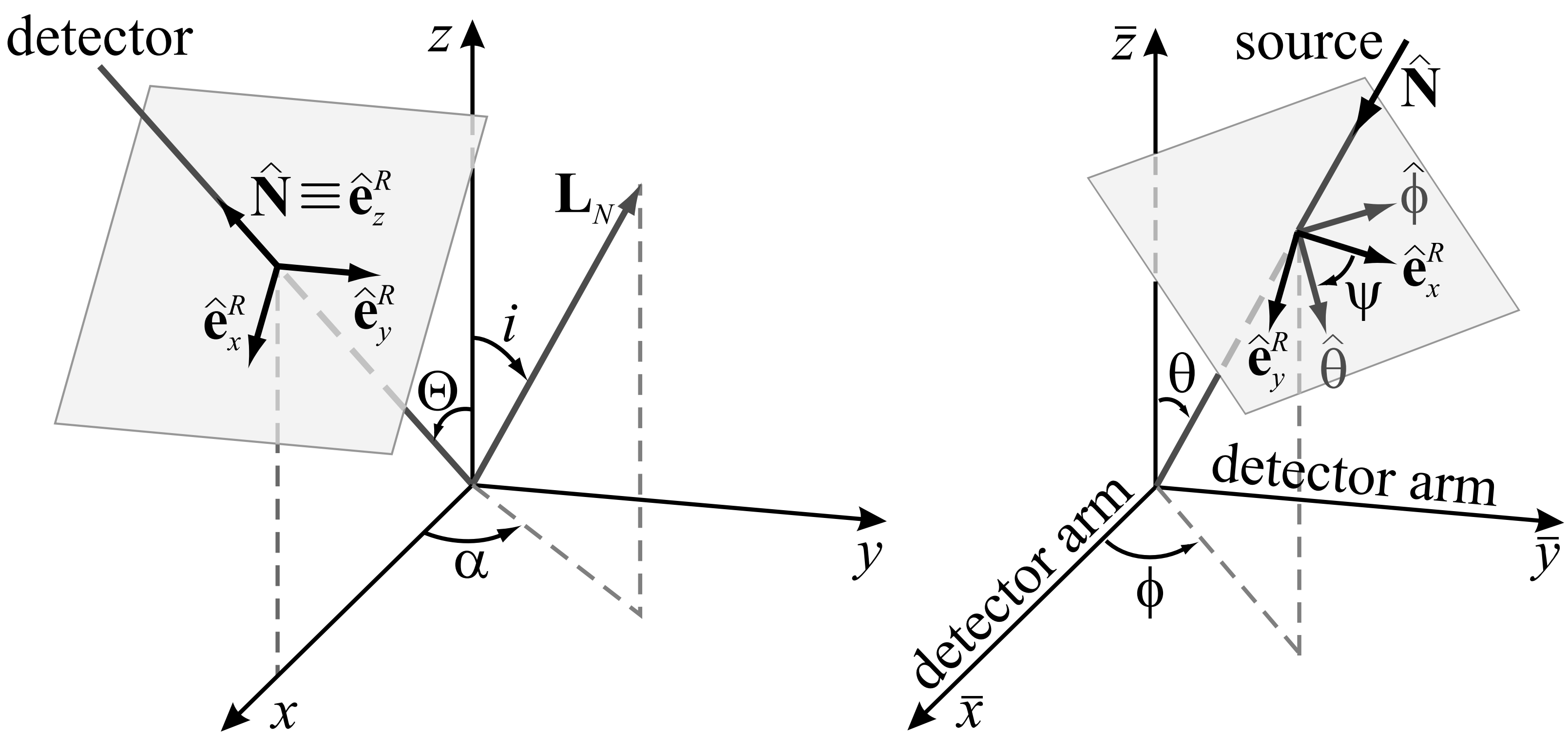
– **Templates to be used in the search of GWs from compact binaries**

– **Post-Newtonian corrections; tidal effects; spin effects, etc.**

– **Extraction of cosmological parameters using binary black holes as standard candles**

– **How to test alternative theories of gravity using GWs**

Binary-detector orientation



Angular distribution of GWs emitted by binaries

Binary frame: $\{\hat{e}^S_x, \hat{e}^S_y, \hat{e}^S_z\}$ Radiative frame: $\{\hat{e}^R_x, \hat{e}^R_y, \hat{e}^R_z = \hat{N}\}$

$$\hat{e}^R_x = -\sin\varphi\,\hat{e}^S_x + \cos\varphi\,\hat{e}^S_y$$

$$\hat{e}^R_y = -\cos\varphi\,\cos\Theta\,\hat{e}^S_x - \sin\varphi\,\cos\Theta\,\hat{e}^S_y + \sin\Theta\,\hat{e}^S_z$$

$$\hat{e}^R_z = \cos\varphi\,\sin\Theta\,\hat{e}^S_x + \sin\varphi\,\sin\Theta\,\hat{e}^S_y + \cos\Theta\,\hat{e}^S_z = \hat{N}$$

Pattern functions for ground-based GW interferometers

$$F_+ = \tfrac{1}{2}(1+\cos^2\theta)\,\cos 2\phi\,\cos 2\psi - \cos\theta\,\sin 2\phi\,\sin 2\psi$$

$$F_\times = \tfrac{1}{2}(1+\cos^2\theta)\,\cos 2\phi\,\sin 2\psi + \cos\theta\,\sin 2\phi\,\cos 2\psi$$

$$\widetilde{F}_+ = \frac{(1+\cos^2\Theta)\,F_+}{[(1+\cos^2\Theta)^2\,F_+^2+4\cos^2\Theta\,F_\times^2]^{1/2}}$$

$$\widetilde{F}_\times = \frac{4\cos^2\Theta\,F_\times}{[(1+\cos^2\Theta)^2\,F_+^2+4\cos^2\Theta\,F_\times^2]^{1/2}}$$

$$\widetilde{F}_+^2 + \widetilde{F}_\times^2 = 1 \quad \Rightarrow \quad \cos\alpha \equiv \widetilde{F}_+ \quad \sin\alpha \equiv \widetilde{F}_\times$$

$$\mathcal{A} = [(1+\cos^2\Theta)^2\,F_+^2 + 4\cos^2\Theta\,F_\times^2]^{1/2}$$

Inspiral signals are "chirps"

- Mass-quadrupole approximation: $h_{\rm GW}^{{\rm TT}\,ij} = \frac{2G}{R_0 c^4}\,\mathcal{P}^{i\ j}_{\ k\ m}\,\ddot{Q}^{km}$

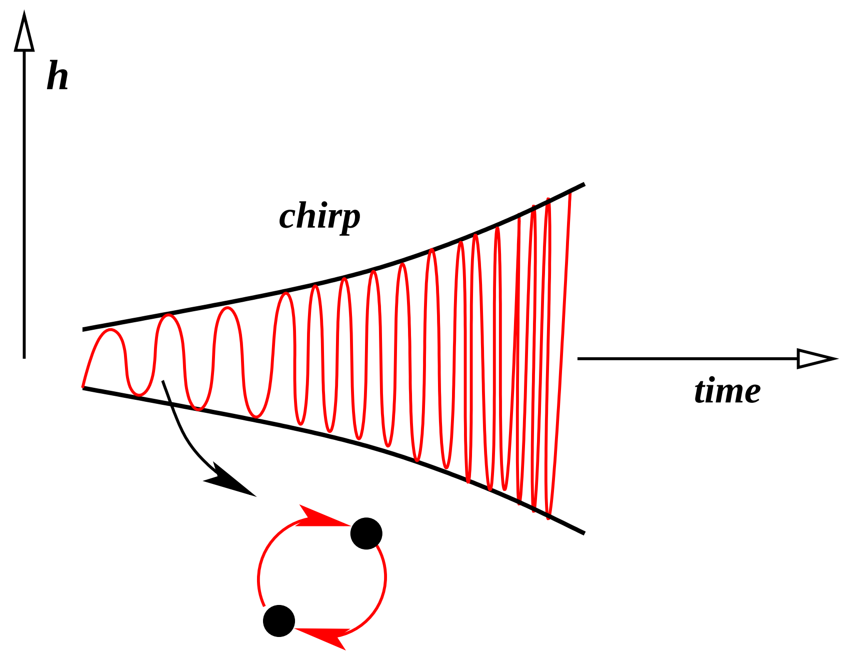
$$Q^{km} = \int d^3x\,\rho\,(x^k\,x^m - r^2\,\delta^{km})$$

$$h_{\rm GW} \propto \omega^{2/3}\,\cos 2\Phi$$

for quasi-circular orbits: $\omega^2 = \dot{\Phi}^2 = \frac{GM}{r^3}$

$$\dot{\omega} \propto \omega^{11/3}$$

Chirp: The signal continuously changes its frequency and the power emitted at any frequency is small



What determines the inspiraling waveform

$h \propto v^2 \cos 2(\Phi + \Phi_0)$ depends on 4 parameters: $\overbrace{\underbrace{(m_1, m_2)}_{\text{slow numerical search}}}^{\text{intrinsic parameters}}$ and $\overbrace{\underbrace{(\Phi_0, t_c)}_{\text{fast analytical search}}}^{\text{extrinsic parameters}}$

Keplerian velocity: $v = (M\dot{\Phi})^{1/3}$ $\qquad M = m_1 + m_2$

Energy-balance equation: $\frac{dE(v)}{dt} = -\mathcal{F}(v)$

$E(v)$ and $\mathcal{F}(v)$ known as a Post-Newtonian expansion in v/c

Two crucial ingredients:

$E(v) \to$ center-of-mass energy $\qquad \mathcal{F}(v) \to$ gravitational flux

Limiting to quasi-circular orbits

It is generally assumed that gravitational radiation reaction will circularize the orbit by the time the binary is close to the final coalescence [Lincoln & Will 90]

Hulse-Taylor binary has currently an eccentricity of $\epsilon_i = 0.617$. In few hundred millions years (!) when the orbital frequency ~ 10 Hz, the eccentricity will be $\epsilon_f \sim 10^{-6}$

Gravitational waveform in adiabatic limit

$$\tilde{h}(f) = \int_{-\infty}^{+\infty} e^{2\pi i f t}\, h(t) = \int_{-\infty}^{+\infty} v^2(t)\, \left[e^{2\pi i f t + 2 i \Phi(t)} + e^{2\pi i f t - 2 i \Phi(t)}\right]$$

PN predictions in the stationary phase approximation (SPA)

$$\tilde{h}_{\rm SPA}(f) = \mathcal{A}_{\rm SPA}(f)\, e^{i\psi_{\rm SPA}(f)} \qquad \mathcal{A}_{\rm SPA}(f) = f^{-7/6}$$

$$\psi_{\rm SPA}(f) = 2\pi f\, t_{\rm ref} - \Phi_{\rm ref} - \pi/4 + 2\int_v^{v_{\rm ref}} (v_{\rm ref}^3 - v^3)\, \frac{E'(v)}{\mathcal{F}(v)}\, dv$$

$$v = (\pi M f)^{1/3}$$

Dynamics of inspiraling compact binaries

Approximating compact bodies (BH and/or NS) with point particles

Two-body equation of motions:

[Damour & Deruelle 81, 82; Damour, Blanchet & Iyer 95; Will & Wiseman 95; Blanchet 96]

[Jaranowski & Schäfer 98,99; Damour, Jaranowski & Schäfer 00, 01; 02; Blanchet & Faye 00,01]

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{GM}{r^2}\left[-\widehat{n} + A_{\rm 1PN} + A_{\rm 2PN} + A_{\rm 2.5PN} + A_{\rm 3PN} + A_{\rm 3.5PN} + \cdots\right]$$

$A_{\rm nPN} \to \mathcal{O}(\epsilon^n)$ relative to Newtonian term

$\epsilon = \frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \to$ post-Newtonian parameter [low velocity, weak gravity]

Gravitational waveforms

Gravitational field far from source:

[Wagoner & Will 76; Wiseman 92; Blanchet, Damour, Iyer, Will & Wiseman 96; Blanchet 96, 98]

[Blanchet, Faye, Iyer & Jouget 01; Blanchet, Damour, Esposito-Farese and Iyer 03-04]

$$h^{ij} = \frac{2GM}{c^4 R_0} \left[Q^{ij} + Q^{ij}_{0.5\mathrm{PN}} + Q^{ij}_{1\mathrm{PN}} + Q^{ij}_{1.5\mathrm{PN}} + Q^{ij}_{2\mathrm{PN}} + Q^{ij}_{2.5\mathrm{PN}} + Q^{ij}_{3\mathrm{PN}} + Q^{ij}_{3.5\mathrm{PN}} \right]$$

