Trieste lectures on AdS/CFT

Massimo Bianchi,[1]
Dipartimento di Fisica and INFN
Università di Roma "Tor Vergata"
00133 Rome ITALY

Abstract

In the first lecture we will give some motivations and review the two sides of the AdS/CFT correspondence between $\mathcal{N} = 4$ SYM theory in $d = 4$ and type IIB superstring on $AdS_5 \times S^5$. In the second lecture we will discuss tests of the correspondence, *i.e.* symmetries, spectrum, two- and three- point functions of protected operators, instantons vs D-instantons and extremal correlators. We also discuss some dynamics emerging from OPE of 4-point functions and the interpretation of logarithmic behaviours in terms of anomalous dimensions of long multiplets. In the third and last lecture we will discuss holographic renormalization in the context of RG flows dual to domain wall solutions of $5-d$ gauged supergravity. Particular attention will be devoted to the GPPZ flow. Some non-local observables (Wilson loops) and a brief mention of the novel double scaling limit will conclude the set of lectures.

[1] `Massimo.Bianchi@roma2.infn.it`

1 Gauge fields or strings?

The Veneziano model was originally introduced as a mean to describe hadronic resonances and strong interactions. Superstring theory that emerged from dual models has become the most promising candidate for the unification of all interactions including gravity. String compactifications and more general vacuum configurations with branes come close to describe the physics seen at low energies. The AdS/CFT correspondence proposed by J. Maldacena is by now a fundamental tool in understanding the interplay between gauge and string theories. In the simplest and most studied case it relates type IIB superstring on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with gauge group $SU(N)$. The correspondence is holographic in that the dynamics in the bulk is coded in the gauge theory that lives on the boundary of AdS. The radial direction transverse to the boundary plays the role of the energy scale in the gauge theory and one gets an UV/IR duality very much reminiscent of the duality between open and closed strings.

The plan of the lectures is as follows. In the first lecture we will give some motivations and review the two sides of the correspondence, N=4 SYM theory in $d = 4$ and type IIB superstring on $AdS_5 \times S^5$. In the second lecture we will discuss tests of the correspondence, *i.e.* symmetries, spectrum, two and three point functions of protected operators, instantons vs D-instantons and extremal correlators. If we have time we will also discuss some dynamics in terms of 4-point and the interpretation of logarithmic behaviours in terms of anomlaos dimensions of long multiplets. In the third and last lecture we will dicsuss other conformal field theories, deformations from the conformal point and some non-local observables (Wilson loops). A brief mention of the novel double scaling limit will conclude the set of lectures.

1.1 String theory and confinement

Confining gauge theories display string-like behaviour in the IR. Colour-electric flux lines are squeezed by a colour-magnetic condensate into narrow flux-tubes that effectively resemble 'fat' strings. A quantity that characterizes the different phases of a gauge theory is the Wilson loop

$$\langle W(\mathcal{C})\rangle_{gauge} = \langle \mathrm{Tr}\ \mathcal{P} \exp i \oint_{\mathcal{C}} A_\mu dx^\mu \rangle \tag{1.1}$$

where $\mathcal{P}$ denotes path ordering. In a confining phase, the static potential between well separated static test charges, such as non-dynamical quarks in QCD, is linear $V = T|x_1 - x_2|$, with $T = 1/2\pi\alpha'$ the effective string tension, and $\langle W(\mathcal{C})\rangle$ decreases as the exponential of the area of the minimal surface Σ bounded by the curve $\mathcal{C}$. In a non-confining phase the perimeter contribution dominates. A. Polyakov has proposed a bosonic string description of confinement whereby $\langle W(\mathcal{C})\rangle$ is given by an integral over the embeddings in spacetime of surfaces Σ bounded by the loop $\mathcal{C}$

$$\langle W(\mathcal{C})\rangle_{string} = \int [DXDg] \exp(-S[X, g]) \tag{1.2}$$

The worldsheet dynamics is governed by the action

$$S[X, g] = \frac{T}{2} \int_\Sigma d^2\sigma \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X \tag{1.3}$$

After quotienting the infinite group of local symmetries, *i.e.* reparametrizations and Weyl rescalings, one ends up with a theory of d scalar fields X^μ coupled to the Liouville field ρ. The latter decouples only in the critical dimension $d = 26$. Few months before Maldacena's proposal A. Polyakov had observed that one of the drawbacks of the string ansatz, *i.e.* the lack of the *zig-zag* symmetry, could be overcome by assuming the flow of the Liouville mode of a non-critical string to a fixed point with $\rho = 0$. Maldacena's proposal than looks like what the Doctor ordered in that it puts on the plate the existence of a fifth coordinate, transverse to the AdS boundary, that could be identified with the Liouville mode.

What sounds surprising of the initial proposal is that $\mathcal{N} = 4$ SYM theory is known not to confine at all. The gauge invariant degrees of freedom at weak and strong coupling are roughly the same and the coupling constant does not run with the energy scale. In recent developments, however, the correspondence deals with confining theories. The hope is that understanding the correspondence in its simplest superconformal case could shed some light on the phenomenologically more interesting cases.

1.2 Large N gauge theories

Another way to expose the emergence of a stringy behaviour in $SU(N)$ gauge theories with a large number of colours was suggested by G. 't Hooft. In order to take the large N limit it is convenient to introduce the double line notation that consists in writing the gauge fields as matrices in color space

$$A^i_{\mu j} = A^a_\mu T^i_{aj} \tag{1.4}$$

so that the propagator becomes

$$\langle A^i_{\mu j}(x) A^l_{\mu k}(y)\rangle = \Delta(x-y)(\delta^i_k \delta^l_j - \frac{1}{N}\delta^i_j \delta^l_k) \tag{1.5}$$

the last term, which is absent in the case of $U(N)$, is subdominant. Then one rescales out N in front of the Yang-Mills action

$$S = \frac{N}{4\lambda}\int d^4x tr(F_{\mu\nu}F^{\mu\nu}) \tag{1.6}$$

where $\lambda = g^2 N$ is the 't Hooft coupling and considers an L loop connected diagrams with V vertices and E propagators. Each loop contributes a factor of N from the trace over colour indices, each vertex – be it trilinear or quartic – N/λ and each propagator λ/N. The overall coefficient is then given by

$$N^L \left(\frac{\lambda}{N}\right)^P \left(\frac{N}{\lambda}\right)^V = N^{L-P+V}\lambda^{P-V} = N^\chi \lambda^{P-V} \tag{1.7}$$

where $\chi = L - P + V$ is the Euler characteristic of the (oriented) surface triangulated by the Feynman diagram. It can be rewritten as $\chi = 2 - 2h$ where h is the number of handles. A generic amplitude thus admits a topological expansion

$$A(\lambda, N) = \sum_{h=0}^{\infty} N^{2-2h} \sum_k a_k^{(h)} \lambda^k = \sum_{h=0}^{\infty} N^{2-2h} A_h(\lambda) \tag{1.8}$$

very much of the same form as in oriented closed string theories. Only "planar" diagramas, *i.e.* the ones with $h = 0$ that can be drawn on the surface of a sphere, survive in the double scaling limit

$$N \to \infty\ , \quad g \to 0 \qquad \text{with} \quad \lambda \quad \text{fixed} \tag{1.9}$$

Intuitively one may think gluon exchange fills in Feynman diagrams completely in this limit so as to effectively produce a string worldsheet.

For theories with fields (*e.g.* quarks) in the fundamental representation the resulting surfaces may have boundaries (single lines) and their Euler characteristic is given by $\chi = 2 - 2h - b$. Theories with fields in symmetric or antisymmetric tensors, *e.g.* theories with orthogonal or symplectic groups, require the addition of unoriented surfaces with $\chi = 2 - 2h - b - c$, where $c = 0, 1, 2$ is the number of "crosscaps", *i.e.* boundaries with diametrically opposite points identified.

EXERCISE

Show that the first non-planar correction to the vector propagator vanishes for any group in theories with only fields in the adjoint representation.

1.3 AdS/CFT correspondence: the baby version

That $AdS_5 \times S^5$ with N fluxes of the self-dual Ramond–Ramond (R-R) 5-form be a maximally supersymmetric solution of the type IIB supergravity was originally observed by J. Schwarz. The motivations that lead Maldacena to conjecture a relation between this solution and $\mathcal{N} = 4$ SYM are to be found in the dynamics of D-branes and open strings. Their massless excitations are vector fields together with their superpartners. All 1/2 BPS Dp-branes at low-energy are governed by a maximally supersymmetric gauge theory that results from the dimensional reduction of $\mathcal{N} = 1$ SYM from $d = 10$ to $d = p + 1$. In particular for a stack of N coincident D3-branes in type IIB theory one has $U(N)$ $\mathcal{N} = 4$ SYM in $d = 4$. Quite remarkably the resulting gauge theory is known to be exactly superconformal invariant at the quantum level. This reflects into the fact that the type IIB D3-brane solution has constant dilaton, $\phi = g_s$ For N coincident D3-branes the metric is given by

$$ds^2 = (1 + L^4/r^4)^{-1/2} dx \cdot dx + (1 + L^4/r^4)^{1/2} dy \cdot dy \qquad (1.10)$$

where x are the four coordinates along the brane, $r^2 = y \cdot y$ is the transverse distance from the stack and $L^4 = 4\pi g_s N \alpha'^2$ with N measuring the 5-form flux. The D3-brane solution is an interpolating soliton between maximally supersymmetric flat 10-D Minkowski spacetime at infinity ($r \to \infty$) and $AdS_5 \times S^5$ near the horizon ($r \to 0$), where the metric reads

$$ds^2 = \frac{r^2}{L^2} dx \cdot dx + \frac{L^2}{r^2} dr^2 + L^2 d\omega_5^2 \qquad (1.11)$$

In the meanwhile 16 out of the 32 Poincaré supercharges at infinity are traded for as many superconformal charges. Thanks to the infinite gravitational red-shift near the horizon, Maldacena has been lead to conjecture that gravity should decouple from the brane at low energies and a perfect equivalence between the two descriptions – strings in the AdS bulk and $\mathcal{N} = 4$ SYM on its boundary – should take place when the following identifications are made

$$g_s = \langle e^{\phi} \rangle = \frac{g^2}{4\pi} \qquad \vartheta_s = \langle \chi \rangle = \frac{\theta}{2\pi} \tag{1.12}$$

The bulk type IIB partition function with prescribed boundary conditions $J(x)$ for the string excitations $\Phi(x, r)$ plays the role of a generating functional for the boundary theory

$$Z_{IIB}[\Phi[J]] = Z_{SYM}[J] = \int D\mathcal{V} \exp\left(iS_{SYM}[\mathcal{V}] + \int J(x)\mathcal{O}(x)\right) \tag{1.13}$$

Here $\mathcal{V}$ collectively denote the gauge fields and their superpartners and $\mathcal{O}(x)$ denote gauge invariant composite operators. Since one has only recently and in a very specific (Penrose) limit started to understand how to quantize string theory on (AdS) spaces with R-R background, so far one has been limited to work in the supergravity approximation $L^2 \gg (\alpha')^2$ which in view of the above identifications describe the regime of strong 't Hooft coupling $\lambda \gg 1$ in the dual SYM theory. In this regime

$$Z_{IIB}[\Phi[J]] \approx \exp(-S_{IIB}[\Phi[J]]) \tag{1.14}$$

where S_{IIB} is the on-shell type IIB action. It admits a double expansion in powers of the string coupling g_s and of α'. The first few terms are roughly of the form

$$S = \frac{1}{g_s^2 \alpha'^4} \int d^{10}X \sqrt{G} \{ (\mathcal{R} + \ldots) + \alpha'^3 (c_0 + c_1 g_s^2 + \sum_k b_k e^{2\pi i k \tau})(\mathcal{R}^4 + \ldots) + \ldots \} \tag{1.15}$$

where $\tau = \chi + i\exp(-\phi)$ is the complexified dilaton and $\mathcal{R}$ represents the curvature. Performing a constant Weyl rescaling of the metric $G_{MN} = L^2 \widehat{G}_{MN}$ and trading g_s and α' for N and λ schematically yields

$$S = \frac{N^2}{4\pi^2} \int d^{10}X \sqrt{\widehat{G}} \{ (\widehat{\mathcal{R}} + \ldots) + (\frac{c_0}{\lambda^{3/2}} + \frac{c_1 \lambda^{1/2}}{N^2} + \sum_k b_k e^{2\pi i k \tau})(\widehat{\mathcal{R}}^4 + \ldots) + \ldots \} \tag{1.16}$$

In principle, at least for the massless excitations and their holographic duals, one can systematically take into account higher derivative ($\sim \lambda^{-n/2}$) as well as string-loops ($\sim g_s^h$) and non-perturbative ($\sim e^{2\pi i k\tau}$) D-instanton corrections.

1.4 Conformal Field Theories

Local quantum field theories are invariant under the Poincaré group that includes translations generated by the momenta P_μ and Lorentz transformations generated by angular momenta and boosts $J_{\mu\nu}$. The (anti-hermitean) generators satisfies the algebra

$$[P_\mu, P_\nu] = 0 \quad [J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\nu\rho} J_{\mu\nu} + \dots \quad [J_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\nu - \dots \tag{1.17}$$

This is not the maximal possible (bosonic) symmetry. At least at the classical level, massless theories are invariant under scale transformations as well as under transformations that preserve the light cone up to a rescaling *i.e.* $ds^2 \to \Omega(x)^2 ds^2$. In d dimensions, infinitesimal transformations $\delta x^\mu = \xi^\mu$ of this kind satisfy

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} \partial \cdot \xi \eta_{\mu\nu} \tag{1.18}$$

that admits solutions of the form

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2b \cdot x x^\mu \tag{1.19}$$

with constant a, λ, ω, b and $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The first three terms correspond to translations, Lorentz and scale transformations generated by $P_\mu = \partial_\mu$, $J_{\mu\nu} = (x_\mu \partial_\nu - x_\nu \partial_\mu)$ and $D = -x^\mu \partial_\mu$, respectively. The latter two combine into a conformal boost generated by $K_\mu = (x^2 \partial_\mu - 2x_\mu x \cdot \partial)$. D and K_μ satisfy the following commutation relations

$$[J_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\nu - \dots \tag{1.20}$$

$$[P_\mu, P_\nu] = 0 \quad [K_\mu, K_\nu] = 0 \quad [P_\mu, K_\nu] = 2J_{\mu\nu} + 2\eta_{\mu\nu} D \tag{1.21}$$

$$[D, J_{\mu\nu}] = 0 \quad [D, P_\rho] = P_\nu - \dots \quad [D, K_\rho] = -K_\nu - \dots \tag{1.22}$$

These relations extend the algebra of the Poincaré group to the (global) conformal algebra in D dimensions[2] that is isomorphic to the algebra of $SO(d, 2)$.

[2]In $d = 2$ there are additional (not globally defined) transformations that make the conformal group infinite dimensional. This fact plays a crucial role in the worldsheet dynamics of string theories.

EXERCISE

The easiest way to see this is to define operators L_{AB} with $A = 0, 1, \dots d-1, d, d+1$ according to

$$L_{\mu\nu} = J_{\mu\nu} \quad L_{d,d+1} = D \quad L_{d,\mu} = P_\mu - K_\mu \quad L_{d+1,\mu} = P_\mu + K_\mu \tag{1.23}$$

and check that they satisfy $[L_{AB}, L_{CD}] = \eta_{BC} L_{AD} + \dots$ with $\eta_{AB} = (\eta_{\mu\nu}; +, -)$.

In conformal field theory a central role is played by the stress energy tensor $\mathcal{T}_{\mu\nu}$ since the generators of the conformal group, whose are essentially 'modes' of $\mathcal{T}_{\mu\nu}$. $\mathcal{T}_{\mu\nu}$ is symmetric $\mathcal{T}_{\mu\nu} = \mathcal{T}_{\nu\mu}$, conserved $\partial^\mu \mathcal{T}_{\mu\nu} = 0$ and classically traceless $\eta^{\mu\nu}\mathcal{T}_{\mu\nu} = 0$. If and only if these properties survive quantum corrections one can fully exploit the implications of (super)conformal symmetry. Unitary representation of the conformal group are infinite dimensional. Highest weight states, usually called primary fields, are characterized by their scaling dimension Δ and their spin. Primary fields satisfy

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x) \qquad [P_\mu, \mathcal{O}(x)] = i\partial_\mu \mathcal{O}(x) \tag{1.24}$$

$$[J_{\mu\nu}, \mathcal{O}(x)] = \{i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}\}\mathcal{O}(x) \tag{1.25}$$

$$[K_\mu, \mathcal{O}(x)] = \{i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\mu + 2x_\mu \Delta) - 2x^\nu S_{\nu\mu}\}\mathcal{O}(x) \tag{1.26}$$

Fields that do not satisfy the above commutation relations are descendants and can be obtained from primary fields by the action of the conformal group.

