Parma 10-12 Sept 2003

Lectures on Supersymmetric Gauge Theories

K. Konishi

I. Dynamics of Susy Gauge Theories

II. Nonabelian Superconductors and Confinement

III. Recent developments

Lectures on Supersymmetric Gauge Theories I:

Introduction: a Review

- **Symmetries and anomalies**
- **Nonrenormalization theorem**
- **NSVZ β functions**
- **Instantons and anomalies**
- **Seiberg's duality**
- **Gluini condensate**
- **Phases of SQCD**

Why Supersymmetry?

- $H = Q^\dagger Q$,

$$Q : \qquad |Boson\rangle \leftrightarrow |Fermion\rangle$$

$$\langle H \rangle \geq 0, \qquad \rightarrow \qquad \Lambda_{Cosm} \ll \Lambda_{QCD}$$

- **Hierarchy (naturalness) problem in the standard model**

$$M_{Higgs}, M_W \ll M_{Planck} \sim 10^{19}\mathrm{GeV}$$

- **Susy GUTs: coupling constant unification at $\mu \sim 10^{16}$ GeV? MSSM $\rightarrow$ LHC ($\geq$ 2007)**

- **Deep results on details of nonperturbative dynamics**

- **Haag-Lopuszhanski-Sohnius: Susy algebra is the only possible nontrivial generalization involving Poincaré and internal symmetry algebra. (*cfr*. Coleman-Mandula)**

- **"A truely beautiful idea never really dies... " (*Y. Nambu*)**

Susy gauge theories

- **Susy algebra**

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha,\dot{\alpha}} P_\mu,$$

- **Superfields**

$$F(x, \theta, \bar{\theta}) = f(x) + \theta\psi(x) + \ldots$$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha,\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha,\dot{\alpha}}\partial_\mu,$$

- **Chiral superfields:** $\bar{D}\Phi = 0$ **(** $D\Phi^\dagger = 0$ **)**

$$\Phi(x, \theta, \bar{\theta}) = \phi(y) + \sqrt{2}\theta\,\psi(y) + \theta\theta\,F(y), \qquad y = x + i\theta\sigma\bar{\theta}$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha,\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha,\dot{\alpha}}\partial_\mu,$$

- **Verctor superfields** $V^\dagger = V$,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V = -i\lambda + \frac{i}{2}\left(\sigma^\mu\,\bar{\sigma}^\nu\right)_\alpha^\beta F_{\mu\nu}\,\theta_\beta + \ldots$$

- **Supersymmetry transformation of fields:**

Chiral superfields

$$[Q_\alpha, \phi] = \sqrt{2}\psi_\alpha; \quad \{Q_\alpha, \psi_\beta\} = \sqrt{2}F; \quad [Q_\alpha, F] = 0,$$

$$[\bar{Q}_{\dot\alpha}, \phi] = 0; \quad \{\bar{Q}_{\dot\alpha}, \psi_\beta\} = i\sqrt{2}\sigma^\mu_{\beta\dot\alpha}\mathcal{D}_\mu A; \quad [\bar{Q}^{\dot\alpha}, F] = i\sqrt{2}(\bar\sigma^\mu)^{\dot\alpha\beta}\mathcal{D}_\mu\psi_\beta,$$

In particular, $\bar{D}\Phi = 0 \Rightarrow [\bar{Q}_{\dot\alpha}, \phi] = 0 : \phi$ **is a "chiral field";**

Vector superfields

$$[Q^\alpha, A^a_\mu] = -i\sqrt{2}\,\bar\lambda^a\,\bar\sigma; \qquad \{Q^\alpha, \lambda^a\} = \sigma^{\mu\nu}F^a_{\mu\nu} + iD^a; \qquad [Q^\alpha, D^a] = -\sigma^\mu\mathcal{D}_\mu\bar\lambda^a;$$

$$[\bar{Q}_{\dot\alpha}, A^a_\mu] = -i\sqrt{2}\,\bar\sigma\lambda^a\,; \quad \{\bar{Q}_{\dot\alpha}, \lambda^a\} = 0; \qquad [Q^\alpha, D^a] = -\mathcal{D}_\mu\lambda^a\sigma^\mu;$$

- **Lagrangian** ($\int d\theta_1\,\theta_1 = 1$, **etc**)

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\,\tau_{cl}\left[\int d^4\theta\,\Phi^\dagger e^V\Phi + \int d^2\theta\,\frac{1}{2}WW\right] + \int d^2\theta\,\mathcal{W}(\Phi) \tag{1}$$

- $\mathcal{W}(\Phi) =$ **superpotential;**

- $$\tau_{cl} = \frac{\theta}{2\pi} + \frac{4\,\pi\,i}{g^2}$$

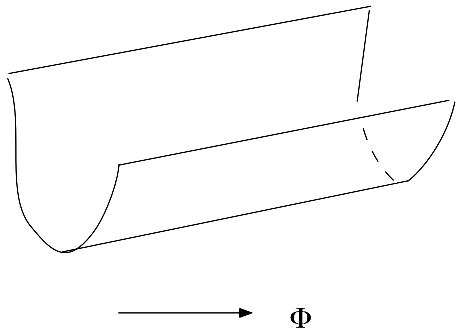
- **Scalar potential**

$$V_{sc} = \sum_{mat}\left|\frac{\partial\mathcal{W}}{\partial\phi}\right|^2 + \frac{1}{2}\sum_a\left|\sum_{mat}\phi^* t^a\phi\right|^2$$

- **For SQCD,** $\{\Phi\} \to Q \sim \underline{N}$, $\tilde{Q} \sim \underline{N}^*$ **of** $SU(N)$

$$G_F = SU(n_f)\times SU(n_f)\times U_V(1)\times U_A(1)\times U_\lambda(1)$$

- **Flat directions (CMS)**



e.g., for $n_f < n_c$,

$$Q = \tilde{Q}^\dagger = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & \ddots & & \\ 0 & \dots & & a_{n_f} \\ 0 & 0 & \dots & 0 \\ \dots & & & \dots \end{pmatrix}$$

Q:
Superpotential generated? CMS modified? Symmetry breaking?

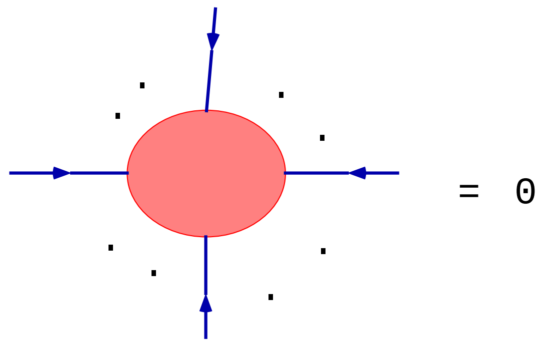
Nonrenormalization theorem

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\,(\bar{\Phi}\Phi + \frac{1}{2}\Phi^2\delta^2(\bar{\theta}) + h.c.)$$

- **Perturbative N.R. theorem**

$$\langle T\Phi(x,\theta,\bar{\theta})\Phi(x',\theta',\bar{\theta}')\rangle$$

$$= -m\,\delta^2(\theta-\theta')\,e^{-i(\theta\sigma^\mu\bar{\theta}-\theta'\sigma^\mu\bar{\theta}')\partial_\mu}\Delta_c(x-x')$$



Only D terms $\propto \int d^2\theta\, d^2\bar{\theta}\,(\ldots)$ generated. No F terms

- If $\exists$ exact **non-anomalous symmetry** $G \rightarrow$ No terms violating G generated;

- **Perturbative anomaly** (West, Grisaru, et. al., SVZ)

$$\Delta L = \int d^2\theta \, d^2\bar{\theta} \; \Phi^2 \frac{D^2}{\Box}\Phi \sim \int d^2\theta \, \Phi^3$$

However, no such nonlocal term simulating F-term, in S_W

- Terms protected only by anomalous (e.g. $U_A(1)$) symmetries **can be generated by instantons**

- **Generalized non-renormalization theorem** (SVZ):

The gauge kinetic term

$$\int d^2\theta \, W_\alpha W^\alpha = \int d^2\theta \, d^2\bar{\theta} \; [\,(e^{-V} D_\alpha e^V) W^\alpha\,]$$

can be generated by 1 loop corrections - **only**.

$\rightarrow$ **NSVZ exact β functions**:

- E.g. $SU(N)$ SQCD:

$$L = \frac{1}{4}\int d^2\theta \left(\frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu}\right) W^a W^a + h.c. + \int d^4\theta \sum_i Z_i(\mu, M)\Phi_i^\dagger e^{2V_i}\Phi_i\,,$$

$$b_0 = -3N_c + \sum_i T_{Fi}; \qquad T_{Fi} = \frac{1}{2} \qquad \text{(quarks)}\,.$$

- Renormalize the fields $\Phi_i \longrightarrow Z_i^{-1/2}\Phi_i = e^{-\frac{1}{2}\log Z_i}\Phi_i$ $\quad(\bar{D}(-\frac{1}{2}\log Z_i) = 0) \Rightarrow$ Anomaly $\propto \frac{1}{16\pi^2}(-\frac{1}{2}\log Z_i(\mu, M))\,WW$

- $$\frac{1}{g^2(\mu)} = \frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\log\frac{M}{\mu} - \frac{1}{8\pi^2}\log Z_i(\mu, M)$$

$$\beta_h(g) \equiv \mu\frac{d}{d\mu}\,g = -\frac{g^3}{16\pi^2}\left(3N_c - \sum_i T_{Fi}(1-\gamma_i(g))\right)\,,$$

where $\gamma_i(g(\mu)) = -\mu\frac{\partial}{\partial\mu}\log Z_i(\mu, M)|_{M,g(M)}$

- Actually by recaling $A_\mu = g_c A_{c\mu}$, $\quad \lambda = g_c\lambda_c$,

$$\frac{1}{g^2} = \frac{1}{g_c^2} + \frac{N_c}{8\pi^2}\log g_c^2, \qquad \beta(g_c) = -\frac{g_c^3}{16\pi^2}\frac{3N_c - \sum_i T_{Fi}(1-\gamma_i(g_c))}{1 - N_c g_c^2/8\pi^2}\,.$$

- $\gamma(g) = -\frac{g^2}{8\pi^2}\frac{N_c^2-1}{N_c} + O(g^4)$

- Zero of the beta function at g^* where

$$\gamma(g^*) = -\frac{3N_c - N_f}{N_f}$$

Susy Identities

- **Susy transf. of** $\Phi(x,\theta,\bar\theta) = \phi(y) + \sqrt{2}\theta\psi + \theta\theta F(y)$**:**

$$[\bar Q^{\dot\alpha}, \phi] = 0, \qquad \{\bar Q^{\dot\alpha}, \psi_\alpha\} = -\sqrt{2}\bar\sigma^\mu \partial_\mu \phi,$$

$$G = \langle T\phi_1(x_1)\phi_2(x_2)\ldots\phi_k(x_k)\rangle$$

$$\bar\sigma^\mu \partial^{x_1}_\mu G = \langle T[\,\bar Q^{\dot\alpha}, (\psi_1(x_1)\phi_2(x_2)\ldots)]\rangle = 0,$$

etc. G **indep. of** $x_i \to = \prod_i \langle\phi_i\rangle$

- **Analytic dep. on** g_i, m_i **etc (**$\mathcal{W}(\Phi) = m\,\Phi^2 + g\,\Phi^3 + \ldots$**)**

$$\frac{\partial G}{\partial m^*} = \langle T[\,\bar Q^{\dot\alpha}, (\bar\Phi^2|_{\bar\theta}\,\phi_1(x_1)\phi_2(x_2)\ldots)]\rangle = 0, \qquad \frac{\partial G}{\partial g_2 *} = 0$$

- **Symmetries**

Fields	Δ	q_V	q_λ	q_X
$Q, \tilde{Q}$	1	$1, -1$	1	$n_c - n_f$
$\psi_Q, \psi_{\tilde{Q}}$	$3/2$	$1, -1$	0	n_c
λ_α	$\frac{3}{2}$	0	1	$-n_f$
g_l	$2 - l$	$-(l+1)$	$1 - l$	2
Λ^{2N}	$2N$	$2N$	$\frac{4N}{3}$	0

Anomalies and Instantons

- $U_A(1)$ **anomaly** (**Steinberger, Schwinger, Adler, Bell, Jackiw**)

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad (\pi_0 \to 2\gamma)$$

- **QCD:**

$$\partial_\mu J_L^\mu = \frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu}$$

$$\Delta Q_5 = 2\, n_f \int d^4x\, \frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu} \neq 0!$$

Axial $U_A(1)$ **broken: solution of "**$U(1)$**" problem (**$m_\eta \gg m_\pi$**? Why NO** $U_A(1)$ **Goldstone boson); But** $\frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu} = \partial_\mu K^\mu$ **!?**

- **Finite energy config. classified by the Pontryagin number**

$$A_\mu \sim U^{-1}(x)\,\partial_\mu U(x), \qquad x \to \infty \qquad \Pi_3(SU(2)) = \mathbb{Z}$$

$$\int d^4x\, \frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu} = n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

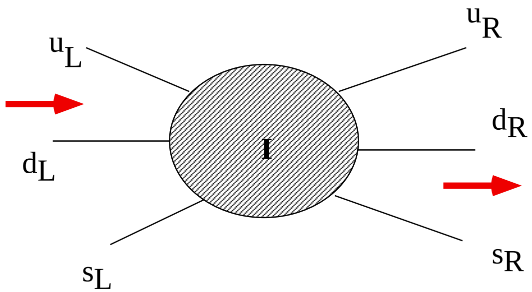
- **Config with $n = 1$: instanton** ([1])

$$A_\mu = -\frac{2i}{g^2}\frac{\tau_{\mu\nu}(x-x_0)_\nu}{(x-x_0)^2+\rho^2}, \qquad \tau_{\mu\nu} = \frac{\tau_\mu\bar{\tau}_\nu - \tau_\nu\bar{\tau}_\mu}{4}$$

- **Instanton effects in QCD ('t Hooft')**

$$\mathcal{L}_{eff} \sim \epsilon^{i_1\ldots i_{n_f}}\epsilon_{j_1\ldots j_{n_f}}\bar{\psi}_L^{j_1}(x)\ldots\bar{\psi}_L^{j_{n_f}}(x)\psi_{R,i_1}(x)\ldots\psi_{R,i_{n_f}}(x)$$

$U_A(1)$ **broken to** Z_{2n_f}; $SU_L(n_f)\times SU_R(n_f)$ **unbroken**



$$\langle\epsilon^{i_1\ldots i_{n_f}}\epsilon_{j_1\ldots j_{n_f}}\bar{\psi}_L^{j_1}(x_1)\ldots\bar{\psi}_L^{j_{n_f}}(x_{n_f})\psi_{R,i_1}(y_1)\ldots\psi_{R,i_{n_f}}(y_{n_f})\rangle \neq 0$$

[1] Belavin, Polyakov, Schwarz, 't Hooft

- θ **term**

$$\mathcal{L} = \theta \frac{g^2}{32\pi^2} G_{\mu\nu^a} \tilde{G}^{a,\mu\nu}$$

renormalizable. Experimentally ($d_n < 10^{-28}$ **e cm** $\rightarrow$

$$|\theta| < 10^{-9}$$

"Strong CP Problem" (Why?)