$R_0 \to$ distance from the source

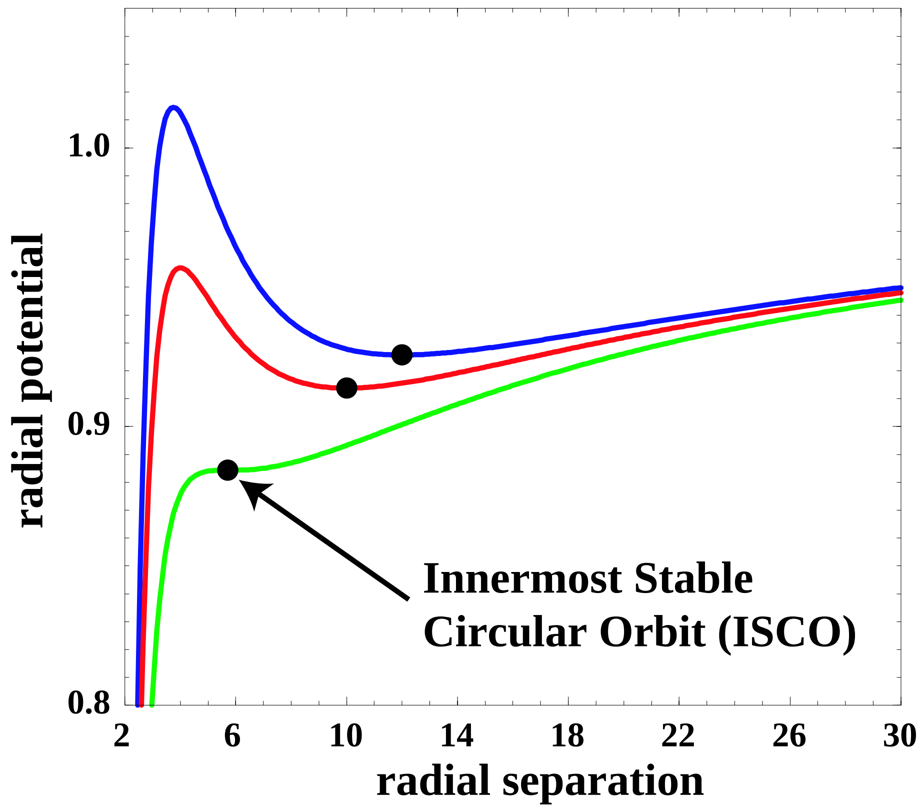
Phasing within Post-Newtoniann expansion in GR

$$\begin{aligned}\psi_{\rm SPA}(f) &= 2\pi f t_c - \phi_c + \frac{3}{128}\,(\pi\mathcal{M}f)^{-5/3}\,\{1+\\ &+ \left(\frac{3715}{756}+\frac{55}{9}\eta\right)\,\eta^{-2/5}\,(\pi\mathcal{M}f)^{2/3} - 16\pi\,\eta^{-3/5}\,(\pi\mathcal{M}f) + 4\beta\,\eta^{-3/5}\,(\pi\mathcal{M}f)\\ &+ \left(\frac{15293365}{508032}+\frac{27145}{504}\eta+\frac{3085}{72}\eta^2\right)\,\eta^{-4/5}\,(\pi\mathcal{M}f)^{4/3} - 10\sigma\,\eta^{-4/5}\,(\pi\mathcal{M}f)^{4/3}\Big\}\end{aligned}$$

$$\beta = \frac{1}{12}\sum_{i=1}^{2}\chi_i\left[113\frac{m_i^2}{M^2}+75\eta\right]\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}_i$$

$$\sigma = \frac{\eta}{48}\chi_1\chi_2\left(-27\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2+721\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}_1\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}_2\right)$$

Radial potential in black-hole spacetime



The issue of spin effects

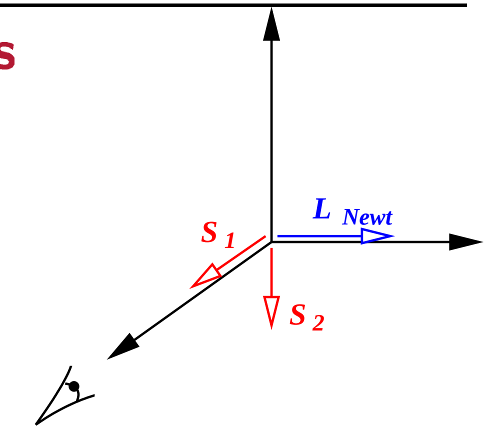
- **Do black holes in binaries carry spin? How large is the spin?**
  30–80% of NS/BH could have tilt angles larger than 30^o

- **Spin-orbit coupling causes orbital plane to precess:**
  $\dot{\boldsymbol{S}} \propto \boldsymbol{S} \times L, \qquad \dot{\boldsymbol{L}} \propto \boldsymbol{S} \times \boldsymbol{L}$

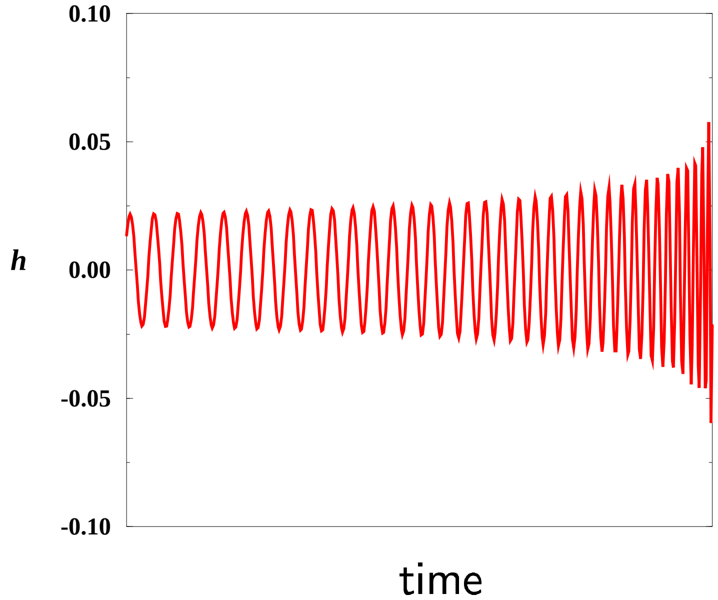
- **If spins large, the precession induces non-negligible, and non-uniform in time, amplitude and phase modulations**

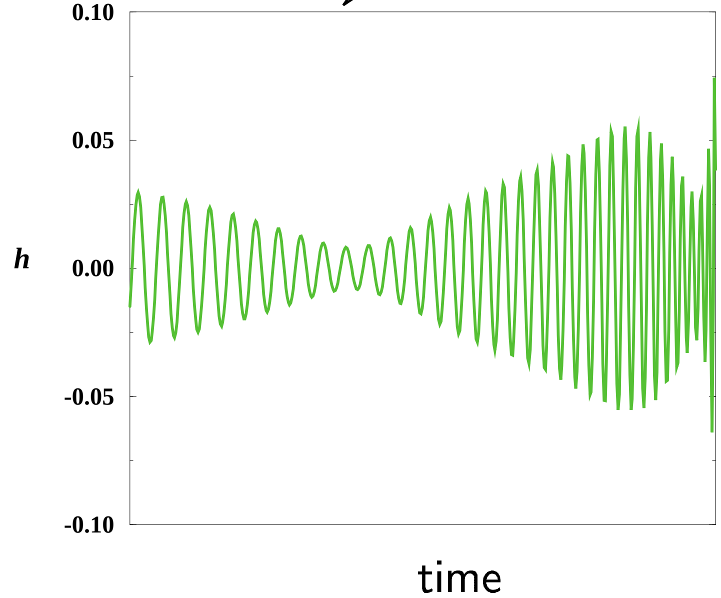
- **Waveforms with spin effects depend on 11 parameters, where 7 are intrinsic $\Rightarrow$ computationally expensive to search over using match-filtering**

Waveforms including spin effects



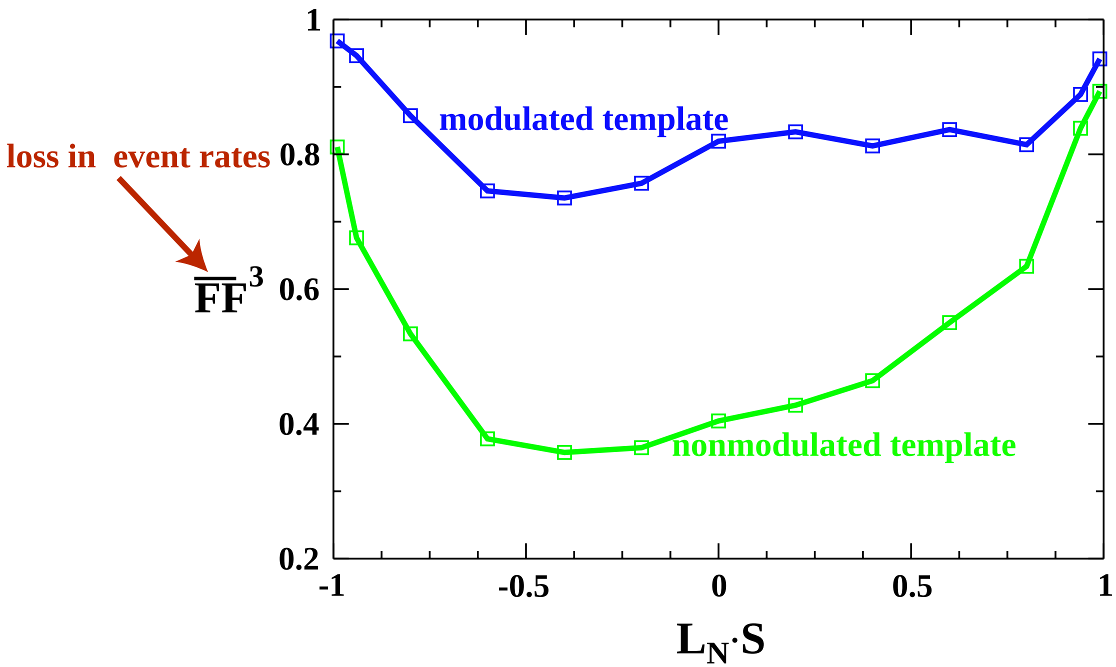
Maximal spins $M = 10M_\odot + 10M_\odot$





Performances for NS/BH binary $(10+1.4)M_\odot$

[AB, Chen & Vallisneri 02]



Phasing in alternative theories of gravity

$$
\begin{aligned}
\psi_{\rm SPA}(f) &= 2\pi f t_c - \phi_c + \frac{3}{128}\,(\pi \mathcal{M} f)^{-5/3}\,\{1+ \\
&- \frac{5\mathcal{S}^2}{84\omega_{\rm BD}}\,\eta^{2/5}\,(\pi\mathcal{M}f)^{-2/3} - \frac{128}{3}\frac{\pi^2 D\,\mathcal{M}}{\lambda_g^2\,(1+z)}\,(\pi\mathcal{M}f)^{2/3} \\
&+ \left(\frac{3715}{756}+\frac{55}{9}\eta\right)\,\eta^{-2/5}\,(\pi\mathcal{M}f)^{2/3} - 16\pi\,\eta^{-3/5}\,(\pi\mathcal{M}f) + 4\beta\,\eta^{-3/5}\,(\pi\mathcal{M}f) \\
&+ \left(\frac{15293365}{508032}+\frac{27145}{504}\eta+\frac{3085}{72}\eta^2\right)\,\eta^{-4/5}\,(\pi\mathcal{M}f)^{4/3} - 10\sigma\,\eta^{-4/5}\,(\pi\mathcal{M}f)^{4/3}\Big\}
\end{aligned}
$$

Gravitational waves: theory and sources

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Content of Lecture 4

– **GWs from black-hole's and neutron-star's ring down**

– **Gws from pulsars, supernovae, low-mass X-ray binaries**

– **Galactic binaries, supermassive black holes, extreme mass ratio binaries**

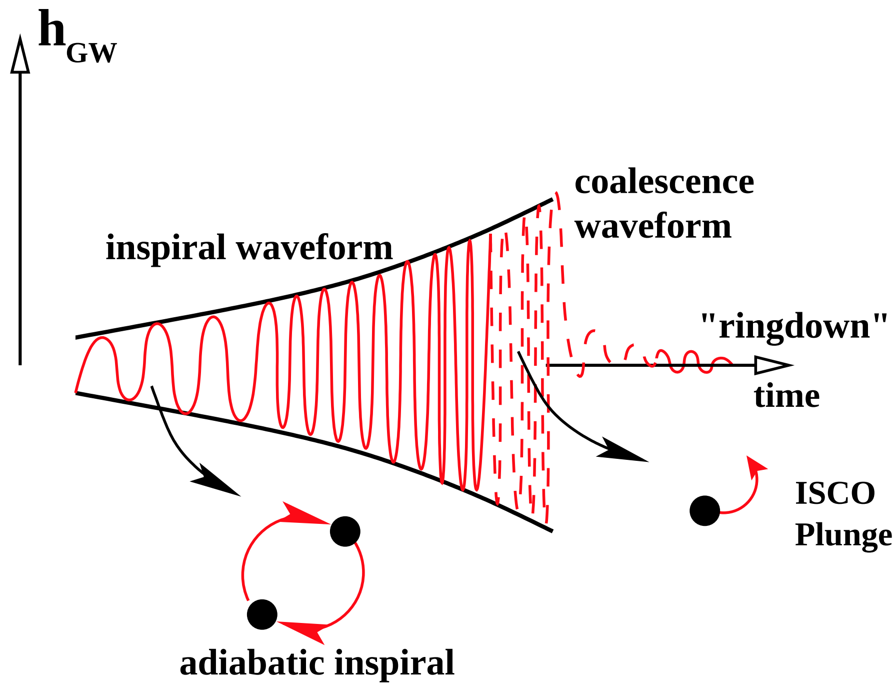
Gravitational waves from compact binaries

- **GW signal: "chirp"** [duration $\sim$ seconds to years] ($f_{\rm GW} \sim 10^{-3}\,{\rm Hz}$–$1{\rm kHz}$)

End-of-inspiral GW frequency

(non-spinning case):

$f_{\rm GW} \sim 4400 M_\odot/M$ Hz



Ring-down phase: quasi-normal modes

Gravitational perturbations of black hole metric

$$ds^2 = g^{(0)}_{\mu\nu}\, dx^\mu\, dx^\nu = -e^{v(r)}\, dt^2 + e^{\lambda(r)}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \qquad \delta G_{\mu\nu} = \frac{8\pi\, G}{c^4}\, \delta T_{\mu\nu}$$

Each $h_{\mu\nu}$ is of the form

$$\Phi = \sum_{l,m} \frac{\Psi_{lm}(r,t)}{r}\, Y_{lm}(\theta, \phi)$$

and perturbation problem reduced to a single wave equation

$$\frac{\partial^2\Psi}{\partial r_*^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = V\,\Psi \qquad r_* = r + 2M\, \log(r/2M - 1)$$

There exist solutions with no-incident radiation and frequencies with positive imaginary part (decay) $\Rightarrow$ quasi-normal modes

Ring-down phase: quasi-normal modes [continued]