Mass is not a Casimir operator of the conformal group. It has thus little meaning to form wave packets and asymptotic states and the relevant observables are not scattering amplitudes but rather correlation functions of scaling operators

$$G(x_1, \dots x_n) = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \tag{1.27}$$

2-point functions of (normalized) primary operators $\mathcal{O}_\Delta$ are completely specified by their scaling dimensions

$$\langle \mathcal{O}^\dagger_\Delta(x) \mathcal{O}_\Delta(y) \rangle = \frac{1}{(x-y)^{2\Delta}} \tag{1.28}$$

Similarly the dependence of 3-point functions on the insertions points is completely fixed by (super)conformal invariance up to trilinear couplings that appear as coefficients in the operator product expansion (OPE)

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) = \sum_k \frac{{C_{12}}^k}{(x_{12})^{\Delta_1+\Delta_2-\Delta_k}} \mathcal{O}_{\Delta_k} + \dots \tag{1.29}$$

OPE is expected to be complete and convergent though involving an infinite number of fields (primary and descendants). In particular

$$\mathcal{T}_{\mu\nu}(x)\mathcal{O}(0) \approx \Delta\mathcal{O}(0)(\partial_\mu\partial_\nu - \delta_{\mu\nu})x^{-2} \tag{1.30}$$

EXERCISE

Show that up to an overall costant 2- and 3-point functions of scalar primary operators are completely fixed by conformal invariance. 4-point functions may depend on a priori arbitrary function of the two independent conformally invariant cross ratioes

$$r = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad s = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} . \tag{1.31}$$

The fundamental relation between operators and states of CFT's

$$|\mathcal{O}\rangle = \lim_{x\to 0} \mathcal{O}(x)|0\rangle \tag{1.32}$$

can be established in radial quantization, whereby the radial direction in $R^d = R^+ \times S^{d-1}$ plays the role of time and the generator of 'time' translation turns out to be $H_{rad} = L_{0,d+1} = (P_0 + K_0)/2$. In particular the vacuum state which is unique corresponds to the identity operator and is annihilated by the generators of the conformal group.

1.5 AdS geometry

Anti De Sitter spaces are maximally symmetric spaces with negative cosmological constant $\Lambda = -d(d-1)/L^2$. One way to represent AdS_d is as the quotient space $SO(d,2)/SO(d,1)$. More intuitively one can picture AdS_d as the hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_i X_i^2 = L^2 \tag{1.33}$$

in flat $d+2$ dimensional spacetime with signature $(-,+,\ldots;+,-)$. The $SO(d,2)$ isometry generated by $L_{AB} = X_A\partial_B - X_B\partial_A$ manifest. A global set of coordinates is given by

$$X_0 = L\cosh(\sigma)\cos(\tau) \quad X_{d+1} = L\cosh(\sigma)\sin(\tau) \quad X_i = L\sinh(\sigma)n_i \tag{1.34}$$

with $\sum_i n_i^2 = 1$. In this coordinate system the metric reads

$$ds^2 = L^2(-\cosh(\sigma)^2 d\tau^2 + d\sigma^2 + \sinh(\sigma)^2 d\omega_{d-1}^2) \tag{1.35}$$

where $d\omega_{D-1}^2$ is the metric on a unit sphere S^{d-1}. The hyperboloid, which has topology $S^1 \times R^d$ is entirely covered by the patch $\sigma > 0$ and $0 < \tau < 2\pi$. In order to avoid closed time-like curves one has to take the universal cover and let $-\infty < \tau < +\infty$. Another parametrization

$$X_{d+1} = \frac{\rho^2 + L^2 - x \cdot x}{2\rho} \quad X_d = \frac{\rho^2 - L^2 - x \cdot x}{2\rho} \quad X_\mu = L\frac{x_\mu}{\rho} \tag{1.36}$$

brings the metric into the manifestly conformally flat form

$$ds^2 = \frac{L^2}{\rho^2}(dx \cdot dx + d\rho^2) \tag{1.37}$$

and makes the location of the boundary at $\rho = 0$ evident. This parametrization is known as the Poincaré patch since it only covers half of the hyperboloid. After Wick rotation ($t \to t_E = -it$), the Poincaré coordinates cover the entire Euclidean AdS that topologically turns out to be a ball. The AdS horizon ($\rho \to \infty$) represents the deep interior region.

EXERCISE

Show that the metric in horospherical (Poincaré) coordinates is invariant under inversions $z^M \to z^M/(z)^2$, where $z^\mu = x^\mu$, $z^5 = \rho$ and $(z)^2 = x \cdot x + \rho^2$.

In the correspondence, the UV regime of the field theory is embodied in the boundary that is at large distances from the centre. The IR is encoded in the interior. A convenient coordinate system where this property is made manifest is

$$ds^2 = dr^2 + \exp(2r/L) dx \cdot dx \tag{1.38}$$

Indeed shifting the variable r which is transverse to the boundary by a constant $r \to r + aL$ requires a compensating rescaling of the coordinates on the boundary $x \to \exp(-a)x$. This meanse that the boundary $r \to \infty$ corresponds to small distance scales, *i.e.* the field theory UV regime, while the point $r \to -\infty$ corresponds to large distance scales, *i.e.* the field theory IR. In an exactly conformal theory, as $\mathcal{N} = 4$ SYM, this rescaling should not change the physics and indeed the transformation is an isometry. For theories which are only asymptotically scale invariant the change in the physics should be reproduced by the change in the geometry. By carefully studying the stability of AdS under perturbations P. Breitenlohner and D. Freedman arrived at setting lower bounds on the masses of propagating particles. For scalar fields

$$(ML)^2 \geq -(d/2)^2 \tag{1.39}$$

Similar bounds obtain for fields of higher spin. Massive particles in AdS can never reach the boundary because of the explosion of the conformal factor there. On the contrary light rays can go to the boundary and back in a finite amount of time if suitable boundary conditions are imposed.

1.6 N=4 SYM theory in $d = 4$

$\mathcal{N}$=4 SYM theory in $d = 4$ is a very special theory. It enjoys maximal supersymmetry for renormalizable theories. It is completely determined by the choice of the gauge group[3]. Elementary fields belong to the adjoint representation and the interactions are proportional to the structure constants f_{abc} of the gauge group. In addition to the vector fields, A_μ, there are six real scalars, φ^i in the **6** of a global $SU(4)$ R-symmetry, four Weyl spinors, λ^A, in the **4** of $SU(4)$. Sometimes it is convenient to put $\varphi^i = \frac{1}{2}\overline{\tau}^i{}_{AB}\varphi^{AB}$ with $\varphi^{AB} = -\varphi^{BA}$ and $\overline{\tau}^i{}_{AB}$ (chiral blocks of) $SO(6)$ γ-matrices. The reality condition translates into $\overline{\varphi}_{AB} = \frac{1}{2}\varepsilon_{ABCD}\varphi^{CD}$.

The $\mathcal{N} = 4$ Lagrangian reads

$$L = -\frac{1}{g^2}tr\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D^\mu\varphi_i D_\mu\varphi^i + \frac{1}{2}[\varphi_i, \varphi_j][\varphi^i, \varphi^j] \tag{1.40}$$

$$+i(\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda + \lambda\sigma^\mu D_\mu\bar{\lambda}_A) + [\bar{\varphi}_{AB}, \lambda^A]\lambda^B - [\varphi^{AB}, \bar{\lambda}_A]\bar{\lambda}_B)\} \tag{1.41}$$

The theory is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta\varphi^i &= \bar{\tau}^i{}_{AB}\lambda^{\alpha A}\eta_\alpha{}^B + \tau^{iAB}\overline{\eta}_{\dot{\alpha}A}\overline{\lambda}^{\dot{\alpha}}_B \\
\delta\lambda_\alpha{}^A &= -\frac{1}{2}F^-_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta\eta_\beta{}^B + i\not{D}_{\alpha\dot{\alpha}}\varphi^{AB}\overline{\eta}^{\dot{\alpha}}_B + \frac{1}{2}[\varphi^i, \varphi^j]\tau_{ij}{}^A{}_B\eta_\alpha{}^B \\
\delta A_\mu &= -i\lambda^{\alpha A}\sigma^\mu{}_{\alpha\dot{\alpha}}\overline{\eta}^{\dot{\alpha}}{}_A - i\eta^{\alpha A}\sigma^\mu{}_{\alpha\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}}{}_A \quad ,
\end{aligned} \tag{1.42}$$

[3]In what follows we will only display group theory factors relevant for the case $\mathcal{G} = SU(N)$. The generalization to an arbitrary gauge group amounts to replacing g^2N with g^2C_A and $N^2 - 1$ with $\dim(\mathcal{G})$, C_A being the quadratic Casimir of the adjoint, that coincides with the Dynkin index ℓ_A for this representation.

where as usual $\tau_{i_1 \cdots i_p}$ denote p-fold antisymmetric products of τ_i.

EXERCISE

Putting $A_M = \{A_\mu, \varphi_i\}$ *and* $\chi^\Lambda = \{\lambda^A_\alpha, \bar{\lambda}^{\dot{\alpha}}_A\}$, *with* $\Gamma_{11}\chi = \chi$, *perform the reduction of the lagrangian*

$$L = -\frac{1}{4g^2} F_{MN} F^{MN} + \frac{i}{2g^2} \bar{\chi} \Gamma^M D_M \chi \tag{1.43}$$

and supersymmetry transformations

$$\delta A_M = \bar{\epsilon} \Gamma_M \chi \qquad \delta \chi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon \tag{1.44}$$

from $d = 10$ *to* $d = 4$. *A convenient choice of* γ*-matrices is* $\Gamma_\mu = \gamma_\mu \otimes 1$, $\Gamma_i = \gamma_5 \otimes i\hat{\gamma}_i$.

The theory admits a quantum moduli space of vacuum configurations $\mathcal{M} = \mathbb{R}^{6N}/\mathcal{S}_N$ parametrized by the vacuum expectation values (VEV's) of the scalar fields $\langle \varphi^i \rangle$ along the Cartan generators, *i.e.* such that $[\varphi^i, \varphi^j] = 0$. $\mathcal{S}_N$ is the Weyl group of $SU(N)$. In the unbroken phase $\langle \varphi^i \rangle = 0$, the theory is believed to be exactly invariant under $\mathcal{N} = 4$ superconformal transformations. In the broken phase $\langle \varphi^i \rangle \neq 0$, electric-magnetic S-duality transforms elementary charges into monoples and dyons leaving the theory invariant. S-duality act by (projective) $SL(2, Z)$ transformations on the complexified coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. Physics is θ dependent since there are no internal chiral anomalies[4]. At the quantum level the absence of UV divergences implies for instance the exact vanishing of the β-function $\beta(g) = \mu \partial g / \partial \mu$.

EXERCISE

Check the vanishing of the one-loop β*-function using*

$$\beta_1 = -\frac{11}{3} C_A + \frac{2}{3} \sum_f \ell(R_f) + \frac{1}{6} \sum_s \ell(R_s) \tag{1.45}$$

where $C_A = \ell_A$ *is the Casimir of the adjoint and* $\ell(R_s)$, *respectively* $\ell(R_f)$, *are the Dynkin indices of the representations scalar fields, respectively (Weyl) fermions, belong to.*

$\mathcal{N} = 4$ superconformal transformations form the supergroup $SU(2, 2|4)$ generated by (generalized) angular momenta $J_{\mu\nu}$, momenta P_μ, conformal

[4] Anomalies can show up when coupling the theory to to external sources/backgrounds.

momenta K_μ, dilation D, Poincaré supercharges Q_A and $\bar{Q}^A$, supercoformal charges S^A and $\bar{S}_A$ and R-symmetry charges $T^A{}_B$. The conformal algebra was given before, the $SU(4)$ generators commute with $SO(d,2)$ while the fermionic charges (in the absence of central charges) satisfy

$$\{Q_A, \bar{Q}_B\} = \delta_A{}^B \sigma^\mu P_\mu \qquad \{Q_A, Q_B\} = 0 \tag{1.46}$$

$$\{S^A, \bar{S}_B\} = \delta^A{}_B \sigma^\mu K_\mu \qquad \{S^A, S^B\} = 0 \tag{1.47}$$

$$\{S^A, Q_B\} = \delta^A{}_B(\frac{1}{2}\sigma^{\mu\nu}J_{\mu\nu} + D) + T^A_B \qquad \{S^A, \bar{Q}^B\} = 0 \tag{1.48}$$

in addition to the usual commutation rules with the bosonic generators

$$\begin{gathered}
[J_{\mu\nu}, Q_A] = \frac{1}{2}\sigma^{\mu\nu}Q_A \quad [D, Q_A] = \frac{1}{2}Q_A \quad [P_\mu, Q_A] = 0 \quad [K_\mu, Q_A] = \sigma_\mu \bar{S}_A \\
[J_{\mu\nu}, S^A] = \frac{1}{2}\sigma^{\mu\nu}S^A \quad [D, S^A] = -\frac{1}{2}S^A \quad [P_\mu, S^A] = \sigma_\mu \bar{Q}^A \quad [K_\mu, S^A] = 0 \\
[T^A{}_B, Q_C] = \delta^A{}_C Q_B - \frac{1}{4}\delta^A{}_B Q_C \quad [T^A{}_B, S^C] = \delta^C{}_B S^A - \frac{1}{4}\delta^A{}_B S^C
\end{gathered} \tag{1.49}$$

The Noether currents associated with the superconformal transformations form the $\mathcal{N} = 4$ supercurrent multiplet. It includes the traceless energy-momentum tensor, $\mathcal{T}_{\mu\nu}$, **15** conserved R-symmetry currents, $\mathcal{J}^\mu{}_A{}^B$, and the γ-traceless supersymmetry currents, $\Sigma^\mu{}_{\alpha A}$ in the $\mathbf{4}^*$ of $SU(4)$. The remaining components consist of three sets of scalars (a complex singlet $\mathcal{C}$, $\mathcal{E}^{(AB)}$ in the $\mathbf{10}_C$, $\mathcal{Q}^{ij}$ in the **20'**), two sets of spin-1/2 fermions ($\widehat{\chi}^C_{AB}$ in the $\mathbf{20_C}$ and $\widehat{\Lambda}^A$ in the **4**) and **6** antisymmetric tensors ($\mathcal{B}^{[AB]}_{\mu\nu}$).

The fields of $\mathcal{N} = 4$ SYM can be packaged into a "twisted chiral superfield"

$$\mathcal{W}^{AB} = \varphi^{AB} + (\theta^{[A}\lambda^{B]} + \theta^A\sigma^{\mu\nu}\theta^B F_{\mu\nu}) + \ldots \tag{1.50}$$

that satisfies $\mathcal{D}_A\mathcal{W}_{BC} + cyclic = 0$ on shell. For lack of a manifestly $\mathcal{N} = 4$ supersymmetric off-shell formalism, it is often useful to decompose the $\mathcal{N}$=4 multiplet in terms of either $\mathcal{N}$=1 or $\mathcal{N}$=2 multiplets. The global symmetry that is manifest in the $\mathcal{N} = 2$ description is $SU(2)_\mathcal{V} \times SU(2)_\mathcal{H} \times U(1)$ and the $\mathcal{N}$=4 elementary supermultiplet decomposes into a $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$ and a hypermultiplet, $\mathcal{H}$. The global symmetry that is manifest in the $\mathcal{N} = 1$ description is $SU(3) \times U(1)$, the $\mathcal{N}$=4 elementary supermultiplet decomposes into a $\mathcal{N} = 1$ vector multiplet V and a three chiral multiplets Φ^I.

1.7 Type IIB superstring on $AdS_5 \times S^5$

We now turn to consider the other side of the correspondence, *i.e.* the type IIB superstring on $AdS_5 \times S^5$. In $d = 10$, this theory enjoys chiral $\mathcal{N}$=(2,0) supersymmetry associated to the presence of a complex spin 3/2 gravitino of definite, say, positive chirality, $\Psi_M^{(L)}$ and a complex spin 1/2 dilatino Λ of negative chirality. The massless bosonic spectrum includes the metric G_{MN}, a scalar dilaton, ϕ, and a two-form potential B_{MN} in the Neveu-Schwarz – Neveu-Schwarz (NS-NS) sector and a pseudo-scalar, χ, another antisymmetric tensor C_{MN} and a four-form potential A_{MNPQ} with self-dual field-strength, in the R–R sector. The classical theory is invariant under $SL(2,\mathbb{R})$ transformations that act projectively on the complex scalar field $\tau = \chi + ie^{-\phi}$. The two antisymmetric tensors form a doublet, while the metric (in Einstein frame!) and the four-form potential are inert. At the quantum level the classical continuous symmetry is expected to be broken to $SL(2,\mathbb{Z})$. Due to the self-duality constraint a covariant action including the four-form potential has a rather complicated expression. With the proviso of imposing the self-duality constraint at the end, the bosonic field equations in the Einstein frame can be obtained by varying

$$S_{IIB} = \frac{1}{8\pi G_N^{(10)}} \int d^{10}X\sqrt{G} \times \tag{1.51}$$
$$\left(\frac{1}{2}R + P_M^* P^M + \frac{1}{6}H_{MNP}^* H^{MNP} + \frac{1}{240}F_{MNPQR}F^{MNPQR}\right)$$

where $G_N^{(10)} = 8\pi^6 g_s^2 (\alpha')^4$ is the 10-D Newton constant, $P = \partial\tau/Im\tau$, $H_3 = (dC_2 - \tau dB_2)/\sqrt{Im\tau}$ and $F_5 = dA_4 + B_2 \wedge dC_2 - C_2 \wedge dB_2 = *F_5$.