PQ symmetry (axions); $m_u = 0$**, etc**

- $\Delta I = \frac{1}{2}$ **problem (Why** $\frac{A(K\rightarrow\pi\pi)^{\Delta I=1/2}}{A(K\rightarrow\pi\pi)^{\Delta I=3/2}} \sim 25$)

Instanton Calculation in Susy QCD

- **Strong coulpling (standard) instanton method**

$$\langle \lambda\lambda(x_1)\lambda\lambda(x_2)\ldots\lambda\lambda(x_{n_c})\rangle = \text{const.}\,\Lambda^{3n_c}$$

$$L.H.S. = \text{const.} = \prod \langle\lambda\lambda\rangle = \langle\lambda\lambda\rangle^{n_c}$$

Require disentangle vac. sum (Z_{2n_c} unbroken)

- **Weak coupling instanton method (svz)**

(i) SQCD with massless $(Q,\tilde{Q})$'s

(ii) Flat direction → Compute instanton corrections at large $\langle Q\rangle \gg \Lambda$;

$$\Delta\mathcal{W}^{(ADS)} = (n_c - n_f)\frac{\Lambda^{(3n_c-n_f)/(n_c-n_f)}}{(\det Q\tilde{Q})^{1/(n_c-n_f)}} \qquad (\#)$$

(iii) Add $\mathcal{W}_{mass} = mQ\tilde{Q}$ → min. of the pot.

(iv) Decouple the quarks $m\to\infty$, $\Lambda^*_{YM} = m\Lambda^*$

$$\langle\lambda\lambda\rangle = \Lambda^3$$

- **Numerical discrepancy ("4/5 puzzle")**

- **Other methods (Compactification on $\mathbf{R}^3 \times S^1$; $\mathcal{N} = 2$ SYM and decoupling the adjoint scalar) give WCI results**

- **For $SU(n_c)$:**

$$\langle \lambda\lambda \rangle = e^{2\pi i k/n_c} \Lambda^3, \qquad k = 1, 2, \ldots n_c$$

- $SU(r+1)$**,** $SO(2r+1)$**,** $USp(2r)$**,** $SO(2r)$ **SYM: (apart from $e^{2\pi i k/T_G}$)**

$$T_G = r+1,\ 2r-1,\ r+1,\ 2r-2,$$

$$\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{SU(r+1)} = \Lambda^3 \,, \qquad \left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{SO(2r+1)} = 2^{\frac{4}{2r-1}-1} \Lambda^3 \,,$$

$$\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{USp(2r)} = 2^{1-\frac{2}{r+1}} \Lambda^3 \,, \qquad \left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{SO(2r)} = 2^{\frac{2}{r-1}-1} \Lambda^3 \,,$$

$U(1)$-Related (Konishi) Anomaly

- $$-\frac{1}{4}\bar{D}^2(\,Q^\dagger e^V Q\,) = m\tilde{Q}Q + \frac{g^2}{16\pi^2}\mathrm{Tr}W_\alpha W^\alpha$$

  Im. part of the F-component = $U_A(1)$ anomaly

- **In SQCD**

  $$\{\bar{Q}_{\dot\alpha}, \bar{\psi}^{\dot\alpha} Q\} = m\,\tilde{Q}Q - \frac{g^2}{16\pi^2}\mathrm{Tr}\lambda_\alpha\lambda^\alpha$$

- **Vacuum alligned with mass perturbation**

  $$\langle m_i\tilde{Q}_i Q_i\rangle = \langle\frac{g^2}{16\pi^2}\mathrm{Tr}\lambda_\alpha\lambda^\alpha\rangle \qquad (\mathrm{no}\quad \mathrm{sum}) \quad i = 1,\ldots n_f$$

  cfr. **Dashen;** $\langle\bar{\psi}_i\psi_i\rangle = -\Lambda^2$ **(** $i = u, d, s$**)**

- **General chiral gauge th with $\mathcal{W}(\Phi_i)$**

  $$-\frac{1}{4}\bar{D}^2(\,\Phi_i^\dagger e^V \Phi_i\,) = \Phi_i\frac{\partial\mathcal{W}}{\partial\Phi_i} + C(\Phi_i)\frac{g^2}{16\pi^2}\mathrm{Tr}W_\alpha W^\alpha \qquad (\S)$$

- **Check of dynamical calculation (Instantons) and general argument**

• **Derivation:** $\delta\Phi_i = i\,A(z)\,\Phi_i$ **(** $A(z)$ **arbitrary)** $\rightarrow$ **Jacobian**

$$J = \det(\delta\Phi'_{z'}/\delta\Phi_z) = \det\langle z'|\, e^{iA(z)}(-\frac{\bar{D}^2}{4})\,|z\rangle = e^{\mathrm{Tr}\, iA(z)\frac{-\bar{D}^2}{4}}$$

• **Regularize the high eigenvalues by**

$$\mathrm{Tr}\,[iA(z)\frac{-\bar{D}^2}{4}] \rightarrow \lim_{M\rightarrow\infty} \mathrm{Tr}\,[iA(z)e^{L/M^2}(\frac{-\bar{D}^2}{4})]$$

$$L \equiv \bar{D}^2 e^{-V} D^2 e^V/16$$

• **Acting on** $\frac{-\bar{D}^2}{4}$

$$L = P^2 - \frac{1}{2}W^\alpha D_\alpha + C^\mu P_\mu + F,$$

where

$$W^\alpha = -\frac{1}{4}\,(\bar{D}^2 e^{-V} D^\alpha e^V),$$

$$C^\mu = -\frac{1}{2}\,\sigma^\mu_{\alpha\dot{\alpha}}(\bar{D}^{\dot{\alpha}} e^{-V} D^\alpha e^V),$$

$$F = (\bar{D}^2 e^{-V} D^2 e^V)/16.$$

- $M \to \infty$;

$$\int d^4p \, e^{-p^2/M^2} \sim M^4;$$

each power of L/M^2 from the exponent; also

$$\langle \theta\bar{\theta} | DD\bar{D}^2 | \theta\bar{\theta} \rangle \neq 0,$$

$\therefore$ **only terms quadratic in $\frac{1}{2}W^\alpha D_\alpha$ contribute $\Rightarrow$ (§)**

- **Pauli-Villars, Supergraph $1-$loop calculation, Point-splitting, BPHZ, (Clark-Love, Gates-Grisaru-Rocek-Siegel, Piguet-Sibold, Konishi, Konishi-Shizuya); All these methods in Component formalism**

- **Functional-integral method particulary elegant for generalization**

Intrilligator, Leigh, Seiberg ('94)

- $\mathcal{N}=1$ Gauge theory G with generic matter ϕ_i with

$$\mathcal{W}_{tree}(\phi_i) = \sum_r g_r X^r(\phi_i)$$

- Set $\mathcal{W}_{tree}(\phi_i)=0$ first. $\rightarrow$ **Flat directions** along ϕ_i. Reinterpret in terms of gauge invariant composites (as (*) for SQCD).

- Turn on g_r and Λ_s. $\mathcal{W}_{eff}$ restricted by

  (i) **holomorphy** (i.e. holomorphic in g_r, X_r, Λ_s.)

  (ii) **invariance under various symmetries.** If some symmetry is broken by $\mathcal{W}$, it can be regarded as exact, by assigning appropriately the charges to g_r, Λ_s

  (iii) **Asymptotics**

- In many cases these are sufficient to determine $\mathcal{W}_{eff}$ exactly.

Phases of SQCD; Seiberg duality

- **Massless SQCD**

  → **Superpot.** (#); **Vacuum runaway ($n_f < n_c$);**

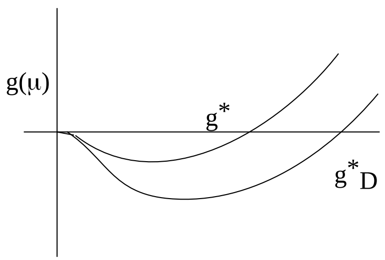
  → **No generation of superpotential for** $n_f > n_c$

- $n_f = n_c$**:**

$$(C.M.S.) \qquad \det M - B\,\tilde{B} = 0 \qquad\qquad (*)$$

$$(Q.M.S.) \qquad \det M - B\,\tilde{B} = \Lambda^{2n_f}$$

- $\frac{3n_c}{2} < n_f < 3n_c$ **(Conformal window), infrared fixed point (SCFT): described either as the original SQCD (with** Q**,** $\tilde{Q}$**) or as dual** $SU(\tilde{n}_c) = SU(n_f - n_c)$ **theory with dual quarks (**$q, \tilde{q}, M$**) (Seiberg,** Kutasov, Schwimmer, ... **)**



N_f	**Deg.Freed.**	**Eff. Gauge Group**	**Phase**	**Symmetry**
0 **(SYM)**	**-**	**-**	**Confinement**	**-**
$1 \leq N_f < N_c$	**-**	**-**	**no vacua**	**-**
N_c	$M, B, \tilde{B}$	**-**	**Confinement**	$U(N_f)$
$N_c + 1$	$M, B, \tilde{B}$	**-**	**Confinement**	**Unbroken**
$N_c + 1 < N_f < \frac{3N_c}{2}$	$q, \tilde{q}, M$	$SU(\tilde{N}_c)$	**Free-magnetic**	**Unbroken**
$\frac{3N_c}{2} < N_f < 3N_c$	$q, \tilde{q}, M$ **or** $Q, \tilde{Q}$	$SU(\tilde{N}_c)$ **or** $SU(N_c)$	**SCFT**	**Unbroken**
$N_f = 3N_c$	$Q, \tilde{Q}$	$SU(N_c)$	**SCFT (finite)**	**Unbroken**
$N_f > 3N_c$	$Q, \tilde{Q}$	$SU(N_c)$	**Free Electric**	**Unbroken**

Parma 10-12 Sept. 2003

Lectures on Supersymmetric Gauge Theories II:

Non-Abelian Superconductors and Confinement

K. Konishi

- **Confinement in QCD**
- **Seiberg-Witten solutions for $\mathcal{N} = 2$ susy gauge theories**
- **Non-Abelian Superconductor: Monopoles, vortices and Confinement**

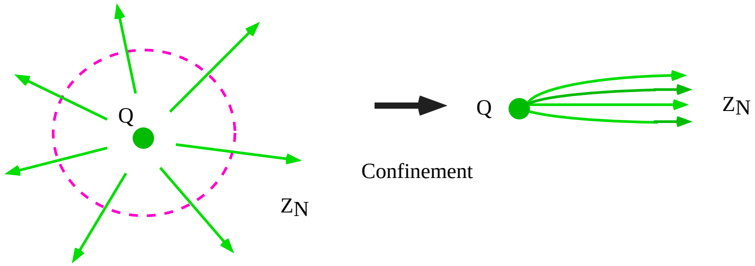
Work with ('98 - '03)

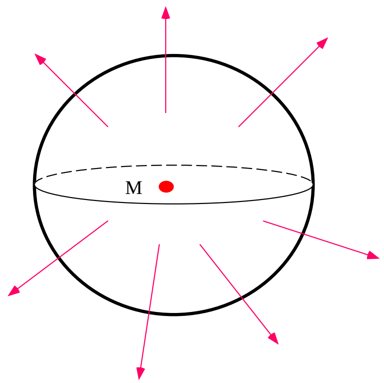
H. Murayama, S. P. Kumar, G. Carlino,

L. Spanu, S. Bolognesi, K. Takenaga,

H. Terao, R. Auzzi, R. Grena, A. Yung

$SU(N)$ **YM**





$\Pi_1(SU(N)/Z_N) \sim Z_N$

$\Longrightarrow$ $(Z_N^{(M)}, Z_N^{(E)})$ **classification of phases ('t Hooft)**

$SU(N)$ YM and Solvable Cousins

- $(Z_N^{(M)}, Z_N^{(E)})$ classification ('t Hooft):

  If field with $x = (a, b)$ condense, particles $X = (A, B)$ with

  $$\langle x, X \rangle \equiv a\,B - b\,A \neq 0 \quad (mod\ N)$$

  are confined. (e.g. $\langle \phi_{(0,1)} \rangle \neq 0 \rightarrow$ Higgs phase.)

- **Quarks are confined if some field χ exist, s.t.**

  $$\langle \chi_{(1,0)} \rangle \neq 0$$

- Softly Broken $N = 4$ (to $N = 1$): all different types of massive vacua, related by $SL(2, Z)$, chiral condensates known

  Donagi-Witten, Strassler, Dorey-Kummer

- Softly Broken $N = 2$ Gauge Theories:

  Dynamics particularly transparent

What is χ ? How do they interact ?

XSB**?** θ**;** $\frac{\epsilon'}{\epsilon}$**;** $\Delta I = \frac{1}{2}$ **?**

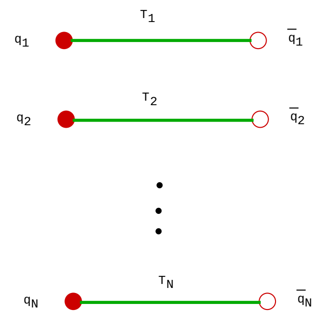
QCD as Dual Superconductor

- $\nexists$ Elementary/soliton monopoles
- Monopoles as topological singularities (lines in $4D$) of Abelian gauge fixing, $SU(3) \to U(1)^2$ ('t Hooft)
- $SU(2) \quad : \quad A^a_\mu = \tilde{\sigma}(x)(\partial_\mu \mathbf{n} \times \mathbf{n})^a + \dots, \qquad \mathbf{n}(\mathbf{r}) = \frac{\mathbf{r}}{r}$

  $\Rightarrow \quad A^a_i = \epsilon_{aij} \frac{r^j}{r^3}$ (Wu-Yang, Cho, Faddeev-Niemi)
- Some evidence in lattice QCD (Di Giacomo, et. al.)
- **Do (Abelian) monopoles carry flavor? ($\mathcal{L}_{eff}$?)**
- **Gauge dependence?**
- **Dynamical $SU(N) \to U(1)^{N-1}$ Breaking? Would imply a richer spectrum of mesons ($T_1 \neq T_2$, etc.)**
- **In Nature and in QCD:**

$$\text{Meson} \sim \sum_{i=1}^{N} |\, q_i \, \bar{q}_i \,\rangle$$

i.e., 1 state vs $\left[\frac{N}{2}\right]$ states ($SU(N) \to U(1)^{N-1} \times Weyl$ not enough).



