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THE AdS/CFT CORRESPONDENCE

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INTRODUCTION

This introduction contains very simple information about CFTs and the large N limit. It may be skipped.

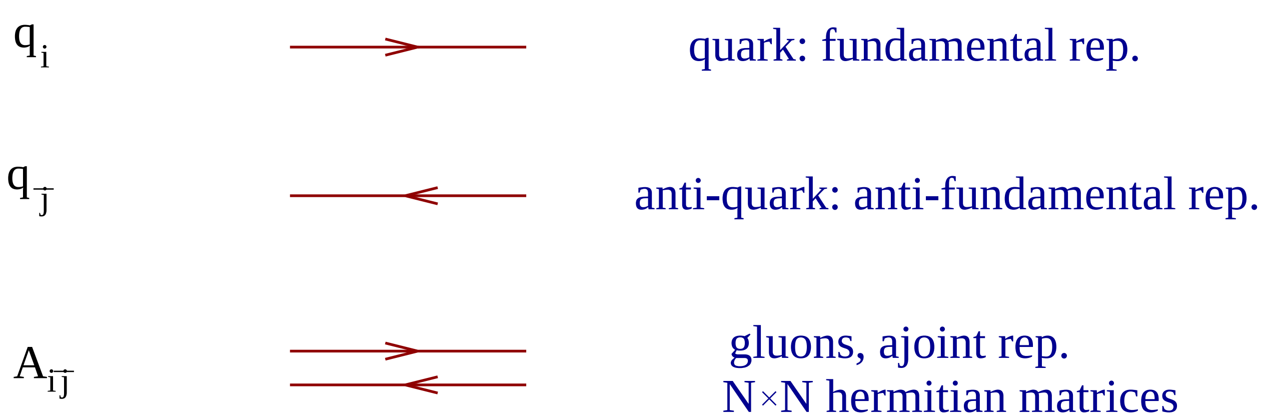
The large N limit for gauge theories

t'Hooft proposal: Consider $SU(N_c)$ Yang-Mills: send $N_c \to \infty$ and do a systematic expansion in $1/N_c$. The large N expansion is quite useful:

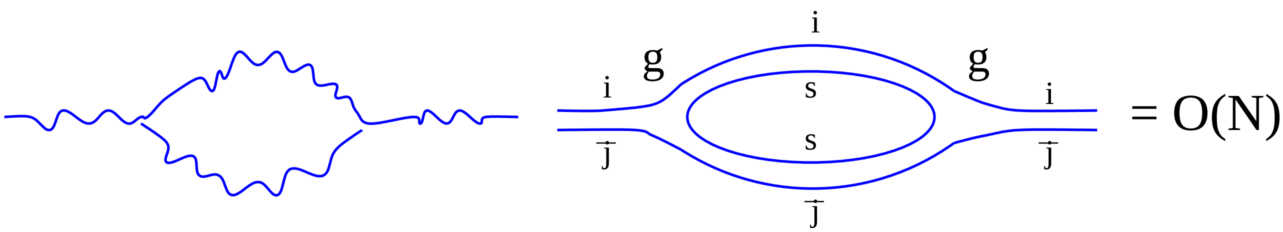
- It is a systematic expansion.
- It simplifies the perturbative computation.
- Solvable in two dimensions.
- Good qualitative agreement with QCD: confinement, weakly coupled Lagrangian for mesons and glueballs, U(1) anomalies, etc...).

For theories like QCD, N is the only dimensionless parameter: the coupling constant runs with the scale $g(\mu)$ and it is better to trade it for the Renormalization Group invariant quantity Λ_{QCD} (dimensional transmutation). We are interested in finite theories, where g does not run and it is a dimensionless parameter, but nevertheless we can consider the large N limit. In the real world $N = 3$ is small, but weighted with factors of π $(1/N(4\pi)^2 = 1/300)$...

Large N expansion for $SU(N)$ YM: For large N, $SU(N) \sim U(N)$. Let us picture:



Consider the self-energy for the gluons:



The only free index is the internal one s, which may take N values: the self-energy diverges as $O(N)$. Many other graphs diverge as well. $N \to \infty$ is not a sensible limit. However, the self-energy contains powers of the coupling constant: $O(g^2 N)$. In the t'Hooft limit

$$\begin{aligned} N &\to \infty, \\ g &\to 0 \qquad x = g^2 N fixed \end{aligned}$$

the self-energy is finite.