Compactification of $d = 10$ type IIB supergravity on S^5 was studied long time ago. For a sphere of radius L the Newton constant in $d = 5$ is given by

$$G_N^{(5)} = G_N^{(10)}/Vol(S^5) = 8\pi^3 g_s^2 (\alpha')^4/L^5 \tag{1.52}$$

In order for the background to be supersymmetric one has to resort to a Freund-Rubin ansatz and turn on a non-vanishing flux for the self-dual R-R 5-from $\int_{S^5} F_5 = 4\pi^3 L^4/g_s$. Setting the dilaton to a constant $\tau = i/g_s$, so that $P_M = 0$, and putting $B_{MN} = C_{MN} = 0$, so that $H_{MNP} = 0$ immediately gives the vanishing of the dilatino variation $\delta\Lambda = 0$. One is left with the gravitino variation

$$\delta\Psi_K = 0 = (D_K + \frac{i\sqrt{8\pi G_N^{(10)}}}{480} F_{MNPQR}\Gamma^{MNPQR}\Gamma_K)\epsilon \tag{1.53}$$

for some chice of $\epsilon = \zeta_{AdS_5} \otimes \kappa_{S^5}$. Decomposing into internal ($i = 5, 6, 7, 8, 9$, sphere) and external ($\mu = 0, 1, 2, 3, 4$) components the Killing spinor equations read

$$D_\mu \zeta = \frac{1}{L}\gamma_\mu \zeta \quad , \quad D_i \kappa = \frac{i}{L}\gamma_i \kappa \tag{1.54}$$

Their integrability condition requires

$$R_{\mu\nu\rho\sigma} = -\frac{4}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad , \quad R_{ijkl} = +\frac{4}{L^2}(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{1.55}$$

that show the existence of an $AdS_5 \times S^5$ vacuum configuration with cosmological constants $\Lambda_{AdS_5} = -\Lambda_{S^5} = -12/L^2$. The ten-dimensional metric turns out to be conformally flat

$$ds^2_{5+5} = \frac{L^2}{\rho^2}(d\rho^2 + dxdx) + L^2 d\omega_5^2 \tag{1.56}$$

Decomposing the $d = 10$ fields in spherical harmonics $Y^I_{(\ell)}(\hat{y})$ on S^5, with I a multi-index running from over the possible projections of $\vec{\ell}$, one can linearize the field equations around the AdS_5 vacuum configuration. For instance, the spectrum of fluctuations of the dilaton can be deduced from the Kaluza–Klein (K-K) ansatz

$$\Phi = \sum_{\ell=0}^{\infty} \sum_{I=1}^{(\ell+1)(\ell+2)^2(\ell+3)/12} \phi_I^{(\ell)}(z) Y^I_{(\ell)}(\hat{y})$$

Using

$$\nabla^2_{S^5} Y^I_{(\ell)} = -\frac{\ell(\ell+4)}{L^2} Y^I_{(\ell)} \quad \forall\, I \tag{1.57}$$

one concludes that the $\phi_I^{(\ell)}$ component field, that transforms in the representation $\mathbf{r} = [0, \ell, 0]$ of $SO(6) \equiv SU(4)$, has AdS mass $(M_\ell L)^2 = \ell(\ell+4)$. For other fields one has to resolve intricate mixings that lead among other things to some scalar "tachyons". At a first look this may sound as an inconsistency. It is easy to check that the scalar tachyons satisfy the Breitenlohner-Freedman bound that arises in spaces with negative cosmological constant. We will see that for scalars this simply imply that their dual operators satisfy a unitarity bound, *i.e.* Δ real and $\Delta \geq 1$. The whole spectrum of linearized fluctuations assembles into representations of the superisometry group $SU(2,2|4) \supset SO(4,2) \times SO(6)$. Notice however that particles in the same multiplet may have different AdS mass, since the latter is not a Casimir operator of $SU(2,2|4)$.

Although a detailed proof is still lacking, it is believed that the full non-linear theory can be consistently truncated to the lowest K-K excitations that give rise to $\mathcal{N} = 8$ gauged supergarvity in $d = 5$. The latter results from the gauging of a subgroup of the global non-compact $E_{6(+6)}$ symmetry of maximally extended Poincaré supergravity that results from compactification of type IIB (or type IIA) on a five-torus (or $d = 11$ supergravity on a six-torus). The starting point is a theory whose bosonic content consists of a graviton $g_{\mu\nu}$, **27** graviphotons $A_\mu^{[ab]]}$ and 42 scalars ϕ^I that parameterize the coset $\mathcal{M} = E_{6(+6)}/Sp(8)$ and can be assembled into a 27-bein $V(\phi)_{ab}^{AB}$. In order to gauge the theory one has to dualize 12 vectors into as many antisymmetric tensors satisfying a self duality condition that reduces their degrees of freedom from 6 to 3 each. The remaining 15 vectors $A_\mu^{[IJ]}$ can be used to gauge a 15-dimensional subgroup, *e.g.* $SO(6)$, of $E_{6(+6)}$. The gauge coupling is given by $g_5 = 2/L$.

The resulting lagrangian is very complicated. We simply display the bosonic terms in a rather sketchy form

$$\mathcal{L}_5 = R + (B+F)^2 + BDB + (D\phi)^2 - V(\phi) + AFF \tag{1.58}$$

Some remarkable features are apparent. The antisymmetric tensors B's have first order kinetic terms, that may come from terms like $F_5(B_2 dC_2 - C_2 dB_2)$ in $d = 10$, when F_5 is set to its background. Moreover, as in Born-Infeld type actions for D-branes, the B's mix with $F \approx dA$. Last but not least, The "topological" Chern-Simons couplings $AFF + ...$ encode the anomalous content of the theory at the boundary and completely fix the lagrangian in combination with supersymmetry. Under $SO(6)$ the 42 scalars decompose into a complex siglet C, $\mathbf{20'}$ real Q and $\mathbf{10}^{\pm}$ complex E. As always in supergravity theories the scalar potential, $V(\phi)$, is given in terms of the spin

1/2 and 3/2 fermionic 'shifts'

$$\delta' \psi_\mu = g_5 W_{3/2}(\phi) \gamma_\mu \varepsilon \quad \delta' \chi_\mu = g_5 W_{1/2}(\phi) \varepsilon \tag{1.59}$$

$V(\phi) = g_5^2 (W_{1/2}^2 - W_{3/2}^2)$ is invariant under the $SL(2)$ in $E_{6(+6)} \supset SL(2) \times SL(6)$ and admits an $SO(6) \subset SL(6)$ invariant extremum at $Q_{\mathbf{20'}} = E_{\mathbf{10}} = 0$ and arbitrary C. There are many other extrema but a complete classification is still lacking and difficult to achieve.

Four-fermi and higher derivative terms can in principle be determined by reduction from $d = 10$. The first higher derivative terms is an $\mathcal{R}^4$ term that only involves the Weyl tensor and thus vanishes in the conformally flat $AdS_5 \times S_5$ background. The first, second and third functional derivatives of $\mathcal{R}^4$ vanish as well, and as a result one finds no corrections to zero, one, two and 3-point amplitudes of the supergravity fields and their K-K excitations. There are many terms related to $\mathcal{R}^4$ by supersymmetry. Schematically one has

$$\frac{1}{\alpha'} \int d^{10} X \sqrt{G}\, e^{-\phi/2} \left[f_4(\tau, \overline{\tau}) \mathcal{R}^4 + \ldots f_8(\tau, \overline{\tau}) H^8 + \ldots + f_{16}(\tau, \overline{\tau}) \Lambda^{16} \right], \tag{1.60}$$

In particular we will be concerned with the last sixteen-fermion interaction involving the complex spin 1/2 dilatino Λ in a totally antisymmetric contraction of its spinor indices.

The functions $f_n(\tau, \overline{\tau})$ are non-holomorphic modular forms of weight $(n - 4, 4 - n)$. At weak coupling $Im\tau \to \infty$, they admit asymptotic expansions that display at most two perturbative contributions (genus zero and one) and an infinite series of non-perturbative D-instanton contributions. Supersymmetry arguments imply that the functions f_n, say f_4 and f_{16}, be related to one another by the action of $SL(2, \mathbb{Z})$ covariant derivatives. Whereas f_4 transforms with modular weight $(0, 0)$, the function f_{16} has weight $(12, -12)$ and therefore transforms with a τ-dependent phase under $SL(2, \mathbb{Z})$. This is precisely cancelled by a compensating anomalous $U(1)_B$ transformation of the 16 negative chirality Λ's, each with charge $+3/2$. Similar cancellation occur for the other supersymmetry related terms if the positive chirality gravitino Ψ_M is assigned charge $+1/2$, the complex 3-form H_{MNP} charge $+1$ and the complex dilaton field-strength P_M charge $+2$.

2 Testing the AdS/CFT correspondence

The matching of global symmetries provides the initial test and motivation of Maldacena's conjecture. We will discuss how $\mathcal{N} = 4$ superconformal symmetry is realized on both sides of the correspondence and then describe the holographic computation of 2-point and 3-point correlations of primary fields. Up to an overall constant these are fixed by superconformal invariance and the tests amount to checking the consistency of the relative normalizations and the (non)-vanishing of some 3-point couplings. More interestingly, SYM instanton effects can be quantitatively put in correspondence with type IIB D-instanton effects. Non-renormalization of extremal and next-to-extremal correlators, though a consequence of superconformal invariance, provides new insights. 4-point functions of protected operators, on the contrary, together with their partial non-renormalization property display some interesting dynamical issues that we will address later.

2.1 Kinematical tests

The superisometry group of $AdS_5 \times S_5$ acts by superconformal transformations on the boundary field theory. The map between gauge invariant composite operators $\mathcal{O}_\Delta$ on the boundary and bulk type IIB fields Φ_M can be made very precise. For instance, the $\mathcal{N} = 4$ supercurrent multiplet is dual to the "massless" $\mathcal{N} = 8$ supergravity multiplet. Higher K-K excitations with spin up to 2 assemble into 1/2 BPS short multiplets of the superisometry group with lowest component fields

$$\mathcal{Q}^{i_1 \ldots i_\ell}_{[0,\ell,0]} = tr(\varphi^{i_1}\varphi^{i_2}\ldots - \delta^{i_1 i_2}\varphi_k\varphi^k\ldots) \tag{2.61}$$

that are chiral primary operators (CPO's) of dimension $\Delta = \ell$ belonging to the ℓ-fold tensor product of the fundamental $\mathbf{6}$ representation of $SO(6)$. 1/2 BPS short multiplets have $32\ell^2(\ell^2-1)/3$ bosonic and as many fermionic components. Operators dual to string excitations with AdS masses of order $1/\sqrt{\alpha'}$ belong to long multiplets with roughly 2^{16} components. The spins of the various components of these supermultiplets range over 8 units and their scaling dimensions are expected to grow like $\Delta \approx \lambda^{1/4}$ at strong coupling. One such example is the $\mathcal{N} = 4$ Konishi multiplet whose lowest component is the scalar $SU(4)$ singlet

$$\mathcal{K}_{\mathbf{1}} = tr(\varphi^i\varphi_i) \tag{2.62}$$

of naive dimension $\Delta_0 = 2$.

Unitary irreducible representations (UIR's) of $SU(2,2|4)$ have been completely classified[5]. A general UIR is specified by the dilation weight Δ, the Lorentz spins (j_1, j_2) and the Dynkin labels $[k,l,m]$ of the $SU(4)$ R-symmetry[6]. There are three unitary series, which are distinguished by different relations between the dilation weight and the other quantum numbers

$$A)\, j_1 j_2 \neq 0 \qquad 2(j_1 - j_2) \leq m - k \qquad \Delta \geq 2 + j_1 + j_2 + k + l + m \tag{2.63}$$
$$B)\, j_2 = 0 \qquad 2 + 2j_1 \leq m - k \qquad \Delta \geq 1 + j_1 + k + l + m \tag{2.64}$$
$$C)\, j_1 = j_2 = 0 \qquad k = m \qquad \Delta \geq \ell + 2k \tag{2.65}$$

When Δ saturates a bound of type (B) or (C) the UIR is short and BPS. When Δ saturates the bound of type (A) multiplet shortening is of the linear type and can be violated by quantum corrections.

Generic UIR's can be obtained by tensoring "singleton" representations. In the harmonic superspace approach, in addition to the usual $\mathcal{N} = 4$ superspace variables $(x^\mu, \theta^\alpha_A, \bar{\theta}^A_{\dot{\alpha}})$ one introduces 4×4 matrices, $u^A_{\hat{A}}$, parametrising the coset $SU(4)/U(1)^3$. Omitting the details of the construction, one defines "Grassmann analytic" superfields $W^{[1\ldots k]}$ with $1 \leq k \leq 3$ that satisfy twisted chirality constraints

$$\begin{aligned} D^A_\alpha W^{[1\ldots k]} &= 0\,, \qquad 1 \leq A \leq k\,, \\ \bar{D}^{\dot{\alpha}}_A W^{[1\ldots k]} &= 0\,, \qquad k+1 \leq A \leq 4\,, \end{aligned} \tag{2.66}$$

with $D^A_\alpha = u^A_{\hat{A}} D^{\hat{A}}_\alpha$ u-projected $\mathcal{N} = 4$ superderivatives. The constraints express that W's depend on half of the spinor coordinates, *i.e.* they are 1/2 BPS objects. The list of singletons includes $\mathcal{N} = 4$ chiral superfields, which - unlike the W's - may have either left or right handed spinor indices, but are $SU(4)$ singlets. For instance a scalar chiral superfield, Ψ, satisfies

$$\bar{D}^{\dot{\alpha}}_A \Psi = 0\,, \quad D^{\alpha(A} D^{B)}_\alpha \Psi = 0\,,$$

where the second (linear type) constraint is a sort of 'field equation'. The above construction is by and large formal in that none of the superfields

[5] In (perturbative) $\mathcal{N} = 4$ SYM theory only UIR's of $PSU(2,2|4)$ are actually relevant. They are characterized by the vanishing of the $U(1)_Z$ central charge that extends $PSU(2,2|4)$ to $SU(2,2|4)$.

[6] The dimension of an irrep of $SU(4)$ with Dynkin lables $[k,l,m]$ is given by $d[k,l,m] = (k+1)(l+1)(m+1)(k+l+2)(l+m+2)(k+l+m+3)/12$

$W^1, W^{[123]}$ and Ψ can be expressed in terms of elementary fields. On the contrary, $W^{[12]}$ is the fundamental $\mathcal{N}=4$ SYM multiplet that squares to give the supercurrent multiplet.

Other 1/2 BPS short multiplets of the form $(W^{[12]})^\ell$ are dual to the K-K excitations of the super-graviton. The lowest component of the multiplet being a scalar $Q^{(\ell)}$ of AdS mass $(ML)^2 = \ell(\ell-4)$ which is a mixture of the trace of the graviton and of the internal R-R 4-form potential and is the obvious candidate dual of the CPO $\mathcal{Q}^{(\ell)}$. At the top level one finds a symmetric tensor $H^{(\ell-2)}_{\mu\nu}$ and a complex scalar $\Phi^{(\ell-2)}$ both with AdS mass $(ML)^2 = (\ell-2)(\ell+2)$, a complex antisymmetric tensor $\tilde{B}^{[2,\ell-3,0]}_{\mu\nu}$ and a complex vector $\tilde{A}^{[1,\ell-4,1]}_\mu$ with AdS mass $(ML)^2 = (\ell-1)(\ell+3)$, and a real scalar $\tilde{Q}^{[2,\ell-4,2]}$ with AdS mass $(ML)^2 = (\ell-2)(\ell+2)$.

The prototype long multiplet is the Konishi supermultiplet. In free theory it can be represented as

$$\mathcal{K}_1|_{g^0} = \Psi\bar{\Psi} \tag{2.67}$$

Using (2.67), one may verify that [7]

$$D^{\alpha(A} D^{B)}_\alpha \mathcal{K}_1|_{g^0} = 0\,, \qquad \bar{D}_{\dot\alpha(A} \bar{D}^{\dot\alpha}_{B)} \mathcal{K}_1|_{g^0} = 0\,. \tag{2.69}$$

These imply that both the singlet and the **15** components of the current

$$K^A_{\mu\,B} \;=\; \bar\sigma^{\dot\alpha\alpha}_\mu [D^A_\alpha, \bar{D}_{\dot\alpha\,B}]\, \mathcal{K}_1|_{g^0,\,\theta,\bar\theta=0} \tag{2.70}$$

are conserved. In the interacting theory the Konishi multiplet has an anomalous dimension, so that one can formally write

$$\mathcal{K}_1 = (\Psi\bar{\Psi})^{(1+\gamma)}\,. \tag{2.71}$$

For $\gamma \neq 0$ it does not satisfy any differential constraint consistently with its being a long multiplet. The moral is that shortening conditions which are not related to BPS bounds can be violated at the quantum level.

[7] In the interacting theory the situation is more complicate. The rhs of (2.69) does not vanish but it rather becomes the $\mathcal{N}=4$ Konishi anomaly

$$D^{\alpha(A} D^{B)}_\alpha \mathcal{K}_1 \propto g\mathrm{tr}([W^{AC}, W^{BD}]\, \bar{W}_{CD})\,. \tag{2.68}$$

2.2 Three $U(1)$'s

There are three $U(1)$'s that play an interesting subtle role in the correspondence.

The first, $U(1)_Z$, is a central extension of $PSU(2,2|4)$. Fundamental fields as well as their composites are neutral with respect to it so that one usually neglects it. It is conceivable that solitonic states (dyons) could carry non-vanishing $U(1)_Z$ charge and form novel $SU(2,2|4)$ multiplets.

The second, $U(1)_C$, is the abelian factor in $U(N)$. From the D3-brane perspective it corresponds to the center of mass degrees of freedom. Its low-energy dynamics on the boundary is not reproduced by the bulk supergravity action. In a sense there is an additional singleton multiplet not captured by the correspondence if not for its contribution to "boundary anomalies".

The third, $U(1)_B$, is a "bonus" symmetry of a restricted class of correlation functions and their dual amplitudes. In SYM it corresponds to a chiral rotation accompanied by a continuous electric-magnetic duality transformation. Its type IIB counterpart is the $U(1)_B$ anomalous chiral symmetry. As originally observed by K. Intriligator, when supergravity loops and higher derivative string corrections are negligible the "bonus" symmetry becomes a true symmetry. Independently of the coupling λ and N, all 2-point correlation functions, 3-point functions with at most one insertion of unprotected operators and 4-point functions of single-trace protected operators seem to respect this symmetry.