Dirac's monopoles

- **QED admits pointlike magnetic monopoles if (Dirac)**

$$g\,e = \frac{n}{2}, \qquad n \in \mathbb{Z}, \tag{1}$$

$$\oint_{\partial\Omega} A_i\,dx^i \to \int_{S^2} d\mathbf{S}\cdot\mathbf{H} = 4\pi g, \qquad \mathbf{H} = \nabla\frac{g}{r}.$$

If A **regular then LHS** $\to 0$. **(!?!) Either Dirac string along** $(0,0,0) \to (0,0,-\infty)$ **(invisible if (1) satisfied), or**

- **Cover** S^2 **by two regions** $a : (0 \le \theta < \frac{\pi}{2} + \epsilon)$ **and** $b : (\frac{\pi}{2} - \epsilon < \theta \le \pi)$ **(Wu-Yang)**

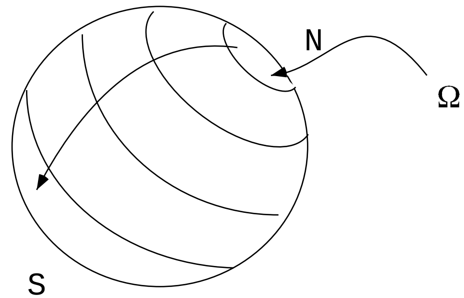
$$(A_\phi)^a = \frac{g}{r\sin\theta}(1-\cos\theta), \qquad (A_\phi)^b = -\frac{g}{r\sin\theta}(1+\cos\theta),$$

$$A_i^a = A_i^b - U^\dagger\frac{i}{e}\partial_i U, \qquad U = e^{2ige\phi}, \quad \text{OK if (1))}$$

- **More generally, for dyons** (e_1, g_1)**,** (e_2, g_2)**,**

$$e_1\,g_2 - e_2\,g_1 = \frac{n}{2}, \qquad n \in \mathbb{Z}, \tag{2}$$

- **Topology:** $\Pi_1(U(1)) = \mathbb{Z}$



Dirac monopoles in NA gauge theories

- **Use the $U(1)$ subgroup**
- **But**

$$SU(2) \sim S^3, \qquad \Pi_1(SU(2)) = \mathbf{1}, \qquad SO(3) \sim \frac{S^2}{Z_2}, \qquad \Pi_1(SO(3)) = \mathbb{Z}_2, \quad (3)$$

$\rightarrow$ **NO monopoles in $SU(2)$, $SU(N)$; one type of monopole in $SO(3)$, and so on.**

’t Hooft-Polyakov

- $$SU(2) \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} U(1)$$
$$\mathcal{D}\phi \stackrel{r\to\infty}{\longrightarrow} 0, \quad \Rightarrow \quad \phi \sim U\cdot\langle\phi\rangle\cdot U^{-1};$$
$$A_i^a \sim U\cdot\partial_i U^\dagger \to \epsilon_{aij}\frac{r_j}{r^3}m, \qquad m = 1,2,\ldots$$
- (ϕ, A_μ) **represents nontrivial elements of** $\Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = \mathbb{Z}$
- **Regular, finite energy configurations (*cfr* Dirac)**
- $$H = \int d^3x \left[\frac{1}{4}(F_{ij}^a)^2 + \frac{1}{2}(D_i\phi^a)^2 + \frac{\lambda}{2}(\phi^2 - v^2)^2\right]$$
$$= \int d^3x \left[\frac{1}{4}(F_{ij}^a - \epsilon_{ijk}D_k\phi^a)^2 + \frac{1}{2}F_{ij}^a\epsilon_{ijk}D_k\phi^a + \frac{\lambda}{2}(\phi^2 - v^2)^2\right]$$
- $\frac{1}{2}F_{ij}^a\epsilon_{ijk}D_k\phi^a = \partial_i S_i; \qquad S_i = \frac{1}{2}\epsilon_{ijk}F_{jk}^a\,\phi^a = B_i^a\phi^a.$ **(Bogomolny equation)**
- $H \geq \int d^3x \nabla\cdot\mathbf{S} = \frac{4\pi v}{g}m, \quad m = 1,2,\ldots$**.**

- $\lambda = 0$ **(BPS):**

$$F^a_{ij} - \epsilon_{ijk} D_k \phi^a = 0, \quad \text{or} \quad B^a_k = D_k \phi^a, \qquad H = \frac{4\pi v}{g} m,$$

$$A^a_i = \epsilon_{aij} \frac{r_j}{r^3} A(r), \qquad \phi^a = \frac{r^a}{r} \phi(r),$$

$A(r),\ \phi(r)$ **known explicitly (**$A(r) \to -\frac{1}{r}, \quad \phi(r) \to v$**),**

Nonabelian monopoles

$$G \overset{\langle\phi\rangle\neq 0}{\longrightarrow} H$$

$$\mathcal{D}\phi \overset{r\to\infty}{\longrightarrow} 0, \quad \Rightarrow \quad \phi \sim U \cdot \langle\phi\rangle \cdot U^{-1} \sim \Pi_2(G/H) = \Pi_1(H);$$

$$t^a A_i^a \sim U \cdot \partial_i U^\dagger \Longrightarrow F_{ij} = \epsilon_{ijk}\frac{r_k}{r^3}\beta_\ell T_\ell, \qquad T_i \in \text{Cartan S.A. of} \quad H$$

Topological quantization $\quad \Longrightarrow \quad 2\,\alpha\cdot\beta \subset \mathbb{Z}$**:**

$$\beta_i = \text{weight vectors of } \tilde{H} = \text{dual of } H.$$

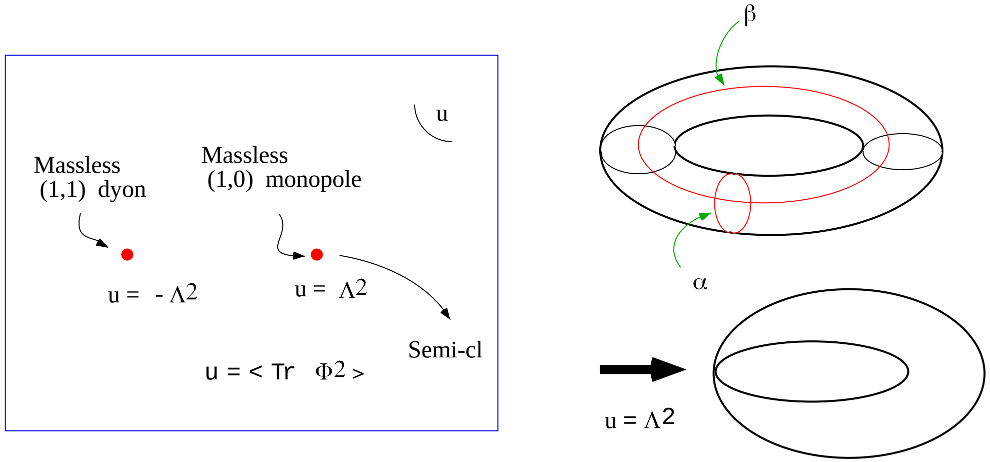
(Goddard-Nuyts-Olive, E. Weinberg)

$$\tilde{H} \Leftrightarrow H$$

$SU(N)/Z_N$	$\Leftrightarrow$	$SU(N)$
$SO(2N)$	$\Leftrightarrow$	$SO(2N)$
$SO(2N+1)$	$\Leftrightarrow$	$USp(2N)$

- **Dirac monopoles (Wu-Yang) for** $|\phi| \to \infty$**;**
- **’t Hooft-Polyakov monopoles for** $G = SU(2),\ H = U(1)$

Seiberg-Witten Solution in $\mathcal{N} = 2$ Gauge Theories



- $SU(2)$: **Lagrangian is Eq.(1) of I with** $\mathcal{W}(\Phi) = 0$

$$\langle \Phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

- $a \neq 0$ **breaks** $SU(2) \to U(1)$**: at IR,**

$$\mathcal{L}_{eff} = \mathrm{Im}\left[\int d^4\theta\, \bar{A}\, \frac{\partial \mathcal{F}_p(A)}{\partial A} + \int \frac{1}{2}\frac{\partial^2 \mathcal{F}_p(A)}{\partial A^2} W_\alpha W^\alpha\right]$$

where W_α, A **describe** $\mathcal{N} = 2\; U(1)$ **theory**

- **Define** $A_D \equiv \frac{\partial \mathcal{F}_p(A)}{\partial A}$**: then**

$$\frac{d\,A_D}{d\,u} = \oint_\alpha \frac{dx}{y}, \qquad \frac{d\,A}{d\,u} = \oint_\beta \frac{dx}{y},$$

where ($u \equiv \mathrm{Tr}\,\langle \Phi^2 \rangle$ **describes QMS)**

$$y^2 = (x-u)(x+\Lambda^2)(x-\Lambda^2)$$

- **Exact mass formula (BPS):** $m_{n_m,n_e} = \sqrt{2}\,|\,n_m A_D + n_e A\,|$

- $\mu\,\Phi^2$ **perturbation (Confinement)**

$$\mathcal{W}_{eff} = \sqrt{2}\,A_D\,M\tilde{M} + \mu\,U(A_D) \Rightarrow \langle M \rangle \sim \sqrt{\mu\,\Lambda}$$

- **At the singularities** $u = \pm\Lambda^2$**, instanton sum diverges**

$$\langle \mathrm{Tr}\Phi^2 \rangle = \frac{a^2}{2} + \frac{\Lambda^4}{a^2} + \ldots = \ldots + 1 + 1 + 1 + \ldots$$

- **Dynamical Abelianization,** $SU(N) \to U(1)^{N-1}$ **(***cfr* **QCD)**
- **These "monopoles" are indeed 't Hooft-Polyakov monopoles**

$$\frac{2}{g} Q_e = n_e + \left[-\frac{4}{\pi}\mathrm{Arg}\,a + \frac{1}{2\pi}\sum_{f=1}^{N_f} \mathrm{Arg}\,(m_f^2 - 2a^2) \right] n_m + \ldots$$

- **Jackiw-Rebbi for** $n_f = 1, 2, 3$**;**
- **Quantum quenching of Quark Numbers (Carlino-Terao-Konishi)**
- **Susy breaking (e.g.** $m_\lambda \lambda \lambda$**)** $\rightarrow$ **nontrivial** θ **dependence**

More general $\mathcal{N} = 2$ models

$SU(n_c)$, $USp(2n_c)$ **or** $SO(n_c)$ **with** n_f **Quarks**

- $$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\,\tau_{cl}\left[\int d^4\theta\,\Phi^\dagger e^V\Phi + \int d^2\theta\,\frac{1}{2}WW\right] + \mathcal{L}^{(quarks)} + \Delta\mathcal{L},$$

  where

  $$\Delta\mathcal{L} = \int d^2\theta\ \mu\,\mathrm{Tr}\,\Phi^2, \qquad \tau_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}$$

- **(**$\mathcal{N}$ =2**) gauge multiplet** $\Phi = \phi + \sqrt{2}\,\theta\,\psi + \ldots;\ W_\alpha = -i\lambda + \frac{i}{2}\,(\sigma^\mu\,\bar{\sigma}^\nu)_\alpha^\beta\,F_{\mu\nu}\,\theta_\beta + \ldots$ **both in the adjoint representation;**

- $$\mathcal{L}^{(quarks)} = \sum_i\,[\int d^4\theta\,\{Q_i^\dagger e^V Q_i + \tilde{Q}_i^\dagger e^{\tilde{V}}\tilde{Q}_i\} + \int d^2\theta\,\{\sqrt{2}\tilde{Q}_i\Phi Q^i + m_i\tilde{Q}_i Q^i\}]$$

- **Asymptotic Freedom** $\rightarrow$

  $$n_f \le 2n_c,\ 2n_c+2,\ n_c-2, \qquad \text{for} \quad SU(n_c),\ USp(2n_c),\ SO(n_c).$$

- **Global Symmetry ($m_i \to 0$):**

$$G_F = \begin{cases} U(n_f) \times Z_{2n_c - n_f} & SU(n_c); \\ SO(2n_f) \times Z_{2n_c + 2 - n_f} & USp(2n_c); \\ USp(2n_f) \times Z_{2n_c - 2n_f - 4} & SO(n_c) \end{cases}$$

Seiberg-Witten curves in general $\mathcal{N} = 2$ theories

$SU(n_c)$ **(**$USp(2n_c)$**)**

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda^{2n_c - n_f} \prod_{j=1}^{n_f} (x + m_j), \qquad SU(n_c), \ \ n_f \leq 2n_c - 2,$$

and

$$y^2 = \prod_{k=1}^{n_c} (x - \phi_k)^2 + 4\Lambda \prod_{j=1}^{n_f} (x + m_j + \frac{\Lambda}{n_c}), \qquad SU(n_c), \ \ n_f = 2n_c - 1,$$

with $\sum_{k=1}^{n_c} \phi_k = 0$**,**

$USp(2n_c)$**:**

$$xy^2 = \left[x \prod_{a=1}^{n_c} (x - \phi_a^2) + 2\Lambda^{2n_c + 2 - n_f} m_1 \cdots m_{n_f} \right]^2 - 4\Lambda^{2(2n_c + 2 - n_f)} \prod_{i=1}^{n_f} (x + m_i^2).$$

$SO(n_c)$**:**

$$y^2 = x \prod_{a=1}^{[n_c/2]} (x - \phi_a^2)^2 - 4\Lambda^{2(n_c-2-n_f)} x^{2+\epsilon} \prod_{i=1}^{n_f} (x - m_i^2),$$

$$y^2 = x \prod_{a=1}^{[n_c/2]} (x - \phi_a^2)^2 - 4\Lambda^{2(n_c-2-n_f)} x^{2+\epsilon} \prod_{i=1}^{n_f} (x - m_i^2), \qquad m_i = 0,$$

where $\epsilon = 1$ if n_c is even; $\epsilon = 0$ if n_c is odd.