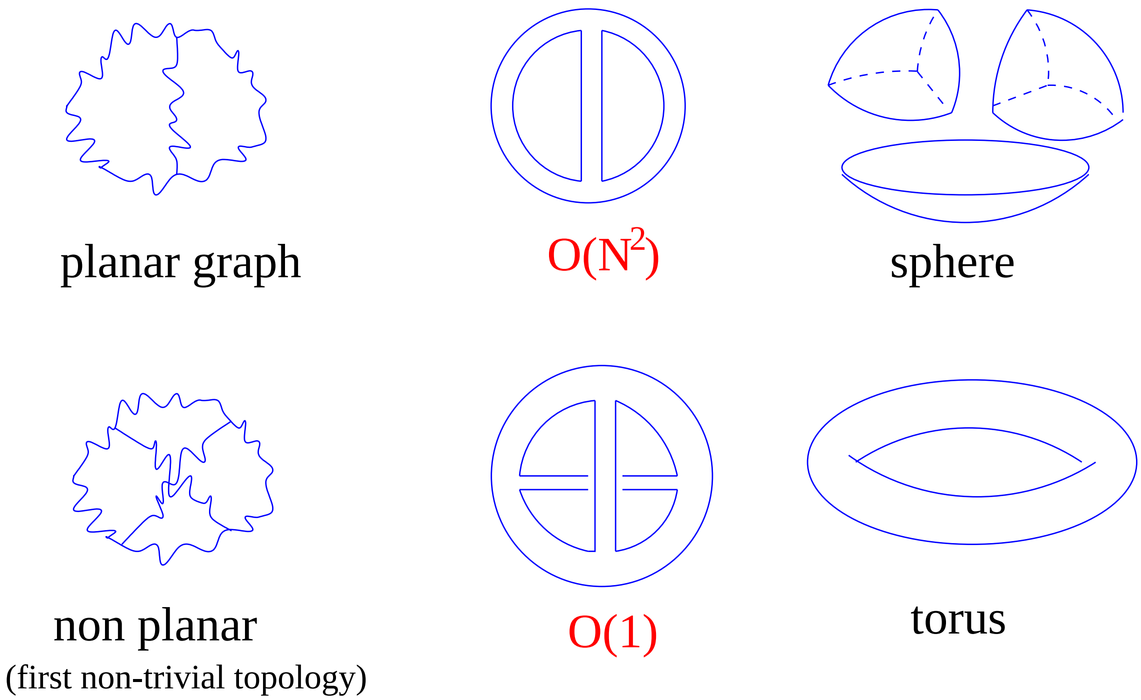
Using the following normalization

$$L = \frac{1}{g_{YM}^2} F_{\mu\nu}^2 = \frac{N}{x} F_{\mu\nu}^2$$

it can be proved that in the t'Hooft limit

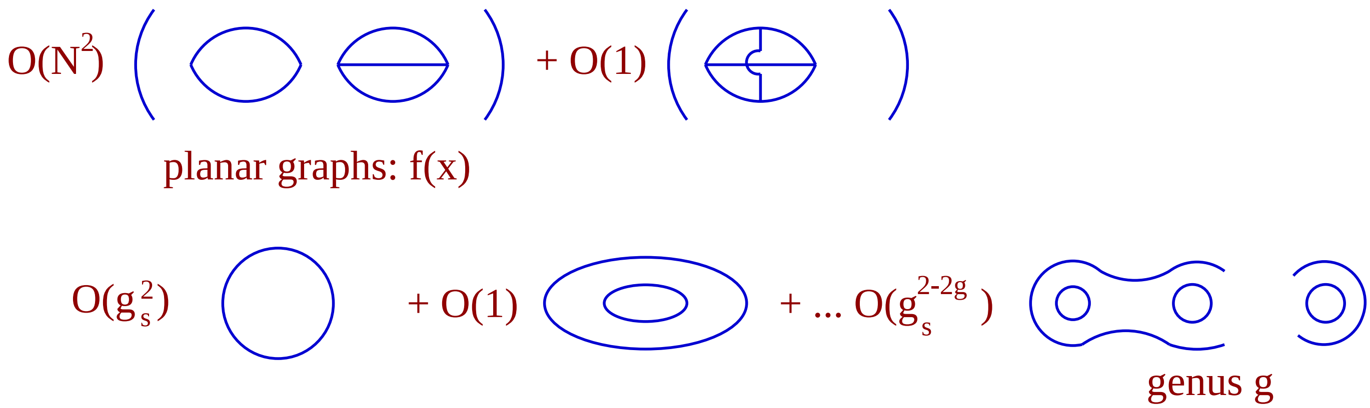
- All the graphs are finite

- The perturbative expansion is organized according to the topology of the graph.



Each graph is a Riemann surface in 2d: classified by the genus. Link to string theory. The expansion for the free-energy

$$F = \sum_{g=0}^{\infty} N^{2-2g} f_g(x)$$



is similar to the loop expansion of a string with coupling $g_s = e^{-\phi} = N$.

- For $N \to \infty$ only the planar graphs (f_0) survive. Still an infinite number of graphs. In 2d the planar theory can be solved; but not in dimension greater than two.

- f_g is a non-trivial function of x. In QCD-like theory, where x is not a parameter, f_g is a function of Λ_{QCD}.

- The planar graphs contributes a $O(N^2)$ to the free-energy. This is of the same order of the Lagrangian

$$F^2/g_{YM}^2 = (x/N)F^2 = NTrF^2 \sim O(N^2)$$

Therefore, it is not a divergent contribution.

Conformal theories

We are interested in theories without scales or dimensionfull parameters. These theories are classically scale invariant. Examples:

Y.M.	$\frac{1}{g^2}F_{\mu\nu}^2$
$\lambda\phi^4$	$(\partial\phi)^2 + \frac{\lambda}{4!}\phi^4$

Invariant under

$$\phi(x) \to e^{\alpha d}\phi(e^{\alpha}x), \quad \text{d} = \text{canonical dimension}$$

Current for dilatation:

$$J_\mu = x_\nu T_{\mu\nu} \quad \text{conserved iff} \quad \partial J = T^\mu_\mu = 0$$

Scale invariance usually implies induces the bigger group of conformal transformation:

$$\delta x_\mu = a_\mu + \omega_{\mu\nu}x_\nu + \lambda x_\mu + (b_\mu x^2 - 2x_\mu(bx))$$

(translation P_μ+Lorentz $M_{\mu\nu}$+dilatation D + special conformal K_μ). These are the transformations that rescale the metric:

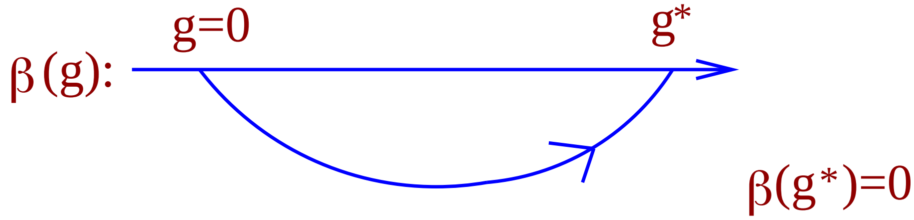
$$x_\mu \to x'_\mu \qquad dx'dx' = \Omega(x)(dx)^2$$

Also satisfied by the inversion I: $Z_2 : x_\mu \to x_\mu/x^2$. The currents for the generators P, M, D, K are constructed with x_μ and the stress-energy tensor and are conserved iff $T^\mu_\mu = 0$. The generators P, M, D, K, I close the conformal group $O(4,2)$.

In a quantum theory, conformal invariance is broken by the introduction of a renormalization scale. The RG or Callan-Symanzick eq. is just the anomalous Ward identity for dilatations

$$\begin{aligned} \mu\frac{d}{d\mu}g &= \beta(g) \to g(\mu), \Lambda_{QCD} \\ T^\nu_\mu &= \beta(g) F^2_{\mu\nu} \end{aligned}$$

The theory recover conformal invariance only at the fixed point of the RG.



At the fixed points, the RG equation is the Ward identity for dilatations, with an anomalous dimension:

$$\Delta = d + \gamma(g^*), \qquad \gamma = \mu \frac{d}{d\mu} \ln Z$$

In a CFT, the Ward identities for $O(4,2)$ give constraints on the Green functions. One can always find a basis of conformal operators $\psi_i(x)$, with fixed scale dimension Δ_i. The set of (ψ_i, Δ_i) gives the spectrum of the CFT. 1,2,3-point functions are fixed by conformal invariance. 1-point functions are zero. 2-point functions equal

$$\langle \psi_i(x) \psi_j(y) \rangle = \frac{A\delta_{ij}}{|x-y|^{2\Delta_i}}$$

The coordinates dependence of 3-point functions is also fixed.

Two classes of CFT

- Endpoint of RG: $g = g^*$, isolated fixed points.

- Finite theories. Since there are no divergences at all, $\beta(g) = 0$ for all values of g. No RG. Line of fixed points.

N=4 SYM An example of finite theory is N=4 SYM. The 4 supercharges Q_a are rotated by the R-symmetry $SU(4) \sim SO(6)$. The theory has

gauge fields A_μ,	
4 Weyl fermions ψ_a	$\underline{4}$ of $SU(4)$
6 real scalars ϕ_i	$\underline{6}$ of $SU(4)$

It is finite at three-loops and believed to be finite at all orders.

It is customary to combine coupling constant and theta angle in a complex parameter

$$\tau = \frac{1}{g^2} + i\theta$$

The theory is finite (and therefore conformal) for all value of τ: $\beta(\tau) = \gamma(\tau) = 0$. Complex line of fixed points.