Oscillations decay in time because of GW emission

$$\omega_{\rm qnm} \sim 3.2{\rm kHz} \left[1 - 0.63 \left(1 - \frac{S}{M^2}\right)^{3/10}\right]$$

$$\text{Quality factor} \sim \frac{\pi \omega_{\rm qnm}}{\tau_{\rm qnm}} \sim 2 \left(1 - \frac{S}{M^2}\right)^{-\frac{9}{20}}$$

$$h^{\rm GW}_{\rm ring-down} = \mathcal{A}\, e^{-t/\tau_{\rm qnm}} \cos(\omega_{\rm qnm}\, t + \varphi)$$

$\mathcal{A}$ depends on BH's initial distortion

Neutron stars can also ring !

$$\omega_{\rm qnm} = \mathcal{G}(m_{\rm NS}, \Omega, R_{\rm NS}, {\rm EOS})$$

$$1/\tau_{\rm qnm} = \mathcal{F}(m_{\rm NS}, \Omega, R_{\rm NS}, {\rm EOS})$$

Gravitational waves from stellar collapse

- **GW signal: "bursts"** [$\sim$ few mseconds] **or (quasi) "periodic"** ($f_{\rm GW} \sim 1\,{\rm kHz}$–$10{\rm kHz}$)

Supernovae:

- Non-axisymmetric core collapse
- Material in the stellar core may form a rapidly rotating bar-like structure
- Collapse material may fragment into clumps which orbit as the collapse proceeds
- Pulsation modes of new-born NS; ring-down of new-born BH

Dynamics of star very poorly understood

- GW amplitude and frequency estimated using mass- and current-quadrupole moments
- Numerical simulations

Correlations with neutrino flux and/or EM counterparts

Event rates in our galaxy and its companions $\lesssim 30$ yrs

Gravitational-wave strain from non-axisymmetric collapse

$$h_{\rm GW} \simeq 2 \times 10^{-17} \sqrt{\eta_{\rm eff}} \left(\frac{1\,{\rm msec}}{\tau}\right)^{1/2} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{10\,{\rm kpc}}{R_0}\right) \left(\frac{1\,{\rm kHz}}{f_{\rm GW}}\right)$$

$\tau \rightarrow$ duration of emission

efficiency $\eta_{\rm eff} = \frac{\Delta E}{M\,c^2} \sim 10^{-10}$–$10^{-7}$

Gravitational waves from spinning neutron stars: pulsars

- **GW signal: (quasi) "periodic"** $(f_{\rm GW} \sim 10\,{\rm Hz}\text{–}1\,{\rm kHz})$

Pulsars: non-zero ellipticity

$$h_{\rm GW} \simeq 7.7 \times 10^{-26}\,\left(\frac{\epsilon}{10^{-6}}\right)\,\left(\frac{I_{zz}}{10^{45}\,{\rm g\,cm}^2}\right)\,\left(\frac{10\,{\rm kpc}}{R_0}\right)\,\left(\frac{f_{\rm GW}}{1\,{\rm kHz}}\right)^2$$

$\epsilon = \frac{I_{xx}-I_{yy}}{I_{zz}} \to$ ellipticity

– **The crust contributes only 10% of total moment of inertia $\Rightarrow \epsilon_C$ is low**

– **Magnetic fields could induce stresses and generate $\epsilon_M \neq 0$**

Expected ellipticity rather low $\leq 10^{-7}$

- **search for known spinning neutron stars: Vela, Crab, ...**
- **all sky search**

Gravitational waves from spinning neutron stars: LMXBs

Low-Mass X-ray binaries

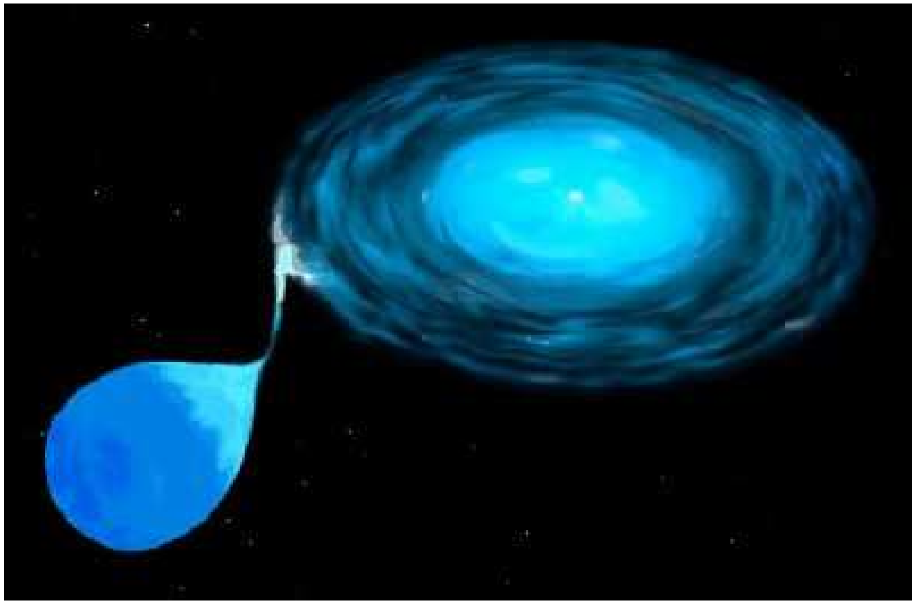
- Observed spin-frequency "locking" between 270–620 Hz, which is lower than break-up frequency ~ 1.5 kHz
- Very old systems, believed to have spun up by accretion torque

Conjecture: gravitational radiation is balancing torque due to accretion

– mass-quadrupole radiation from deformed NS crusts: $f_{\rm GW} = 2 f_{\rm S}$

– mass-current radiation from pulsation modes in NS core: $f_{\rm GW} = 4/3 f_{\rm S}$

Low-mass X-ray Binary



Gravitational radiation balancing torque due to accretion

- **GW strain:** $h_{\rm GW} \propto Q\, f_S^2$
- **Angular-momentum gain by accretion**: $\left(\frac{dJ}{dt}\right)_A = \dot{M}\,(G M R)^{1/2}$

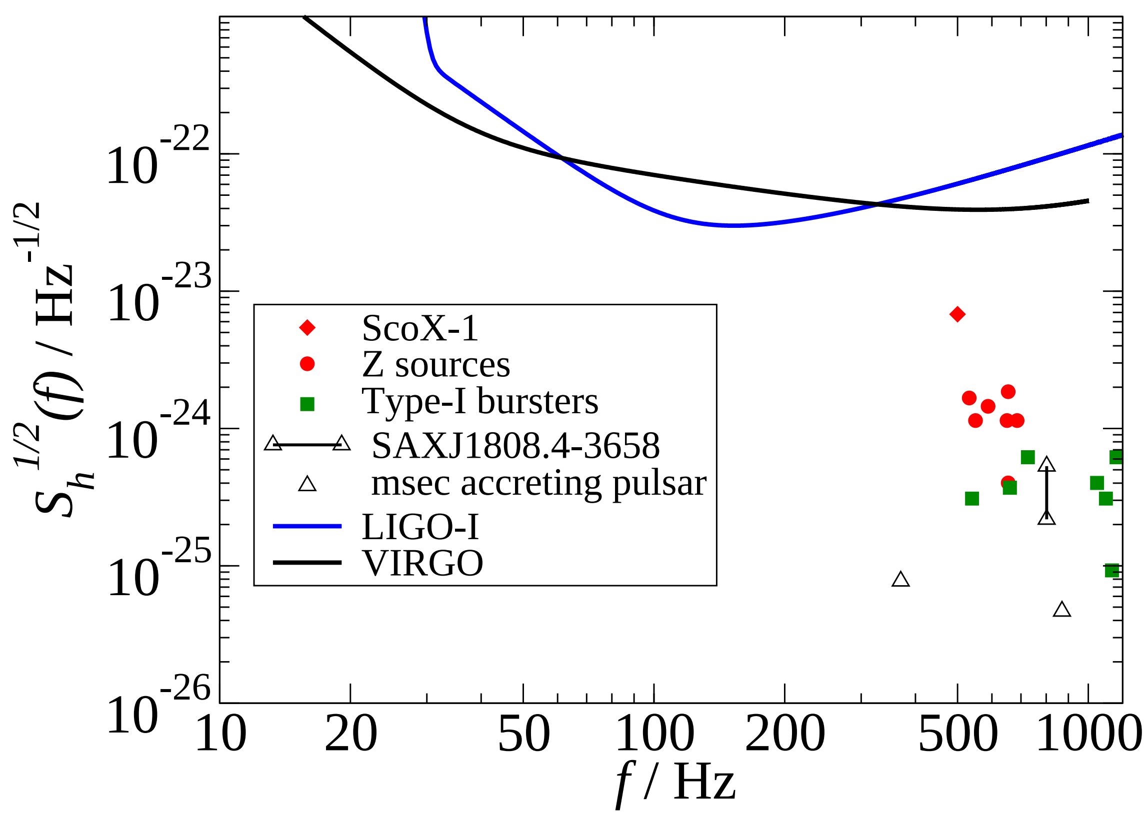
$$\text{Flux}_{\rm X-ray} = \frac{G\,\dot{M}\,M}{4\,\pi\,R\,R_0}$$

- **Angular-momentum loss by GR**: $\left(\frac{dJ}{dt}\right)_{GW} \propto Q^2\, f_S^5$

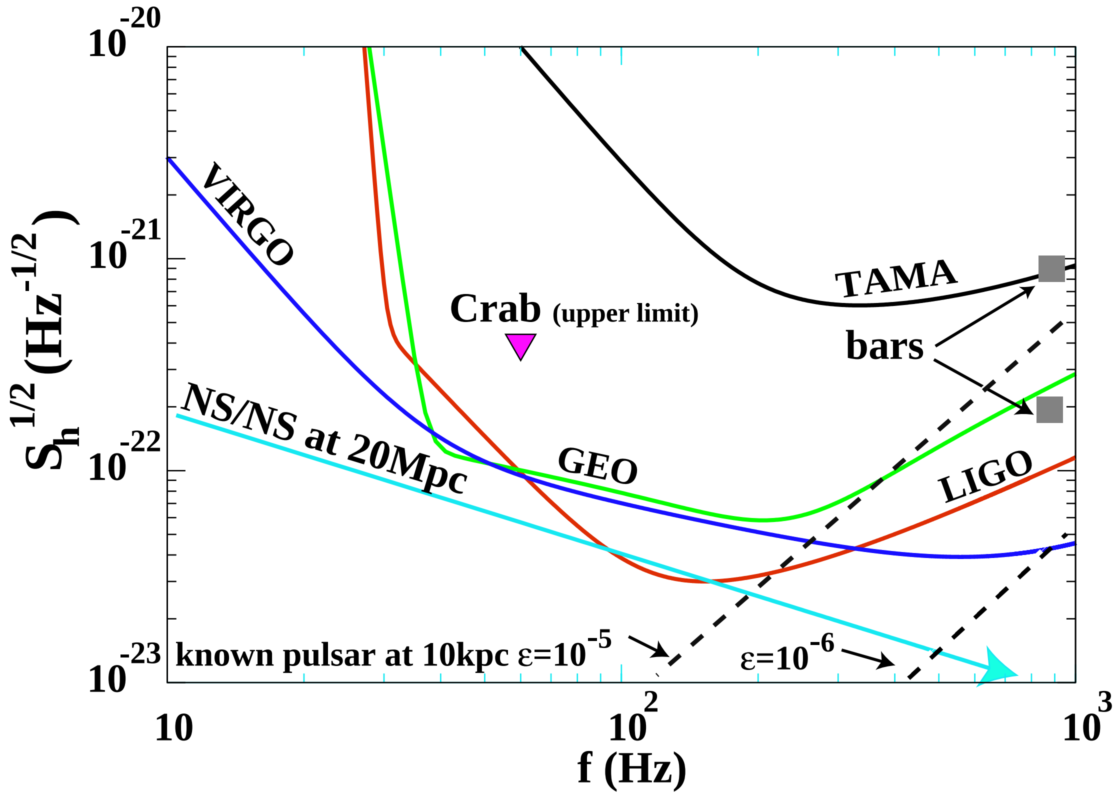
Imposing balance we get quadrupole moment Q and GW strain $h_{\rm GW}$:

$$h_{\rm GW} \simeq 3.5 \times 10^{-27} \left(\frac{300}{f_S}\right)^{1/2} \left(\frac{\text{Flux}_{\rm X-ray}}{10^{-8}\,{\rm erg\,cm^{-2}\,sec^{-1}}}\right)^{1/2}$$