Genus K ($n_c - 1, n_c, [n_c/2]$ for the above groups) hypertorus

$\mu \, \mathrm{Tr}\, \Phi^2$ perturbation and $\mathcal{N}=1$ vacua

The effective action near the

$$(n_{m1}, n_{m2}, \ldots, n_{mK}; n_{e1}, n_{e2}, \ldots, n_{eK}) = (1, 0, \ldots, 0; 0, \ldots), \;\; \ldots, \;\; (0, 0, \ldots, 1; 0, \ldots)$$

singularity is:

$$\mathcal{W} = \sum_{i=1}^{K} \tilde{M}_i \{\sqrt{2}\, a_{Di} \, + \sum_{k=1}^{n_f} S_k^i \, m_k\} M_i + \mu \, u_2(a_D, a)$$

The vacuum equations:

$$-\frac{\mu}{\sqrt{2}} \; = \; \sum_{i=1}^{K} \frac{\partial a_{Di}}{\partial u_2} \tilde{M}_i M_i \,; \qquad 0 = \sum_{i=1}^{K} \frac{\partial a_{Di}}{\partial u_j} \tilde{M}_i M_i, \quad j = 3, 4, \ldots, K+1; \qquad (4)$$

$$\begin{aligned} (\sqrt{2}\, a_{D1} + \sum_{k=1}^{n_f} S_k^1 \, m_k) \, \tilde{M}_1 \; &= \; (\sqrt{2}\, a_{D1} + \sum_{k=1}^{n_f} S_k^1 \, m_k) \, M_1 = 0 \,; \\ \ldots \qquad \ldots &= \ldots \qquad \ldots \\ (\sqrt{2}\, a_{DK} + \sum_{k=1}^{n_f} S_k^2 \, m_k) \, \tilde{M}_K \; &= \; (\sqrt{2}\, a_{DK} + \sum_{k=1}^{n_f} S_k^2 \, m_k) \, M_K = 0 \,. \end{aligned} \qquad (5)$$

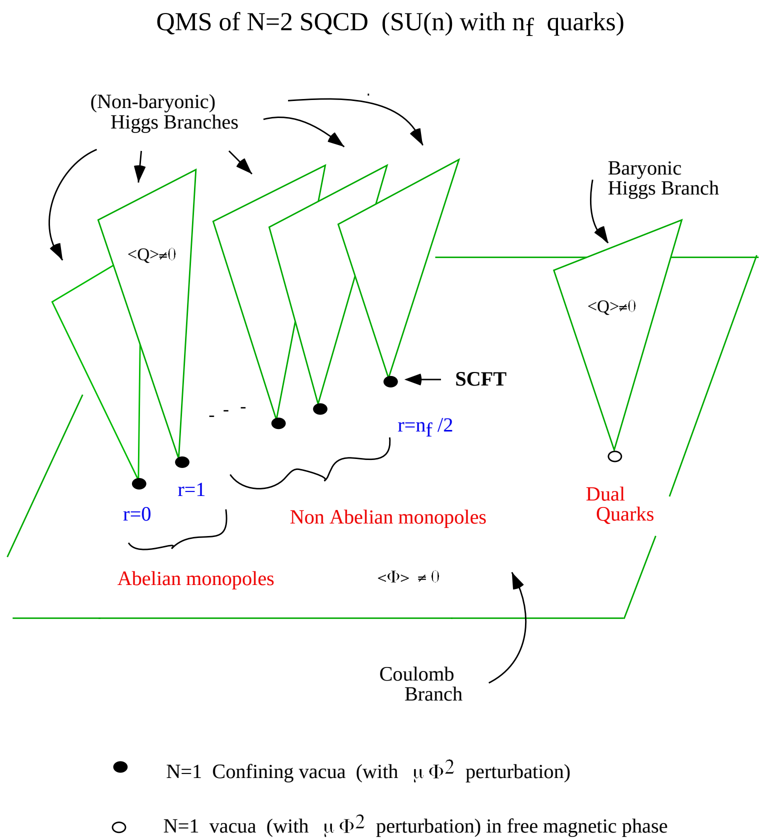
For generic m_i, Eqs.(4) $\to \tilde{M}_i \neq 0$; $M_i \neq 0$, $\forall i$ ($\frac{\partial a_{Di}}{\partial u_j}$ and $\frac{\partial a_{Di}}{\partial u_2}$ satisfy no special relations.) This means that

$$\sqrt{2}\, a_{Di} + \sum_{k=1}^{n_f} S_k^i \, m_k = 0 \qquad \forall i \tag{6}$$

i.e., **all K monopoles are simultaneously massless, and condense**. Confinement à la 't Hooft-Mandelstam, corresponding to **dual superconductor** in the maximal Abelian subgroup.

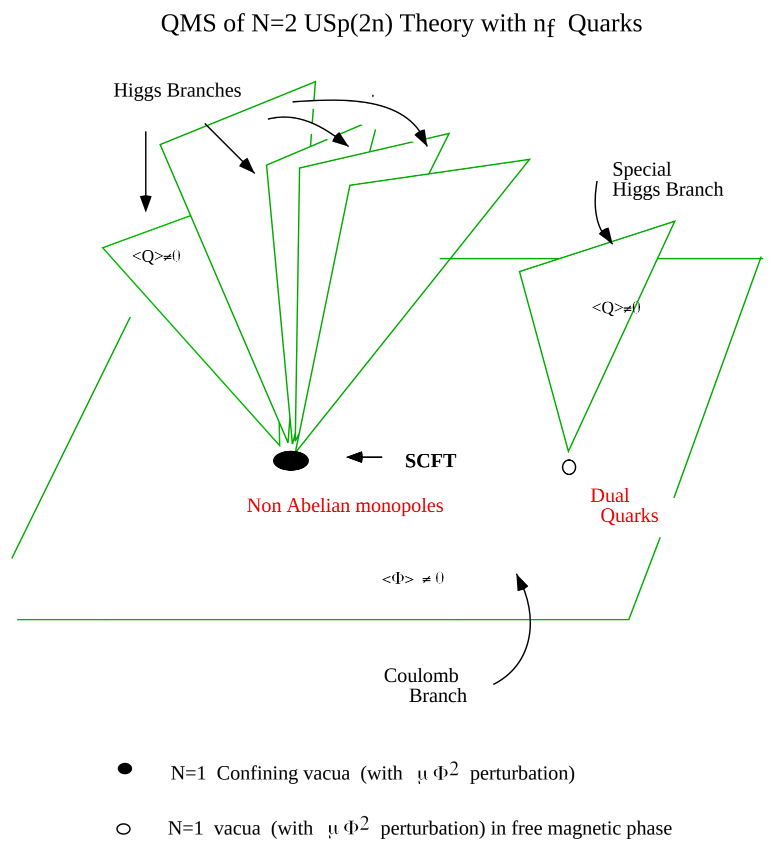
Actually, non-abelian dual superconductors in the $m_i \to 0$ limit

Vacua of general $\mathcal{N} = 2$ models



- $G_{eff} \sim SU(r) \times U(1)^{n_c-r-1}$; n_f dual quarks* in $\underline{r}$
- Confining for $r \leq \frac{n_f}{2}$; SCFT at $r = \frac{n_f}{2}$ **

$^{*)}$ **What are they?** $^{**)}$ **What DoF?**



- All r vacua (at finite m) collapse into a single SCFT at $m \to 0$;
- All confining vacua (with $\mu\,\Phi^2$) are of this type;
- Global $SO(2\,n_f) \to U(n_f)$ Symmetry Breaking
  $(cfr\ \langle\bar{\psi}\,\psi\rangle^{(QCD)} \neq 0)$

label (r)	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
0 (NB)	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f)$
1 (NB)	monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f-1)\times U(1)$
$2,..,[\frac{n_f-1}{2}]$ (NB)	dual quarks	$SU(r)\times U(1)^{n_c-r}$	Confinement	$U(n_f-r)\times U(r)$
$n_f/2$ (NB)	rel. nonloc.	-	Almost SCFT	$U(n_f/2)\times U(n_f/2)$
BR	dual quarks	$SU(\tilde{n}_c)\times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$U(n_f)$

Table 1: Phases of $SU(n_c)$ gauge theory with n_f flavors. $\tilde{n}_c \equiv n_f - n_c$.

	Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
1st Group	rel. nonloc.	-	Almost SCFT	$U(n_f)$
2nd Group	dual quarks	$USp(2\tilde{n}_c)\times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$SO(2n_f)$

Table 2: Phases of $USp(2n_c)$ gauge theory with n_f flavors with $m_i \to 0$. $\tilde{n}_c \equiv n_f - n_c - 2$.

Q.N. of the NA Monopoles

$$SU(3) \xrightarrow{\langle\phi\rangle} \frac{SU(2)\times U(1)}{\mathbb{Z}_2}, \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

't Hooft-Polyakov solutions in $SU_U(2), SU_V(2) \subset SU(3)$

$\Rightarrow$ **two $SU(3)$ solutions* with** $\Pi_1(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}$

monopoles	$\tilde{SU}(2)$	$\tilde{U}(1)$
$\tilde{q}$	$\underline{2}$	1

$$SU(n) \xrightarrow{\langle\phi\rangle} SU(r)\times U^{n-r}(1), \qquad \langle\phi\rangle = \begin{pmatrix} v_1 \mathbf{1}_{r\times r} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & v_2 & 0 & \dots \\ \mathbf{0} & 0 & \ddots & \dots \\ \mathbf{0} & 0 & \dots & v_{n-r+1} \end{pmatrix}$$

- **Degenerate r-plet of monopole solutions** (q);**
- **The same charge structure in the r-vacua of $\mathcal{N}=2$ SQCD**

monopoles	$\tilde{SU}(r)$	$\tilde{U}_0(1)$	$\tilde{U}_1(1)$	$\tilde{U}_2(1)$	$\dots$	$\tilde{U}_{n-r-1}(1)$
q	$\underline{r}$	1	0	0	$\dots$	0
e_1	$\underline{1}$	0	1	0	$\dots$	0
e_2	$\underline{1}$	0	0	1	0	0
$\vdots$	$\underline{1}$	0	$\dots$			0
e_{n-r-1}	$\underline{1}$	0	0	$\dots$	$\dots$	1

• **Flavor q.n. of N.A. monopoles? $\Leftarrow$ Jackiw-Rebbi**

BPS monopoles

* $SU(3)$

$$SU(3) \xrightarrow{\langle\phi\rangle} SU(2) \times U(1), \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}.$$

A broken $SU_U(2)$ subgroup $\rightarrow$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad \frac{t^3 + \sqrt{3}t^8}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a solution

$$\phi(\mathbf{r}) \;=\; \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3\,v(t_4, t_5, \frac{t_3}{2} + \frac{\sqrt{3}t_8}{2}) \cdot \hat{r}\phi(r),$$

$$\vec{A}(\mathbf{r}) \;=\; (t_4, t_5, \frac{t_3}{2} + \frac{\sqrt{3}t_8}{2}) \wedge \hat{r}A(r),$$

where $\phi(r)$ and $A(r)$ are BPS- 't Hooft's functions with $\phi(\infty) = 1,\ \phi(0) = 0,\ A(\infty) = -1/r$.

A second solution with the same energy by using another $SU_V(2)$ group.

Nonabelian Monopoles Are Subtle

- $\nexists$ "Colored dyons" (?!) [1] (Abouelsaood, Coleman, E. Weinberg, Balachandran, ...)

  i.e. In the background of a non-Abelian monopole not possible to construct globally defined $T^1 - T^3$, isomorphic to unbroken $SU(2)$

- Monopoles are multiplets of the dual $\tilde{H}$ group, not of H. The no-go theorem $\rightarrow$

$$G_{gauge} \neq H \otimes \tilde{H}$$

- Not justified to study $G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H$ as a limit of max.ly broken cases;

- NA monopoles never really semi-classical, even if $\langle\phi\rangle \gg \Lambda_H$:

  - If H broken $\Rightarrow$ approximately degenerate set of monopoles *e.g.*, Pure $\mathcal{N}=2$, $SU(3)$

  - If H unbroken $\Rightarrow$ N.A. monopoles in irreps of $\tilde{H}$. ♡

- ♡ **realized in the r vacua of $\mathcal{N}=2$ SQCD with $SU(r) \times U(1)^{n_c - r+1}$ gauge group.**

[1]No charge fractionalization (Goldstone-Wilczek, Niemi-Paranjape-Semenoff, Witten effect) for non-Abelian charges.

It occurs only for $r < \frac{n_f}{2}$ $\Leftarrow$ **Sign-flip of the beta function:**

$$b_0^{(dual)} \propto -2\,r + n_f > 0, \qquad b_0 \propto -2\,n_c + n_f < 0.$$

- When sign flip not possible (pure $\mathcal{N}=2$ YM, generic vacua of $\mathcal{N}=2$ theories) $\Rightarrow$ Dynamical Abelianization!

- **QM'ly, NA monopoles requires massless fermions**

Vortices

$\mathbb{Z}_N$ Vortices

Spanu, Konishi

- **Condensation of NA monopoles $\Leftrightarrow$ Confinement**
- $SU(N) \Rightarrow \mathbb{Z}_N$ **(in general, $G \Rightarrow \mathcal{C}$, a discrete center)**
- $\exists$ **Vortex if** $\Pi_1(G/\mathcal{C})$ **nontrivial, e.g.** $\Pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$

$$A_i \sim \frac{i}{g} U(\phi)\partial_i U^\dagger(\phi); \quad \phi_A \sim U \phi_A^{(0)} U^\dagger, \qquad U(\phi) = \exp i \sum_j^r \beta_j T_j \phi$$

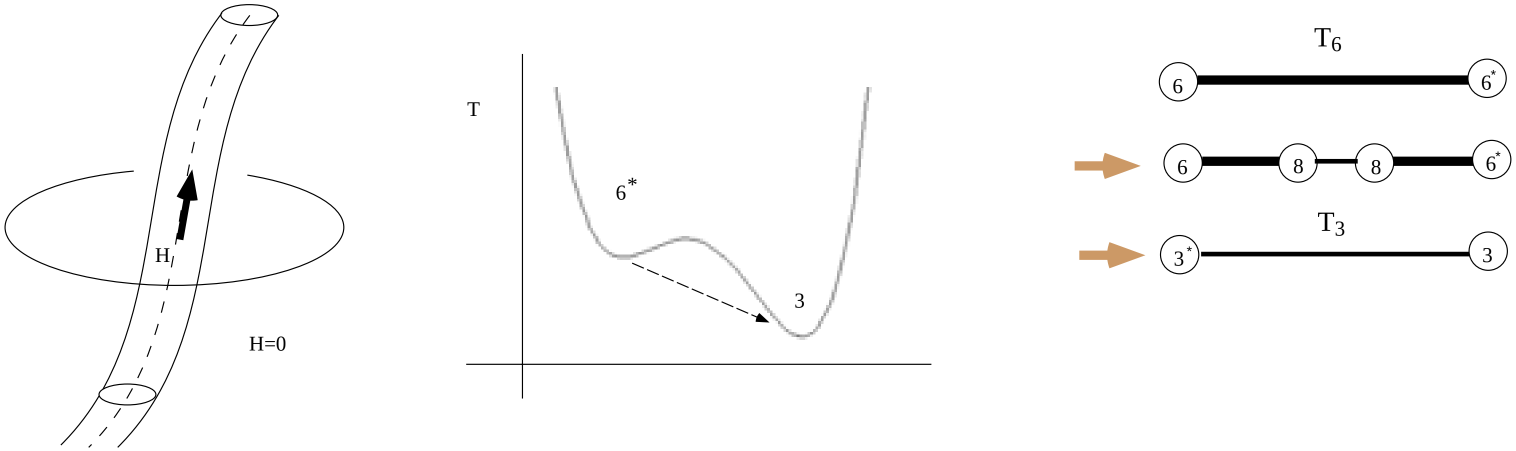
- **Quantization**

$$U(2\pi) \in \mathbb{Z}_N, \qquad \alpha \cdot \beta \in \mathbb{Z},$$

- **Solutions are irreps of** $\tilde{G} = SU(N)$: **carry** $\mathbb{Z}_N$ **(N-ality)**
- $\mathbb{Z}_N$ **vortices are non-BPS (*cfr.* ANO)**
- **Tension ratios**[2]

$$T_k \propto \sin \frac{\pi k}{N} \qquad ? \qquad T_\ell + T_m < T_{\ell+m} \quad ?$$