2.3 Mass to dimension relation

Following S. Gubser, I. Klebanov, A. Polyakov and E. Witten, the standard prescription for computing correlation functions of gauge-invariant local composite operators using the correspondence starts with solving

$$\nabla^2\Phi - M^2\Phi = 0 \tag{2.72}$$

with generalized Dirichlet boundary conditions

$$\Phi(x,\rho) \to \rho^\nu \phi(x) \tag{2.73}$$

as $\rho \to 0$. In (Euclidean) Poincaré coordinates the scalar laplacian becomes:

$$\nabla^2_{AdS}\Phi = \frac{1}{\sqrt{G}}\partial_\mu\left(\sqrt{G}G^{\mu\nu}\partial_\nu\Phi\right) = \rho^5\partial_\rho\left(\rho^{-3}\partial_\rho\Phi\right) + \rho^2\partial\cdot\partial\Phi \tag{2.74}$$

The solution may be expressed in terms of the bulk-to-boundary propagator

$$K_\Delta(\rho, x; x_0) = c_\Delta \frac{\rho^\Delta}{(\rho^2 + (x - x_0)^2)^\Delta} \tag{2.75}$$

Quite strikingly $K_2(\rho, x; x_0)$ resembles the profile of YM instanton! Guess why.

EXERCISE

Show that K satisfies the Laplace equation and fix the constant $c_\Delta = \Gamma(\Delta)/\pi^2\Gamma(\Delta - 2)$ so that $K_\Delta(\rho, x; x_0) \to \rho^{4-\Delta}\delta(x - x_0)$ at the boundary.

Plugging K into the equation yields the mass-to-dimension relation

$$(ML)^2 = \Delta(\Delta - 4) \tag{2.76}$$

and its inverse

$$\Delta = 2 \pm \sqrt{4 + (mL)^2} \tag{2.77}$$

Two remarks are in order. First, Δ is real as expected in any unitary theory once the Breitenlohner - Freedman bound $(mL)^2 \geq -4$, required for the stability of AdS under pertubations, is enforced. Second, one could in principle use both branches of the square root. The first, say the plus branch, for which

$$\Phi(x, \rho) \approx \rho^{4-\Delta_+}\phi(x) \tag{2.78}$$

is non-normalizable in AdS, corresponds to operator deformations of the fixed point action as required in the computation of correlation functions. The second, say the minus branch, for which

$$\Phi(x, \rho) \approx \rho^{\Delta_+}\phi(x) \tag{2.79}$$

is a normalizable state, corresponds to studying the theory in a background with non-vanishing VEV for the field. The boundary conditions may be switched by a Legendre transform. Not without some effort one can establish the mass-to-dimension relations and their inverse for symmetric tensors

$$(ML)^2 = \Delta(\Delta - 4) \qquad \Delta = 2 \pm \sqrt{4 + (ML)^2} \tag{2.80}$$

p-forms

$$(ML)^2 = (\Delta + p)(\Delta + p - 4) \qquad \Delta = 2 \pm \sqrt{(2 - p)^2 + (ML)^2} \tag{2.81}$$

and fermions

$$(ML)^2 = (\Delta - 2)^2 \qquad \Delta = 2 \pm |ML| \tag{2.82}$$

Vectors and scalars correspond to p-forms with $p = 1, 0$ respectively.

2.4 Witten diagrams

In order to compute n-point functions using holography, one has to expand the type IIB action to the desired order and substitute $K(\rho, x; x')$ for each external insertion and the (rather unwieldly) expression for the bulk-to-bulk propagator $G(z, z')$ for each internal exchange. The resulting diagrams (trees and loops) are termed "Witten diagrams".

The computation of the simplest Witten diagrams amounts to integrating expressions of the form

$$\mathcal{I}_{\{\Delta_i\}}(x_1, \ldots x_n) = \int \frac{d^4x d\rho}{\rho^5} \prod_{i=1,\ldots n} \frac{\rho^{\Delta_i}}{[\rho^2 + (x - x_i)^2]^{\Delta_i}} \tag{2.83}$$

For $n = 3$ the result is of the form

$$\mathcal{I}_{\{\Delta_i\}}(x_1, x_2, x_3) = \frac{C_{\{\Delta_i\}}}{(x_{12})^{\Sigma_{12}}(x_{23})^{\Sigma_{23}}(x_{13})^{\Sigma_{13}}} \tag{2.84}$$

where $\Sigma_{12} = \Delta_1 + \Delta_2 - \Delta_3$ and cyclic. Notice the latter integral diverge for $\Delta_1 = \Delta_2 + \Delta_3$ and cyclic. We will return to these "extremal" cases in due course.

EXERCISE

Compute the above integrals either using standard Feynman parametrization and determine $C_{\{\Delta_i\}}$.

Diagrams with internal exchange are much more involved and we will not discuss them in these lectures.

2.5 Two-point functions

For 2-point functions the naive procedure gives the correct functional dependence on the insertion points, that however is completely fixed by conformal invariance. The correct normalization can be found in several ways, *e.g.* by imposing Ward identities or by working in momentum space. Eventually we will describe the procedure of holographic renormalization that allows one to consistently dispose of the divergences that appear due to the infinite volume of AdS. For the time being let us proceed rather naively.

Integrating by parts and imposing field equations the on-shell action for a real scalar field is given by

$$S[\Phi] = \frac{1}{2} \int d^5z \sqrt{g} \nabla_\mu (\Phi \nabla^\mu \Phi) = \lim_{\epsilon \to 0} \int \frac{d^4x}{2\epsilon^3} \Phi(x, \epsilon) \partial_\epsilon \Phi(x, \epsilon) \tag{2.85}$$

Plugging the solution of the boundary problem

$$\Phi(x,\rho) = \int d^4x' K(x,\rho;x')j(x') \tag{2.86}$$

into $S[\Phi]$ yields

$$S[j(x)] = \lim_{\epsilon\to 0}\int \frac{d^4xd^4x'}{2\epsilon^3}\epsilon^{4-\Delta}j(x)\partial_\epsilon K(x,\epsilon;x')j(x') \tag{2.87}$$

where the behaviour $K(x,\epsilon;x') \approx \epsilon^{4-\Delta}\delta(x-x')$ near-boundary has been used. In the same limit

$$\partial_\epsilon K(x,\epsilon;x') \approx \frac{c_\Delta \Delta \epsilon^{\Delta-1}}{(x-x')^{2\Delta}} \tag{2.88}$$

where $c_\Delta = \frac{\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}$. Differentiating twice w.r.t. the sources j's, yields

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{\Delta\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}(x-x')^{-2\Delta} \tag{2.89}$$

In momentum space with $\Phi(x,\rho) = f_p(\rho)\exp(ip\cdot x)$, the Klein-Gordon equation reduces to a Bessel equation

$$u^2F'' + uF' - (u^2 + (ML)^2 + 4)F = 0 \tag{2.90}$$

where $F(p\rho) = \rho^{-2}f_p(\rho)$ and primes denote derivatives w.r.t. $u = p\rho$. Using the mass-to-dimension relation $(ML)^2 = \Delta(\Delta-4)$, the solution which is regular in the interior (large u) is $\mathcal{K}_{\Delta-2}(u)$. For small u it behaves as

$$\mathcal{K}_{\Delta-2}(u) \approx \frac{1}{2}\Gamma(\Delta-2)\left(\frac{u}{2}\right)^{2-\Delta} \tag{2.91}$$

Plugging

$$\Phi(x,\rho) = \frac{\rho^2\mathcal{K}_{\Delta-2}(p\rho)}{\epsilon^2\mathcal{K}_{\Delta-2}(p\epsilon)}\exp(ip\cdot x) \tag{2.92}$$

into the action and differentiating twice w.r.t. to the (Fourier transformed) sources yields

$$\langle \mathcal{O}_\Delta(p)\mathcal{O}_\Delta(q)\rangle = \delta(p+q)(2\Delta-4)\frac{\Gamma(3-\Delta)}{\Gamma(\Delta-1)}\left(\frac{p}{2}\right)^{2\Delta-4}\log(p\epsilon) \tag{2.93}$$

where only the leading (in ϵ) non-analytic (in p) term has been displayed. Terms analytic in p produce local countertems in correlation functions. In order for the subtractions in different correlations functions to be consistent with one another one has to resort to a systematic procedure such as holographic renormalization that will be the subject of the third lecture. Fourier anti-transforming to position space produces the correction factor $(2\Delta - 4)/\Delta$.

In all above computations we have negelected a factor of $N^2/2\pi^2$ arising from reduction on S^5 and a constant rescaling of the metric. Once this is included one finds for CPO's and conserved currents the same normalization as in free field theory at large N. This leads to the conjecture that operators in 1/2 BPS short multiplet not only have protected scaling dimension (typically $\Delta \approx \ell$) but also protected normalization. The resulting non-renormalization theorem has been shown to hold in perturbation theory and even non-perturbatively relying on the $U(1)_B$ "bonus" symmetry.

2.6 Three-point functions

Three point functions of normalized operators are not completely fixed by conformal invariance. The dependence on the insertion points, at least for scalar operators is determined by their dimensions, but there are a priori undetermined coefficients that carry dynamical information and appear in OPE's.

In particular the CP-odd part of the 3-point functions of the $SU(4)$ R-symmetry currents encode the anomalous content of the theory. Using

$$\langle \bar{\lambda}_B(x) \lambda^A(y) \rangle = i \delta^A{}_B \frac{1 - \gamma_5}{4\pi^2} \frac{\not{x} - \not{y}}{(x-y)^4} \tag{2.94}$$

for the Wick contractions of the Weyl fermions in the fundamental $\mathbf{4}$ of the R-symmetry group yields

$$\begin{aligned} &\langle J^a_\mu(x) J^b_\nu(y) J^c_\rho(z) \rangle_{odd} = \\ &-i \frac{N^2-1}{36\pi^6} d^{abc} \frac{Tr_s(\gamma_5 \gamma_\mu (\not{x} - \not{y}) \gamma_\nu (\not{y} - \not{z}) \gamma_\rho (\not{z} - \not{x}))}{(x-y)^4 (y-z)^4 (z-x)^4} \end{aligned} \tag{2.95}$$

where $d^{abc} = 2Tr_c(T^a\{T^b, T^c\})$ is the three-index symmetric tensor of $SU(4)$.

The Adler-Bardeen theorem implies that the external anomaly

$$D^\mu J^a_\mu = i\frac{N^2-1}{384\pi^2} d^{abc}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^b_{\mu\nu}\mathcal{F}^c_{\rho\sigma} \tag{2.96}$$

is one-loop exact. The holographic computation confirms that this is the case. Indeed the supergravity counterpart of the trilinear CP-odd couplings that encode the "external" R-symmetry anomaly are the Chern-Simons couplings that emerge after gauging. The supergravity tree-level result is renormalized by a one-loop anomaly that can be thought of as living on the boundary and implies a finite shift $N^2 \to N^2-1$. The physical origin of the anomaly is the absence of $U(1)_C$ singleton multiplet, that live on the boundary. Including its contribution to the anomaly, one would get a coefficient N^2 as appropriate for $U(N)$.

Another class of particularly interesting 3-point functions is

$$\langle Q^{I_1}_{\ell_1} Q^{I_2}_{\ell_2} Q^{I_3}_{\ell_3}\rangle = \frac{C^{I_1I_2I_3}_{(\ell_1,\ell_2,\ell_3)}(g,\theta;N)}{|x_12|^{\ell_1+\ell_2-\ell_3}|x_23|^{\ell_2+\ell_3-\ell_1}|x_31|^{\ell_3+\ell_1-\ell_2}} \tag{2.97}$$

where I is a multindex that runs over the $d[0,\ell,0] = (\ell+1)(\ell+2)^2(\ell+3)/12$ components of $Q^I_\ell = S^I_{i_1\ldots i_\ell} tr_c(\varphi^{i_1}\ldots\varphi^{i_\ell})$ with $tr_f(S^I S^J) = \delta^{IJ}$. The Wick contractions of the scalar fields

$$\langle\varphi^i(x)\varphi^j(y)\rangle = \frac{\delta^{ij}}{4\pi^2(x-y)^2} \tag{2.98}$$

account for the dependence on the insertion points. The coefficients $C^{I_1I_2I_3}_{(\ell_1,\ell_2,\ell_3)}$ contain a group theory factor and a dynamical factor. The group theory factor is essentially a Clebsch Gordan coefficient. The dynamical factors can been computed at weak coupling (free theory). At large N the result is

$$C^{I_1I_2I_3}_{(\ell_1,\ell_2,\ell_3)}(g,\theta;N) = \frac{\sqrt{\ell_1\ell_2\ell_3}}{N} tr_f(S^I_1 S^I_2 S^I_3) \tag{2.99}$$

This can be compared with the prediction of the AdS/CFT correspondence in the supergravity approximation that should capture the strong coupling regime. In order to compute the supergravity counterpart of $C^{I_1I_2I_3}_{(\ell_1,\ell_2,\ell_3)}(g,\theta;N)$ one has to go beyond the linearized approximation briefly described above. Quadratic terms in the field equations or equivalently cubic couplings in the action are necessary. Since a precise mapping between gauged supergravity and type IIB fields is still missing one has to work at the level of the field equations. This is in general very complicated but for CPO's the computation turns out to be feasible. After performing a field redefinitions that brings the quadratic terms in the field equations into an integrable Lagrangian form one finds – quite remarkably! – the same result as in free-field theory at large N. The exact matching between weak and strong coupling results and the absence of instanton corrections suggest the validity of a non-renormalization theorem for any λ and N. This has been tested at one-loop and to two-loops. As for 2-point functions, barring contact terms, $U(1)_B$ "bonus" symmetry in combination with $\mathcal{N}$=2 harmonic superspace techniques gives a demonstration of the non-renormalization of 3-point functions of CPO's. The extremal case, $\ell_1 = \ell_2 + \ell_3$, is subtler. We will return to this issue after discussing non-perturbative effects.

2.7 Instanton vs D-instantons

As any non-abelian gauge theory, $\mathcal{N} = 4$ SYM admits instanton solutions. Contrary to familiar cases, *e.g.* QCD, $\mathcal{N} = 1$ or $\mathcal{N} = 2$ SYM, instantons do not violate chiral symmetry and their effects interfere with perturbation theory. The remarkable feature of the $SU(2)$ one-instanton solution

$$F^{a-}_{\mu\nu} = -\frac{4}{g_{_{YM}}} \frac{\eta^a_{\mu\nu}\rho_0^2}{\left(\rho_0^2 + (x - x_0)^2\right)^2} , \tag{2.100}$$

where $\eta^a_{\mu\nu}$ is the 't Hooft symbol, is that its moduli space, parameterized by the position x^μ_0 and size ρ_0, coincides with AdS_5! At the classical level the instanton solution breaks the (Euclidean) conformal group $SO(5,1)$ to $SO(5)$. The latter is generated by

$$\Sigma_a = T_a + \frac{1}{2}\eta^{\mu\nu}_a J_{\mu\nu} , \quad \bar{\Sigma}_a = \frac{1}{2}\bar{\eta}^{\mu\nu}_a J_{\mu\nu}, \quad \Pi_\mu = P_\mu + \rho^2 K_\mu . \tag{2.101}$$

The quotient $SO(5,1)/SO(5)$ is is exactly (Euclidean) AdS_5. The same is true for a single type IIB D-instanton on AdS_5. This strongly indicates a correspondence between these sources of non-perturbative effects whereby the fifth radial coordinate transverse to the boundary plays the role of the YM instanton size ρ.

The correspondence can be made more quantitative. Using the AdS/CFT dictionary (**??**), the charge-k type IIB D-instanton action $S^{(k)}_{IIB} = 2\pi k/g_s$ coincides with the action of a charge-k YM instanton $S^{(k)}_{SYM} = 8\pi^2 k/g^2_{YM}$. Moreover, the super-instanton measure contains an overall factor $g^8_{_{YM}}$ that arises from the combination of bosonic and fermionic zero-mode norms and exactly matches the power expected on the basis of the AdS/CFT correspondence.

The computation of the one-instanton contribution to the SYM correlation function $G_{16} = \langle \Lambda(x_1) \dots \Lambda(x_{16}) \rangle$, where $\Lambda^A = Tr(F^-_{\mu\nu}\sigma^{\mu\nu}\lambda^A)$, is the fermionic composite operator dual to the type IIB dilatino, and its comparison with the D-instanton contribution to the dual type IIB amplitude has represented the first truly dynamical test of the correspondence. Correlation functions of this kind are almost completely determined by the systematics of fermionic zero-modes in the instanton background. Performing (broken) superconformal transformations on the instanton field-strength

$$F^-_{\mu\nu} = K_2(\rho, x; x')\sigma_{\mu\nu} \tag{2.102}$$

yields the relevant gaugino zero-modes

$$\lambda^A_{(0)} = \frac{1}{2} F^-_{\mu\nu}\sigma^{\mu\nu}\zeta^A \tag{2.103}$$

where $\zeta^A = \eta^A + x \cdot \sigma\bar{\xi}^A$, with η^A and $\bar{\xi}^A$ constant Weyl spinors of opposite chirality.

EXERCISE *Using Bianchi identities and anti self-duality of $F^-_{\mu\nu}$, show that* $\not{D}\lambda^A_{(0)} = 0$.