No divergences for fundamental fields in the Lagrangian, but composite operators have divergences, and therefore anomalous dimensions. Examples, the bilinear operators: fields with canonical dimension even at quantum level (protected by susy)

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu}(x) \qquad & \rightarrow \Delta = 4 \\ \psi_a \psi_b(x) \qquad & \rightarrow \Delta = 3 \\ \phi_i \phi_j(x) - \text{traces} \qquad & \rightarrow \Delta = 2 \end{aligned}$$

fields with anomalous dimensions (non-zero also at weak coupling):

$$\phi_i \phi_i(x) \qquad \rightarrow \Delta = 2 + O(g^2)$$

Correlation functions are not easy to compute even using conformal invariance. For large N we can expand:

$$\langle O \cdots O \rangle = \sum N^{2-2g} f_g(x)$$

CFT/AdS CORRESPONDENCE

The large N limit of gauge theories gives a good qualitative description of the dynamics (weakly coupled Lagrangian for mesons or glue-balls, compatibility with confinement,...)

$$N \to \infty, \qquad x = g_{YM}^2 N \text{fixed}$$

(t'Hooft limit)

It is an old idea (Polyakov) that the large N limit should be described by a string theory. Certainly not a conventional one: a 4d non-critical string, maybe. The anomalous Weyl mode of the 2d world-sheet metric does not decouple, it must be included as an extra field and the theory develops a fifth coordinate.

It is a recent idea that critical strings may be used as well. This works at the best for CONFORMAL GAUGE THEORIES.

Our favorite example:

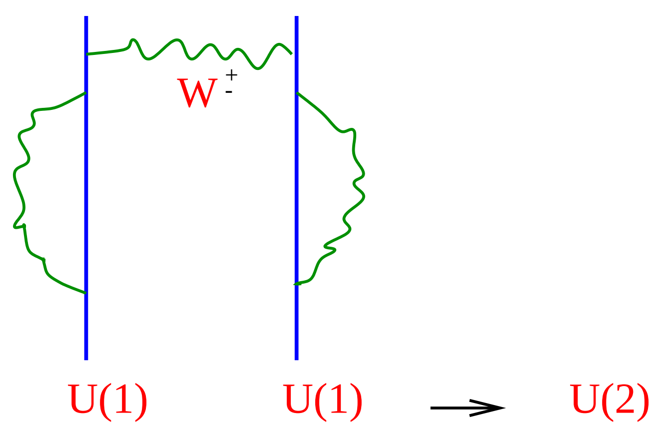
N=4 SYM in 4d with gauge group U(N)

The theory is FINITE

$$\beta(g) = \gamma(g) = 0$$

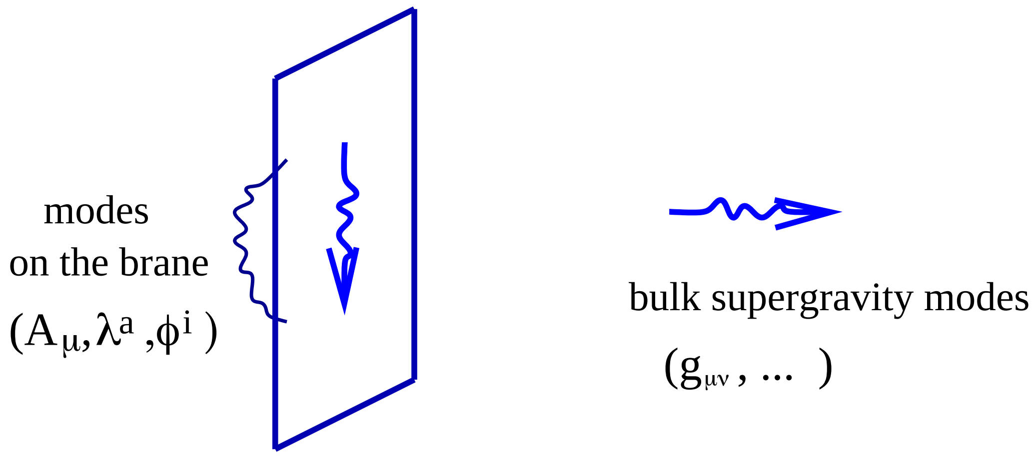
and therefore CONFORMAL, for all values of the complex coupling constant parameter $\tau = \frac{1}{g^2} + i\theta$. There is no dimensional transmutation and a line of fixed points.

1. Motivation from branes The theory can be realized on the world-volume of N parallel D3-branes in Type IIB



in the low energy limit $\alpha' \to \infty$.

Main motivation from branes:



$$\frac{1}{g_s}\int d^4x F_{\mu\nu}^2 + \frac{1}{\alpha'^4}\int d^{10}x\sqrt{g}Re^{-2\phi} + \cdots$$

$$(g_{YM}^2 = g_s)$$

For $\alpha' \to 0$, we decouple gravity and obtain pure SYM on the brane

$$\alpha' \to 0$$

$$g_s \quad \text{fixed}$$

$$N \quad \text{fixed}$$

$$\phi^i = \frac{r^i}{\alpha'} \quad \text{fixed}$$

Distances between branes are rescaled in order to maintain the entire moduli space of N=4 SYM, i.e. to keep finite VEVs of scalar fields and masses of W bosons.

A D3-brane is also a solution of the eqs of motion of type IIB supergravity/string:

$$\begin{aligned} ds^2 &= H^{-1/2} dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2\Omega_5) \\ F^{(5)} &= \text{flux of a charged object} \\ \phi &= \text{constant} \\ H &= 1 + \frac{g_{YM}^2 N \alpha'^2}{r^4} = 1 + \frac{g_{YM}^2 N}{\alpha'^2 \phi^4} \end{aligned}$$

Metric and RR field induced by a 3+1 tensionfull object charged under the RR four-form

In the $\alpha' \to 0$ limit (with g_s, N, ϕ fixed)

$$ds^2 = \alpha' \left\{ R^2 \frac{(d\phi)^2}{\phi^2} + \frac{\phi^2}{R^2}(dx)^2 + R^2 \Omega_5 \right\}$$

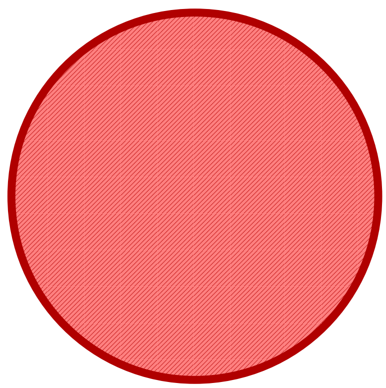
that is the direct PRODUCT of two spaces of constant curvature: $AdS_5 \times S^5$.

The radius of both AdS_5 and S^5 is

$$R^2 = \sqrt{x}, \qquad\qquad x = \sqrt{g_{YM}^2 N}$$

AdS_5 and S^5 are Einstein spaces $R_{\mu\nu} = \Lambda g_{\mu\nu}$. They solve the Einstein equations $\int d^5x\sqrt{g}(R+\Lambda)$ with cosmological constant Λ, positive for S^5, negative for AdS_5.