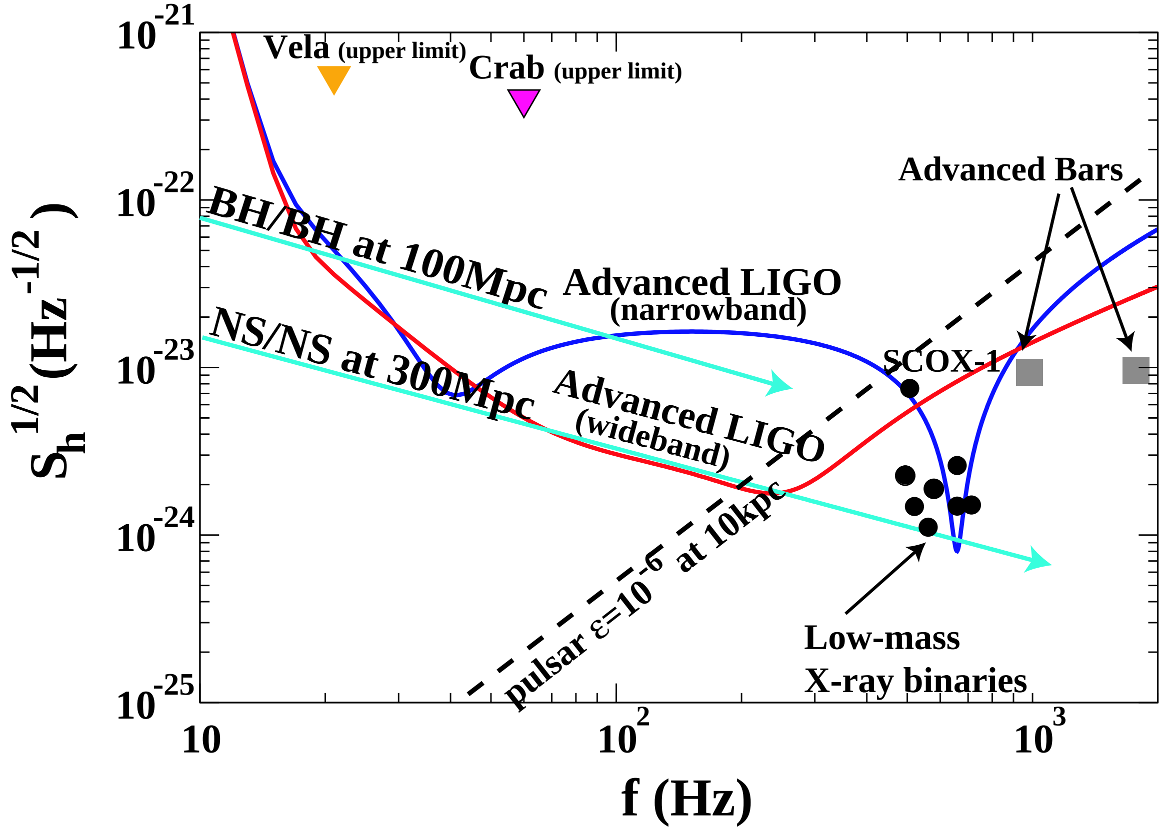
LMXBs and first generation detectors



Sources for first generation detectors



Sources for advanced detectors



Known, short-period binary stars in our galaxy

There are currently a dozen galactic binaries with GW frequencies above 0.1 mHz

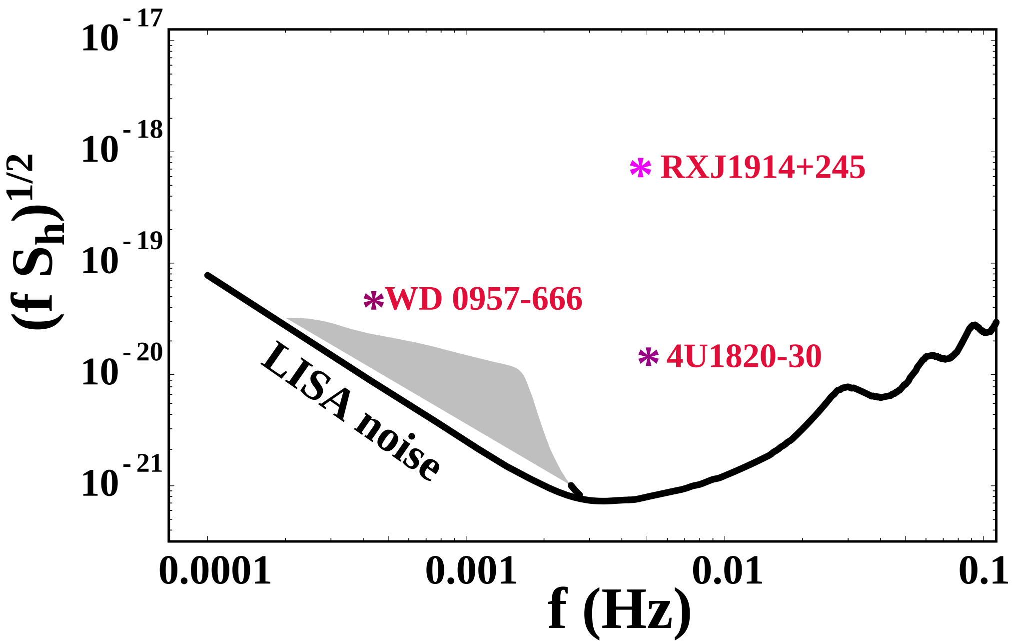
Monochromatic gravity-wave signals

– **These sources will provide an important test LISA is functioning as expected (verification binaries)**

– **Test of general relativistic theory of GW emission in weak-gravity limit**

– **Compare with predictions from optically measured orbital motion**

Three known galactic binaries



– **WD 0957-666:** WD-WD binary with masses $0.37 M_\odot$ and $0.32 M_\odot$, at 100 pc from Earth and time to merger 2×10^8 years

– **RXJ1914+245 (Am CVn binary):** WD $(0.6 M_\odot)$ accreting from low-mass, helium-star companion $(0.07 M_\odot)$ at 100 pc from Earth

– **4U1820-30:** star $(< 0.1 M_\odot)$ orbiting a NS at 100 pc from Earth

Astrophysical stochastic background

- **There are an estimated 10^8–10^9 galactic WD-WD binaries with GW frequencies $f > 0.1$ mHz**
- **At frequencies below ~ 2mHz, there are too many binaries per resolvable frequency bin $\Delta f = 1/T_{\rm obs} \sim 10^{-8}$ Hz, to be fit and removed from LISA data $\Rightarrow$ confusion noise**

WD, NS, BH binaries in Milky Way and nearby galaxies

Above ~ 2 mHz the density of WD-WD sources decreases and we expect to measure individually all these binaries

There should be $\sim 10^4$ WD binaries, but also many NS, BH binaries

- **we could make a 2-D map of compact binaries in our galaxy**
- **catalog of periods, eccentricities, inclination angles**

Supermassive BH binaries (10^5 - $10^8 M_\odot$)

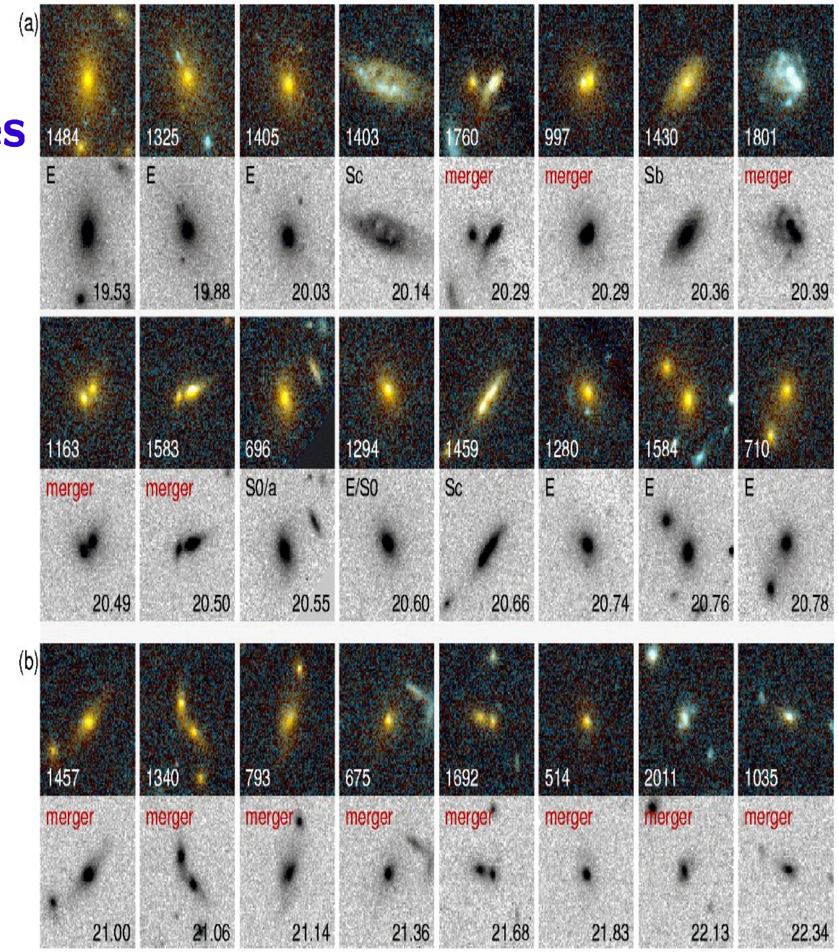
Supermassive BHs at center of galaxies

Supermassive BH binaries form following the mergers of galaxies and pregalactic structures

MS 1054-03 (cluster of galaxies) at $z = 0.83$: about 20% are merging!

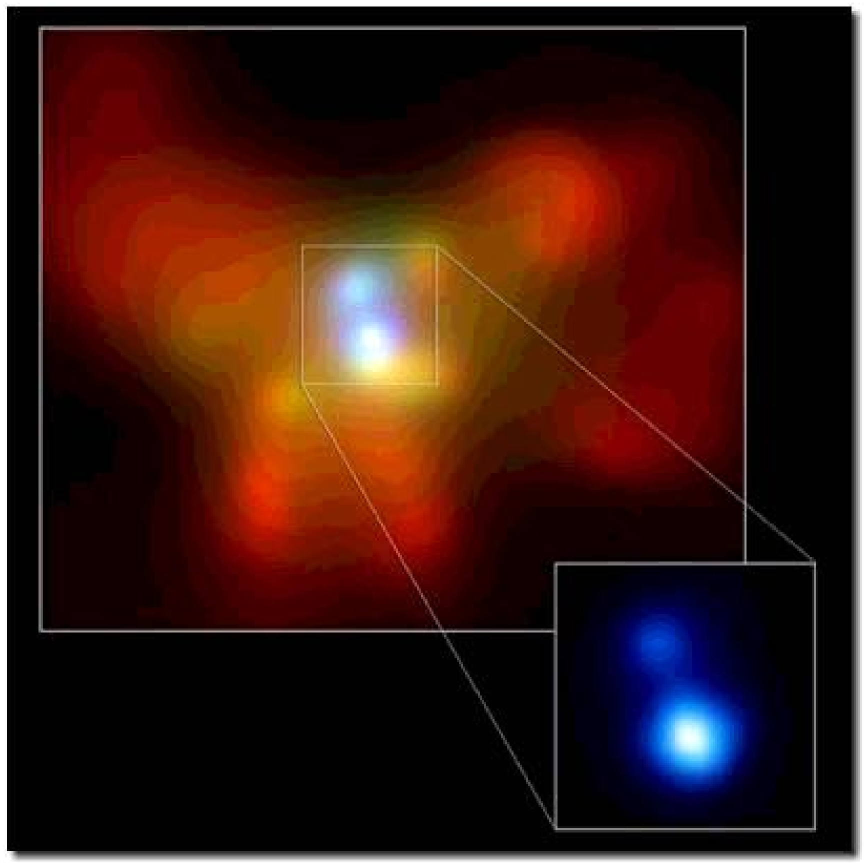
(a) $\rightarrow$ 16 most luminous galaxies in the cluster

(b) $\rightarrow$ 8 fainter galaxies



van Dokkum et al., ApJ Letters, in press (astro-ph/9905394)

Image of NGC6240 taken by Chandra showing a butterfly shaped galaxy product of two smaller galaxies (two active giant BHs)



Supermassive BH binaries

Relativity:

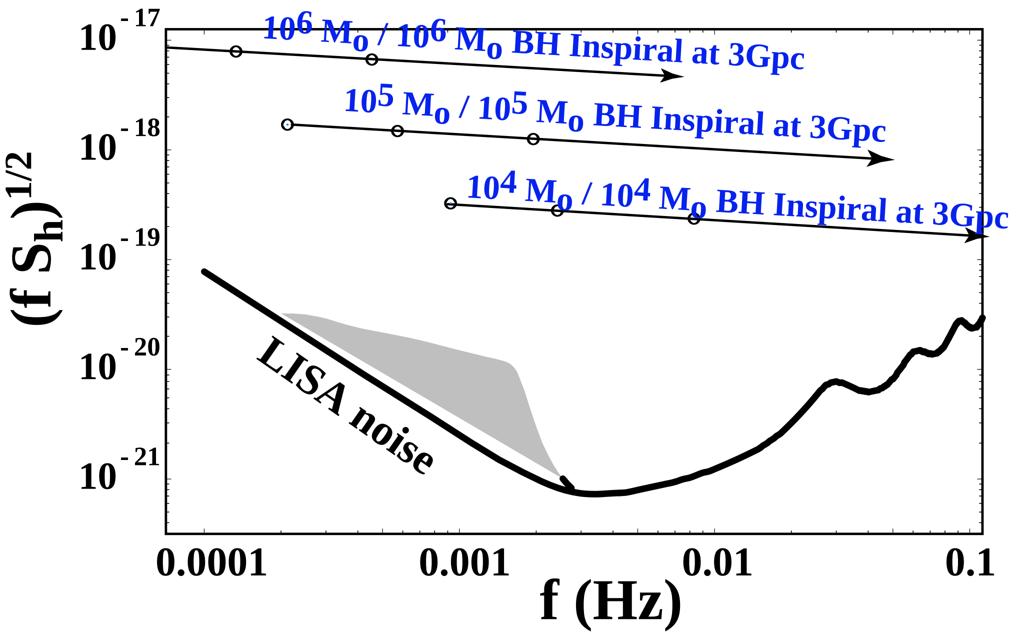
- From inspiraling post-Newtonian waveforms $\rightarrow$ precision tests of GR
- From merger waveforms (num. relativity) $\rightarrow$ tests of non-linear gravity

Astrophysics:

- Cosmic history of supermassive black holes
- What causes two SMBH's to get close enough for merging?