[2] Douglas-Shenker, Hanany-Strassler-Zaffaroni, Herzog-Klebanov, Del Debbio-Panagopoulos-Rossi-Vicari, Lucini-Teper, Auzzi-Konishi


H
H=0
T
6*
3
T6
6
6*
6
8
8
6*
T3
3*
3

Non-Abelian Vortices Hanany-Tong, Auzzi-Boplognesi-Evslin-Konishi-Yung '03

$SU(3) \to SU(2) \times U(1)/\mathbb{Z}_2$:

$\mathcal{N} = 2$ **theory with** $4 \le n_f \le 5$ **with large bare mass** m **(with adj mass** $\mu\,\Phi^2$**):**

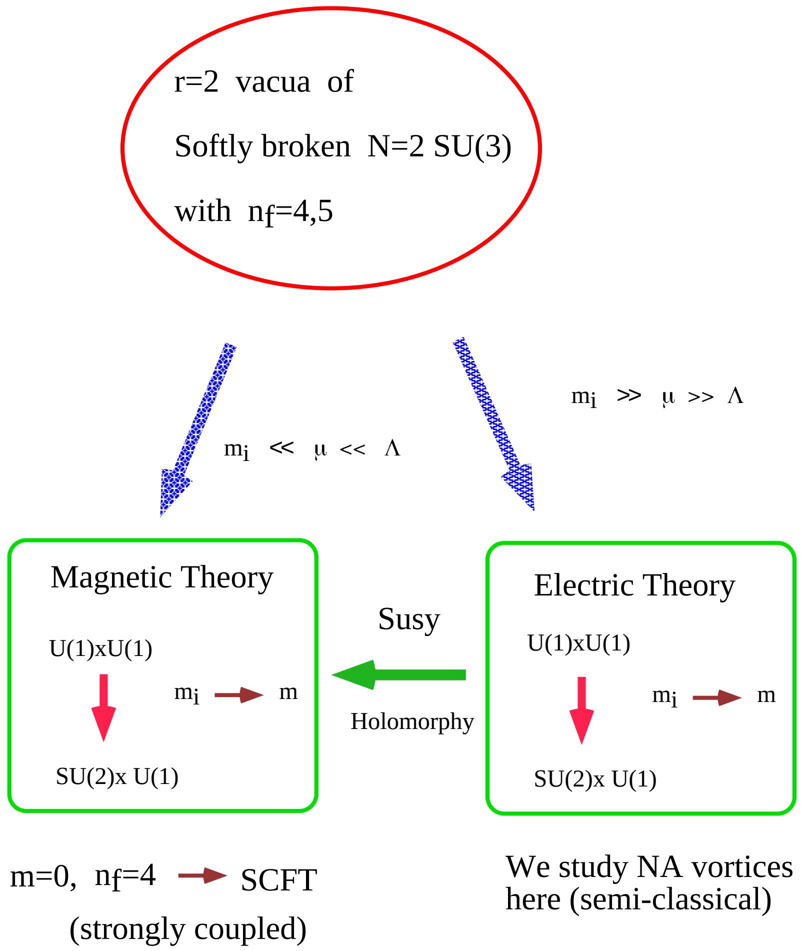
$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix}, \qquad < q^{kA} >=< \bar{\tilde{q}}^{kA} >= \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $\xi = \sqrt{\mu\, m} \ll m$. **For vortex solution, set** $\Phi = \langle \Phi \rangle$; $q = \tilde{q}^{\dagger}$; **and** $q \to \frac{1}{2} q$**:**

$$S = \int d^4x \left[\frac{1}{4g_2^2} \left(F^a_{\mu\nu} \right)^2 + \frac{1}{4g_1^2} \left(F^8_{\mu\nu} \right)^2 + \left| \nabla_\mu q^A \right|^2 + \frac{g_2^2}{8} \left(\bar{q}_A \tau^a q^A \right)^2 + \frac{g_1^2}{24} \left(\bar{q}_A q^A - 2\xi \right)^2 \right],$$

$$\begin{aligned} T \;&=\; \int d^2x \,(\sum_{a=1}^{3} \left[\frac{1}{2g_2} F^{(a)}_{ij} \pm \frac{g_2}{4} (\bar{q}_A \tau^a q^A) \epsilon_{ij} \right]^2 + \left[\frac{1}{2g_1} F^{(8)}_{ij} \pm \frac{g_1}{4\sqrt{3}} \left(|q^A|^2 - 2\xi \right) \epsilon_{ij} \right]^2 \\ &+\; \frac{1}{2} \left| \nabla_i\, q^A \pm i\epsilon_{ij} \nabla_j\, q^A \right|^2 \pm \frac{\xi}{2\sqrt{3}} \tilde{F}^{(8)} \,) \end{aligned} \tag{7}$$

Example of Non-Abelian Vortices in $\mathcal{N} = 2$ SQCD



Non-Abelian Bogomolny equations (Auzzi, Bolognesi, Evslin, Konishi, Yung, hep-th/0307287)

$$\frac{1}{2g_2}F_{ij}^{(a)} \pm \frac{g_2}{4}(\bar{q}_A \tau^a q^A)\epsilon_{ij} = 0, \qquad (a=1,2,3); \qquad \frac{1}{2g_1}F_{ij}^{(8)} \pm \frac{g_1}{4\sqrt{3}}\left(|q^A|^2 - 2\xi\right)\epsilon_{ij} = 0,$$

$$\nabla_i\, q^A + i\varepsilon\epsilon_{ij}\nabla_j\, q^A = 0, \qquad A = 1,2.$$

Abelian (particular) solutions of $SU(3) \to U(1) \times U(1)$ by *e.g.* setting $A_\mu^1 = A_\mu^2 = 0$, and with squark fields of the 2×2 color-flavor diag. form:

$$q^{kA}(x) = \bar{\tilde{q}}^{kA}(x) \neq 0, \qquad \text{only for} \quad k = A = 1,2.$$

$$q^{kA}(x) = \begin{pmatrix} e^{i\,n\,\varphi}\phi_1(r) & 0 \\ 0 & e^{i\,k\,\varphi}\phi_2(r) \end{pmatrix},$$

$$A_i^3(x) = -\varepsilon\epsilon_{ij}\frac{x_j}{r^2}\;\left((n-k) - f_3(r)\right), \qquad A_i^8(x) = -\sqrt{3}\;\varepsilon\epsilon_{ij}\frac{x_j}{r^2}\;\left((n+k) - f_8(r)\right)$$

where

$$r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_1(r) - \frac{1}{2}\left(f_8(r) + f_3(r)\right)\phi_1(r) = \;0, \qquad r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_2(r) - \frac{1}{2}\left(f_8(r) - f_3(r)\right)\phi_2(r) = \;0,$$

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f_8(r) + \frac{g_1^2}{6}\;\left(\phi_1(r)^2 + \phi_2(r)^2 - 2\xi\right)\;=\;0, \qquad -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f_3(r) + \frac{g_2^2}{2}\;\left(\phi_1(r)^2 - \phi_2(r)^2\right)\;=\;0.$$

with boundary conds for the gauge fields:

$$f_3(0) = \varepsilon_{n,k}\left(n-k\right), \quad f_8(0) = \varepsilon_{n,k}\left(n+k\right), \qquad f_3(\infty) = 0, \quad f_8(\infty) = 0$$

and the requirement that the squark fields be everywhere regular. Also

$$\phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}$$

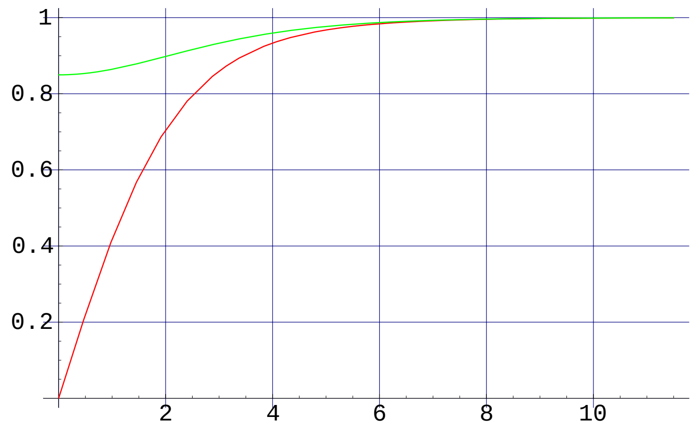


Figure 1: Vortex profile functions $\phi_1(r)$ and $\phi_2(r)$ of the $(1,0)$-string. Note $\phi_1(0) = 0$.

Figure 2: The profile functions $f_3(r)$ and $f_8(r)$ for the $(1,0)$-string.

Exact Symmetry

The $SU(3) \to SU(2) \times U(1) \to \emptyset$ theory has unbroken (both by inter. and by VEVS) global symmetry, $SU(2)_{C+F}$.

$SU(2)_{C+F}$ broken (to a $U(1)$) by a vortex configuration $\Rightarrow$ continuous vortex zero modes (moduli) of

$$SU(2)/U(1) = S^2 = \mathbf{CP}^1$$

Minimum vortex of generic orientation:

$$q^{kA} = U \begin{pmatrix} e^{i\varphi}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = e^{\frac{i}{2}\varphi(1+n^a\tau^a)}\, U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1},$$

$$\mathbf{A}_i(x) = U[-\frac{\tau^3}{2}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)]]U^{-1} = -\frac{1}{2}\,n^a\tau^a\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)],$$

$$A_i^8(x) = -\sqrt{3}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_8(r)]$$

where[3]

$$U \in SU(2)_{C+F}$$

The tension

$$T = 2\pi\xi$$

independent of U.

[3]Explicitly, if $n^a = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$, the rotation matrix is given by $U = \exp -i\beta\,\tau_3/2\,\exp -i\alpha\,\tau_2/2$.

Remarks

- **Reduction of the vortex spectrum (meson spectrum): (Fig)**

$$\Pi_1(\frac{U(1)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}^2$$

  to

$$\Pi_1(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}$$

- Transition from abelian ($m_i \neq m_j$) to nonabelian ($m_i = m$) superconductivity **reliably** and **quantum mechanically** analysed

- (Indirect) solution for the "existence problem" of nonabelin **monopoles**

- Dynamics of vortex zero modes

$$\mathbf{n} \to \mathbf{n}(z,t)$$

$$S_\sigma^{(1+1)} = \beta \int d^2x\, \frac{1}{2}\,(\partial\, n^a)^2 + \text{fermions.}$$

  $O(3) = \mathbf{CP}^1$ sigma model! Dual (Shifman et.al.; Vafa-Hori) to a chiral theory with two vacua $\longrightarrow$ No spontaneous breaking of $SU(2)_{C+F} \Leftrightarrow$ confining, dual $SU(2)$ (Witten index = 2).

Reduction of the vortex spectrum (meson spectrum)

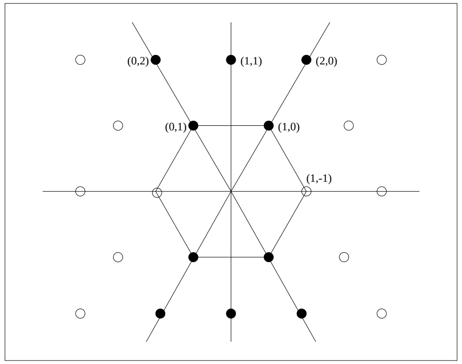


Figure 3: Lattice of (n, k) vortices in the theory $SU(3) \to U(1)^2$.

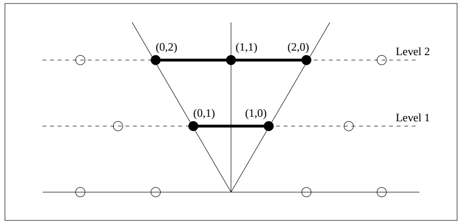


Figure 4: Reduced lattice of $\mathbb{Z}$ vortices $SU(3) \to SU(2) \times U(1)$.

Subtle are Non-Abelian Vortices (too)

Auzzi, Bolognesi, Evslin, Konishi, Yung; Hanany, Tong

- General setting: **Gauge** group broken as

$$G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H \stackrel{\langle\phi'\rangle\neq 0}{\longrightarrow} \emptyset, \qquad \langle\phi\rangle \gg \langle\phi'\rangle,$$

- An exact **global** symmetry $H_{C+F} \subset H \otimes G_F$ (not spontaneously broken), but broken by the vortex to G_0

  $\Rightarrow$ **Vortex zero modes (moduli)** $\sim H_{C+F}/G_0$

- $SU(N) \to \frac{SU(N-1)\times U(1)}{\mathbb{Z}_{N-1}} \to \emptyset$ system with $2N > N_f \geq 2(N-1)$

  $\Rightarrow$ Vortex with $2(N-2)$-parameter family of zeromodes

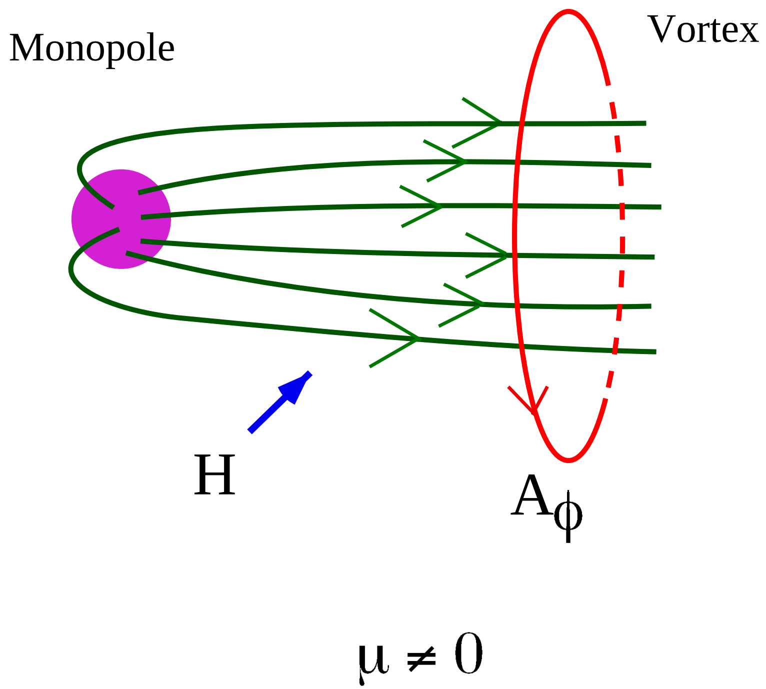
$$\frac{SU(N-1)}{SU(N-2)\times U(1)} \sim \mathbf{CP}^{N-2}.$$

- Vortices with non-Abelian quantum numbers

- Monopoles of G/H are confined by magnetic vortices of $H \to \emptyset$;

- **Both** described by $\Pi_1(H)$

- **Q-M'ly, non-Abelian vortices also requires massless quark flavors!**
- Monopoles can be attached at the ends of the vortex (Figure) ($\mathbb{Z}_{N-1}$ factor crucial)



Almost Superconf. Confining Vacua

Auzzi, Grena, Konishi

Sextet Vacua of $SU(3)$, $n_f = 4$ Model

$$y^2 = \prod_{i=1}^{3}(x-\phi_i)^2 - \prod_{a=1}^{4}(x+m_a) \equiv (x^3 - Ux - V)^2 - \prod_{a=1}^{4}(x+m_a).$$

For *equal bare quark masses* ($m_a = m$), it simplifies:

$$y^2 = \prod_{i=1}^{3}(x-\phi_i)^2 - (x+m)^4 \equiv (x^3 - Ux - V)^2 - (x+m)^4.$$

The sextet vacua: $diag\,\phi = (-m, -m, 2m)$, i.e.,

$$U = \langle \mathrm{Tr}\,\Phi^2 \rangle = 3\,m^2; \qquad V = \langle \mathrm{Tr}\,\Phi^3 \rangle = 2\,m^3,$$

where the curve exhibits a singular behavior, $y^2 \propto (x+m)^4$ corresponding to the unbroken $SU(2)$.