The exact matching with the corresponding type IIB amplitude is quite impressive and somewhat surprising. Indeed the SYM computation initially performed for an $SU(2)$ gauge group at weak coupling, *i.e.* in a regime which is clearly far from the regime of validity of the supergravity approximation, has since then been extended to the $k = 1$ instanton sector for $SU(N)$ and to any k in the large N limit. The resulting 16-point functions have the same dependence on the insertion points. In the large N limit the overall coefficients of the dominant terms are those predicted by the analysis of type IIB D-instanton effects. For $N \neq 2$ all but the 16 superconformal zero-modes (2.103) are lifted by Yukawa interactions. Additional bosonic coordinates parameterizing S^5 appear in the large N limit as bilinears in the lifted fermionic zero-modes.

Other correlators that are related by supersymmetry to the Λ^{16} function and can thus saturate the 16 exact zero-modes have been computed. Correlators that cannot absorb the exact zero-modes receive vanishing contributions. This is the case for 2- and 3-point functions of CPO's as well as for extremal and next-to-extremal correlators to which we now turn our attention.

2.8 Extremal and Next-to-Extremal Correlators

The correlator of CPO's

$$G(x,\ldots,x_n) = \langle \mathcal{Q}^{(\ell)}(x)\mathcal{Q}^{(\ell_1)}(x_1)\ldots\mathcal{Q}^{(\ell_n)}(x_n)\rangle \ , \tag{2.104}$$

is said to be "extremal" when $\ell = \ell_1 + \ell_2 + \ldots + \ell_n$. It is easy to check that (2.104) contains only one $SU(4)$ tensor structure so that computing (2.104) is equivalent to computing

$$\langle Tr[(\phi)^{\ell}(x)]Tr[(\phi^{\dagger})^{\ell_1}(x_1)]\ldots Tr[(\phi^{\dagger})^{\ell_n}(x_n)]\rangle \ . \tag{2.105}$$

where ϕ is $\phi^I = \varphi^I + i\varphi^{I+3}$ with, say, $I=1$. The tree-level contribution corresponds to a "flower" diagram with ℓ lines exiting from the point x, which form n different "petals" connecting x to the points x_i, the i-th petal containing ℓ_i lines. The result is of the form

$$G(x,x_1,\ldots,x_n) = c(g,N)\prod_{i=1}^{n}(x-x_i)^{-2\ell_i} \tag{2.106}$$

The dual supergravity computation is very subtle in that the relevant AdS integrals are divergent but at the same time extremal trilinear couplings are formally vanishing. If one carefully analytically continue the computation away from extremality, one finds a non-vanishing result of the same form as at tree-level in SYM theory. One is thus lead to conjecture extremal correlators should satisfy a non-renormalization theorem of the same kind as 2- and 3-point functions of CPO's. This has been tested both at one-loop and non-perturbatively and later shown to be a consequence of $SU(2,2|4)$ invariance.

At one loop, there are two sources of potential corrections. The first corresponds to the insertion of a vector lines connecting the chiral lines of the same petal. Its vanishing is in some sense related to the vanishing of one-loop corrections to 2-point functions of CPO's. The second corresponds to insertion of vector lines connecting lines belonging to different petals. Its vanishing is in the same sense as above related to the vanishing of one-loop correction to 3-point functions of CPO's. The same analysis can be repeated step by step in the case of extremal correlators involving multi-trace operators in short multiplets.

As far as the instanton contributions are concerned, it is easy to check that (2.105) cannot absorb the relevant 16 zero-modes. The induced scalar zero-modes read

$$\varphi^i_{(0)} = \frac{1}{2}\tau^i_{AB}\zeta^A F^-_{\mu\nu}\sigma^{\mu\nu}\zeta^B \tag{2.107}$$

and the 4 exact zero-modes with flavour $I = 1$ could only be possibly absorbed at x. Since however $\zeta(x)^4 = 0$, the non perturbative corrections to (2.105) vanishes for any instanton number and for any gauge group in the leading semiclassical approximation.

EXERCISE

Using $[D_\mu, D_\nu] = F^-_{\mu\nu}$, *show that* $D^2\varphi^i_{(0)} = \tau^i_{AB}[\lambda^A_{(0)}, \lambda^A_{(0)}]$.

Other correlators, involving only one $SU(4)$ singlet projection, enjoy similar non-renormalization properties. 2- and 3-point functions of CPO's belong to this class. The identification of $U(1)_B$-violating nilpotent super-invariants beginning at five points prevents one from generically extending the same argument to higher-point functions. However the absence of the relevant nilpotent super-invariants for next-to-extremal correlators with $\ell = (\sum_i \ell_i) - 2$ allows one to include them in the list of protected observables. The absence of one-loop and instanton corrections in this case can be verified along the same lines as for the extremal ones. Supergravity computations confirm the weak coupling result and suggest that near-extremal correlators, with $\ell = (\sum_i \ell_i) - 4$, satisfy a sort of "partial" non renormalization. The *a priori* independent contributions to a given correlation function are functionally related to one another. Functional relations of this form easily emerge in instanton computations. Some additional effort allows one to derive them in perturbation theory.

2.9 Four-point functions and logs

The dynamics of a conformal field theory is elegantly encoded in the 4-point functions. The simplest ones have been computed both at weak coupling, up to order g^4 as well as in the semiclassical instanton approximation, and at strong coupling from the AdS perspective. At short distance they generically display logarithmic behaviours that are to be interpreted in terms of anomalous dimensions. At first sight this might seem surprising in a theory, such as $\mathcal{N}=4$ SYM, that is known to be finite. Indeed, operators which belong to short BPS multiplets have protected scaling dimensions and cannot contribute to the logarithmic behaviours. Completeness of the operator product expansion (OPE) requires however the inclusion of "unprotected" operators in addition to the "protected" ones. Single-trace operators in Konishi-like multiplets contribute to the logarithms at weak coupling but are expected to decouple at strong coupling. On the contrary unprotected multi-trace operators that are holographically dual to multi-particle states appear both at weak and at strong coupling since their anomalous dimensions are at most of order $1/N^2$.

To clarify the point in a simpler setting, consider the 2-point function of a primary operator of scale dimension $\Delta = \Delta^{(0)} + \gamma$. In perturbation theory $\gamma = \gamma(g_{YM})$ is expected to be small and to admit an expansion in the coupling constant g_{YM}. Expanding in γ yields

$$\langle \mathcal{O}^\dagger_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{a_\Delta}{(x-y)^{2\Delta}} = \frac{a^{(0)}_\Delta}{(x-y)^{2\Delta^{(0)}}} \times \left(1-\gamma\log[\mu^2(x-y)^2] + \frac{\gamma^2}{2}(\log[\mu^2(x-y)^2])^2 + \dots\right) . \tag{2.108}$$

Although the exact expression (**??**) given above is conformally invariant, at each order in γ (or in g_{YM}) (2.109) contains logarithms that are an artifact of the perturbative expansion.

Similar considerations apply to arbitrary correlation functions. Assuming the convergence of the OPE, a 4-point function of primary operators can be schematically expanded as

$$\langle \mathcal{Q}_A(x)\mathcal{Q}_B(y)\mathcal{Q}_C(z)\mathcal{Q}_D(w)\rangle = \sum_K \frac{C_{AB}{}^K(x-y,\partial_y)}{(x-y)^{\Delta_A+\Delta_B-\Delta_K}} \frac{C_{CD}{}^K(z-w,\partial_w)}{(z-w)^{\Delta_C+\Delta_D-\Delta_K}} \langle \mathcal{O}_K(y)\mathcal{O}_K(w)\rangle , \tag{2.109}$$

where K runs over a (possibly infinite) complete set of primary operators. Descendants are implicitly taken into account by the presence of derivatives in the Wilson coefficients, C's. To simplify formulae we assume that $\mathcal{Q}$'s are protected operators, *i.e.* they have vanishing anomalous dimensions. In general the operators $\mathcal{O}_K$ may have anomalous dimensions, γ_K, so that $\Delta_K = \Delta_K^{(0)} + \gamma_K$. Similarly ${C_{IJ}}^K = {C^{(0)}_{IJ}}^K + {\eta_{IJ}}^K$. Indeed, although 3-point functions of single-trace CPO's are not renormalized beyond tree level, *a priori* nothing can be said concerning corrections to 3-point functions involving unprotected operators.

Neglecting descendants and keeping the lowest order terms in γ and η

$$\langle \mathcal{Q}_A(x)\mathcal{Q}_B(y)\mathcal{Q}_C(z)\mathcal{Q}_D(w)\rangle_{(1)} = \sum_K \frac{\langle \mathcal{O}_K(y)\mathcal{O}_K(w)\rangle_{(0)}}{(x-y)^{\Delta_A+\Delta_B-\Delta_K^{(0)}}(z-w)^{\Delta_C+\Delta_D-\Delta_K^{(0)}}} \times$$
$$\left[{\eta_{AB}}^K {C^{(0)}_{CD}}^K + {C^{(0)}_{AB}}^K {\eta_{CD}}^K + \frac{\gamma_K}{2} {C^{(0)}_{AB}}^K {C^{(0)}_{CD}}^K \log\frac{(x-y)^2(z-w)^2}{(y-w)^4}\right] .$$

whence one can extract both corrections to OPE coefficients and anomalous dimensions.

In order to exemplify the above considerations, let us consider the 4-point function of the lowest CPO's in the $\mathcal{N}$=4 current multiplet

$$\mathcal{Q}^{ij}_{\mathbf{20'}} = \mathrm{tr}(\varphi^i\varphi^j - \frac{\delta^{ij}}{6}\varphi_k\varphi^k) . \tag{2.110}$$

Due to the lack of a manifestly $\mathcal{N} = 4$ off-shell superfield formalism, perturbative computations have to be either performed in components or in one of the two available off-shell superfield formalisms. Although the number of diagrams is typically larger in the $\mathcal{N} = 1$ superfield approach its simplicity makes it more accessible than the less familiar $\mathcal{N} = 2$ harmonic superpace.

For illustrative purposes let us consider the one-loop contribution to 4-point functions of lowest CPO's $G_4 = \langle \mathcal{QQQQ}\rangle$. The $\mathcal{N} = 1$ chiral superfield propagator reads

$$\langle \Phi^I(x_1,\theta_1)\Phi^\dagger_J(x_2,\theta_2)\rangle = e^{(\xi_{11}+\xi_{11}-2\xi_{12})\cdot\partial}\frac{{\delta^I}_J}{4\pi^2 x^2_{12}} \tag{2.111}$$

where $\xi^\mu_{ij} = i\theta_i\bar\sigma^\mu\bar\theta_j$. Out of the six $SU(4)$ singlet projections of G_4, consider one that has no connected tree level contribution. The only relevant vertex at one-loop comes from the superpotential $W = \sqrt{2}gtr(\Phi^1[\Phi^2,\Phi^3])$ and the result reads

$$G_H(x_1, x_1, x_3, x_4) = \langle (\phi^1)^2(x_1)(\phi_1^\dagger)^2(x_2)(\phi^2)^2(x_3)(\phi_2^\dagger)^2(x_4)\rangle = \tag{2.112}$$
$$\langle (\Phi^1)^2(z_1)(\Phi_1^\dagger)^2(z_2)(\Phi^2)^2(z_3)(\Phi_2^\dagger)^2(z_4)\rangle|_{\theta_i=0} = -\frac{2g^2_{YM}N(N^2-1)\pi^2}{(2\pi)^{12}x^2_{12}x^2_{34}x^2_{13}x^2_{24}}\ B(r,s)$$

where $B(r,s)$ is a box-type integral, that only depends on the two independent conformally invariant cross ratios r and s and their combination

$$p = 1 + r^2 + s^2 - 2r - 2s - 2rs\ .$$

and can be expressed as

$$B(r,s) = \frac{1}{\sqrt{p}}\left\{\ln(r)\ln(s) - \left[\ln\left(\frac{r+s-1-\sqrt{p}}{2}\right)\right]^2 + \right.$$
$$\left. -2\mathrm{Li}_2\left(\frac{2}{1+r-s+\sqrt{p}}\right) - 2\mathrm{Li}_2\left(\frac{2}{1-r+s+\sqrt{p}}\right)\right\}\ .$$

The other *a priori* independent 4-point function

$$G_V(x_1, x_1, x_3, x_4) = \langle (\phi^1)^2(x_1)(\phi_1^\dagger)^2(x_2)(\phi^1)^2(x_3)(\phi_1^\dagger)^2(x_4)\rangle \tag{2.113}$$

The non-perturbative contributions are quite involved and we refrain to display them. We simply notice that the relation

$$x^2_{13}x^2_{24}G_H(x_1, x_1, x_3, x_4) = x^2_{14}x^2_{23}G_V(x_1, x_1, x_3, x_4) \tag{2.114}$$

which embodies the content of "partial non-renormalization", can be easily derived from the systematics of the fermionic zero-modes. Some additional effort allows one to derive it in perturbation theory The AdS computation is even more involved and the final result is quite uninspiring.

In order to extract some physics one has to perform an OPE analysis. Restricting for brevity our attention to the sectors **1**, **20′**, **84**, and **105** the results can be summarized as follows[8].

In the **105** one finds only subdominant logarithms, consistently with the expected absence of any corrections to the dimension of 1/2 BPS single- and double-trace operators of dimension $\Delta = 4$ in the **105**.

[8]Recall that in addition to these irreps, $\mathbf{20'}\times\mathbf{20'}$ contains $\mathbf{15}+\mathbf{175}$ in the antisymmetric part.

In the **84** channel, the dominant contribution at one and two loops is purely logarithmic and consistent with the exchange of the operator $\mathcal{K}_{\mathbf{84}}$ in the Konishi multiplet. The absence of dominant logarithmic terms in the instanton as well as AdS results suggests confirms the absence of any corrections to the dimension and trilinear of a 1/4 BPS operator $\hat{\mathcal{D}}_{\mathbf{84}}$ of dimension 4, defined by subtracting the Konishi scalar $\mathcal{K}_{\mathbf{84}}$ from the projection on the **84** of the naive normal ordered product of two $\mathcal{Q}_{\mathbf{20'}}$.

In the $\mathbf{20'}$ sector, there is no dominant logarithm suggesting a vanishing anomalous dimension for the unprotected operator $:\mathcal{Q}_{\mathbf{20'}}\mathcal{Q}_{\mathbf{20'}}:_{\mathbf{20'}}$. This striking result seems to be a consequence of the partial non-renormalization of 4-point functions of CPO's that is valid not only at each order in perturbation theory (beyond tree level!) but also non-perturbatively and at strong coupling (AdS). In order to disentangle the various scalar operators of naive dimension 4 exchanged in this channel it is necessary to compute other independent 4-point functions involving the insertions of the lowest Konishi operator $\mathcal{K}_{\mathbf{1}}$.

The analysis of the singlet channel is very complicated by the presence of a large number of operators. In perturbation theory one has logarithmically-dressed double pole associated to the exchange of $\mathcal{K}_{\mathbf{1}}$ with

$$\gamma_{\mathcal{K}}^{(1)} = 3\frac{g_{YM}^2 N}{4\pi^2} \qquad \gamma_{\mathcal{K}}^{(2)} = -3\frac{g_{YM}^2 N}{16\pi^2} \tag{2.115}$$

Non-perturbative and strong coupling results only show a logarithmic singularity that is associated to the exchange of some double-trace unprotected operator $\mathcal{O}_{\mathbf{1}}$ with $\gamma \approx 1/N^2$.

The picture that emerges is very interesting. In addition to protected single an multi-trace operators satisfying shortening conditions of BPS type as well as single- and multi-trace operators in long multiplets there seems to be a new class of operators that satisfy a linear type (non BPS!) shortening condition and have vanishing anomalous dimensions.

Konishi-like operators decouple both from non-perturbative (instanton) correlators as well as from the strong coupling AdS results but they represent the only available window on genuine string dynamics. The OPE algebra at strong coupling requires the inclusion of multi-trace operators of three kinds. Those dual to multi-particle BPS states, those dual to non BPS-states with gravitational corrections to their binding energy and those dual to non BPS states without mass corrections. A deeper understanding of the last two classes of operators would help clarifying profound issues in the AdS/CFT correspondence such as the string exclusion principle that is expected to play a role at finite N.

2.10 Other conformal theories

Much of what have been said so far can be easily generalized to other (super)conformal SYM theories that emerge by considering the dynamics of D3-brane near orbifold and conifold singularities or in F-theory backgrounds (orientifolds). In general the near horizon geometry looks like some warped product of AdS_5 and a 5-dimensional Einstein manifold E^5. For instence the choice $E^5 = T^{11} \equiv SU(2) \times SU(2)/U(1)$, which is the base of the conifold, gives rise to an $\mathcal{N} = 1$ superconformal theory with global $SU(2) \times SU(2) \times U(1)$ flavour / baryon symmetry. Deforming or resolving the conifold determines a logarithmic evolution of the theory on the brane.

Not all (super)conformal theories admit holographic duals. By studying external Weyl anomalies M. Henningson and K. Skenderis were able to show that only when the coefficients c and a that appear in the trace anomaly satisfy

$$c = a = \frac{N^2 - 1}{4} \frac{V(S^5)}{V(E^5)} = \frac{1}{G_N^{(5)} \Lambda^{3/2}} \qquad (2.116)$$

the theory under consideration has a chance of admitting a holographic dual at least in the large N limit. We will discuss Ward identities and holographic anomalies in the next lecture. For the time being In general the coefficients c, a are given by

$$c = \frac{1}{120}(N_s + 6N_f + 12N_v) \quad , \quad a = \frac{1}{360}(N_s + 11N_f + 62N_v) \qquad (2.117)$$

where N_s, N_f, N_v, are the number of scalars, (Weyl) fermions, and vectors, respectively.