Very high S/N (very large z); high accuracy in determining binary parameters, but event rates uncertain

Supermassive BH binaries

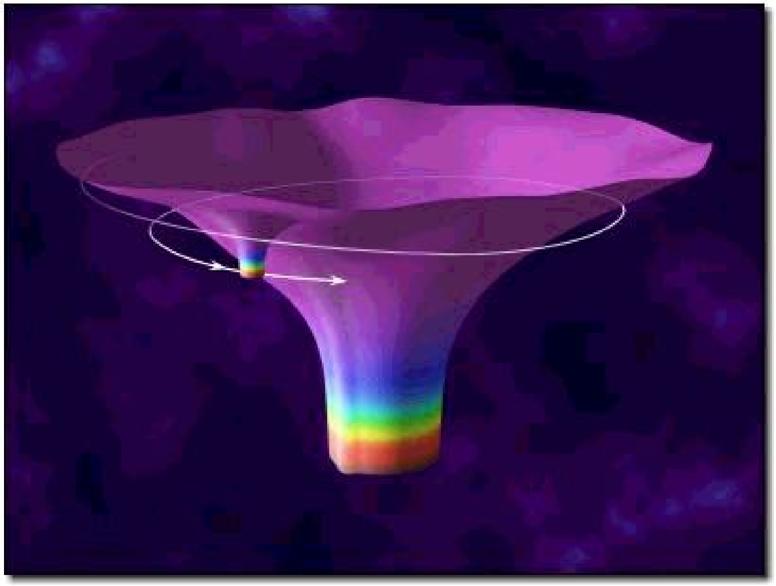


- *Non-precessing binaries:* **distance determined to a few percent, sky position to tens of arcminutes, chirp mass $1/N_{\rm cycles}$, reduced mass to few percents**

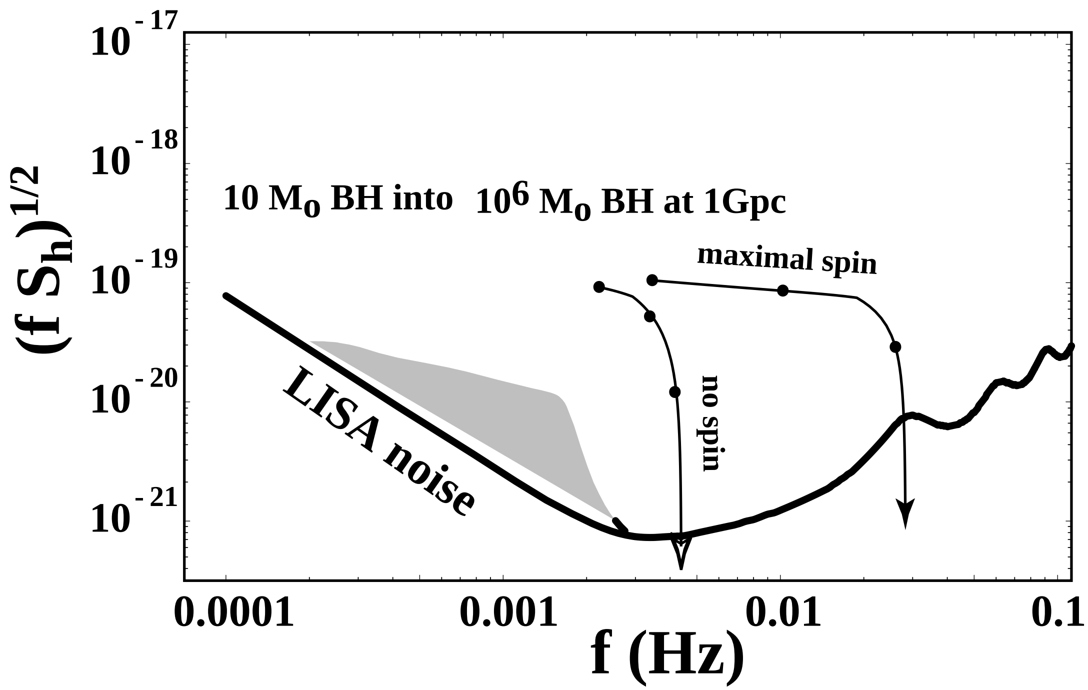
Extreme mass-ratio inspiraling binaries

Small body spiraling into central body of $\sim 10^5$–$10^7\,M_\odot$ out to $\sim$ Gpc distance

- **Relativistic orbits (test of GR)**
- **Map of massive body's external spacetime geometry. Extract multiple moments**



Extreme mass-ratio binaries



Gravitational waves: theory and sources

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Content of Lectures 5 & 6

- **GWs from the early Universe: typical frequencies and amplitudes**
- **Amplification of quantum-vacuum fluctuations**
- **Stochastic GW background from standard inflationary models**
- **Examples of stochastic GW background from non-standard inflation**
- **GWs from first order phase transitions and cosmic strings**

References

B. Allen, Lectures at the Les Houches School 1996 [gr-qc/9604033]

M. Maggiore, Phys. Rep. 331, 283 (2000) [gr-qc/9909001]

AB, TASI Lectures on GWs from the early Universe [gr-qc/0303...]

Characteristic intensity and frequency of relic gravitational waves

- **The intensity**

$$\Omega_{\rm GW}(f) = \frac{1}{\rho_c}\frac{d\rho_{\rm GW}}{d\log f} \qquad \left[\Omega_{\rm GW}(f) = \frac{4\pi^2}{3H_0^2}\, f^3\, S_h(f) \qquad h_{\rm rms} = \sqrt{S_n(f)\,\Delta f} = \frac{\Delta L}{L}\right]$$

$$\rho_{\rm GW} = \frac{1}{32\pi G}\,\overline{\dot{h}_{ij}(t)\,\dot{h}_{ij}(t)}$$

  – **Phenomenological bounds**

- **Two features determine the typical frequencies: the *dynamics* of production mechanism which is model dependent, and the *kinematics*, i.e. the redshift from the production era**

  – **Suppose a graviton is produced at time t_* with frequency f_* during RD or MD era**

$$f_0 = f_*\, a(t_*)/a(t_0), \quad g\, a^3\, T^3 = \text{const.}, \quad 1/f_* = \lambda_* = \epsilon\, H_*^{-1}$$

$$f_0 \simeq 10^{-7}\,\frac{1}{\epsilon}\,\left(\frac{T_*}{1\,\text{GeV}}\right)\,\left(\frac{g_*}{100}\right)^{1/6}\,\text{Hz}$$

Phenomenological bounds

- BBN bound

$\int h_0^2\,\Omega_{\rm GW}(f)\,d\log f \leq 5.6\times 10^{-6}\,(N_\nu - 3)$

- COBE bound

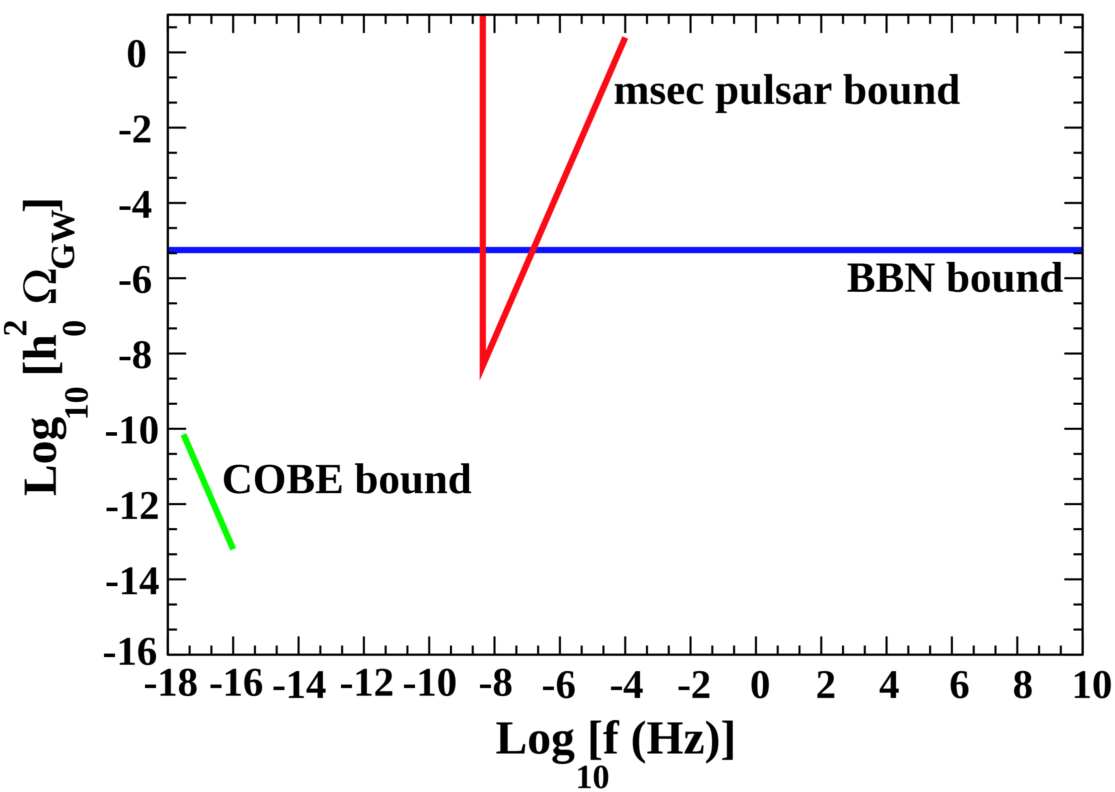
$h_0^2\,\Omega_{\rm GW}(f) \leq 7\times 10^{-11}\,\left(\frac{H_0}{f}\right)^2$

$H_0 \leq f \leq 10^{-16}$ Hz

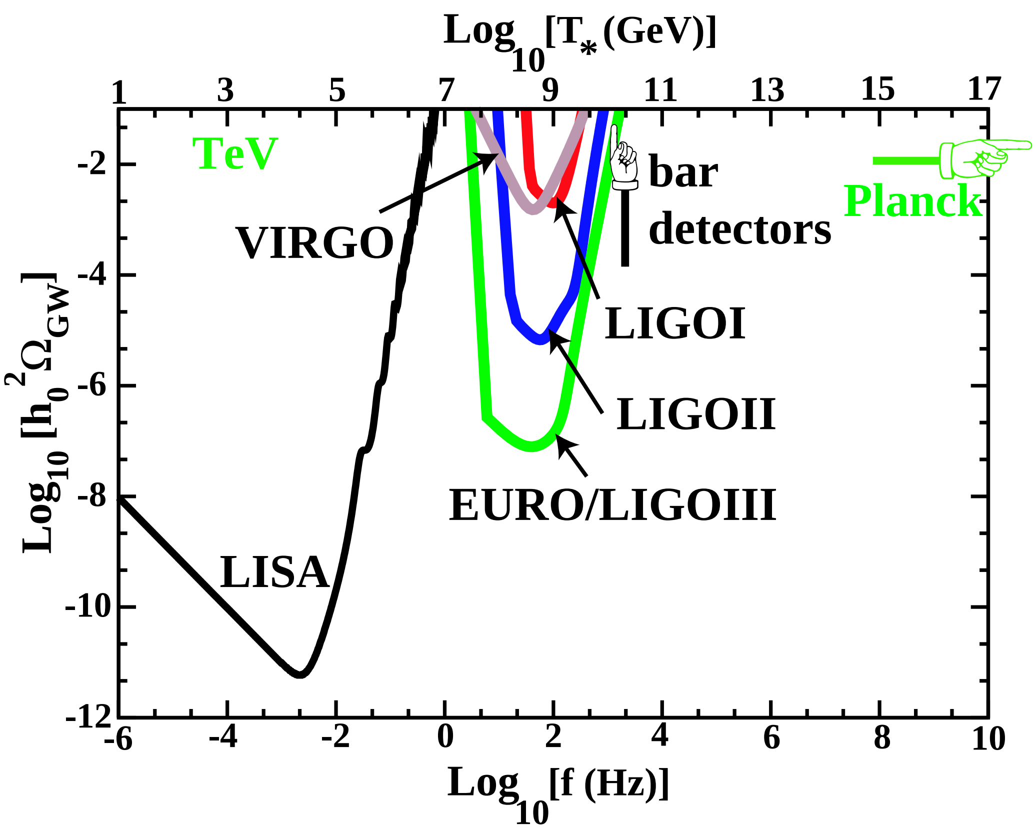
- msec pulsar bound

$h_0^2\,\Omega_{\rm GW}(f) \leq 4.8\times 10^{-9}\left(\frac{f}{f_*}\right)^2$

$f > f_* \equiv 4.4\times 10^{-9}$ Hz



Typical temperatures probed by GWs produced by *causal* mechanisms



Semiclassical point of view

Introducing “canonical field” $\psi_k(\eta) = a\, h_k(\eta)$:

$$\psi_k'' + \left[k^2 - U(\eta)\right]\, \psi_k = 0 \qquad U(\eta) = \frac{a''}{a}$$

de Sitter inflationary era: $a = -1/(\eta\, H_{\rm dS})$ $\left[|U(\eta)| \sim 1/\eta^2,\ (a\, H_{\rm dS}) \sim 1/\eta\right]$

• If $k^2 \gg |U(\eta)|$

$\left[k\eta \gg 1\,, k/a \gg H_{\rm dS}\,, \lambda_{\rm phys} \ll H_{\rm dS}^{-1} \to \text{the mode is inside the Hubble radius}\right]$