Mass Formula

$$M_{(g_1,g_2;q_1,q_2)} = \sqrt{2}\,|g_1\,a_{D1} + g_2\,a_{D2} + q_1\,a_1 + q_2\,a_2|.$$

$$a_{D1} = \oint_{\alpha_1} \lambda, \qquad a_{D2} = \oint_{\alpha_2} \lambda, \qquad a_1 = \oint_{\beta_1} \lambda, \qquad a_2 = \oint_{\beta_2} \lambda,$$

where the (meromorphic) one-form λ is given by

$$\lambda = \frac{x}{2\pi} d \log \frac{\prod(x-\phi_i) - y}{\prod(x-\phi_i) + y}.$$

Expansion near the SCFT Point

$$U = 3m^2 + u, \qquad V = 2m^3 + v,$$

The discriminant of the curve factorizes as

$$\Delta = \Delta_s \, \Delta_+ \, \Delta_-, \qquad \Delta_s = (m\,u - v)^4$$

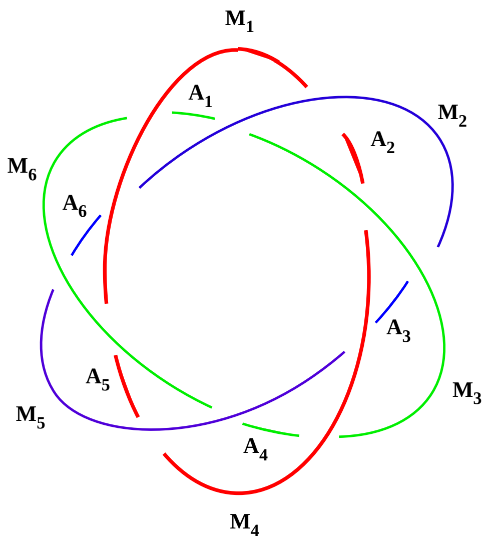
and the loci of $\Delta = 0$ are

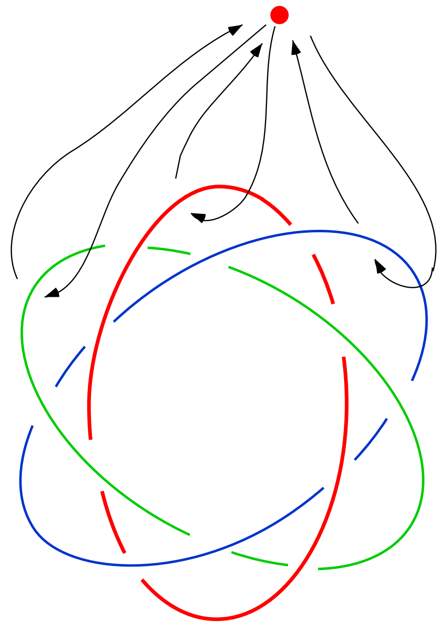
$$v = m\,u, \qquad v = m\,u + \frac{u^2}{4}, \qquad v = m\,u - \frac{u^2}{4}.$$

By rescaling $u = m\,\tilde{u},\ v = m^2\,\tilde{v}$, intersecting them with a S^3

$$|\tilde{u}|^2 + |\tilde{v}|^2 = 1.$$

and making a stereographic projection from $S^3 \to R^3$





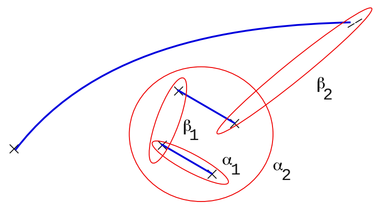
Monodromy and Charges

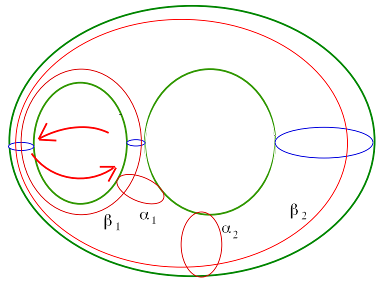
Monodromy around M_1 $\Rightarrow$

$$\alpha_1 \to \alpha_1, \qquad \beta_1 \to \beta_1 - 4\alpha_1, \qquad \alpha_2 \to \alpha_2, \qquad \beta_2 \to \beta_2.$$

The monodromy transformation:

$$\begin{pmatrix} a_{D1} \\ a_{D2} \\ a_1 \\ a_2 \end{pmatrix} \to M_1 \begin{pmatrix} a_{D1} \\ a_{D2} \\ a_1 \\ a_2 \end{pmatrix}, \qquad M_1 = \tilde{M}_1^4, \quad \tilde{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$





From

$$M = \begin{pmatrix} \mathbf{1} + \vec{q} \otimes \vec{g} & \vec{q} \otimes \vec{q} \\ -\vec{g} \otimes \vec{g} & \mathbf{1} - \vec{g} \otimes \vec{q} \end{pmatrix} \tag{9}$$

the (four) massless particles at the singularity $\tilde{v} = \tilde{u}$ have charges

$$(g_1, g_2; q_1, q_2) = (1, 0; 0, 0).$$

Analogously:

$$M_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}, \quad etc.$$

Conjugations:

$$\begin{aligned} &M_1 = M_6^{-1} A_5 M_6, \quad A_2 = M_2^{-1} M_1 M_2, \quad M_4 = M_3^{-1} A_2 M_3, \quad A_5 = M_5^{-1} M_4 M_5, \\ &M_2 = M_1^{-1} A_6 M_1, \quad A_3 = M_3^{-1} M_2 M_3, \quad M_5 = M_4^{-1} A_3 M_4, \quad A_6 = M_6^{-1} M_5 M_6, \\ &M_3 = M_2^{-1} A_1 M_2, \quad A_4 = M_4^{-1} M_3 M_4, \quad M_6 = M_5^{-1} A_4 M_5, \quad A_1 = M_1^{-1} M_6 M_1 \end{aligned}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}, \qquad M_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 1 & 0 \\ 4 & -4 & 0 & 1 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 3 & 0 \\ 4 & -4 & -2 & 1 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 4 & 5 & 0 \\ 4 & -1 & -1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & -1 & 0 \\ 4 & -4 & 2 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} -3 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 12 & 5 & 0 \\ 12 & -9 & -3 & 1 \end{pmatrix}, \qquad A_5 = \begin{pmatrix} -3 & 4 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 5 & 0 \\ 4 & -4 & -4 & 1 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

N.B.

$$M_1 = (\tilde{M}_1)^4, \quad M_4 = (\tilde{M}_4)^4, \quad A_2 = (\tilde{A}_2)^4, \quad A_5 = (\tilde{A}_5)^4,$$

with

$$\tilde{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

$$\tilde{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_5 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

$\Rightarrow$ **Charges**

$$M_1 : (1,0;0,0)^4, \quad M_4 : (-1,1;0,0)^4, \quad M_2 : (-2,0;1,0), \quad M_5 : (2,-2;-1,0),$$
$$A_2 : (-1,0;1,0)^4, \quad A_5 : (1,-1;-1,0)^4, \quad A_3 : (-2,2;-1,0), \quad A_6 : (2,0;1,0),$$
$$M_3 : (0,1;-1,0), \quad M_6 : (0,1;1,0), \quad A_4 : (4,-3;-1,0), \quad A_1 : (-4,1;1,0),$$

(A) **Which q.n. with respect to $SU(2) \times U(1)$?**

(B) **Which of them are actually there at the SCFT Point as LEEDF ?**

(C) **How do they give $\beta = 0$?**

(D) **How do they interact ?**

Ans. to (A):

$$\tilde{m}_1 = m_1; \quad \tilde{q}_1 = q_1 - \frac{1}{2} q_2; \qquad U_1(1) \subset SU(2);$$

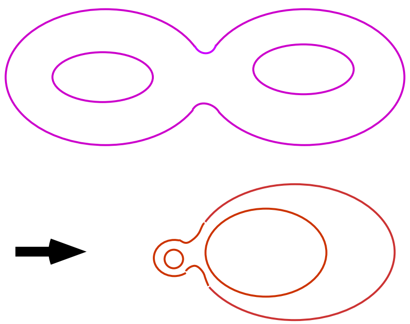
and

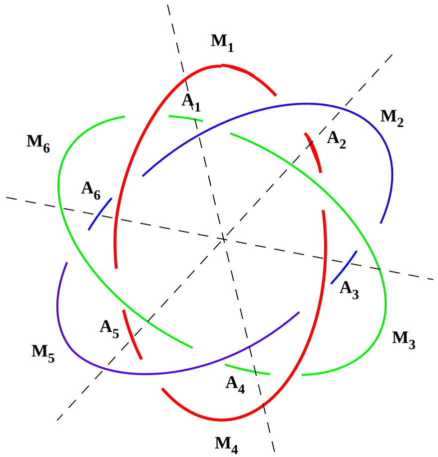
$$\tilde{m}_2 = m_1 + 2\,m_2; \quad \tilde{q}_2 = \frac{1}{2} q_2, \qquad U_2(1);$$

$\Rightarrow$

Matrix	Charge
M_1, M_4	$(\pm 1, 1, 0, 0)^4$
A_2, A_5	$(\pm 1, -1, \mp 1, 0)^4$
M_2, M_5	$(\pm 2, 2, \mp 1, 0)$
A_3, A_6	$(\pm 2, -2, \pm 1, 0)$
M_3, M_6	$(0, 2, \pm 1, 0)$
A_1, A_4	$(\pm 4, -2, \mp 1, 0)$

Superconformal Limit ($u = 0, v = 0$) :





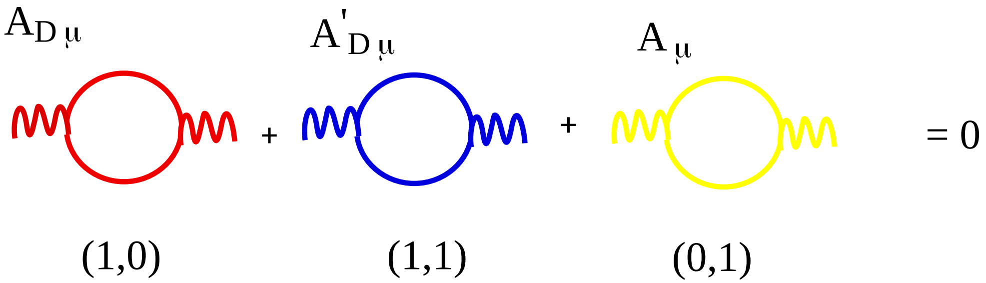
- Large torus: $\tau_{22} \to 1$ (Weakly interacting $U(1)$ theory);

- Small torus ($SU(2)$) : τ_{11} depends on the way u, $v \to 0$!
- τ_{11} depends only on ρ where $v = \epsilon^2$; $u = \epsilon\rho$
- At different phase of $\epsilon \Rightarrow$ different sections ($SU(2, Z)$-related descriptions of the same physics!) (**Ans. to (B)**)

$+2i$	$(0,1)$	$(4,-1)$	$(0,1)$	$(0,-1)$	$(-4,1)$	$(0,-1)$	$\ldots$
2	$(2,1)$	$(2,-1)$	$(2,-1)$	$(-2,-1)$	$(-2,1)$	$(-2,1)$	$\ldots$
$-2i$	$(0,-1)$	$(-4,1)$	$(0,-1)$	$(0,1)$	$(4,-1)$	$(0,1)$	$\ldots$
-2	$(-2,-1)$	$(-2,1)$	$(-2,1)$	$(2,1)$	$(2,-1)$	$(2,-1)$	$\ldots$
∞	$(\pm1,0)^4$	$(\mp1,0)^4$	$(\pm1,\mp1)^4$	$(\mp1,0)^4$	$(\pm1,0)^4$	$(\mp1,\pm1)^4$	$\ldots$

♡ **Define SCFT in the limit,** $\quad \epsilon \to 0, \quad \rho \to 0.$

Renormalization-Group Fixed Point



- Inversion formula

$$\frac{1}{2} = \frac{\theta_{00}^4(0,\tau_{11})}{\theta_{10}^4(0,\tau_{11})} \qquad \rightarrow \qquad \tau_{11} = \frac{\pm 1 + i}{2}, \quad \frac{\pm 3 + i}{10}, \quad \ldots$$

  Other solutions by $SL(2,Z)$ transformations $\tau \to \tau + 2; \ \tau \to \frac{\tau}{1-2\tau}$

- **Cancellation of b_0 (Consider $U_1(1) \subset SU(2)$)**

  i) $(\mp 1, 0)^4$ cancel the contr. of the gauge multiplets ;

  ii) $(\pm 2, \pm 1)$ and $(0, \pm 1)$ cancel (*cfr*: Argyres-Douglas)

$$\sum_i (q_i + m_i \tau)^2 = 1 + (2\,\tau + 1)^2 = 0, \qquad \text{for} \qquad \tau^* = \frac{-1+i}{2}$$

  iii) In the second section $(\pm 4, \mp 1)$ and $(\pm 2, \mp 1)$ cancel for $\tau^* = \frac{3+i}{10}$!

  iv) Different sections $\Rightarrow$ Different description of the Same physics

- **Low Energy Theory is an interacting SCFT with**

  $SU(2) \times U(1)$ **Gauge Group and 4 magnetic monopole doublets, one dyon doublet and one electric doublet.** (**Ans. to (C)**)