Orbifold theories are associated to quiver diagrams that codify the relevant gauge groups and representation of matter fields. Restricting our attention to the case of C^3/Z_k singularities that preserve at least $\mathcal{N} = 1$ supersymmetry *i.e.* such that $Z_k \in SU(3)$ one has to embed the group action into the Chan-Paton group. For unitary groups one gets the breaking of $U(N)$ into $\prod U(N_i)$ with matter fields in the $(\mathbf{N}_i, \mathbf{N}_{i+1})$ representation. The resulting theory is superconformal only when $N_i = N_{i+1}$ for any $i = 1, \ldots, k$.

EXERCISE

Check that indeed for $\mathcal{N} = 4$ SYM and the other SCFT's from D3-branes at orbifolds the relation $c = a$ is satisfied.

When the condition $Z_k \in SU(3)$ is relaxed one gets theories that are only conformal in the large N limit. The presence of Yukawa and quartic scalar coupling that are only related to the gauge coupling in the UV regime makes the evolution of the theory with the scale quite involved. Some insight on RG flows can thus be gained by means of open-closed string duality. A restricted class of perturbative open-string backgrounds (often related to F-theory at constant and weak coupling) show an interesting relation between NS-NS tadpoles and running of the gauge coupling on the brane much in the same way as chiral anomalies are associated to R-R tadpoles. Alternatively, by deforming $\mathcal{N} = 4$ SYM by relevant operators it is possible to flow from the superconformal point to phenomenologically more interesting gauge theories possibly with a dynamically generated mass gap. As we will see, however the typical energy scale of the resulting (super)glueballs is $M \sim 1/L$ and coincides with the scale of the K-K excitations. This may be cured by uplifting the solution to $d = 10$ or equivalently studying branes wrapping (supersymmetric) cycles in (non-compact) Calabi-Yau (CY) manifolds. We will have no time at all to discuss these very interesting issues here.

Before discussing holographic RG flows and their supergravity duals, *i.e.* $5 - D$ domain wall solutions, let us comment to what happens at finite temperature. By conformal invariance the entropy of $\mathcal{N} = 4$ SYM should scale as $S \approx N^2 V T^3$ although in qualitative agreement with the entropy of the $D3 - brane$ the exact coefficients do not match. In the near horizon limit the relevant supergravity solution is the AdS-Schwarschild black hole. Computing the free energy $F = E - TS$ in (free) field theory, with $T = T_H$ the Hawking temperature of the dual supergravity solution, and comparing it with the supergravity prediction $\mathcal{I} = \beta F$, with β the inverse temperature and $\mathcal{I}$ the (regulated) action of the (Euclidean) solution, one finds a (in)famous discrepancy of a factor of 3/4. Indeed $F = -\pi^2 f(\lambda) N^2 V T^4/6$ where $f(\lambda) = 1$ for $\lambda = 0$ and $f(\lambda) = 3/4$ for $\lambda = \infty$. First order corrections in thermal field theories on the boundary as well higher derivative corrections to the action of the classical supergravity solution, which is not conformally flat!, tend to reduce the discrepancy pointing towards a reduction of degrees of freedom from weak to strong coupling.

EXERCISE

Count the number of bosonic and fermionic species and compute internal energy, entropy and free energy for a gas of non-interacting $\mathcal{N} = 4$ *particles in a volume* V *at temperature* T *using Bose-Einstein and Fermi-Dirac distributions with* $\mu = 0$.

3 Holographic renormalization

Once the correspondence has been "established" in the superconformal case, one may envisage the possibility of introducing relevant mass-deformations $S \to S_* + g_i \int d^4x \mathcal{O}^i$ that trigger renormalization group (RG) flows in the boundary theory. Their holographic counterparts are domain wall solutions of $5-d$ gauged supergravity that interpolate from one critical point (say the maximally symmetric $SO(6)$ point) of the scalar potential to another critical point or to the "hades". In some cases the IR endpoint of the RG flow is a confining theory. It would be interesting to get predictions for the bound states that appear in the IR as a result of the RG flow. This amounts to extracting the spectrum of mass poles in 2- or higher-point functions. In order to do this one has to handle the divergences that plague the calculations. A systematic procedure has been developped that goes under the name of holographic renormalization and consists in the following steps. The first step is finding a consistent truncation that allows for an asymptotically AdS domain wall solution, typically preserving some fraction of the original supersymmetries but breaking conformal invariance. Then one performs the near boundary analysis of the coupled field equations in a convenient coordinate system in order to identify sources and induced VEV's. Plugging the symtotic solutions into the action allows one to isolate a finite number of counterterms that one has to subtract in order to cancel the large volume divergences dual to UV divergences in the boundary theory. The resulting renormalized action has to be expressed in terms of field living at the regulated boundary. In principle, differentiating with respect to the sources one finds the desired correlation functions that satisfy the correct Ward identities including anomalies. In practice this would require the knowledge of the non-local relation between sources and induced VEV's that is not determined by the near-boundary analysis. For 2-point functions, one can achieve the goal by linearizing the field equations around the domain wall background. After describing the general procedure in detail, we will illustrate it for the simple but interesting holographic RG flow found by Girardello, Petrini, Porrati and Zaffaroni (GPPZ).

3.1 Domain walls and RG flows

The method generally used to obtain asymptotically AdS domain wall solutions of 5-dimensional supergravity relies on symmetry arguments to consistently truncate the full bulk theory, usually $d = 5, \mathcal{N} = 8$ gauged supergravity, to a small number of scalar fields interacting with gravity[9]

$$S_0 = \frac{N^2}{2\pi^2} \int_M \mathrm{d}^5 x \sqrt{G} \left[\frac{1}{4} R + \frac{1}{2} G^{\mu\nu} H_{IJ} \partial_\mu \Phi^I \partial_\nu \Phi^J + V(\Phi) \right] - \frac{1}{2} \int_{\partial M} \sqrt{\gamma} \mathcal{K} \tag{3.118}$$

where $\mathcal{K}$ is the trace of the second fundamental form. We work in Euclidean signature and assume that the potential $V(\Phi)$ has a stationary point at $\Phi = 0$. By a constant Weyl transformation one may set $V(\Phi{=}0) = -3/L^2$, so that the action admits a pure AdS solution. We then look for solutions of the coupled field equations with $d = 4$ Poincaré symmetry. The most general ansatz is of the form

$$ds^2 = e^{2A(r)} dx \cdot dx + dr^2 \qquad , \qquad \Phi^I = \Phi^I(r) \tag{3.119}$$

The asymptotic boundary AdS region is at $r \to \infty$, and the scale factor is exponential in this region, *i.e.* $\exp(2A(r)) \to \exp(2r/L)$. We set $L = 1$, *i.e.* $g_5 = 2$ henceforth. The asymptotic behavior of the scalar field distinguishes solutions which are dual to flows governed by operator deformations from those triggered by VEV's.

Even within the ansatz (3.119) it is generally difficult to solve the second order field equations of (3.118). However, if the potential $V(\Phi)$ is derivable from a superpotential $W(\Phi)$,

$$V = \frac{1}{2} H^{IJ} \frac{\partial W}{\partial \Phi^I} \frac{\partial W}{\partial \Phi^J} - \frac{4}{3} W^2, \tag{3.120}$$

then any solution of the first order equations

$$\frac{dA}{dr} = -\frac{2}{3} W \qquad , \qquad \frac{d\Phi^I}{dr} = H^{IJ} \frac{\partial W}{\partial \Phi^J} \tag{3.121}$$

provides a domain wall solution for the action (3.118). In a supersymmetric context the superpotential is associated with Killing spinors via $W_{ab}\epsilon^b = W\epsilon_a$, and has a critical point at $\Phi^I = 0$.

[9]Our curvature conventions are as follows $R_{\mu\nu\kappa}{}^\lambda = \partial_\mu \Gamma_{\nu\kappa}{}^\lambda + \Gamma_{\mu\rho}{}^\lambda \Gamma_{\nu\kappa}{}^\rho - \mu \leftrightarrow \nu$ and $R_{\mu\nu} = R_{\mu\lambda\nu}{}^\lambda$.

In order to interpret the radial evolution in terms of RG flows it is convenient to identify the warping factor A with the renormalization scale and define holographic β and c functions according to

$$\beta_H^I = \frac{d\Phi^I}{dA} = -\frac{3H^{IJ}}{2W}\frac{\partial W}{\partial \Phi^J} \tag{3.122}$$

and

$$c_H = \frac{\pi}{8G_N^{(5)}(A')^3} = -\frac{27\pi}{64G_N^{(5)}W^3} \tag{3.123}$$

the latter is normalized so that $c_H = N^2/4$ at the UV boundary, thus measuring the number of degrees of freedom. Using the first order form of the radial evolution it is easy to check that c_H is not increasing along the flow.

EXERCISE

Check that indeed

$$\frac{dc_H}{dA} = -2c_H H_{IJ}\beta_H^I\beta_H^J \tag{3.124}$$

This is known as the holographic c-theorem. Its extension to theories that do not admit a holographic description is still a matter of debate and intense investigation.

3.2 Field equations

In order to evaluate the on-shell action, including its divergences, as a functional of the sources one has to solve the coupled field equations. For the sector that includes the stress tensor, certain R-symmetry currents and scalar operators, the bulk theory is that of gravity coupled to gauge and scalar fields. For the (final) purpose of computing 2-point functions, it is sufficient to keep only bilinear terms in the vectors since they have vanishing background. In this approximation the action for the $SO(6)$ gauge fields reduces to a sum of uncoupled abelian sectors and it is convenient to use the Stückelberg formalism involving gauge invariant combinations $B_\mu = A_\mu + \partial_\mu\alpha$, with α bulk Goldstone fields.

The bulk Lagrangian that describes this system of fields is given by

$$S = S_0 + \int_M \mathrm{d}^5x\sqrt{G}\left[\frac{1}{4}K(\Phi)F_{\kappa\lambda}F_{\mu\nu}G^{\kappa\mu}G^{\lambda\nu} + \frac{1}{2}M^2(\Phi)G^{\mu\nu}B_\mu B_\nu\right] \tag{3.1}$$

where S_0 is given in (3.118) and $K(\Phi)$ and $M^2(\Phi)$ are positive semi-definite functions of the active scalar Φ.

The field equations that follow from this action are

$$\nabla_\mu \left(K(\Phi)F^{\mu\nu}\right) = M^2(\Phi)B^\nu \ , \tag{3.2}$$

$$\Box_G \Phi = \frac{\partial V}{\partial \Phi} + \frac{1}{4}\frac{\partial K}{\partial \Phi} G^{\kappa\mu}G^{\lambda\nu}F_{\kappa\lambda}F_{\mu\nu} + \frac{1}{2}\frac{\partial M^2}{\partial \Phi} G^{\mu\nu}B_\mu B_\nu \ , \tag{3.3}$$

$$R_{\mu\nu} = -2\left[T_{\mu\nu} - \frac{1}{3}G_{\mu\nu}\mathrm{Tr}\, T\right], \tag{3.4}$$

where $T_{\mu\nu}$ is the matter stress energy tensor and $\mathrm{Tr}\, T = G^{\mu\nu}T_{\mu\nu}$.

In holographic renormalization it is most convenient to work in the coordinate system where the bulk metric takes the form

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho)dx^i dx^j \tag{3.5}$$

Any asymptotically AdS metric can be brought to this form near the boundary. The radial variables ρ and r are related by $\rho = \exp(-2r/L)$. The boundary is located at $\rho = 0$ and the regularized action will be defined by restricting to $\rho \geq \epsilon$.

EXERCISE

Write the coupled field equations in the coordinate system (3.5) and check that they look as follows.

Scalar fields

$$\begin{gathered} 4\rho^2\Phi'' - 4\rho\Phi'(1 - \frac{1}{2}\rho(\log g)') + \rho\Box_g\Phi - \frac{\partial V}{\partial \Phi} = \\ \frac{\rho^2}{4}\frac{\partial K}{\partial \Phi}[g^{mn}g^{kl}F_{mk}F_{nl} + 8\rho g^{kl}F_{k\rho}F_{l\rho}] + \frac{\rho}{2}\frac{\partial M^2}{\partial \Phi}[g^{ij}B_iB_j + 4\rho B_\rho B_\rho] \end{gathered} \tag{3.6}$$

where prime indicates derivative with respect to ρ, $g = \det g_{ij}(x,\rho)$, and $\Box_g$ is the scalar Laplacian in the metric $g_{ij}(x,\rho)$.

Vector fields

$$\partial_i(\sqrt{g}Kg^{ij}F_{j\rho}) = M^2\frac{\sqrt{g}}{\rho}B_\rho \tag{3.7}$$

$$\partial_i(\sqrt{g}Kg^{ij}g^{kl}F_{jl}) + 4\rho\partial_\rho(\sqrt{g}Kg^{kl}F_{\rho l}) = M^2\frac{\sqrt{g}}{\rho}g^{ki}B_i \tag{3.8}$$

Gravity

$$\rho\,[2g''_{ij} - 2(g'g^{-1}g')_{ij} + \mathrm{Tr}\,(g^{-1}g')\,g'_{ij}\,] + R_{ij}[g] + 2g'_{ij} - \mathrm{Tr}\,(g^{-1}g')\,g_{ij} =$$
$$-2\left(\partial_i\Phi\partial_j\Phi + \frac{2}{3\rho}[V(\Phi) - V(0)]g_{ij} + M^2B_iB_j + \rho K[g^{kl}F_{ik}F_{jl} + 4\rho F_{i\rho}F_{j\rho}]\right.$$
$$\left.-\frac{1}{6}\rho K g_{ij}[g^{mn}g^{kl}F_{mk}F_{nl} + 8\rho g^{kl}F_{k\rho}F_{l\rho}]\right) \tag{3.9}$$
$$\nabla_i\mathrm{Tr}\,(g^{-1}g') - \nabla^j g'_{ij} = -2\left(\partial_i\Phi\partial_\rho\Phi + M^2B_iB_\rho + \rho K g^{kl}F_{ik}F_{\rho l}\right) \tag{3.10}$$
$$\mathrm{Tr}\,(g^{-1}g'') - \frac{1}{2}\mathrm{Tr}\,(g^{-1}g'g^{-1}g') = -2\left(\partial_\rho\Phi\partial_\rho\Phi + \frac{1}{6\rho^2}[V(\Phi) - V(0)] + M^2B_\rho B_\rho\right.$$
$$\left.+\frac{2}{3}\rho K g^{kl}F_{k\rho}F_{l\rho} - \frac{1}{24}K g^{mn}g^{kl}F_{mk}F_{nl}\right) \tag{3.11}$$

.

3.3 Near boundary analysis

Near the boundary each field $\mathcal{F}(x,\rho)$ admits an asymptotic expansion of the form

$$\mathcal{F}(x,\rho) = \rho^m\left(f_{(0)}(x) + f_{(2)}(x)\rho + ... + \rho^n(f_{(2n)}(x) + \log\rho\ \tilde{f}_{(2n)}(x) + ...) + ...\right) \tag{3.12}$$

For instance, the expansion of the metric reads[10]

$$g_{ij}(x,\rho) = g_{(0)ij} + g_{(2)ij}\rho + \rho^2[g_{(4)ij} + h_{1(4)ij}\log\rho + h_{2(4)ij}(\log\rho)^2] + ... \tag{3.13}$$

Field equations are second order differential equations in ρ, so there are two independent solutions. Their asymptotic behaviors are ρ^m and ρ^{m+n}, respectively. For bosonic fields in domain wall solutions dual to flows driven by protected operators, n and $2m$ are non-negative integers. The boundary field $f_{(0)}$ that multiplies the leading behavior, ρ^m, is interpreted as the source for the dual operator. In the near-boundary analysis one solves the field equations iteratively by treating the ρ-variable as a small parameter. This

[10] In the Fefferman-Graham framework the most general expansion may contain half-integrals powers of ρ, or integral powers of the coordinate $U, \rho = U^2$. In the case of pure gravity in $d = 4$ and for the GPPZ flow, the coefficients with odd powers of U can be shown to vanish.

yields *algebraic* equations for $f_{(2k)}$, $k < n$, that uniquely determine $f_{(2k)}$ in terms of $f_{(0)}(x)$ and derivatives up to order $2k$. These equations leave $f_{(2n)}(x)$ undetermined. This was to be expected: the coefficient $f_{(2n)}(x)$ represents the boundary condition for a solution which is linearly independent from the one that starts as ρ^m. As we will shortly see, $f_{(2n)}$ is related to the VEV of the corresponding operator. The logarithmic term in (3.12), necessary in order to solve the equations, is also fixed in terms of $f_{(0)}(x)$ and is related to conformal anomalies of the dual theory. The latter emerge from bulk diffeomorphisms that preserve the form of the coordinate system (3.5), but induce a conformal transformation at the boundary. Correlation functions will eventually be expressed in terms of certain coefficients in (3.12). It follows that the (local) RG equations are coded in the transformations of the coefficients under bulk diffeomorphisms of this kind.