The Euclidean version of AdS_5 is a five-dimensional ball

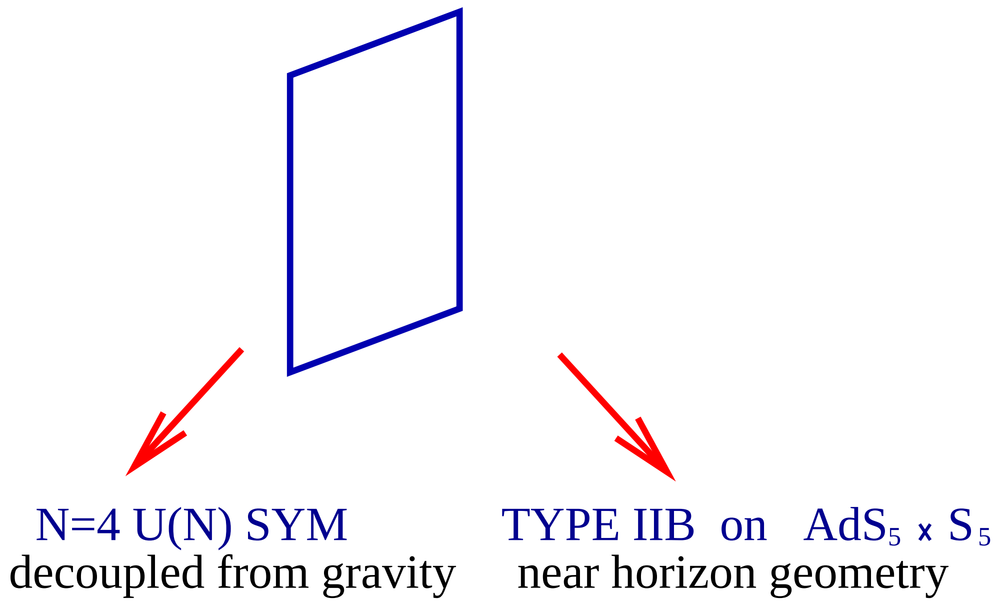


$$R^5 : \qquad x_1^2 + \cdots x_5^5 \leq R^2, \qquad ds^2 = \frac{(dx)^2}{1-|y|^2}$$

It has a BOUNDARY: S^4, the compactification of flat four-dimensional space-time.

This is a generalization of the standard notion of boundary: notice that the metric has a pole on the "boundary".

Taking the $\alpha' \to 0$ limit we obtain two different theories: a QFT and a string theory:



which are conjecture to be dual.

Relation between the parameters $(x = g^2_{YM} N, N)$ of SYM and $(\frac{R^2}{\alpha'}, g_s)$ of the string theory:

$$
\begin{aligned}
g_s &= \frac{x}{N} \\
\frac{R^2_{AdS}}{\alpha'} &= \sqrt{g^2_{YM} N}
\end{aligned}
$$

The t'Hooft limit $N \to \infty, x = g^2_{YM} N$ fixed, is naturally implemented in the correspondence.

2. The symmetries

On the two side:

N=4 SYM	Type IIB on $AdS_5 \times S^5$
Conformal group SO(4,2)	Isometry group of AdS_5 SO(4,2)
Supersymmetries 8 = 4 linear +4 conformal	Type IIB on $AdS_5 \times S^5$ has N=8 susy
R-symmetry SU(4)	Isometry of S^5 SO(6)= SU(4)

In a word, the symmetry on both sides is the superconformal group $SU(2,2|4)$.

3. Precise definition of the correspondence

The AdS/CFT correspondence relates

CFT in 4d	Critical string in 10d on $AdS_5 \times H$ H compact$\to$ 5d theory

The conformal group SO(4,2) is

group of conformal transformation in 4d	Isometry group of the AdS_5 space-time

Two way of realizing theories with SO(4,2) symmetry

CFT in 4d	relativistic theories in AdS_5 including gravity

To define the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. The correspondence is via HOLOGRAPHY: the boundary of AdS_5 is Minkowski space-time.

THE PLAYERS:

- The CFT is specified by a complete set of conformal operators
- The fields in AdS are the excitations of the string background. They certainly contain the metric. Their interaction is described by an effective action $S_{AdS_5}(g_{\mu\nu}, A_\mu, \phi, ...)$: typically, some supergravity.
- The fields h in AdS know about the CFT operators O via the boundary coupling

$$L_{YM} + \int d^4 x h O$$

The restriction of the 5d field $h(x, x_5)$ to the boundary is the source that couples to the operator O_h

- We need a map between CFT operators and AdS fields. The field that couples to an operator is often found using symmetries

  $$L_{YM} + \int d^4x \sqrt{g}(g_{\mu\nu} T_{\mu\nu} + A_\mu J_\mu + \phi F^2_{\mu\nu} + \cdots)$$

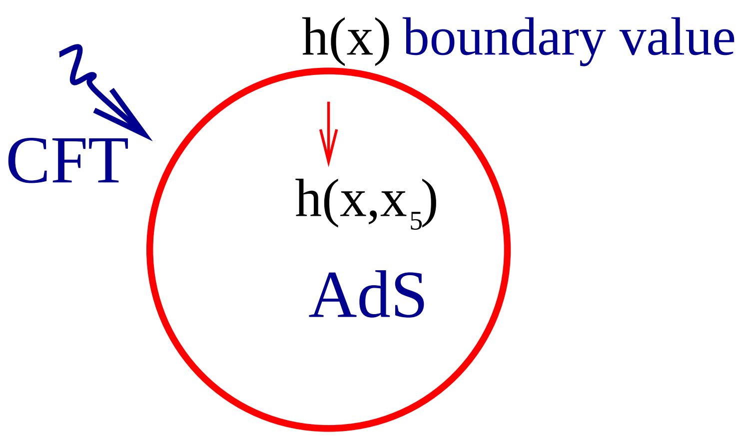
  Global symmetries in CFT corresponds to gauge symmetries in AdS.

THE PRESCRIPTION:

In CFT we define the functional generator $W(h)$ for the connected Green functions for the operator O. $h(x)$ is a source, depending on 4 coordinates.

Extend the source $h(x)$ in AdS_5 using the equations of motion derived from S_{AdS}:

$$h(x) \to \hat{h}(x, x_5) \tag{1}$$



We claim that

$$e^{W(h)} = \left\langle e^{\int hO} \right\rangle = e^{-S_{AdS_5}(\hat{h})}$$

- Eqs of motion in AdS are second order, but the extension of the boundary value inside the space is unique (we impose regularity at the center of the ball)

- In full string theory, the right-hand side of the last equation is replaced by some S-matrix element for the state h

- An off-shell theory in 4d corresponds to an on-shell theory in 5d

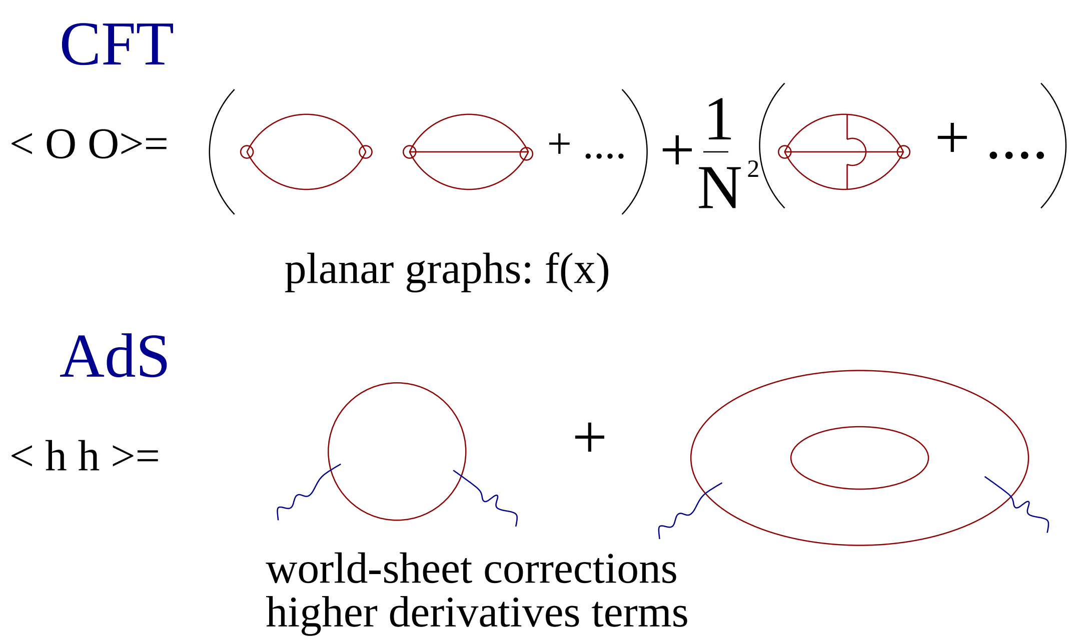
VALIDITY AND PREDICTION

The dual string theory is useful when weakly coupled: supergravity (defined as a classical theory). $g_s \to 0$ (string loops suppressed) and $\alpha'/R^2 \to 0$ (higher derivatives terms and massive string modes suppressed). This corresponds to the large N limit and strong coupling ($x \to \infty$) of the CFT.