Unequal masses: Six Colliding $\mathcal{N} = 1$ Local Vacua:

- **Each $\mathcal{N} = 1$ theory is a local $U(1)^2$ theory with $M_i, \tilde{M}_i$, ($i = 1, 2$) $\Rightarrow$ 12 hypermultiplets (as in the SCFT);**
- **Effect of $\mathcal{N} = 1$ perturbation $\mu \operatorname{Tr} \Phi^2$ in terms of an effective Lagrangian:**

$$\mathcal{P} = \sum_{i=1}^{2} \sqrt{2}\, A_{D_i}\, M_i \tilde{M}_i + \mu\, U(A_{D1}, A_{D2}) + \text{mass terms}$$

$\Rightarrow \langle M_i \rangle \neq 0$**,** $\langle \tilde{M}_i \rangle \neq 0$ **(Confinement);**

- **But in the $m_i \to m$ (SCFT Limit)**

$$\langle M_i \rangle \to 0, \quad \langle \tilde{M}_i \rangle \to 0,$$

!!?? Deconfinement? (cfr. Gorski, Yung, Vainshtein)

- **We do know (the large μ analysis, vacuum counting, and holomorphic dependence of physic on μ) that**

$$G_F = SU(4) \times U(1) \Rightarrow U(2) \times U(2)$$

- *Order parameter of the symmetry breaking?*

ANS: Condensation of $SU(2)$ doublets $\mathcal{M}^i_\alpha$, ($\alpha = 1, 2$, $i = 1, \dots, 4$)

$$\langle \mathcal{M}^i_\alpha \, \mathcal{M}^j_\beta \rangle = \epsilon_{\alpha\beta} \, C^{ij} \neq 0, \quad \text{or} \quad \langle \mathcal{M}^i_\alpha \rangle = \delta^i_\alpha \, v \neq 0$$

due to the $SU(2)$ interactions. (Probable Ans. to (D))

Summarizing:

Softly broken $\mathcal{N} = 2$, $SU(n_c)$ gauge theories with n_f quarks $\Rightarrow$ confining vacua with:

- Physics quite different for

  (i) $r = 0, 1 \Rightarrow$ Weakly coupled Abelian monopoles;

  (ii) $r < \frac{n_f}{2} \Rightarrow$ Weakly coupled non-Abelian monopoles;

  (iii) $r = \frac{n_f}{2} \Rightarrow$ Strongly coupled non-Abelian monopoles,

- Both at generic r - vacua and at the SCFT ($r = \frac{n_f}{2}$) vacua,

$$\langle \mathcal{M}_\alpha^i \rangle = \delta_\alpha^i \, v \neq 0, \qquad (\alpha = 1, 2, \dots, r; \quad i = 1, 2, \dots, n_f)$$

  ("Color-Flavor-Locking")

- Gauge invariant condensates are

$$\epsilon^{\alpha_1 \alpha_2 \dots \alpha_r} \mathcal{M}_{\alpha_1}^{i_1} \mathcal{M}_{\alpha_2}^{i_2} \dots \mathcal{M}_{\alpha_r}^{i_r} \sim U(1) \text{ monopole?}$$

QCD

- **No dynamical Abelianization**
- **QCD with n_f flavor ($\tilde{n}_c = 2, 3$, $n_f = 2, 3$)**

$$b_0 = 11\, n_c - 2\, n_f \qquad \Rightarrow \qquad \tilde{b}_0 = 11\, \tilde{n}_c - n_f$$

No sign flip (no weakly-coupled nonabelian monopoles)

- **Strongly-interacting nonabelian superconductor**
- **Hint from r-vacua and from the almost SCF vacua**

$$\langle \mathcal{M}^i_{L,\alpha} \rangle = \delta^i_\alpha \, v_R \neq 0, \quad \langle \mathcal{M}^\alpha_{R,i} \rangle = \delta^\alpha_i \, v_L \neq 0, \quad (\alpha = 1, 2, \ldots \tilde{n}_c; i = 1, 2, \ldots n_f)$$

- **A better picture might be**

$$\langle \mathcal{M}^i_{L,\alpha} \, \mathcal{M}^\alpha_{R,j} \rangle = \mathrm{const.}\, \delta^i_j \neq 0;$$

for $\tilde{n}_c = 2, \; n_f = 2$

$$G_F = SU_L(2) \times SU_R(2) \Rightarrow SU_V(2)$$

Parma 11-12 Sept. 2003

Lectures on Supersymmetric Gauge Theories III:

Recent developments

(Dijkgraaf-Vafa hep-th/0208048, Cachazo-Douglas-Seiberg-Witten hep-th/0211170)

K. Konishi

- **Chiral Rings of Operators**
- **Generalized Konishi anomaly and Determination of** W_{eff}
- **Confinement Index**
- $\mathcal{N}=2$ **vs** $\mathcal{N}=1$ **approaches**
- **Phases and Multiplication Maps**

Veneziano-Yankielovicz Eff. Action

- $\mathcal{N}=1$ **susy** $SU(N)$ **Yang-Mills (** $W_\alpha = -i\lambda + \frac{i}{2}(\sigma^\mu\,\bar\sigma^\nu)_\alpha^\beta\,F_{\mu\nu}\,\theta_\beta + \,\ldots$ **)**

$$\mathcal{L}^{bare} = \int d^2\theta\,\frac{1}{g_0^2}WW = \int d^2\theta\,\frac{1}{g_0^2}\,S$$

-

$$S \equiv W^\alpha W_\alpha = -\lambda\lambda + \ldots - \frac{1}{2}F_{\mu\nu}^2 - i\lambda\sigma^\nu\mathcal{D}_\nu\bar\lambda +$$

$$\mathcal{L}^{VY} = kin.term - \int d^2\theta\,S\,[\log\frac{S^N}{\Lambda^{3N}} - N] + h.c. \qquad (\&)$$

- 1**-loop renormalization**

$$[\,\frac{1}{g_0^2} + b_0\,\log\frac{M}{S^{1/3}}\,]\,S = \frac{1}{g(S)^2}\,S = b_0\,S\log\frac{S^{1/3}}{\Lambda}, \qquad b_0 = 3N$$

- $\mathcal{L}^{VY} \to \langle S\rangle = \Lambda^3\,\exp 2\pi ik/N,$ **with** $k = 1,2,\ldots N$ **(**$Z_{2N}\subset U_A(1)$ **broken to** Z_2**)**

- **Under** $\lambda \to e^{i\alpha}\lambda$

$$\Delta\mathcal{L}^{VY} = 2N\,\alpha\,F_{\mu\nu}\tilde F^{\mu\nu}$$

- $\int e^{i\,S}$ **invariant under** Z_{2N} **while (&) not invariant under** Z_{2N} **!?!**

  $\to$ **Chirally symmetric vacuum (**Kovner, Shifman**) ? No.**

Chiral Rings in Theory with adjoint field Φ

- $\mathcal{N}=1$ **susy** $U(N)$ **gauge theory**

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\,\tau_{cl}\left[\int d^4\theta\,\Phi^\dagger e^V \Phi + \int d^2\theta\,\frac{1}{2}WW\right] + \int d^2\theta\,\mathcal{W}(\Phi) + h.c.$$

$$\Delta\mathcal{L} = \int d^2\theta\ \mu\,\mathrm{Tr}\,\Phi^2 + h.c., \qquad \tau_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}$$

- **(**$\mathcal{N}=1$**) multiplets** $\Phi = \phi + \sqrt{2}\,\theta\,\psi + \ldots;\ W_\alpha = -i\lambda + \frac{i}{2}\,(\sigma^\mu\,\bar{\sigma}^\nu)_\alpha^\beta\,F_{\mu\nu}\,\theta_\beta + \ldots$

$$\mathcal{W}(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1}\mathrm{Tr}\,\Phi^{k+1}. \tag{1}$$

- **Gauge inv. chiral composites, modulo** $\{\bar{Q},\ldots\} \to$

$$\{\ \mathrm{Tr}\,\Phi^k, \qquad \mathrm{Tr}\,W_\alpha\,\Phi^k, \qquad \mathrm{Tr}\,W^\alpha W_\alpha\,\Phi^k\} \qquad (\%)$$

- **Perturbatively (for** $k>N$**), e.g.,**

$$\mathrm{Tr}\,\Phi^k = \mathcal{P}(\{u_i\}), \qquad u_i = \mathrm{Tr}\,\Phi^j, \qquad j \le N,$$

$$\frac{\partial}{\partial\Phi}\mathcal{W}(\Phi) = \bar{D}^2(\ldots) = 0, \qquad S^N = 0, \qquad (\$)$$

Problem

- **Classically** $\{a_i\}$ = **eigenvalues of** Φ**,**

$$\mathcal{W}'(z) = g_n \prod_i^n (z - a_i) \quad \rightarrow \qquad U(N) \Rightarrow \prod_i U(N_i)$$

- **Low-energy eff. degrees of freedom are**

$$S_i = \frac{1}{16\pi^2} \mathrm{Tr}\, W_i^\alpha W_{\alpha i}, \qquad w_{\alpha i} = \frac{1}{4\pi} \mathrm{Tr}\, W_{\alpha i} :$$

$$\mathcal{L}_{eff} = \int d^2\theta\, \mathcal{W}_{eff}(S_i, w_{\alpha i}, g_k) + \ldots$$

-

$$\int \mathcal{D}\Phi\, e^{i\,S} = e^{i \int \mathcal{L}_{eff}} = \exp i \int d^6 z\, \mathcal{W}_{eff}(S_i, w_{\alpha i}, g_k)$$

- **Problem: Compute** $\mathcal{W}_{eff}(S_i, w_{\alpha i}, g_k)$
- **Idea**

$$\frac{\partial}{\partial g_k} \mathcal{W}_{eff}(S_i, w_{\alpha i}, g_k) = \langle \mathrm{Tr}\, \frac{\Phi^{k+1}}{k+1} \rangle, \tag{2}$$

etc. Determine all chiral condensates as fns of $S_i, w_{\alpha i}, g_k$

Symmetries

Fields	Δ	Q_Φ	Q_R	Q_θ
Φ	1	1	$\frac{2}{3}$	0
W_α	$\frac{3}{2}$	0	1	1
g_l	$2-l$	$-(l+1)$	$\frac{2}{3}(2-l)$	2
Λ^{2N}	$2N$	$2N$	$\frac{4N}{3}$	0

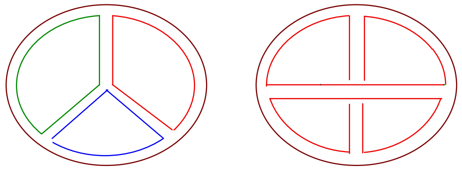
- $U_\Phi(1)$**:** $\Phi \to e^{i\alpha}\,\Phi$ **is anomalous at one loop only** $\Rightarrow$ **for** $\geq 2\,loops$

$$\mathcal{W}_{eff} = W_\alpha^2\, F(g_k\, W_\alpha^{k-1}/g_1^{(k+1)/2}), \quad \text{or}$$

$$\left[\sum_k (2-k)\, g_k \frac{\partial}{\partial g_k} + \frac{3}{2}\, W_\alpha \frac{\partial}{W_\alpha}\right] \mathcal{W}_{eff} = 3\,\mathcal{W}_{eff}$$

- **Index loops (L), vertices (k_i), genus (g)**

$$L = 2 - 2g + \frac{1}{2}\sum_i (k_i - 1)$$



- $\therefore$ **Only planar diagrams contribute to** $\mathcal{W}_{eff}$

  (Proof of the conjecture by Dijkgraaf-Vafa)

- $U(1) \subset U(N)$ **is free:** $\mathcal{W}_{eff}(S_i, w_{\alpha i}, g_k)$ **inv under**

$$W_\alpha \to W_\alpha - 4\pi\psi_\alpha$$

  $\Rightarrow$ **General form of** $\mathcal{W}_{eff}$

$$U(N) \Rightarrow \prod_i U(N_i),$$

$$\mathcal{W}_{eff} = \sum N_i \frac{\partial \mathcal{F}_p(S_k, g_k)}{\partial S_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 \mathcal{F}_p(S_k, g_k)}{\partial S_i \, \partial S_j} w_{\alpha i} w_j^\alpha .$$
$$= \int d^2\psi \; \mathcal{F}_p(\mathcal{S}_i, g_k);$$
$$\mathcal{S}_i = -\frac{1}{2}\mathrm{Tr}\,(\frac{1}{4\pi}W_{\alpha i} - \psi_\alpha)(\frac{1}{4\pi}W_i^\alpha - \psi^\alpha) = S_i + \psi_\alpha w_i^\alpha - N_i \psi_\alpha \psi^\alpha \qquad (3)$$

- **(2) $\Rightarrow$**

$$\frac{\partial \mathcal{W}_{eff}}{\partial g_k} = \int d^2\psi \ \frac{\partial \mathcal{F}_p(\mathcal{S}_i, g_k)}{\partial g_k} = \langle \ \frac{\Phi^{k+1}}{k+1} \ \rangle_\Phi$$

$$= -\frac{1}{2(k+1)} \int d^2\psi \, \langle \, \mathrm{Tr} \, (\frac{1}{4\pi} W_\alpha - \psi_\alpha)^2 \, \Phi^{k+1} \rangle_\Phi$$

- **So**

$$\frac{\partial \mathcal{F}_p(\mathcal{S}_i, g_k)}{\partial g_k} = -\frac{1}{2(k+1)} \int d^2\psi \langle \mathrm{Tr}(\frac{1}{4\pi} W_\alpha - \psi_\alpha)^2 \, \Phi^{k+1} \rangle_\Phi$$

- **Problem: find the right hand side**

Generalized Konishi Anomaly

- **Konishi Anomaly**

$$\bar{D}^2 \mathrm{Tr}\{\bar{\Phi} e^V \Phi\} = \mathrm{Tr}\,\Phi \frac{\partial \mathcal{W}}{\partial \Phi} + \frac{1}{32\pi^2} \mathrm{Tr}\,(ad\,W_\alpha\,ad\,W^\alpha) \tag{4}$$

$$\bar{D}^2 \mathrm{Tr}\{\bar{\Phi} e^V \Phi\} = \mathrm{Tr}\,\Phi \frac{\partial \mathcal{W}}{\partial \Phi} + \frac{N}{16\pi^2} \mathrm{Tr}\,(W_\alpha W^\alpha) - \frac{1}{16\pi^2} \mathrm{Tr}\,W_\alpha \mathrm{Tr}\,W^\alpha$$

- **Supersymmetrized form of $U_\Phi(1)$ anomaly**

- **Taking the VEVS**

$$\langle \mathrm{Tr}\,\Phi \frac{\partial \mathcal{W}}{\partial \Phi} \rangle = -\frac{N}{16\pi^2} \langle \mathrm{Tr}\,(W_\alpha W^\alpha) \rangle.$$

But

$$L.H.S. = \langle \mathrm{Tr}\,\sum_k g_k \Phi^{k+1} \rangle = \sum_k (k+1) g_k \frac{\partial}{\partial g_k} \mathcal{W}_{eff}$$

- **(4) à la Fujikawa (Shizuya-Konishi)** $\delta\Phi = \alpha\Phi$.