3.4 Counterterms

Asymptotic solutions can be inserted in the regulated action and a finite number of terms which diverge as $\epsilon \to 0$ can be isolated. The on-shell action takes the form

$$S_{\rm reg}[f_{(0)};\epsilon] = \int_{\rho=\epsilon} \mathrm{d}^4x\sqrt{g_{(0)}}[\epsilon^{-\nu}a_{(0)} + \epsilon^{-(\nu+1)}a_{(2)} + ... - \log\epsilon\; a_{(2\nu)} + \mathcal{O}(\epsilon^0)] \tag{3.14}$$

where ν is a positive number that only depends on the scale dimension of the dual operator and $a_{(2k)}$ are local functions of the source(s) $f_{(0)}$. The counterterm action is defined as

$$S_{\rm ct}[\mathcal{F}(x,\epsilon);\epsilon] = -\text{divergent terms of } S_{\rm reg}[f_{(0)};\epsilon] \tag{3.15}$$

where divergent terms are expressed in terms of the fields $\mathcal{F}(x,\epsilon)$ 'living' at the regulated surface $\rho = \epsilon$ and the induced metric there, $\gamma_{ij} = g_{ij}(x,\epsilon)/\epsilon$. This is required for covariance and entails an "inversion" of the expansions (3.13),(3.12) up to the required order.

To obtain the renormalized action we first define a subtracted action at the cutoff

$$S_{\rm sub}[\mathcal{F}(x,\epsilon);\epsilon] = S_{\rm reg}[f_{(0)};\epsilon] + S_{\rm ct}[\mathcal{F}(x,\epsilon);\epsilon]. \tag{3.16}$$

The subtracted action has a finite limit as $\epsilon \to 0$, and the renormalized action is a functional of the sources defined by this limit, *i.e.*

$$S_{\text{ren}}[f_{(0)}] = \lim_{\epsilon \to 0} S_{\text{sub}}[\mathcal{F}; \epsilon] \tag{3.17}$$

The distinction between S_{sub} and S_{ren} is needed because the variations required to obtain correlation functions are performed before the limit $\epsilon \to 0$ is taken.

The procedure above amounts to a "minimal" scheme in which the divergences of S_{reg} are subtracted. As in standard quantum field theory, one still has the freedom to add finite invariant counterterms. These correspond to a change of scheme. For example, such finite counterterms may be needed in order to restore some symmetry (*e.g.* supersymmetry).

Given a bulk action there is a universal set of counterterms that makes the on-shell action finite for *any* solution of the bulk field equations with given Dirichlet boundary data. The counterterms are different for different bulk actions, *i.e.* for different truncations that lead to different potentials $V(\Phi)$.

3.5 Correlation functions

Having obtained the renormalized on-shell action one can compute correlation functions by functionally differentiating S_{ren} with respect to the sources. The variation of (3.1) reads

$$\begin{aligned} \delta S_{\text{ren}}[g_{(0)ij}, \phi_{(0)}, A_{(0)i}, a_{(0)}] &= \int d^4x \sqrt{g_{(0)}} [\frac{1}{2} \langle T_{ij} \rangle \delta g^{ij}_{(0)} + \langle O_\Phi \rangle \delta \phi_{(0)} \\ &\quad + \langle J^i \rangle \delta A_{(0)i} + \langle O_\alpha \rangle \delta a_{(0)}] \end{aligned} \tag{3.18}$$

where $g^{ij}_{(0)}, A_{(0)i}, \phi_{(0)}, a_{(0)}$ are sources for the dual operators and appear as the leading coefficients in the near boundary expansions of the bulk metric $G_{\mu\nu}$, gauge field A_μ, active scalar Φ and Stückelberg field α, respectively.

The expectation value of a scalar operator is defined by

$$\langle O_\Phi \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta \phi_{(0)}} \tag{3.19}$$

It can be computed by rewriting it in terms of the fields living at the regulated boundary[11]

$$\langle O_\Phi \rangle = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{\Delta/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\rm sub}}{\delta\Phi(x,\epsilon)}\right) \tag{3.20}$$

where $\gamma_{ij}(x) = g_{ij}(x,\epsilon)/\epsilon$ is the induced metric on the boundary and $\gamma = \det(\gamma_{ij})$. In general one can prove that

$$\langle O_\Phi \rangle = (2\Delta - 4)\phi_{(2n)} + \text{local} \tag{3.21}$$

where the local terms are completely fixed by the choice of $\phi_{(0)}$.

The expectation value of the stress-energy tensor of the dual theory is given by

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{\rm ren}}{\delta g^{ij}_{(0)}} = \lim_{\epsilon\to 0}\frac{2}{\sqrt{g(x,\epsilon)}}\frac{\delta S_{\rm sub}}{\delta g^{ij}(x,\epsilon)} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}T_{ij}[\gamma]\right) \tag{3.22}$$

where $T_{ij}[\gamma]$ is the stress-energy tensor of the theory at $\rho = \epsilon$. From the gravitational point of view this is the Brown-York stress energy tensor supplemented by appropriated counterterms contributions

$$T_{ij}[\gamma] = T^{\rm reg}_{ij} + T^{\rm ct}_{ij}, \tag{3.23}$$

$T^{\rm reg}_{ij}$ comes from the regulated bulk action and it is equal to

$$\begin{aligned} T^{\rm reg}_{ij}[\gamma] &= -\frac{1}{2}(\mathcal{K}_{ij} - \mathcal{K}\gamma_{ij}) \qquad (3.24)\\ &= -\frac{1}{2}\left(-\partial_\epsilon g_{ij}(x,\epsilon) + g_{ij}(x,\epsilon)\,\mathrm{Tr}\,[g^{-1}(x,\epsilon)\partial_\epsilon g(x,\epsilon)] + \frac{3}{\epsilon}g_{ij}(x,\epsilon)\right) \end{aligned}$$

where $\mathcal{K}_{ij}$ is the extrinsic curvature tensor. $T^{\rm ct}_{ij}$ is the contribution due to the counterterms.

Similarly, the one-point function of R-symmetry currents reads

$$\langle J^i \rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{\rm ren}}{\delta A_{(0)i}} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^2}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\rm sub}}{\delta A_i(x,\epsilon)}\right) \tag{3.25}$$

The one-point functions $\langle \mathcal{O}_\Phi \rangle$, $\langle T_{ij} \rangle$ and $\langle J_i \rangle$ depend on all sources. Field theory VEV's can be obtained by setting the sources to zero.

[11] For scalars of dimension 2, an additional $\log\epsilon$ is needed in this formula, see (**??**).

In order to obtain higher point functions we need to functionally differentiate one-point functions with respect to the sources. In pratice one has to restrict at most to 2-point functions.

For example one can define and compute the connected 2-point functions of the stress tensor as

$$\langle T_{ij}(x)T_{kl}(y)\rangle = -\frac{2}{\sqrt{g_{(0)}(y)}}\frac{\delta\langle T_{ij}(x)\rangle}{\delta g_{(0)}^{kl}(y)}, \tag{3.26}$$

and the connected 2-point function of the currents as

$$\langle J_i(x)J_k(y)\rangle = -\frac{1}{\sqrt{g_{(0)}(y)}}\frac{\delta\langle J_i(x)\rangle}{\delta A_{(0)}^{k}(y)}. \tag{3.27}$$

In terms of the projectors

$$\pi_{ij} = \delta_{ij} - \frac{p_i p_j}{p^2}, \qquad \Pi^{TT}_{ijkl} = \frac{1}{2}(\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk}) - \frac{1}{3}\pi_{ij}\pi_{kl} \tag{3.28}$$

the (Fourier transform of the) former decomposes into

$$\langle T_{ij}(p)T_{kl}(-p)\rangle = \Pi^{TT}_{ijkl}\mathcal{G}_T(p^2) + \pi_{ij}\pi_{kl}\mathcal{G}_L(p^2) \tag{3.29}$$

while the latter into

$$\langle J_i(p)J_k(-p)\rangle = \pi_{ik}\mathcal{F}_T(p^2) + \frac{p_i p_k}{p^2}\mathcal{F}_L(p^2) \tag{3.30}$$

Supersymmetry Ward identities imply $\mathcal{G}_T = \mathcal{F}_T$ and $\mathcal{G}_L = \mathcal{F}_L$ when J is an R-symmetry current.

3.6 Ward identities

Bulk gauge fields couple to boundary currents associated to global symmetries. It follows that bulk gauge invariance translates into Ward identities of the boundary quantum field theory. The holographically computed one-point functions do satisfy these Ward identities including anomalies.

Using (3.18), invariance of (3.1) under diffeomorphisms,

$$\begin{aligned} &\delta g_{(0)}^{ij} = -(\nabla^i\xi^j + \nabla^j\xi^i), \quad \delta\phi_{(0)} = \xi^i\nabla_i\phi_{(0)}, \\ &\delta a_{(0)} = \xi^i\nabla_i a_{(0)}, \quad \delta A_{(0)i} = \xi^j\nabla_j A_{(0)i} + \nabla_i\xi^j A_{(0)j}, \end{aligned} \tag{3.31}$$

implies the Ward identity for the conservation of the stress tensor[12],

$$\nabla^i \langle T_{ij} \rangle = -\langle O_\Phi \rangle \nabla_j \phi_{(0)} - \langle O_\alpha \rangle \nabla_j a_{(0)} - F_{(0)ij} \langle J^i \rangle + A_{(0)j} \nabla_i \langle J^i \rangle \qquad (3.32)$$

where $F_{(0)ij}$ is the field strength of $A_{(0)i}$.

Invariance under Weyl transformations,

$$\begin{aligned} \delta g^{ij} &= -2\sigma g^{ij}, \qquad \delta \phi_{(0)} = -(4-\Delta)\sigma \phi_{(0)} \\ \delta A_{(0)i} &= -\sigma A_{(0)i}, \qquad \delta a_{(0)} = -\sigma a_{(0)} \end{aligned} \qquad (3.33)$$

leads to the conformal Ward identity

$$\langle T^i_i \rangle = -(4-\Delta)\phi_{(0)} \langle O_\Phi \rangle - a_{(0)} \langle O_\alpha \rangle - \langle J^i \rangle A_{(0)i} + \mathcal{A}. \qquad (3.34)$$

where we have allowed for the conformal anomaly $\mathcal{A}$. By Wess-Zumino consistency conditions, $\mathcal{A}$ is conformally invariant and obtains directly from the logarithmic counterterm of the the bulk action. When the only external source is the background metric $\mathcal{A} = \mathcal{A}_{grav}$ with

$$\mathcal{A}_{grav} = \frac{N^2}{16\pi^2}(R_{ij}R^{ij} - \frac{1}{3}R^2) \qquad (3.35)$$

Comparing with the standard formula

$$\langle T^i_i \rangle = \frac{c}{16\pi^2} W_{ijkl} W^{ijkl} - \frac{a}{16\pi^2} \tilde{R}_{ijkl} \tilde{R}^{ijkl} \qquad (3.36)$$

one immediately concludes that $c = a$ by the absence of terms bilinear in the Riemann tensor.

3.7 Two-point functions and superglueballs

The near boundary analysis does not fix certain asymptotic coefficients, associated with operator VEV's, in terms of the corresponding sources. One needs a solution of the field equations which is valid beyond the asymptotic region of small ρ. For the purpose of computing 2-point functions a linearized solution around a given background is enough.

[12]Notice that these Ward identities are valid in the presence of sources. In particular, $\langle T_{ij} \rangle$ depends on sources. In field theory one usually expresses the Ward identities in terms of the stress energy tensor with the sources set equal to zero, $\langle T_{ij} \rangle_{QFT} = \langle T_{ij} \rangle|_{\rm sources=0}$.

For supersymmetric flows, transverse modes of the metric and vectors [13] can be expressed in terms of an auxiliary "massless scalar field" $f(r)$. For transverse traceless tensor fluctuations one has

$$h_{ij}^{TT}(r,x) = e_{ij}(p)e^{ip\cdot x}f_p(r) \tag{3.37}$$

while for transverse vectors one finds

$$a_i^T(r,x) = v_i(p)e^{ip\cdot x}K(\Phi(r))^{-1/2}e^{2A(r)}f_p'(r) \tag{3.38}$$

Longitudinal and radial modes are less universal and must be studied on a case by case basis.

Once the non-local relation between sources and VEV's has been established, one can relatively easily compute 2-point functions. In some cases, the resulting mass poles form a discrete spectrum and deserve the name of (super)glueballs. The typical mass scale is $\Lambda \approx 1/L$, *i.e.* $(M_nL)^2 = f(n)$. In some cases the spectrum is continuous with or without a mass gap of order $M_{gap} \approx 1/L$. Notice that all mass scales are of the same order as those of the K-K excitations. This is the main drawback of the supergravity approximation. In order to go beyond this point one would need to understand how to quantize string theory in (asymptotically) AdS spaces. Some progress has been made very recently in a double scaling limit which is dual to a pp-wave background obtain by a Penrose limit of $AdS_5 \times S^5$.

3.8 GPPZ flow: a case study

Let us exemplify the procedure of holographic renormalization for the case of the GPPZ flow[14]. This solution corresponds to adding an operator of dimension $\Delta = 3$ to the Lagrangian that gives a common mass to the three

[13] Let us mention that by rescaling the vector fields so that their kinetic term is canonically normalized, one obtains that all transverse vector fluctuations have a universal mass

$$M_{eff}^2 = \frac{1}{2}\frac{\partial_r^2 K}{K} + A'\frac{\partial_r K}{K} - \frac{1}{4}\left(\frac{\partial_r K}{K}\right)^2 + \frac{M^2}{K} = -2\partial_r^2 A \geq 0.$$

This has been checked for any vector field in all analytically known supersymmetric flows with one active scalar.

[14] For simplicity we only consider the flow with one active scalar. More general solutions with VEV's for the bulk field dual to the gaugino bilinear have been considered by GPPZ.

$\mathcal{N} = 1$ chiral multiplets appearing in the decomposition of the $\mathcal{N} = 4$ vector multiplet. The solution was proposed as the holographic dual of pure $\mathcal{N} = 1$ SYM theory. Although it does not capture all of the expected properties of the field theory, it is particularly simple and still displays some interesting features.

The active scalar is a singlet under an $SO(3)$ subgroup of $SO(6)$. A consistent truncation to the $SO(3)$ singlets yields $\mathcal{N} = 2$ gauged supergravity coupled to two hypermultiplets describing a $G_{2(2)}/SO(4)$ coset. After lengthy calculation one gets the $5 - d$ superpotential that reads

$$W(\Phi) = -\frac{3}{4}\left[1 + \cosh\left(\frac{2\Phi}{\sqrt{3}}\right)\right] \tag{3.39}$$

Near $\Phi = 0$, the potential has an expansion,

$$V(\Phi) = -3 - \frac{3}{2}\Phi^2 - \frac{1}{3}\Phi^4 + \mathcal{O}(\Phi^6) \tag{3.40}$$

The mass of Φ is $M^2 = -3$, (in units such that $L = 1$) indicating that the dual scalar operator has indeed dimension $\Delta = 3$ in the UV. The domain-wall solution is given by

$$\Phi = \frac{\sqrt{3}}{2}\log\frac{1 + \sqrt{1-u}}{1 - \sqrt{1-u}} \qquad e^{2A} = \frac{u}{1-u}, \tag{3.41}$$

where $u = 1 - \exp(-2r)$. The boundary is at $u = 1$ and the solution has a naked singularity of "good" type at $u = 0$. Since $\Phi \approx \sqrt{3}\exp(-r)$ near the boundary, we are dealing with an operator deformation, namely the top component of the superpotential $\Delta W = \sum_{I=1}^{3}\Phi_I^2$, rather than a VEV.

EXERCISE

Solve the field equations in first order form and check that indeed $\Phi = \Phi(u)$ *and* $A = A(u)$ *is a solution.*

Up to a numerical rescaling, the kinetic term of the graviphoton is canonical and its mass is given by

$$M^2 = \sinh^2\left(\frac{2\Phi}{\sqrt{3}}\right) = \frac{(1-u)}{u^2}. \tag{3.42}$$

The bulk Stückelberg gauge invariance implies a corresponding Ward identity. We find convenient to use normalizations such that $B_\mu = A_\mu -$

$3\delta_\mu\alpha/2$. Then the bulk gauge invariance implies

$$\delta A^i_{(0)} = \nabla^i \lambda, \qquad \delta a_{(0)} = \frac{2}{3}\lambda \tag{3.43}$$

and

$$\nabla^i \langle J_i \rangle = \frac{2}{3}\langle O_\alpha \rangle \tag{3.44}$$

follows.

When expressed in terms of the radial variable u, the "massless" scalar field equation turns out to reduce to a hypergeometric equation that admits the solution

$$f_p(u) = (1-u)^2 F(2 + i\frac{p}{2}, 2 - i\frac{p}{2}; 2; u) \ . \tag{3.45}$$

For the transverse components of the $U(1)_R$ graviphoton one has

$$b^{(t)}(u) = u(1-u)F(2 + i\frac{p}{2}, 2 - i\frac{p}{2}; 3; u) \tag{3.46}$$

after dropping an irrelevant p-dependent factor.

For the longitudinal and radial components one finds

$$C_p(u) = -8B_\rho = uF\left(\frac{3}{2} + \frac{1}{2}q, \frac{3}{2} - \frac{1}{2}q; 3; u\right) \tag{3.47}$$

where

$$q = \sqrt{1-p^2} \tag{3.48}$$

We are now ready to combine results of near boundary analysis with the fluctuations to obtain correlation functions of the $U(1)_R$ current and the scalar operator O_α.