- We can compute all Green functions of the CFT and predict the spectrum of conformal operators in the large N limit and strong coupling using a classical theory.

- The correspondence is valid for all N and x, but for any computation we need the full string theory.

- We can organize a t'Hooft expansion in 1/N and 1/x: it is the loop and world-sheet expansion of string theory.



CHECKS AND PREDICTIONS

Since the perturbative regime of the CFT and of the string theory on AdS are opposite, we can only check quantities that are protected by supersymmetry and do not depend on the value of the coupling x.

1) Green function

- General structure of correlation function is dictated by conformal invariance

- Some Green functions are protected: central charges, etc...

2) Wilson loop

The CFT prediction is $E = 1/L$. The precise coefficient is renormalized.

3) Spectrum of operators

All operators with finite dimension for $x \to \infty$ should correspond to KK modes on $AdS \times H$. In N=4 SYM:

- Finite dimension operators are those protected by susy: short multiplets.

- All KK modes computed in the eighties.

In all the computations done: PERFECT AGREEMENT.

1. 2-point Green function

Use the metric

$$ds^2 = (dr)^2 + e^{2r/R}(dx_\mu dx^\mu)$$

boundary $r \to +\infty$

center $r \to -\infty$

Given a scalar field $\phi = \phi(r)$ with action

$$\begin{aligned} S = \quad & \int r dx_\mu \sqrt{g}((\partial\phi)^2 + m^2\phi^2) = \\ & \int r e^{4r}((\partial\phi)^2 + m^2\phi^2) \quad R = 1 \end{aligned}$$

the eqs. of motion $\phi'' + 4\phi' = m^2\phi$ have solution

$$\phi = \phi_+ e^{-(4-\Delta)r} + \phi_- e^{-\Delta r}, \qquad m^2 = \Delta(\delta - 4) \tag{2}$$

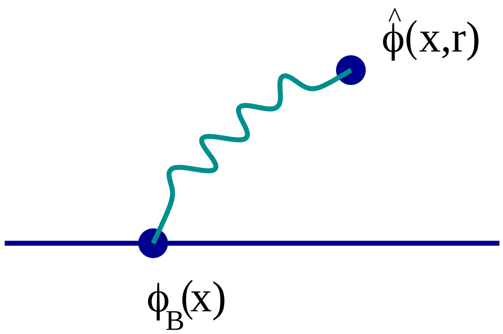
Δ is identified with the conformal dimension of the operator O_ϕ. ϕ_+ is normalizable at the boundary, while ϕ_- is not.

Extend ϕ inside AdS using prescription in eq. (1). Regularity at the center of AdS selects ϕ_-. Against naive expectations, we select the non-normalizable (at the boundary) solution. ϕ_+ diverges at the center.

The requirement $\hat{\phi}(x, r \to \infty) \to \phi(x)$ is too naive and can be applied only when $m^2 = 0, \Delta = 4$. The solutions of 5d eqs of motion generically vanish on the boundary (eq. (2)). We should impose

$$\hat{\phi}(x, r \to \infty) \to e^{-(4-\Delta)r}\phi(x)$$

$\hat{\phi}$ can be analytically found



$$\hat{\phi}(x,r) = \int dx' \, K(x-x',r) \;\; \phi_B(x)$$

(see hepth/9803131). Plugging it into the 5d Lagrangian

$$W[\phi_B(x)] = S_{AdS}(\hat{\phi}(x,r)) = \int dx dx' \frac{\phi_B(x)\phi_B(x')}{(x-x')^{2\Delta}}$$

The computation can be simplified using a standard trick: by integrating by parts

$$S = \int_{BOUND} \phi \partial_n \phi + \int \phi(-(\partial)^2 + m^2)\phi \quad (3)$$

the second term is zero on the eqs. of motion: the action reduces to a boundary contribution.

We only need ϕ and its first normal derivative at the boundary.

The two point function is

$$\langle \phi(x)\phi(x')\rangle = \frac{\delta S}{\delta\phi(x)\delta\phi(x')} = \frac{1}{(x-x')^{2\Delta}}$$

in agreement with CFT expectations.

Standard example of scalar field is the dilaton: $\phi \to F^2_{\mu\nu}$, with $m^2 = 0, \Delta = 4$.

Checks and prediction for the Green functions

- All 2- and 3-point functions computed from AdS satisfy the requirement of conformal invariance. 2-point functions of conserved currents often define central charges (es, energy-momentum tensor:

  $$\langle T(x)T(0)\rangle = c/x^8$$

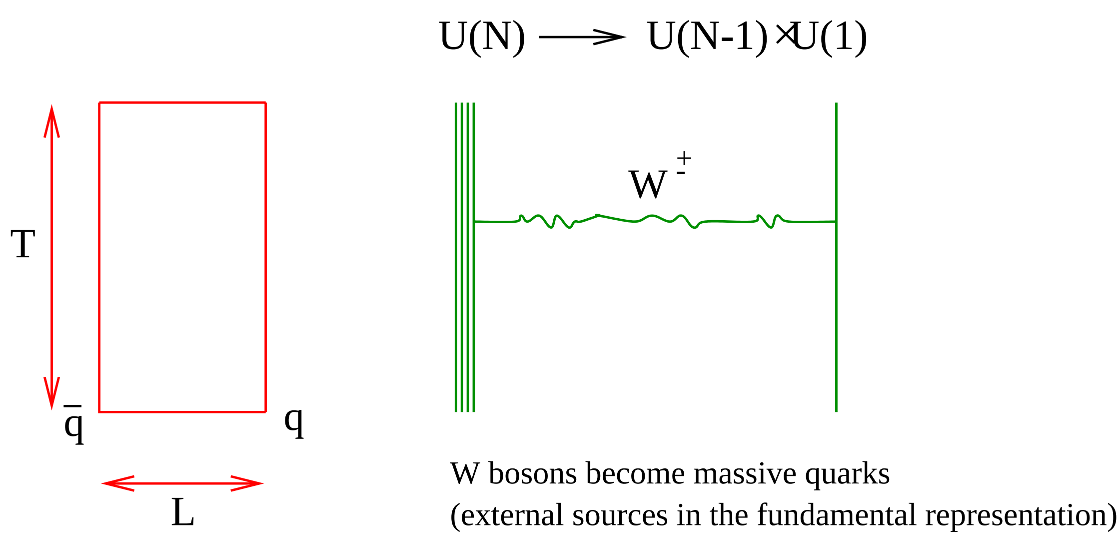
  which are not renormalized. AdS computation ($x = \infty$) coincides with free-field computation ($x = 0$).