- **Generalization ($J_f = \mathrm{Tr}\{\bar{\Phi} e^V f(\Phi, W_\alpha)\}$):**

$$\delta\Phi = f(\Phi, W_\alpha) \tag{5}$$

$$\bar{D}^2 J_f = \mathrm{Tr}\, f(\Phi, W_\alpha)\frac{\partial \mathcal{W}}{\partial \Phi} + \frac{1}{32\pi^2}\sum_{ij}[W_\alpha, [W^\alpha, \frac{\partial f}{\partial \Phi_{ij}}]]_{ij} \tag{6}$$

$$\langle R.H.S.\rangle = 0.$$

- **Define**

$$\mathcal{R}(z, \phi) = -\frac{1}{2}\,\mathrm{Tr}\,(\frac{1}{4\pi}W_\alpha - \psi_\alpha)^2\frac{1}{z - \Phi}, \tag{7}$$

$$= R(z) + \psi_\alpha w^\alpha(z) - \psi^1\psi^2 T(z),$$

where generating functions are

$$T(z) = \mathrm{Tr}\,\frac{1}{z - \Phi}, \qquad w^\alpha = \frac{1}{4\pi}\mathrm{Tr}\,W_\alpha\,\frac{1}{z - \Phi},$$

$$R(z) = -\frac{1}{32\pi^2}\mathrm{Tr}\,W_\alpha W^\alpha\,\frac{1}{z - \Phi},$$

- **By choosing $f(\Phi) = W_\alpha W^\alpha \frac{1}{z-\Phi}$ in (5):**

$$\langle -\frac{1}{32\pi^2} \sum_{ij} [W_\alpha, [W^\alpha, \frac{\partial}{\partial \Phi_{ij}} (W_\beta W^\beta \frac{1}{z-\Phi})]]_{ij} = \langle \operatorname{Tr} [\frac{\partial \mathcal{W}}{\partial \Phi} W_\alpha W^\alpha \frac{1}{z-\Phi}] \rangle.$$

By identity

$$\sum_{ij} [\chi_1, [\chi_2, \frac{\partial}{\partial \Phi_{ij}} \frac{\chi_1 \chi_2}{z-\Phi}]]_{ij} = (\operatorname{Tr} \frac{\chi_1 \chi_2}{z-\Phi})^2$$

(valid if $\chi_1^2 = \chi_2^2 = 0,\ [\chi_i, \Phi] = 0$) one gets

$$R(z,\psi)^2 = \operatorname{Tr} (\mathcal{W}'(\Phi) R(z,\psi))$$

- **Analogously, with $f(\Phi) = \mathcal{R}$ (r.h.s of (7) without trace)**

$$\mathcal{R}(z,\psi)^2 = \operatorname{Tr} (\mathcal{W}'(\Phi) \mathcal{R}(z,\psi))$$

which can be rewritten as

$$\mathcal{R}(z,\psi)^2 = \operatorname{Tr} (\mathcal{W}'(z) \mathcal{R}(z,\psi)) + \frac{1}{4} f(z,\psi), \tag{8}$$

or

$$\begin{aligned} R^2(z) &= \mathcal{W}'(z)R(z) + \frac{1}{4}f(z); \\ 2\,R(z)\,w^\alpha(z) &= \mathcal{W}'(z)w^\alpha(z) + \frac{1}{4}\rho^\alpha; \\ 2\,R(z)\,T(z) + w_\alpha(z)w^\alpha(z) &= \mathcal{W}'(z)T(z) + \frac{1}{4}c(z). \end{aligned}$$

with

$$\begin{aligned} f(z,\psi) &= \frac{1}{8\pi^2}\mathrm{Tr}\,\frac{(\mathcal{W}'(z)-\mathcal{W}'(\Phi))(W_\alpha - 4\pi\psi_\alpha)\,(W^\alpha - 4\pi\psi^\alpha)}{z-\Phi}, \\ &= f(z) + \psi_\alpha\,\rho^\alpha(z) - \psi_1\psi_2\,c(z) \end{aligned}$$

where $f(z)$ **is an** n**th order polynomial in** z

- **Solving the quadratic euation**

$$2\,\mathcal{R}(z,\psi) = \mathcal{W}'(z) - \sqrt{\mathcal{W}'(z)^2 + f(z,\psi)}$$

or

$$2\,R(z) = \mathcal{W}'(z) - \sqrt{\mathcal{W}'(z)^2 + f(z)}$$

etc.

• $R(z) = -\frac{1}{32\pi^2}\mathrm{Tr}\, W_\alpha W^\alpha \frac{1}{z-\Phi}$ **determined in terms of** f_i **where**

$$f(z) = \sum_{i=0}^{n-1} f_i \, z^i$$

• **(3) and def of** $\mathcal{R}(z,\psi) \Rightarrow$

$$\mathcal{S}_i = S_i + \psi_\alpha w_i^\alpha - N_i \psi_\alpha \psi^\alpha = \frac{1}{2\pi i}\oint_{C_i} dz\, \mathcal{R}(z,\psi)$$

$\Rightarrow$ **Relations** $\{f_i\} \leftrightarrow (S_i, w_i^\alpha)$

• **Finally**

$$\begin{aligned}\frac{\partial \mathcal{F}_p(\mathcal{S}_i, g_k)}{\partial g_k} &= -\frac{1}{2(k+1)}\int d^2\psi \langle \mathrm{Tr}(\frac{1}{4\pi}W_\alpha - \psi_\alpha)^2\, \Phi^{k+1}\rangle_\Phi \\ &= -\frac{1}{2(k+1)}\oint dz\, z^{k+1}\mathcal{R}(z,\phi)\end{aligned}$$

• **By integrating over** g_k **and adding integration constant -** g_k **independent,** $1-loop$**, contribution (VY), we get** $\mathcal{W}_{eff}$ **in terms of** $(S_i, w_i^\alpha, \Lambda_i)$ ♡

Matrix Model (Dijkgraaf-Vafa)

- **Integral over $\hat{N} \times \hat{N}$ Hermitian matrices M**

- **Free energy (*cfr* $\mathcal{W}(\Phi)$)**

$$\exp -\frac{\hat{N}^2}{g_m^2} F_{m.m.} = \int d^{\hat{N}^2} M \;\; \exp -\frac{\hat{N}}{g_m} \operatorname{Tr} \mathcal{W}(M)$$

- $\delta M = \epsilon\, M^{n+1} \longrightarrow 0 = \int d^{\hat{N}^2} M \;\, e^{-\frac{\hat{N}}{g_m} \operatorname{Tr} \mathcal{W}(M)} \, [\mathrm{Tr} \frac{\partial}{\partial M} M^n - \frac{\hat{N}}{g_m} \mathrm{Tr} \mathcal{W}' M^n]$

$$\langle R_m(z)^2 \rangle = \langle W'(z) R_m(z) \rangle + \frac{1}{4} f_m(z), \qquad \text{where} \qquad R_m(z) = \frac{g_m}{\hat{N}} \langle \mathrm{Tr} \frac{1}{z - M} \rangle$$

- **Take now $\hat{N} \to \infty$: $\to$ factorization**

$$\langle R_m(z) \rangle^2 = \langle W' \rangle \langle R_m(z) \rangle + \frac{1}{4} f_m(z),$$

: relation identical to Eq. (6) !!!

- $$S_i = \frac{1}{2\pi i} \oint_{C_i} R_m(z) \, dz \qquad \frac{\partial F_{m.m}}{\partial g_k} = \langle \frac{\operatorname{Tr} M^{k+1}}{k+1} \rangle$$

$F_{m.m}(S_i, g_k) \Rightarrow$ **identify with** $\mathcal{F}_p(\mathcal{S}_i, g_k) \to \mathcal{W}_{eff}(S_i, w_i^\alpha, \Lambda_i)$

Further development

(Cachazo-Seiberg-Witten, hep-th/0301006)

$$\mathcal{L}^{U(N)} = \frac{1}{8\pi}\mathrm{Im}\,\tau_{cl}\left[\int d^4\theta\,\Phi^\dagger e^V\Phi + \int d^2\theta\,\frac{1}{2}WW\right] + \int d^2\theta\,\mathcal{W}(\Phi)$$

where $\mathcal{W}(\Phi) = \sum_{r=0}^{k}\frac{g_r}{r+1}\mathrm{Tr}\,\Phi^{r+1}$ **(superpotential)**

- $\mathcal{W}(\Phi) = 0 \Rightarrow \mathcal{N} = 2 \Rightarrow G \sim U(1)^{N-1}$ **on a generic point of QMS.**
- **Special points where some** $N-n$ **monopoles become massless (condensation and Higgs mech. for** $N-n$ **dual gauge bosons)** $\Rightarrow G \sim U(1)^n$
- **Curve factorizes (cond. on QMS)**

$$y^2 = P_N^2(x) - 4\Lambda^2 = F_{2n}\,H_{N-n}^2(x) \tag{9}$$

- **Classically** $\{a_i\}$ = **eigenvalues of** Φ**,**

$$\mathcal{W}'(z) = g_k\prod_i^k(z-a_i) \qquad \mathrm{diag}\,\Phi = \{a_1,\ldots a_1, a_2,\ldots,\ldots,a_n,\ldots,a_n\}$$

$$U(N) \Rightarrow \prod_i^n U(N_i) \Rightarrow U(1)^n \qquad n \le k \tag{10}$$

- **Problem: find QM'ly the relation**

$$\text{Vacua (10)} \Longleftrightarrow \mathcal{W}(\Phi)$$

- **Ans: the Factorization condition** (9) **with (for** $k = n$**)**

$$F_{2n}(x) = \frac{1}{g_n^2}\mathcal{W}'(x)^2 + f_n(x)$$

$f_n(x) = O(x^{n-1})$ **with** n **unknown coefficients.**

- **Generalized Konishi anomaly (electric variables) from** $\mathcal{N} = 2$ **curves (whose singularities related magnetic variables)**

- **Use of Konishi anomaly in** $\mathcal{N} = 2$, $SU(2)$ **theory with** $n_f = 1 \Leftrightarrow$ **the knowledge of the curve (**Gorsky-Vainshtein-Yung**)**

- **Detailed knowledge about the decpoupling of** Φ

Vacua of $\mathcal{N} = 2$ **theory** $\rightarrow$ **Vacua of** $\mathcal{N} = 1$ **theory**

Confinement Index

$\equiv$ **Smallest possible $r \in Z_N^{(E)}$ for which Wilson loop displays no area law**

- $SU(N)$ **YM:** $r = N$ **completely confining;** $r = 1$ **totally Higgs**
- $r = 1$ **in a theory with**

$$SU(N) \to SU(N-1) \times U(1)$$

- $\mathcal{N} = 1$ **Susy** $SU(N)$ **theory broken (by adjoint VEV) as**

$$SU(N) \to SU(N_1) \times SU(N_2) \times U(1)$$

Then

$$r = l.c.d\,\{N_1, N_2, r_1 - r_2\}$$

where

$$r_1 = 0, 1, 2, \ldots N_1 - 1, \qquad r_2 = 0, 1, 2, \ldots N_2 - 1$$

label the vacua in which $(n_m, n_e) = (1, r_1)$ **and** $(n_m, n_e) = (1, r_2)$ **are condensed**

Multiplication Map

$U(N) \Leftrightarrow U(tN)$ **with the same superpotential** $\mathcal{W}(\Phi)$

- **Vacua with** $\Pi_{i=1}^{n} U(N_i)$ **of** $U(N) \Leftrightarrow$ **vacua with** $\Pi_{i=1}^{n} U(tN_i)$ **of** $U(tN)$**;**
- **Confinement Index** r **gets simply multiplied by** t**;**
- *All* **confining vacua with** $r = t$ **in the** $U(tN)$ **theory, arise from the Coulomb vacua of** $U(N)$ **theory**
- **Map of the chiral condensates**

$$\langle \mathrm{Tr} \frac{1}{x - \Phi} \rangle = t \, \langle \mathrm{Tr} \frac{1}{x - \Phi_0} \rangle$$

- $USp(N) \Leftrightarrow U(N+2n)$ **map in the theory with order** $n+1$ $\mathcal{W}(\Phi)$ **(**Cachazo**)**

$$\Pi_{i=1}^{n} USp(N_i) \Leftrightarrow \Pi_{i=1}^{n} U(N_i + 2) :$$

$$\langle \mathrm{Tr} \frac{1}{z - \Phi} \rangle = \langle \mathrm{Tr} \frac{1}{z - \Phi_U} \rangle - \frac{d}{dz} \log(W'(z)^2 + f(z))$$

Authors

- Matrix model

  Dijkgraaf-Vafa, Cachazo-Intrilligator-Vafa, Argurio-Campos-Ferretti-Heise, Suzuki.H, Bena-Roiban, Tachikawa,T. Itoh, Kraus-Ryzkov-Shigemori, Abbaspur-Imaanpur-Parvizi, Berenstein, Gorsky, ...;

- Field Theory

  Cachazo-Douglas-Seiberg-Witten, Cachazo-Seiberg-Witten, Alday-Cirafici, Brandhuber-Ita-Nieder-Oz-Römelsberger, Eguchi-Sugawara, Matone, Feng, Ahn-Ookouchi, Feng, Huang-Naqvi, Ahn-Nam, Shih, Svrcek, Merlatti, Gripaios-Wheater, David-Gava-Narain, ...

Summary

- $\mathcal{W}_{eff}$ **of** $U(N)$ **theory with superpotential** $\mathcal{W}(\Phi)$ **in terms of** $(S_i, w_i^\alpha, \Lambda_i)$
- **Only planar diagrams contribute to the effective superpotential**
- **Complete set of Chiral ring relations (Ward-Takahashi Identities)**
- **e.g. for** $SU(N)$ **SYM**

$$S^N = \Lambda^{3N} + \{\bar{Q}_{\dot{\alpha}}, X^{\dot{\alpha}}\}$$

  as an *operator relation* $\rightarrow$ **No chirally symmetric vacuum with** $\langle \lambda\lambda \rangle = 0$.
- **Whole results elegantly summarized by matrix model bookkeeping**
- **Addition of matter; other gauge groups (** $SO(N), USp(2N)$ **)**
- **More precise relations** $\mathcal{N} = 2$ *vs* $\mathcal{N} = 1$ **theories**

  (Generalized K anomaly relations from $\mathcal{N} = 2$ **!)**
- **Phases, new duality (continuous and discrete maps among the vacua of the same or different theories), etc.**

Symmetry, Quantization, Phase Factor

$\sim$ Melodies of Theoretical Physics of the 20th Century

C.N. Yang

Do we need something else ?