It is again useful to 'integrate' the first order variation δS_{ren} to obtain a quadratic action from which the 2-point correlators may be immediately read. Analytic continuation as $u \to 1$ of the hypergeometric function that appear in (3.46) yields gives

$$B_i = A_{(0)i}(1 - \rho[(1 + \frac{p^2}{4})[(\bar{K} - \frac{1}{2}) - \log\rho)] + 1] + ...) \tag{3.49}$$

for the transverse components, where $\bar{K} = \psi(3) + \psi(1) - \psi(2 + ip/2) - \psi(2 - ip/2)$. One then finds

$$\tilde{B}_{(2)i} = A_{(0)i}(1 + \frac{p^2}{4}) \qquad B_{(2)i} = -A_{(0)i}[(1 + \frac{p^2}{4})(\bar{K} - \frac{1}{2}) + 1] \tag{3.50}$$

Similarly, for the radial component, (3.47) gives

$$B_{(0)} = -\tilde{B}_{(0)}\bar{J}, \tag{3.51}$$

where $\bar{J} = 2\psi(1) - \psi(3/2 + q/2) - \psi(3/2 - q/2)$ and from the near-boundary analysis we know that

$$\nabla^i B_{(0)i} = -4\tilde{B}_{(0)} \tag{3.52}$$

The renormalized action to quadratic order in the sources thus reads (in momentum space)

$$\begin{aligned} S_{\text{ren}} &= \frac{N^2}{12\pi^2}\int d^4p \left(A_{(0)i}\pi^{ij}A_{(0)j}[-(1+\frac{p^2}{4})(\bar{K}-\frac{1}{2})+\frac{p^2}{4}] \right. \\ &\left. -\frac{1}{2p^2}(p_iA^i_{(0)}+\frac{2}{3}ip^2a_{(0)})^2\bar{J}\right) \end{aligned} \tag{3.53}$$

The fact that the the longitudinal part of the $A_{(0)i}$ and the $a_{(0)}$ appear as a total square is a consequence of the β-function operator relation (3.58) between the divergence of J_i and O_α. An analogous phenomenon takes place in the graviton-scalar sector.

The transverse 2-point function reads

$$\langle J_i(p)J_k(-p)\rangle_{(t)} = \frac{N^2}{6\pi^2}\pi_{ik}\left((1+\frac{p^2}{4})(\bar{K}-\frac{1}{2})-\frac{p^2}{4}\right) \tag{3.54}$$

It has poles at $p^2 = -4(n+2)^2$ (we are using Euclidean signature) with $n = 0, 1, ...,$ as expected, but also a disturbing massless pole whose residue is $-(N^2/6\pi^2)(\bar{K}(p{=}0) - 1/2) = (N^2/6\pi^2)$. Happily, as we now show, the longitudinal 2-point function also contains a massless pole and the two contributions cancel each other!

The remaining correlators are

$$\langle J_i(p)J_j(-p)\rangle_{(l)} = \frac{N^2}{12\pi^2}\frac{p_ip_j}{p^2}\bar{J} \tag{3.55}$$

$$\langle J_i(p)O_\alpha(-p)\rangle = \frac{N^2}{18\pi^2}ip_i\bar{J} \tag{3.56}$$

$$\langle O_\alpha(p)O_\alpha(-p)\rangle = -\frac{N^2}{27\pi^2}p^2\bar{J} \tag{3.57}$$

The residue of the zero mass pole in $\langle J_i(p)J_k(-p)\rangle_{(l)}$ is $N^2/(6\pi^2)$, and indeed the zero mass poles cancel.

These correlation functions are consistent with the operator relation

$$\nabla^i J_i = \frac{2}{3}O_\alpha = -\frac{2}{3}\beta O_\Psi \tag{3.58}$$

where $\beta = -\sqrt{3}$. The same β-function can be found in the graviton-scalar sector since the Ward identity for the R-symmetry current is related by supersymmetry to the trace Ward identity, $T_i^i = \beta O_\Phi$. Similar results obtain in the stress energy – active scalar sector where the mixing is rather intricate and we leave its study to the interested reader. Let us simply summarize the three distinct glueball spectra for the operators dual to bulk fields which are singlets under the $SO(3)$ subgroup of $SU(4)$ that leaves ΔW invariant.

For the $\mathcal{N}=1$ supercurrent multiplet $\mathcal{J}_{\alpha\dot\alpha} = \mathrm{Tr}\,(W_\alpha \bar W_{\dot\alpha} + \ldots)$ dual to the transverse components of the bulk supergravity multiplet $\{h_{\mu\nu}, \psi_\mu^{1,2}, B_\mu\}$, we have states with momenta:

$$(pL)^2 = 4(n+2)^2 \qquad n = 0,1,2,\ldots\,. \tag{3.59}$$

For the $\mathcal{N}=1$ chiral anomaly multiplet $\mathcal{A} = \mathrm{Tr}\,(\Phi^2)$ dual to the active hypermultiplet $\{\rho, \xi^{1,2}, m\}$:

$$(pL)^2 = 4(n+1)(n+2) \qquad n = 0,1,2,\ldots\,. \tag{3.60}$$

For the $\mathcal{N}=1$ chiral "Lagrangian" multiplet $\mathcal{S} = \mathrm{Tr}\,(W^2 + \ldots)$ dual to the dilaton hypermultiplet $\{\sigma, \xi^{3,4}, \tau\}$,

$$(pL)^2 = 4n(n+3) \qquad n = 0,1,2,\ldots\,, \tag{3.61}$$

including a zero-mass pole for the lowest component operators dual to σ.

This pattern agrees with physical expectations, but it emerges in a subtle way from the interwoven symmetries and dynamics of the bulk supergravity theory. In order to deal with the dilaton and axion fields τ correctly one is forced to confront oneself with the complexity of a $G_{2(2)}/SO(4)$ coset.

4 Wilson loops

In the introduction we have seen that Wilson loops are the basic non-local observables in gauge theories. Their scaling with the area was presented as a suggestive hint to an underlying string description of confinement.

In supersymmetric theories the concept of the gauge connection generalizes to a superfield that contains other components beyond the usual vector potential. In $\mathcal{N}=4$ SYM the natural generalization of the Wilson loop is

$$\langle W(\mathcal{C})\rangle = \frac{1}{N}\langle \mathrm{Tr}\mathcal{P}\exp\{i\int_{\mathcal{C}}(A_\mu \dot{x}^\mu + [\bar{\theta}_A \dot{x}_\mu \sigma^\mu \lambda^A + \theta^A \dot{y}_i \widehat{\Gamma}^i{}_{AB}\lambda^B + h.c.] + i\varphi_i \dot{y}^i)ds\}\rangle, \tag{4.62}$$

The curve $\mathcal{C}$ now represents the worldline in 'superspace' — in other words this kind of Wilson loop depends not only on the curve $x^\mu(s)$ but also on $y^i(s)$ and $\theta^A(s)$. The six variables y^i are in a sense conjugate to the central charges Z_i in the $\mathcal{N}=4$ supersymmetry algebra and θ^A contains the sixteen odd (Grassmann) variables of $\mathcal{N}=4$ on-shell superspace. The expression (4.62) is appropriate to euclidean signature whereas the factor of i in the last term in the exponent is absent with Minkowski signature and arises from the Wick rotation. Its presence is important, among other reasons, because it implies that the exponential is not purely a phase. The expression (4.62) can be motivated by considering the holonomy of a supersymmetric test particle of infinite mass that is generated by breaking the gauge group $U(N)$ to $U(N-1)\times U(1)$.

In the context of the AdS/CFT correspondence the Wilson loop is interpreted as the functional integral over all world-sheets embedded in AdS_5 and bounded by the loop. In the supergravity limit (the small α' limit of the string theory) this integration over fluctuating surfaces is dominated by the surface of minimum area in AdS_5. The behaviour of the loop is therefore

$$\langle W(C)\rangle \sim \exp(-\mathrm{TA}_{\min}(\Sigma))$$

where $T=1/2\pi\alpha'$ is the string tension. Since the metric is singular near the boundary of AdS_5 an infinite perimeter term arises that is eliminated by a suitable choice of boundary conditions. In this manner one ends up with results applicable to the strong 'tHooft coupling limit of the $\mathcal{N}=4$ gauge theory that in general (for non BPS configurations) differ from the weak coupling ones.

Wilson loops that satisfy a BPS condition, *i.e.* that are invariant under a fraction (say one half) of the 32 superconformal supersymmetries enjoy special properties. Setting for simplicity $\theta^A(s)=0$, the sixteen residual supersymmetries are defined by spinor parameters κ^A_α, $\bar{\kappa}^{\dot{\alpha}}_A$ that are related by

$$\dot{x}_\mu \sigma^\mu \bar{\kappa}_A = \dot{y}_i \widehat{\Gamma}^i{}_{AB}\kappa^B\,. \tag{4.63}$$

More generally, the BPS condition allows θ^A to be freely shifted by any Killing spinor κ^A. This BPS condition has a close connection with the presence of κ symmetry of the massless ($p^2 = 0$) superparticle. The condition (4.63) implies that $|\dot{x}| = |\dot{y}|$.

EXERCISE

Show that thanks to cancellations between scalar and vector exchanges there in no one-loop perimeter divergence for a Wilson loop with $|\dot{x}| = |\dot{y}|$.

A particularly symmetric example of a BPS Wilson loop is the circular loop of radius R. Superconformal invariance implies that the expectation value of such a loop cannot depend on R so that $\langle W(R)\rangle$ is a constant. Indeed extremizing the area for a loop in the (1,2) plane with the ansatz

$$X^\mu = f(\rho)(e_1^\mu \cos(\phi) + e_2^\mu \sin(\phi)) \tag{4.64}$$

one finds $f(\rho) = \sqrt{R^2 - \rho^2}$. Plugging into the string action on AdS_5 yields

$$S = TA_{min}(\Sigma) = \frac{L^2}{2\pi\alpha'} \int d\phi \int_\varepsilon^R \frac{Rd\rho}{\rho^2} = \frac{L^2}{\alpha'}\left(\frac{R}{\varepsilon} - 1\right) \tag{4.65}$$

Subtracting the divergent term, one finally gets

$$\langle W\rangle_{circle} \approx \exp(-S) = \exp(L^2/\alpha') = \exp(\sqrt{\lambda}) \tag{4.66}$$

that is indeed independent of R. A class of perturbative contributions to Wilson loops of this kind has been calculated to all orders in the coupling constant and argued to be the only relevant ones at least in the large N limit. This consists of the 'rainbow diagrams', *i.e.* planar diagrams in which all propagators begin and end on the loop (there are no internal interaction vertices). A suggestion has been made for extending this to all orders in the $1/N$ expansion by use of an 'anomaly' argument that relies on the fact that a Wilson loop that is a straight line has no perturbative contributions. The conformal transformation that maps the line to a circle is singular at a point on the loop and it was argued that this induces an anomalous behaviour that gives rise to a nontrivial correction to $\langle W(R)\rangle$. The form of this correction was determined in terms of a zero-dimensional gaussian matrix model. It coincides with the results found above in the $\lambda >> 1$.

EXERCISE

Sow that inversions of the plane generically map circles and straight lines onto circles. When does a circle map onto a line? And when a line onto a line?

Recently the one-instanton contribution to a circular Wilson loop in $SU(N)$ $\mathcal{N}=4$ Yang–Mills in semi-classical approximation – to lowest order in the Yang–Mills coupling constant, g_{YM} has also been computed. As usual, the instanton computation boils down to an integral over the supermoduli space spanned by eight bosonic and sixteen fermionic collective coordinates. The presence of the loop breaks the $SO(4,2)$ conformal invariance but for a circular loop there a residual unbroken $SO(2,2)$ subgroup allows one to map an arbitrary instanton to one that is located at the centre of the loop. Thus effectively abelianizing the instanton connection. $\mathcal{N}=4$ superconformal symmetry implies the existence of sixteen fermionic generatorsan that enhancement $SO(2,2)$ to an $OSp(2,2|4)$ subgroup of $SU(2,2|4)$. After lenghty and computer-aided manipulations and subtraction of a perimeter divergence which is an artifact of the cutoff procedure one gets a unique finite result. The problem of reconciling the instanton contributions to the Wilson loop with the $SL(2,Z)$ Montonen–Olive duality and with strong coupling supergravity predictions is still open and may not forgo a better understanding of D-instanton effects in this context.

5 Penrose as a novel double scaling limit

The AdS/CFT correspondence has recently received a sudden twist connected with the double scaling limit that corresponds to type IIB superstring around a pp-wave supported by a R-R 5-form flux. From the supergravity perspective this corresponds to performing a Penrose limit around a null geodesic at the center of $AdS_5\times S^5$. In the Penrose limit the $SU(2,2|4)$ super-isometry of $AdS_5\times S^5$ undergoes an Inonü-Wigner contraction. In particular $SO(4,2)\to SO(4)_X\times U(1)_\Delta$ and $SO(6)\to SO(4)_Y\times U(1)_J$ but at the same time a Heisenberg group, $H(8)$, emerges so that the total number of generators remains equal to 30 as for $AdS_5\times S^5$. A similar rearrangement takes place for the 32 supersymmetry charges.

The resulting maximally supersymmetric geometry[15]

$$ds^2=-4dx^+dx^- - \mu^2(|\vec{X}|^2+|\vec{Y}|^2)(dx^+)^2+(|d\vec{X}|^2+|d\vec{Y}|^2) \tag{5.67}$$

$$F_{+2356}=F_{+1478}=\mu \quad , \quad e^\phi = g_s\,, \tag{5.68}$$

[15]In our conventions, the indices of $\vec{X}$ run over $1,4,7,8$ and those of $\vec{Y}$ over $2,3,5,6$.

with all other fields set to zero, admits an exactly solvable worldsheet description in the Green-Schwarz formalism. Indeed, in the light-cone gauge, one simply has 8 massive bosons and as many massive fermions. The former getting their mass from the $\mu^2(|\vec{X}|^2 + |\vec{Y}|^2)(dx^+)^2$-term in the metric. The latter from their coupling to the R-R 5-form flux. The action read

$$S^{(l.c.)}_{GS} = \frac{1}{2}\int d^2\sigma\{\partial\vec{X}\cdot\bar{\partial}\vec{X} + \partial\vec{Y}\cdot\bar{\partial}\vec{Y} + \nu^2(|\vec{X}|^2 + |\vec{Y}|^2) + S\bar{\partial}S + \bar{S}\partial\bar{S} + \nu\bar{S}\Gamma S\} \tag{5.69}$$

where $m = \alpha' p^+ \mu$ and $\Gamma = \Gamma_{2356}$. Expanding in normal modes one gets

$$\omega(n) = \pm\sqrt{n^2 + \nu^2} \tag{5.70}$$

Modes with $n > 0$ are left-movers, those with $n < 0$ are right-movers. Notice that the "zero-modes" with $n = 0$ have $\omega = \nu$ thus contributing to the mass of the string excitation. Level matching requires $\sum_n nN_n = 0$.

In addition to the standard holographic identifications

$$g_s = \frac{g^2}{4\pi} \quad , \qquad \frac{L^2}{\alpha'} = \sqrt{g^2 N} \quad , \tag{5.71}$$

at large N and large J, with $J \approx \sqrt{N}$, the relevant coupling turns out to be

$$\lambda' = \frac{g^2 N}{4\pi J^2}\,, \tag{5.72}$$

where J is the $U(1)_J$ charge that appears in the above decomposition of $SO(6)$ [16] and can be identified with the light cone momentum P^+

$$J \approx \mu p^+ L^2 = \frac{\Delta + J}{2}\,. \tag{5.73}$$

Operators with $\Delta = J$ and $\Delta = J + 1$ are known to be protected, as a consequence of $SU(2,2|4)$ shortening conditions of BPS type that survive the relevant Inonü-Wigner contraction. The simplest nearly protected operators, that are expected to correspond to the lowest type IIB superstring excitations $Y^a_n Y^b_{-n}|p^+\rangle$, are of the form

$$\mathcal{O}^{ab}_n = \sum_{\ell=0}^{J} q^\ell_n tr(Z^{J-\ell}Y^a Z^\ell Y^b) \tag{5.74}$$

[16]This $U(1)_J$ does not coincide with the $U(1)_R$ in the $\mathcal{N} = 1$ decomposition of $\mathcal{N} = 4$ SYM used so far.

where $q_n = \exp(2\pi i n/(J+1))$ and, in our previous notation, $Z = \phi^1$ and $Y^a = \varphi^a$ for $a = 2, 3, 5, 6$

The knowledge of the free spectrum of the light cone Hamiltonian P^- gives a prediction for the 'planar' contributions to the anomalous dimensions of the operators $\mathcal{O}_n^{ab}$, *i.e.*

$$p^- = \frac{\mu}{2}(\Delta - J) = \frac{\mu}{2}\sqrt{1 + \frac{4\pi g_s N n^2}{J^2}} \,. \tag{5.75}$$

this has recently been tested using $\mathcal{N} = 1$ superfield techniques. The effective string loop counting parameter is J^2/N. The first 'non-planar' corrections in SYM theory have been explicitly computed and matched with string one-loop corrections. An intricate operator mixing problem remains to be solved at this order.

Despite the success of the proposal, the way holography is realized in the pp-wave background is still a matter of debate. This prevents a naive application of the procedure that for (asymptotically) AdS spaces has lead to 'holographic renormalization'. Moreover even the simplest amplitudes are very laborious to compute in the light-cone gauge. However conformal flatness of the background, that is made manifest by the coordinate transformation

$$u = \tan(x^+)\,,\ v = x^- - \frac{1}{2}(|\vec{x}|^2 + |\vec{y}|^2)\tan(x^+)\,,\ \vec{x}' = \frac{\vec{x}}{\cos(x^+)}\,, \quad \vec{y}' = \frac{\vec{y}}{\cos(x^+)}\,, \tag{5.76}$$

guarantees the absence of higher derivative corrections to amplitudes with fewer than four insertions, much in the same way as in $AdS_5 \times S^5$, and makes one hope that a viable, *i.e.* covariant, superstring description could be not far from reach. Hopefully this should open the way to more interesting superstring backgrounds with R-R fluxes.

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