- 3-point functions have been computed for all operators corresponding to KK modes: free-field result. Non-renormalization theorem?

- A 4-point function have been computed completely. They are renormalized. Logarithms and multi-trace operators.

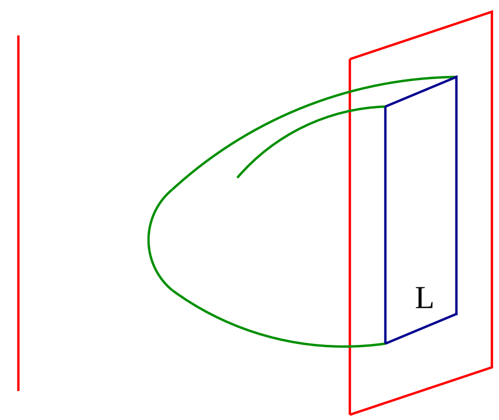
2. Wilson loop

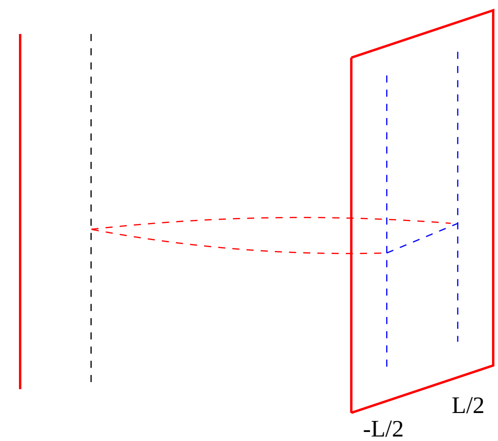
In QFT $W(C) = Tr_R e^{i\int_C A} = e^{-TE(L)}$ computes energies between external sources in the representation R. We can get the analogous of an external source in AdS by moving a brane:



Consider a string whose endpoint lies on C (a contour on the boundary) and free to move in

10d. The action for the string is the Nambu-Goto $\int dx^2 \sqrt{g}$: the string chooses the minimal area configuration.





The natural prescription in the AdS/CFT is

$W(C) = S_{AdS}$(Minimal Area Surface with boundary C)

In a flat 5d space-time, the surface of minimal area with boundary C would lie entirely on the boundary, giving an obvious $S = LT \to E = L$, good for a confining theory, not certainly for a CFT. The point is that the AdS metric diverges on the boundary $U = \infty$:

$$ds^2 = \frac{U^2}{R^2}(dx_\mu dx_\mu) + R^2 \frac{(dU)^2}{U^2}$$

and it is energetically favorable for the string to enter inside AdS. The string goes really near the center where the metric is zero.

For a time invariant configuration

$$S = T \int dx \sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}}$$

To find the minimal area is just a classical exercise with Euler-Lagrange equation: $\partial_x \frac{\delta S}{\delta \partial_x U} - \frac{\delta S}{\delta U}$. The result is (hepth/9803002)

$$E = \frac{(g_{YM}^2 N)^{1/2}}{L}$$

Right behavior 1/L for a CFT (dimensional analysis: no scales). The coefficient is the (prediction for the) strong coupling result. At weak coupling $E = \frac{g_{YM}^2 N}{L}$

3. Spectrum of operators

CFT Operators and AdS fields are classified by the quantum number under $SO(4,2) \times SU(4)$. The maximal compact subgroup of $SO(4,2)$ is $SU(2) \times SU(2) \times U(1)$: this gives quantum number (j_1, j_2, Δ). Δ is the conformal dimension in the CFT and the *energy* ($m^2 = \Delta(\delta - 4)$) in AdS.

The summary:

KK modes on S^5	N=8 short multiplet
	maximum spin 2
	mass of order 1
CFT operators	canonical dimension
	protected by susy

on the other hand,

stringy states	long multiplets
	maximum spin 4
CFT ops	anomalous dimensions
	$\gamma = \sqrt{x} \to \infty$
	they decouple

We now compare the supergravity KK modes with CFT operators. In the supergravity limit, all stringy states are very massive and decouple. In the CFT these states should correspond to operators with anomalous dimension $\sqrt{x}$ (as can be seen from $m^2 = \Delta(\delta - 4)$ when $m^2 =$ integer$/\alpha'$). Since they have infinite dimension, they decouple from all the OPE and Green functions.

3a. The spectrum of excitations of $AdS_5 \times S^5$:

The bosonic massless modes in 10d are

$$(g_{\mu\nu}, B^{NS-NS}_{\mu\nu}, B^{R-R}_{\mu\nu}, \phi, \tilde{\phi}, A^{+}_{\mu\nu\rho\sigma})$$

They give rise to 5d modes by expansion on spherical harmonics on the sphere

for a scalar : $\phi(x,y) = \sum \phi^I(x) Y_I(y)$

where Y_I are the eigenfunctions of the Laplacian on S^5.

This is the generalization of the Fourier expansion on S^1: $\phi(x,y) = \sum_k \phi_k(x) e^{iky/R}$. The masses of the modes ϕ_k are given by the eigenvalues of the Laplacian: k^2/R^2.

The zero-modes are massless and SEPARATED from massive modes by a quantity of order $1/R^2$. Massive modes decouple for large R.

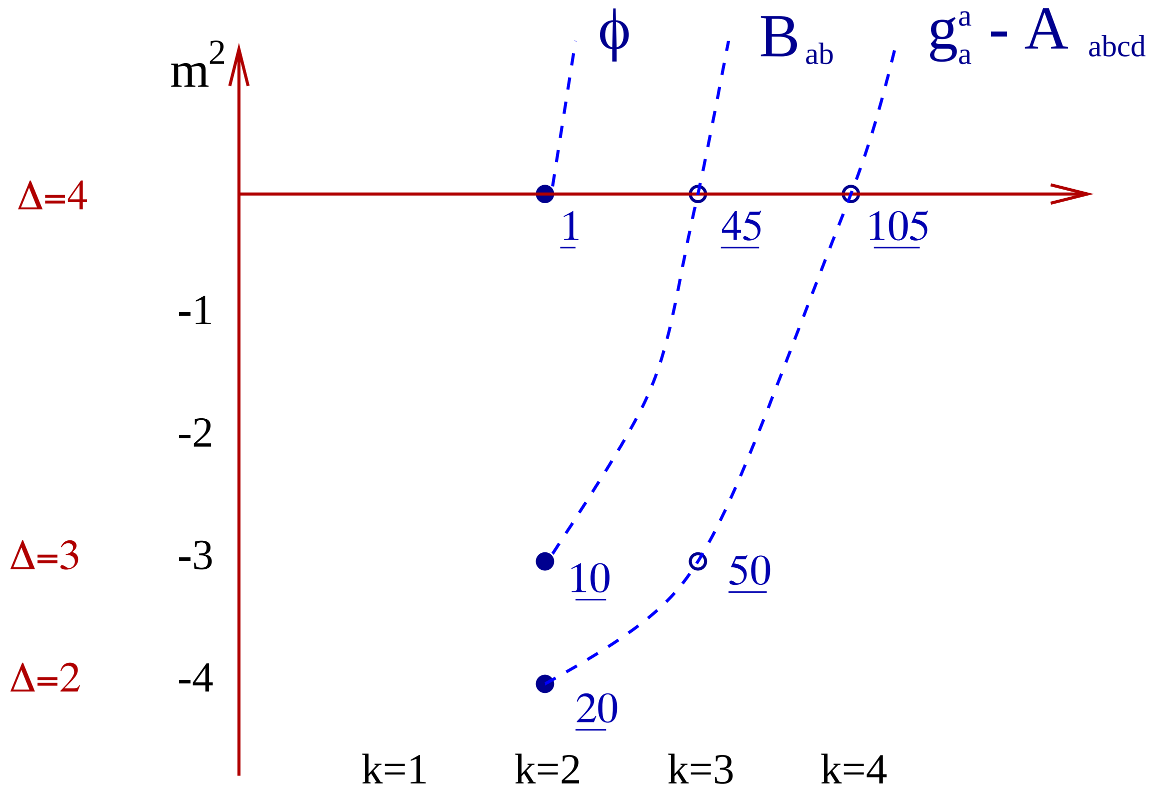
Due to the non-zero curvature, in AdS there is no separation: all the KK modes have a mass of the same order of the zero-modes.