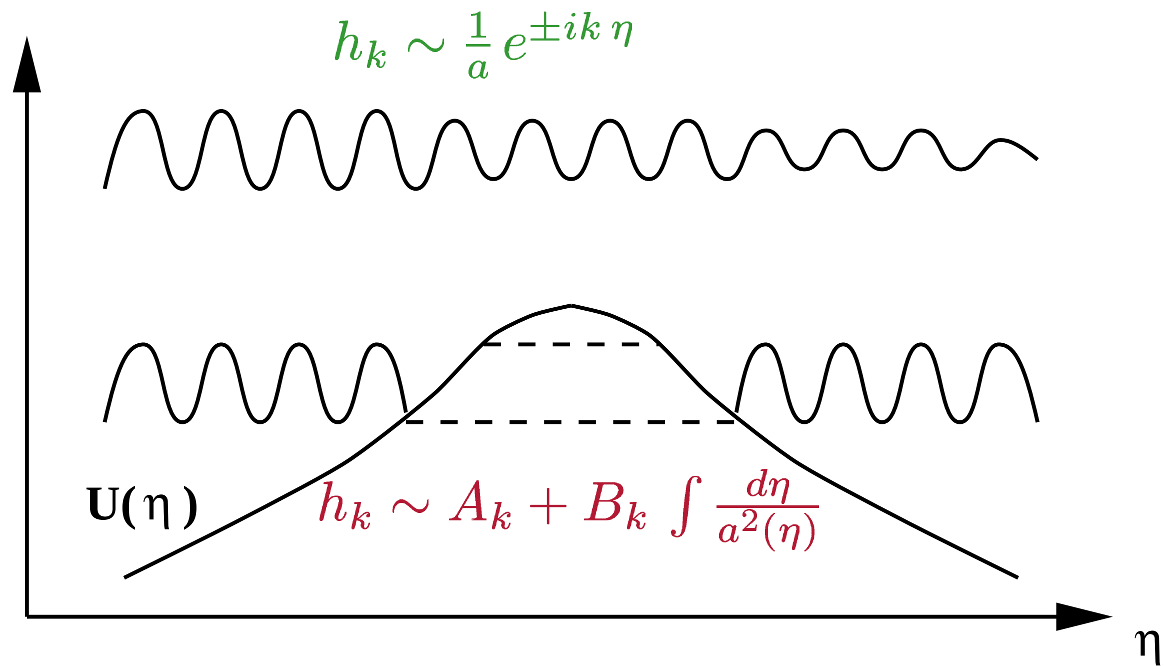
$$\psi_k \sim e^{\pm i k\, \eta} \Rightarrow \quad h_k \sim \frac{1}{a}\, e^{\pm i k\, \eta}$$

• If $k^2 \ll |U(\eta)|$:

$\left[k\eta \ll 1\,, k/a \ll H_{\rm dS}\,, \lambda_{\rm phys} \gg H_{\rm dS}^{-1} \to \text{the mode is outside the Hubble radius}\right]$

$$\psi_k \sim a \left[A_k + B_k \int \frac{d\eta}{a^2(\eta)}\right] \quad \Rightarrow \quad h_k \sim A_k + B_k \int \frac{d\eta}{a^2(\eta)}$$

Semiclassical point of view [continued]



Quantum point of view

$$S = \frac{1}{16\pi\, G_N} \int d^4x \sqrt{|g|}\, \mathcal{R} + S_{\rm matter}$$

- **Isotropic and spatially homogenous FLRW background**

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + a^2(t)\, d\vec{x}^2 = a^2(\eta)\, (-d\eta^2 + d\vec{x}^2)$$

- **Metric perturbations** ($\delta g_{\mu\nu} = h_{\mu\nu}$):

$$h''_k(\eta) + \frac{2a'}{a}\, h'_k(\eta) + k^2\, h_k(\eta) = 0$$

- **Two cosmological phases I and II: two scale factors** $a_{\rm I,II}$ **and two vacua** $|0\rangle_{\rm I,II}$

$$h_i^{\ j} = \sqrt{8\pi\, G_N}\, \sum_{\rm P} \int \frac{d^3k}{\sqrt{2k}\,(2\pi)^3}\, \left[b_{\rm P}^{\rm I}(k)\, h_k(\eta)\, \epsilon_i^{\rm P\ j}(\hat{\Omega})\, e^{i\vec{k}\cdot\vec{x}} + {\rm h.c.}\right]$$

$$h_i^{\ j} = \sqrt{8\pi\, G_N}\, \sum_{\rm P} \int \frac{d^3k}{\sqrt{2k}\,(2\pi)^3}\, \left[b_{\rm P}^{\rm II}(k)\, H_k(\eta)\, \epsilon_i^{\rm P\ j}(\hat{\Omega})\, e^{i\vec{k}\cdot\vec{x}} + {\rm h.c.}\right]$$

Quantum point of view [continued]

$$b_k^{\mathrm{I}} = \sum_{k'} \left[\alpha_{kk'}\, b_{k'}^{\mathrm{II}} + \beta_{kk'}^*\, b_{k'}^{\mathrm{II}\,\dagger}\right]$$

$$b_k^{\mathrm{II}} = \sum_{k'} \left[\alpha_{kk'}^*\, b_{k'}^{\mathrm{I}} - \beta_{kk'}^*\, b_{k'}^{\mathrm{I}\,\dagger}\right]$$

$$N_k^{\mathrm{II}} \equiv \langle \{N^{\mathrm{I}}\}\, | b_k^{\mathrm{II}}\, b_k^{\mathrm{II}\,\dagger} |\, \{N^{\mathrm{I}}\} \rangle = N_k^{\mathrm{I}}\, (1 + 2|\beta_k|^2) + |\beta_k|^2$$

Stochastic GW background from *standard* inflation

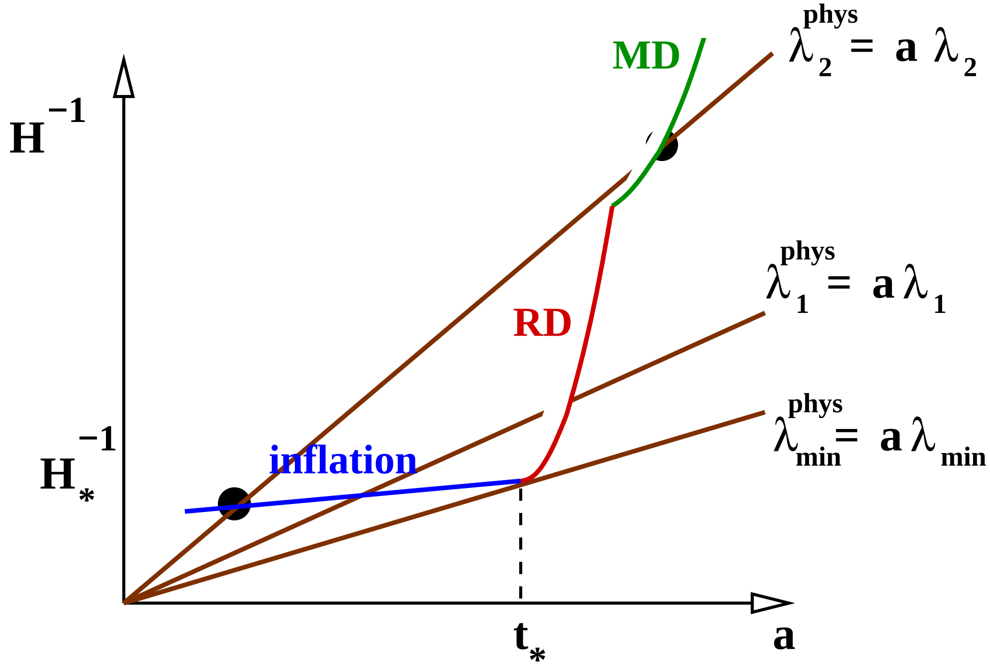
In *standard* inflationary models Hubble radius almost constant

- $2\pi f_* H_*^{-1} \ll 1 \Rightarrow$ **abrupt transition**
  $\Rightarrow$ production of particles out of vacuum

- $2\pi f_* H_*^{-1} \gg 1 \Rightarrow$ **adiabatic transition**
  $\Rightarrow$ no production of particles

$h_0^2\,\Omega_{\rm GW}(f) \sim H_\bullet^2\, f^{n_T} \quad |n_T| \ll 1$

cutoff frequency $f_*^{\rm max} \sim H_*/2\pi$



Inflation: $H^{-1} \simeq$ const.

RD: $H^{-1} \propto a^2$

MD: $H^{-1} \propto a^{3/2}$

Example: Slow-roll inflation

$$n_T = -\frac{M_{\rm Pl}^2}{8\pi}\left(\frac{V'_\bullet}{V_\bullet}\right)^2$$

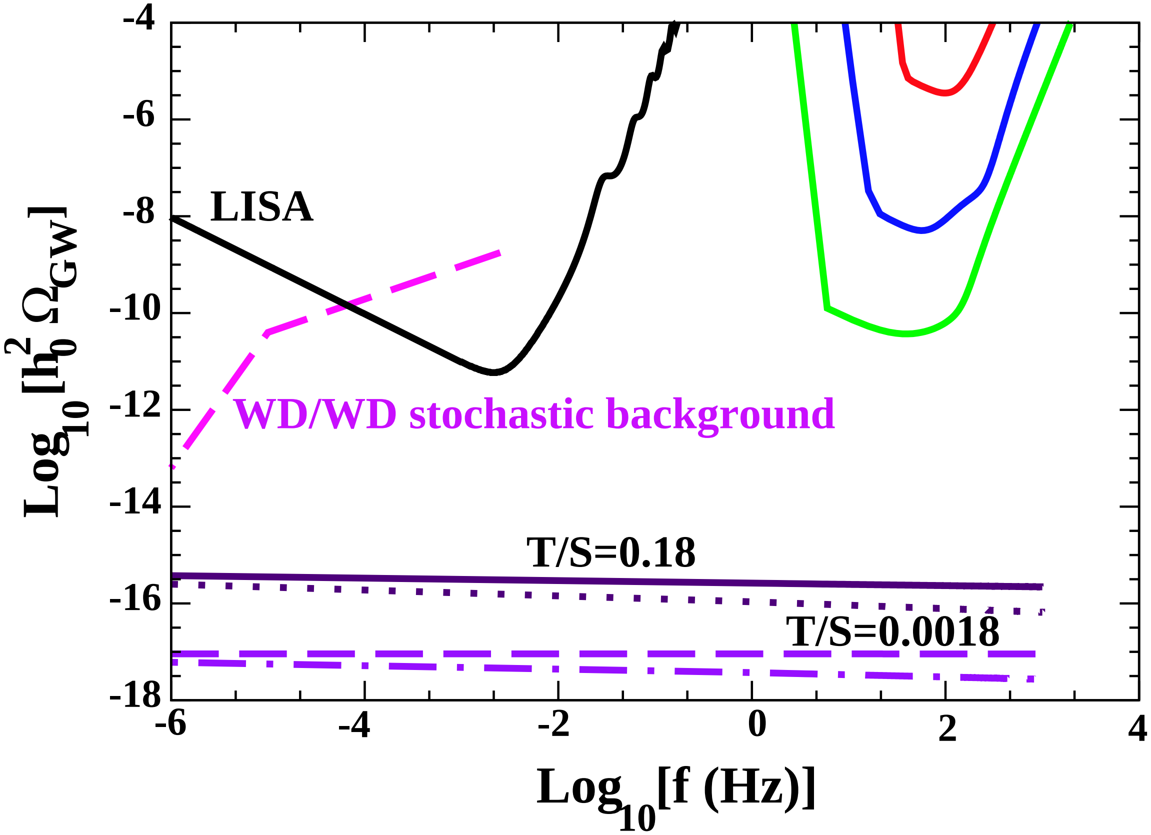
$$S \equiv \frac{5\langle|a_{2m}^S|^2\rangle}{4\pi} \propto \frac{V_\bullet/M_{\rm Pl}^4}{\left(M_{\rm Pl}\,V'_\bullet/V_\bullet\right)^2}$$

$$T \equiv \frac{5\langle|a_{2m}^T|^2\rangle}{4\pi} \propto \left(\frac{V_\bullet}{M_{\rm Pl}^4}\right)$$

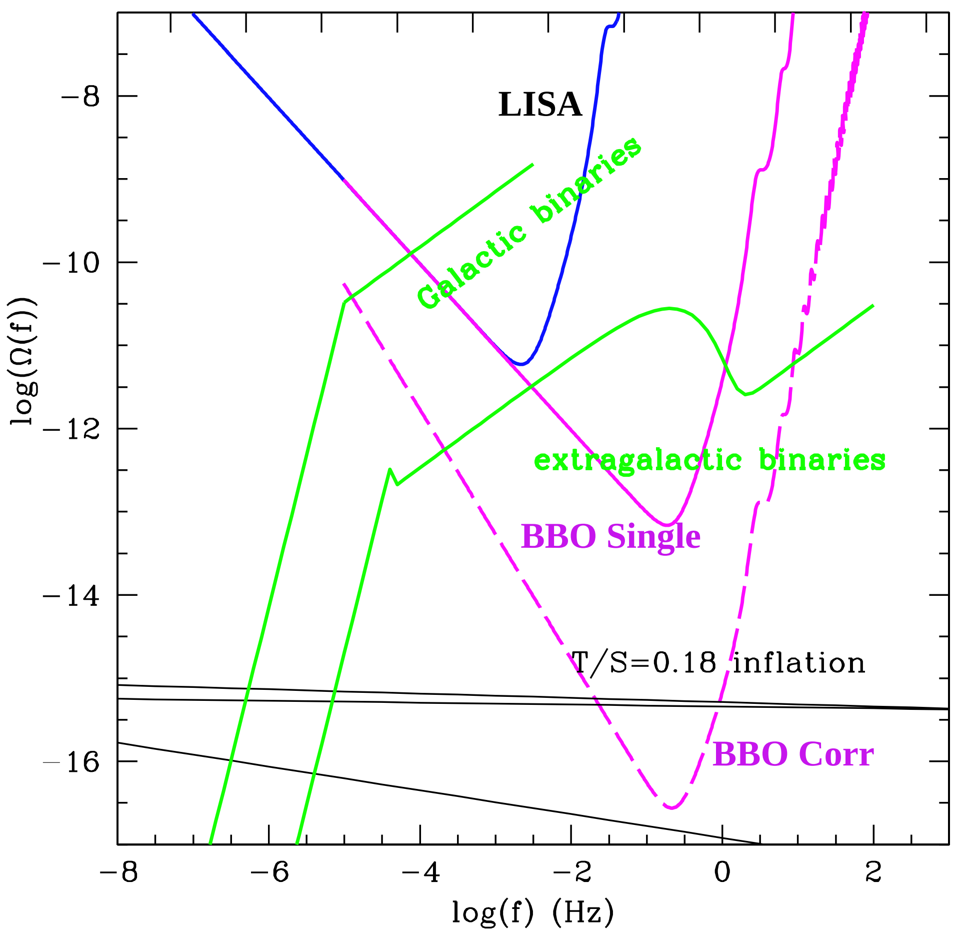
consistency relation:

$$n_T = -\frac{1}{7}\,\frac{T}{S}$$

$$\frac{dn_T}{d\log k} \sim -n_T\left(\frac{V'_\bullet}{V_\bullet}\right)'$$



Post-LISA mission?