Strictly massless modes are those with gauge invariance (graviton,etc...). But even some of their susy partners (the scalars partners of the graviton, for example), not protected by gauge invariance, are massive. KK modes have masses of the same order and can not be decoupled by taking the internal manifold large.

The KK spectrum was computed in the eighties. It consists of a series of N=8 multiplets A_k labelled by an integer $k \geq 2$. $k = 2$ is the graviton multiplet:

$$(g_{\mu\nu},\ \psi^i_\mu :\ \text{in the}\ \underline{4}, A_\mu :\quad \underline{15},\quad B_{\mu\nu} :\ \underline{6} + \underline{\bar{6}},$$
$$\lambda :\ \underline{4} + \underline{10}_c,\quad \text{scalars} :\ \underline{1}_c + \underline{10}_c + \underline{20})$$

KK scalars



These are all the scalars with negative (or zero) mass.
(relevant and marginal operators).
many others scalars have positive mass.

We indicated the $SO(6)$ repr. The lowest state in A_k is a scalar in the k-fold symmetric rep. of $SO(6)$ with mass $m^2 = k(k-4)/R^2$.

A_2 is the multiplet of the gauged N=8 supergravity in 5d.

3b. The spectrum of operators of N=4 SYM:

Using the fact that global symmetries in CFT correspond to gauge symmetries in AdS, we identify $g_{\mu\nu}$ with $T_{\mu\nu}$ and the 15 gauge fields with the $SO(6)$ R-currents J_μ. They belong to a supermultiplet of currents:

$$(Tr\phi_i\phi_j - \text{trace}, \cdots, J_\mu, T_{\mu\nu})$$

Conserved currents in CFT have canonical dimensions. Their scalar partners are not conserved, but they also have canonical dimensions by supersymmetry. The supercurrent-multiplet corresponds to A_2 on the AdS side.

The multiplets corresponding to A_k are obtained by applying supersymmetries to the operator

$$Tr\phi_{\{i_1} \cdots \phi_{i_k\}} - \text{traces}$$

of dimension k. One can prove that this multiplet is short and therefore has protected dimensions.

Exercise: identify all scalars in the previous figure, by using susy, mass/dimension relation and $SU(4)$ quantum numbers. Result:

SU(4) rep.	operator	multiplet/dim.
$\underline{20}$	$\mathrm{Tr}\phi_{\{i}\phi_{j\}} - \mathrm{traces}$	A_2 $\Delta = 2$
$\underline{50}$	$\mathrm{Tr}\phi_{\{i}\phi_j\phi_{k\}} - \mathrm{traces}$	A_3 $\Delta = 3$
$\underline{10_c}$	$\mathrm{Tr}\lambda_a\lambda_b + \phi^3$	A_2 $\Delta = 3$
$\underline{105}$	$\mathrm{Tr}\phi_{\{i}\phi_j\phi_k\phi_{p\}} - \mathrm{traces}$	A_4 $\Delta = 4$
$\underline{45_c}$	$\mathrm{Tr}\lambda_a\lambda_b\phi_i + \phi^4$	A_4 $\Delta = 4$
$\underline{1_c}$	$\mathrm{on-shell\,Lagrangian}$	A_2 $\Delta = 4$

There is a complete correspondence with the KK spectrum. It is believed that these are ALL the short multiplets of N=4 SYM.

Examples of stringy states: the traces in the previous formula. Simplest example: $Tr(\phi_i\phi_i)$.

As noticed in the eighties, some of the masses of AdS fields are negative. Due to the negative curvature, these modes are not *tachionic*. A mode is stable if $M^2 \geq -4$ (Breitenloner-Freedman bound). Using $m^2 = \Delta(\Delta - 4)$, we have

- Negative mass modes $\rightarrow$ Relevant operators ($\Delta < 4$)

- Massless modes $\rightarrow$ Marginal operators ($\Delta = 4$)

- Positive mass modes $\rightarrow$ Irrelevant operators ($\Delta > 4$)

In N=4, all operators have $\Delta \geq 2$ and therefore $m^2 \geq -4$. Stability is a consequence of supersymmetry. Non-supersymmetric solutions might be unstable (CFT not non-unitary. subtleties for $1 \leq \Delta \leq 2$).

4. Anomalies

The conformal group has a global anomaly, harmless for the CFT, which appears only when the CFT is coupled to external gauge fields, for example gravity. Given a conserved current J, the source h is the corresponding (external) gauge field. Anomaly means that $W(h)$ is not gauge invariant. Two examples:

$SU(4)$ anomaly:

N=4 SYM: Weyl fermions ψ_a in the $\underline{4}$ of $SU(4)$. Chiral rep. $\to SU(4)$ is anomalous. Can we see this from supergravity?

The 5d effective action for the massless multiplet on AdS_5 is completely fixed by N=8 supersymmetry: N=8 gauged supergravity. It contains a Chern-Simon coupling for the $SU(4)$ gauge fields: $L_{CS} = \int A \wedge dA \wedge dA$. Under a gauge transformation, this gives a boundary term:

$$L_{CS} \to \int_{AdS} d(F \wedge \tilde{F}) = \int_{\partial AdS} F \wedge \tilde{F}$$

That is, $W(A)$ is not invariant, but has the standard non-abelian anomaly.

Weyl anomaly: Central charges measure the Weyl rescaling/dilatation anomaly of a CFT. The effective action for an external metric explicitly depends on the Weyl factor σ; correspondingly, the stress-energy tensor is not traceless:

$$T_\mu^\nu = \frac{\delta W}{\delta \sigma} = aI_a + cI_c$$

where I_a, I_c are two particular invariants made with the Riemann tensor. The number of independent central charges in a 4d CFT is two, a and c.

Central charges do not depend on the coupling and can be computed in the free theory, where a formula is known in terms of the number N_i of fields of spin i:

$$c = \frac{12N_1 + 2N_{1/2} + N_0}{120} \qquad a = \frac{124N_1 + 11N_{1/2} + 1N_0}{720}$$

Regularizing $W(g)$ and removing divergences, T can be computed in supergravity: the functional form predicted by conformal invariance is reproduced. Moreover, we have the prediction that for all CFT described by AdS:

$$c = a$$

For N=4 SYM, $c = a = (N^2 - 1)/2$.