GW spectrum in more generic (inflationary) background

$$\Gamma_{\rm eff} \propto \int d^4x \sqrt{|g|}\, e^{-\phi}\, [\mathcal{R} + g_{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \cdots]$$

$$\psi_k'' + \left[k^2 - U(\eta)\right]\, \psi_k = 0 \qquad U(\eta) = \frac{\tilde{a}''}{\tilde{a}}$$

$\tilde{a} \to$ pump field (e.g., $\tilde{a} = a\, e^{-\phi/2}$; depends on equation of state)

Two cosmological phases: $\tilde{a}_{\rm I} \sim |\eta|^{\gamma_{\rm I}}$ and $\tilde{a}_{\rm II} \sim |\eta|^{\gamma_{\rm II}}$

$\psi_k(\eta) = \sqrt{|\eta|}\, C\, H^{(1)}_{\nu_{\rm I}}(|k\,\eta|)$ with $\nu_{\rm I} = |\gamma_{\rm I} - 1/2|$

$\psi_k(\eta) = \sqrt{|\eta|}\, [\alpha_k\, H^{(1)}_{\nu_{\rm II}}(|k\,\eta|) + \beta_k\, H^{(2)}_{\nu_{\rm II}}(|k\,\eta|)]$ with $\nu_{\rm II} = |\gamma_{\rm II} - 1/2|$

$|\beta_f|^2 \sim f^{2\epsilon_T}$ with $\epsilon_T = \epsilon_T(\gamma_{\rm I}, \gamma_{\rm II}) \to n_T = \frac{d\log\Omega_{\rm GW}}{d\log f} = 4 + 2\epsilon_T$

- **The spectral slope of GW spectrum depends on the** *kinematics*
- **The GW spectrum amplitude (and special features as, e.g., oscillations) depends** on *dynamics* **and physical assumptions (initial state, reheating, entropy produced, ...)**

Stochastic GW background from *non standard* post–big-bang phases

In some models the inflationary era is not followed immediately by the radiation era but rather by an expanding phase whose equation of state is stiffer than radiation

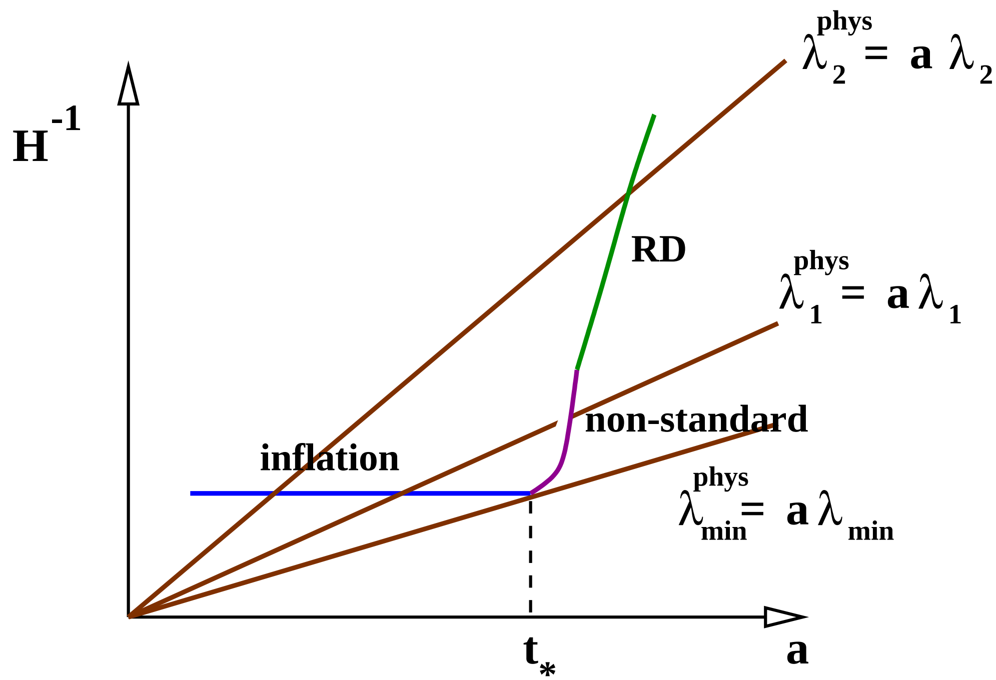
stiff era: $H^{-1} \propto a^3$

GW spectrum at high frequency can be blue

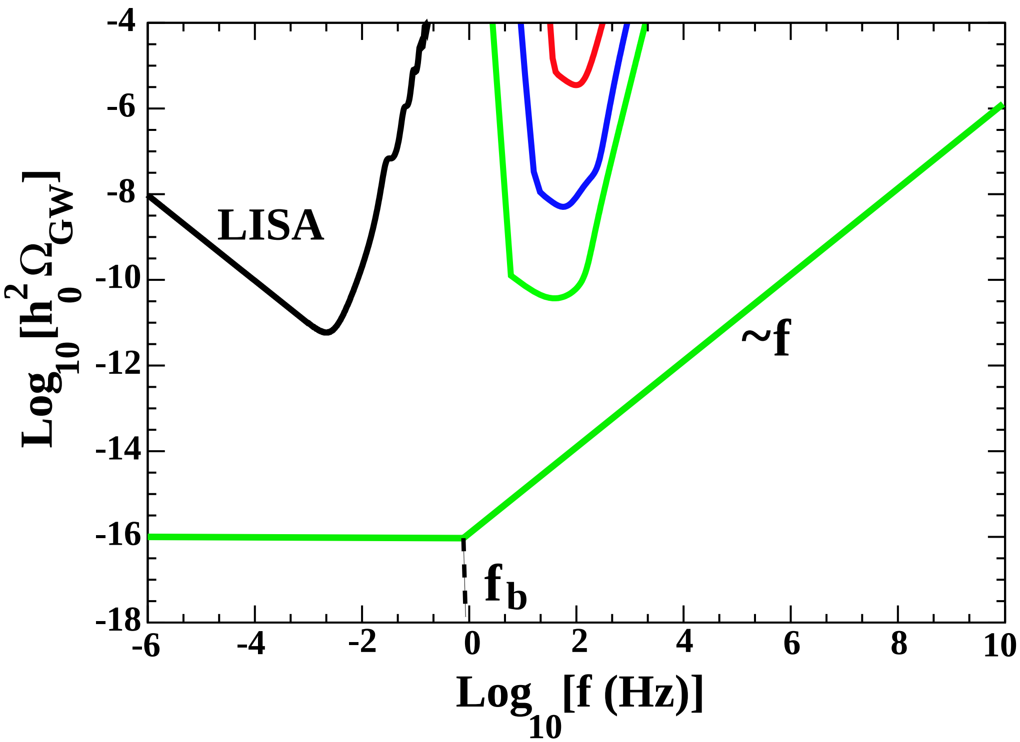
$h_0^2\,\Omega_{\rm GW}(f) \sim f$

cutoff frequency $f_*^{\rm max} \sim H_*/2\pi$

— Quintessential inflation



Example: quintessential inflation



Electromagnetic detectors in MHz or GHz region?

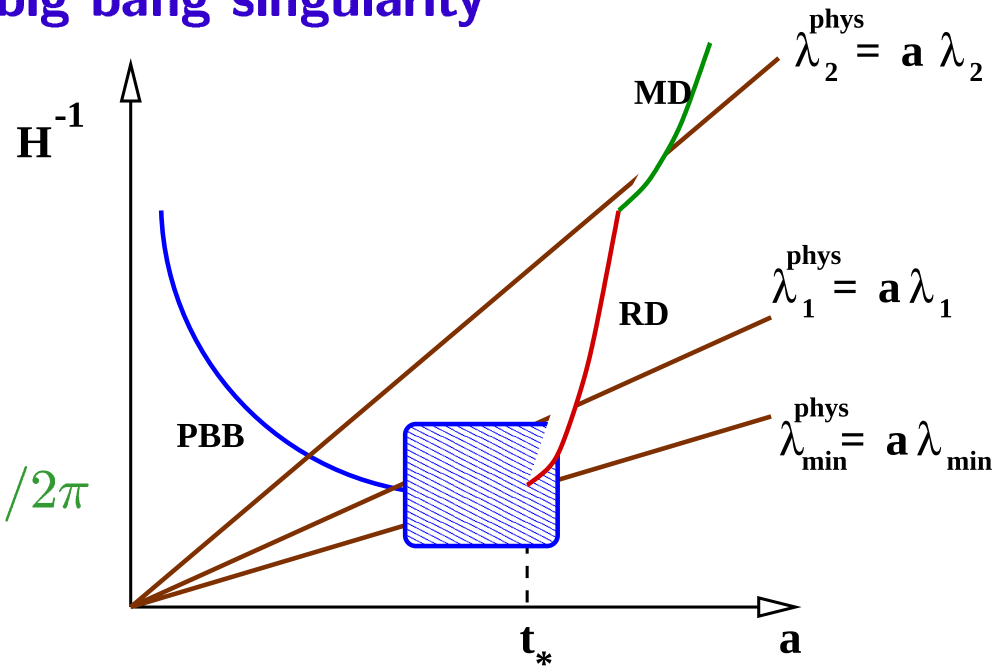
Stochastic GW background from string-theory– inspired models

In some string-inspired inflationary models or bounce Universe models, such as the pre-big bang the Hubble radius grows toward the would-be big bang singularity

GW spectrum is blue at low frequency $\Rightarrow$ no contribution to CMB

$h_0^2\,\Omega_{\rm GW}(f) \sim f^n$

cutoff frequency $f_*^{\rm max} \sim H_*/2\pi \sim H_s/2\pi$

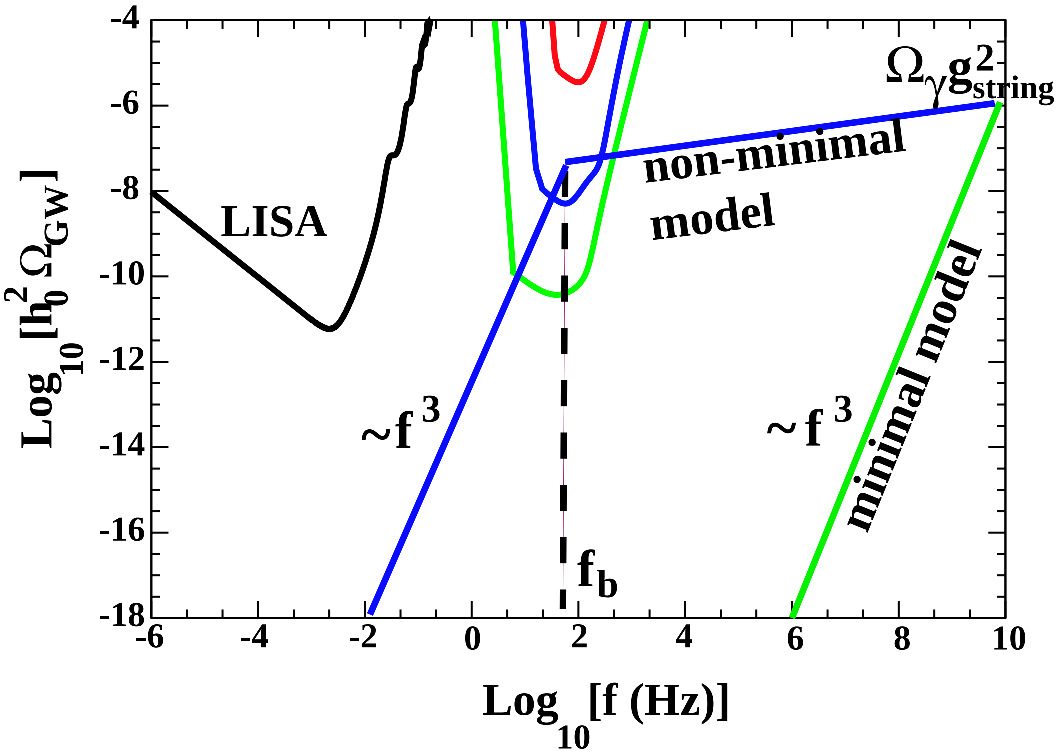


Warnings:

- **Those models do not provide** *reliable* **description of transition from pre to post eras**
- **The observability of GW spectrum depends on details of the transition**

Example: pre-big bang scenario

In non-minimal models the spectrum at high frequency can be red, flat or blue



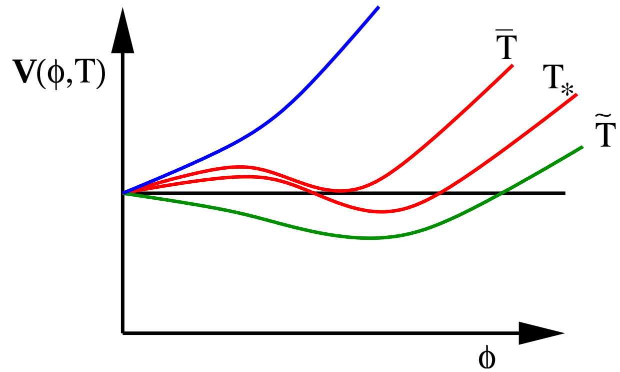
Gravitational waves from first-order phase transitions

Via quantum tunnelling true vacuum bubbles nucleates

When bubbles collide $\Rightarrow$ emission of gravitational waves

$\beta \longrightarrow$ bubble nucleation rate per unit volume

$\alpha \longrightarrow$ jump in energy density experienced by order parameter



EW phase transition: $T_* \simeq 100\,\mathrm{GeV}$ and $\beta/H_* \simeq 10^2$–10^3

$\Rightarrow f_{\rm peak} \simeq 10^{-8}(\beta/H_*)\,(T_*/1\mathrm{GeV}) \simeq 10^{-4}$–$5 \times 10^{-3}$Hz

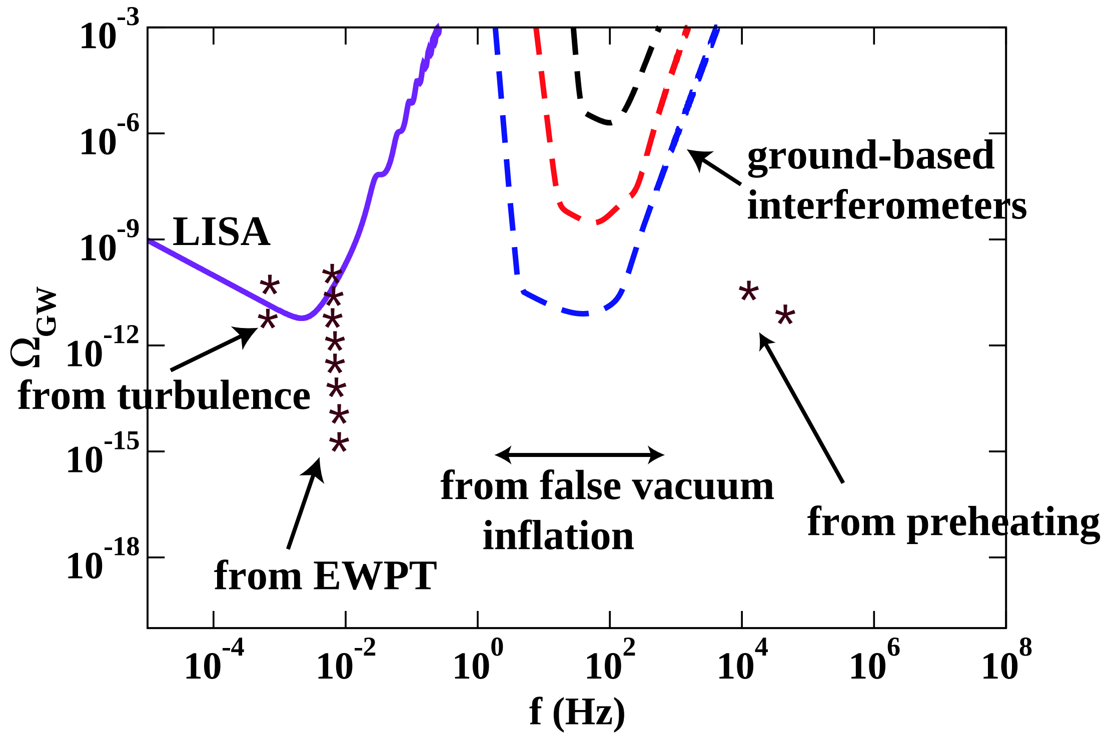
Intensity of GW spectrum: $h_0^2\,\Omega_{\rm GW} \simeq 10^{-6}\,(H_*/\beta)^2\,f(\alpha, v)$

– In SM there is *no* first-order EW phase transition for Higgs mass larger then M_w

– In MSSM, for certain values of Higgs mass, there are possibilities but $h_0^2\,\Omega_{\rm GW} \leq 10^{-16}$

– In NMSSM: $h_0^2\,\Omega_{\rm GW} \leq 10^{-15}$–$10^{-10}$ with $f_{\rm peak} \simeq 10$mHz

Probing the early Universe



Gravitational waves from cosmic strings

Topological defects formed at phase transitions

- Contribution of topological defects to structure formation $< 10\%$
- Cosmic strings have large tension μ, they oscillate relativistically and emit GWs
- Small loops (smaller than Hubble radius) oscillate, emit GWs and disappear, but are replaced by small loops broken off very long loops (longer than Hubble radius)

$r \rightarrow$ characteristic loop's radius

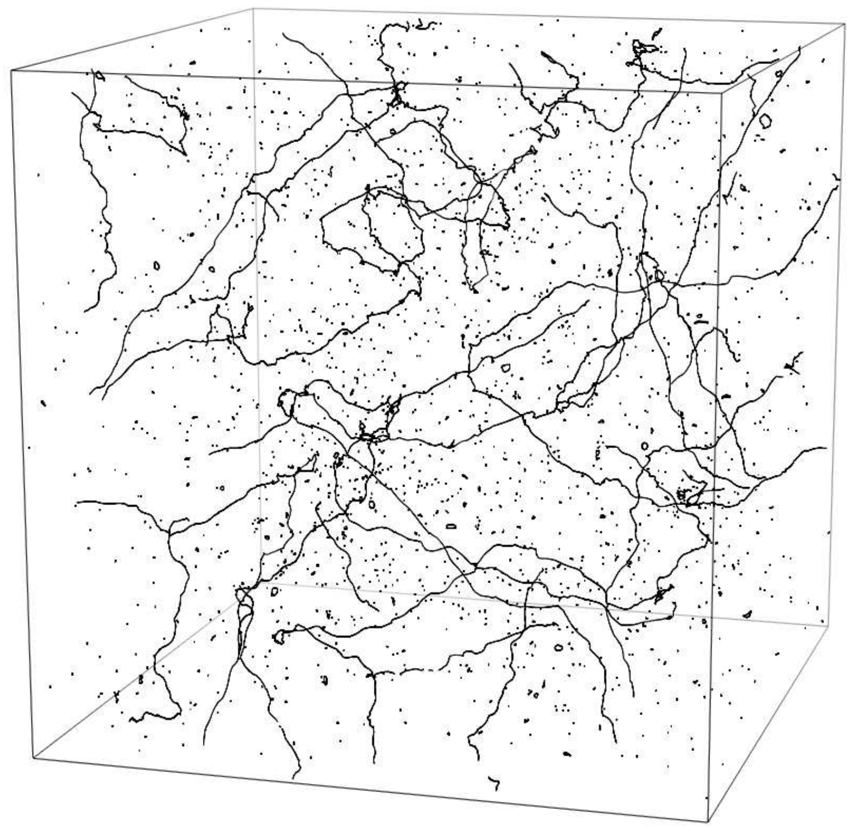
$\tau \rightarrow$ oscillation period $(\tau \sim r)$

Quadrupole moment $Q \sim \mu\, r^3$

Loop radiates with power: $dE/dt = P \sim G_N \dddot{Q}^2 \sim \Gamma\, G_N\, \mu^2$

$\Omega_{\rm GW} \sim P/\rho_c < 10^{-9}$–$10^{-8}$ for cosmic strings with $G_N\,\mu < 10^{-7}$ and $\Gamma \sim 50$

Snapshot of a cosmic string network



GW spectrum from cosmic strings

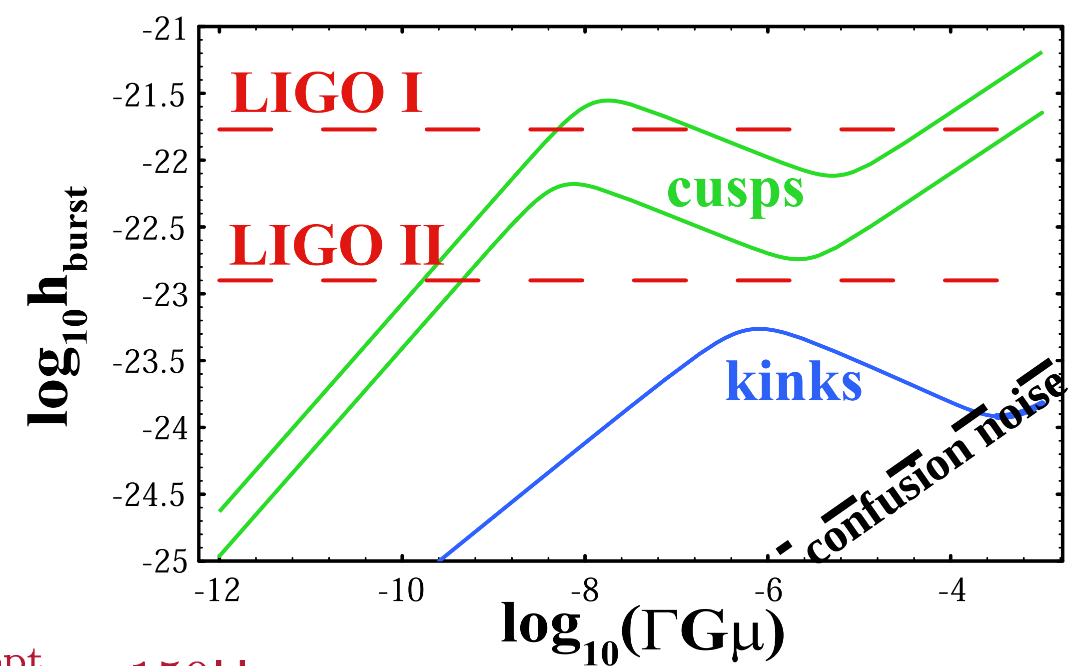
The stochastic ensemble of GWs from network of oscillating loops is strongly non-Gaussian and include occasional, sharp GW bursts emanating from cusps and kinks

Strongly non-Gaussian "burst" part

+ nearly Gaussian "background"

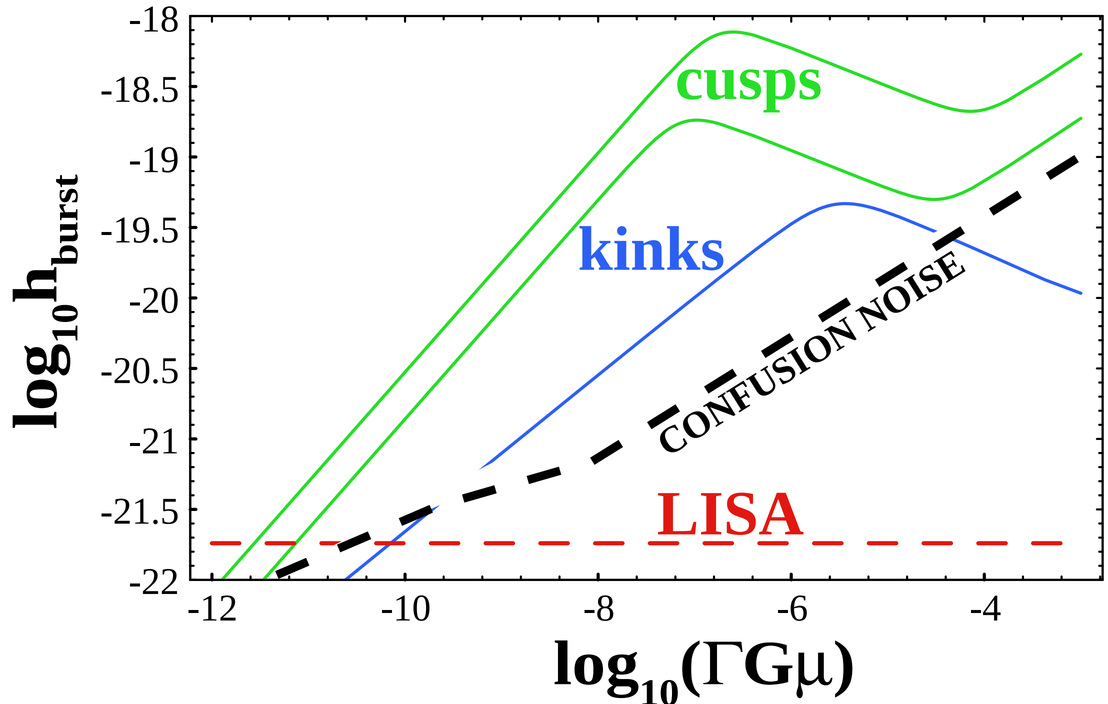
Individual bursts stand out above the background

Signal detectable for a large range of values of the string tension μ



$$f_{GW}^{opt} = 150\text{Hz}$$

GW spectrum from cosmic strings for LISA

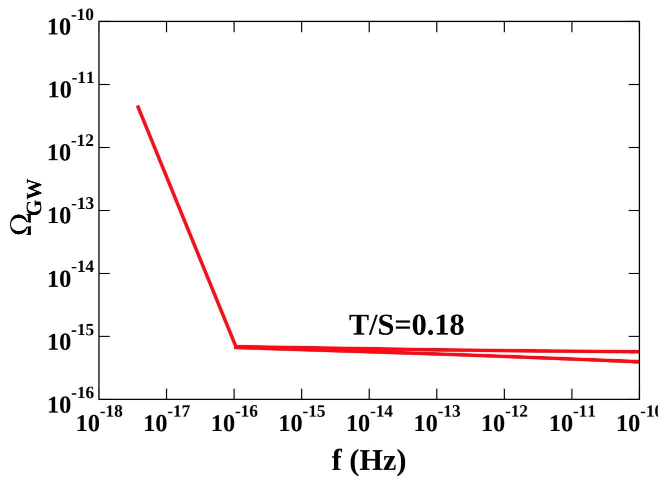


$f_{\rm GW}^{\rm opt} = 3.9 \times 10^{-3}$Hz

Imprints of relic gravitational waves on CMB

$$S \equiv \frac{5\langle |a^S_{2m}|^2\rangle}{4\pi} \propto \frac{V_\bullet/M^4_{\rm Pl}}{\left(M_{\rm Pl}\,V'_\bullet/V_\bullet\right)^2}$$

$$T \equiv \frac{5\langle |a^T_{2m}|^2\rangle}{4\pi} \propto \left(\frac{V_\bullet}{M^4_{\rm Pl}}\right)$$



- **Measuring polarization of CMB**

– Tensor fluctuations give different polarization patterns than scalar fluctuations

– Contamination due to weak-gravitational lensing of CMB along the line of sight

–Minimum detectable inflation-energy $V_\bullet^{1/4} > 10^{15}$ GeV with $s = 1\mu{\rm K}\,\sqrt{\rm sec}$