The computation only uses

$$L = \frac{1}{k^2} \int d^5x \sqrt{g}(R - \Lambda)$$

c=a also appear in the 2-point function:

$$< T(x)T(0) > \sim c/|x|^8$$

From $g = R^2 \hat{g}$, $\Lambda \sim 1/R^2$:

$$c \sim \Lambda^{-3/2} k^{-2} \tag{4}$$

Ratios between c of different CFT are easy to compute. For a spacetimes of the form $AdS_5 \times H$, k and Λ can be computed by dimensional reduction, and the ratios of c can be determined from the ratios of internal manifold volumes.

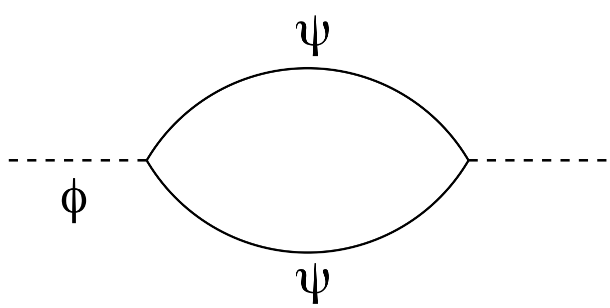
NON-CONFORMAL THEORIES

1. Finite temperature and QCD_3

Consider N=4 SYM on $R^3 \times S^1$, with anti-periodic boundary conditions for fermions:

$$\psi(y) = -\psi(y+2\pi R) \to \psi = \sum_{Z+1/2} \psi_k e^{iky/R}$$
$$m^2_{\psi_k} = \frac{k^2}{R^2} > 0 \quad (k>0) \text{ no zero mode}$$

All 3d fermions are massive, and scalars get mass at one-loop level:



At low energy (much below $T = 1/R$), the theory contains only massless gluons: it flows to QCD_3 in the IR.

The right limit:

$$\int d^4x \frac{1}{g_4^2} F_{\mu\nu}^2 = \int d^3x \frac{R}{g_4^2} F_{\mu\nu}^2 \;\rightarrow\; \frac{1}{g_3^2} = \frac{R}{g_4^2}$$

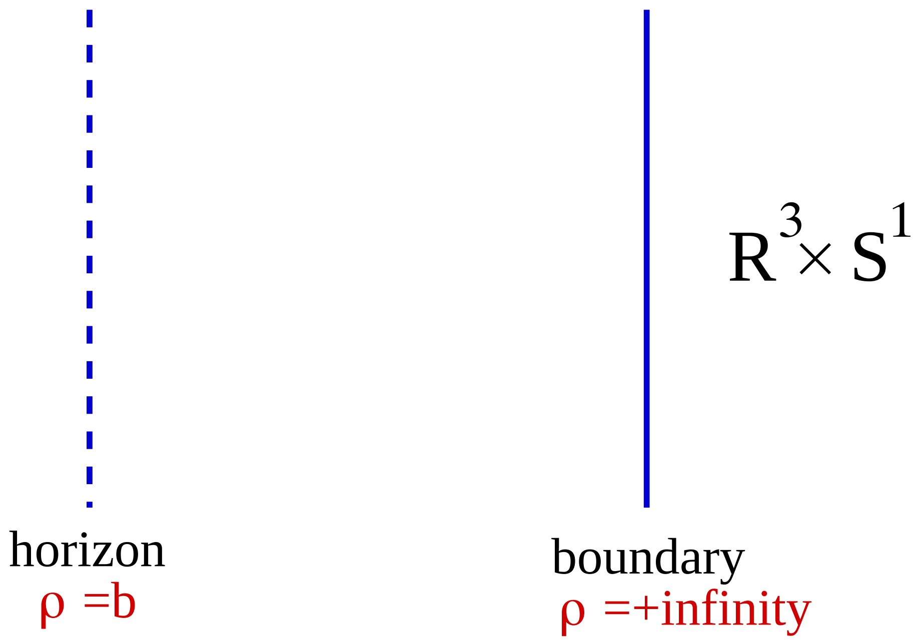
We need: $R \to \infty$ to decouple fermions and scalars, g_3 finite to describe QCD$_3$. This implies $g_4 \to 0$, a limit where supergravity is not valid.

Let us use supergravity anyway. We are considering the theory with a finite cut-off. Analogous to the strong coupling limit in lattice gauge theories. We can obtain all the qualitative features of QCD_3 (confinement, mass gap, ...) and compute the glue-ball spectrum.

We need a solution of type IIB supergravity where AdS$_5$ is deformed to another space with boundary $R^3 \times S^1$. Black hole solution:

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2}\right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2}} + \rho^2 \sum_{i=1}^{3} dx_i^2$$

τ is periodic of period πb.



The space admits only one spin structure: antiperiodic fermions.

2. Wilson loop

A crucial difference with respect to AdS: the 3d part of the metric is not zero at the center of the space: g is bounded.

$$\rho^2 \geq b^2 : \qquad \rho^2 \sum_{i=1}^{3} dx_i^2$$

Therefore,

$$S = \int \sqrt{g} \geq b^2 A$$

AREA LAW, with tension $T_{QCD} \sim b^2$.

3. The spectrum

Usual problem: how to extend fields from the boundary to the interior of the space. Near $\rho = b$,

$$\begin{aligned} ds^2 &= \frac{(d\rho)^2}{\rho - b} + (\rho - b)(d\tau)^2 + \sum (dx_i)^2 \\ &= (dr)^2 + r^2 (d\tau)^2 + \cdots \end{aligned}$$

$r = \sqrt{\rho - b}$. R^2 in polar coordinates: regular space. We impose also on fields regularity at the center $\rho = b$.

Expand a massive field

$$\phi(x, \tau, \rho) = \phi(\rho) e^{in\tau} e^{ikx}$$

$M^2 = -k^2$ is the mass of the 3d field. We are interested in the lowest states: keep only the zero-mode on S^1 (n=0).

$$S = \int_b^\infty d\rho \rho^3 \left\{ \left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right) \left(\frac{\partial \phi}{\partial \rho} \right)^2 + m^2 \phi^2 + \frac{k^2}{\rho^2} \phi^2 \right\}$$

Near $\rho = \infty$ the metric is AdS: $(d\rho)^2/\rho^2 + \rho^2 (dx_\mu)^2$ ($\rho = e^r$). Two asymptotic solutions

$$\phi_+ = \rho^{\Delta - 4}, \phi_- = \rho^{-\Delta}$$

In AdS regularity at the center selects ϕ_+. With a more complicated metric:

$$\phi \sim A(k) \rho^{\Delta - 4} + B(k) \rho^{-\Delta}$$

From the explicit solution of the eqs. of motion we can compute all Green functions. In a theory with mass gap and discrete spectrum, we expect poles in the two point functions corresponding to the physical states (mesons, glueballs,...)

$$\langle \phi(q) \phi(o) \rangle = \sum_i \frac{A_i}{q^2 - M_i^2}$$

M_i^2 are the masses of the glueballs. The lowest spin glueball corresponds to $\phi = TrF_{\mu\nu}^2, \Delta = 4$.

Normalizing fields as in AdS:

$$\hat{\phi}_k(\rho) = \phi_k \left[\rho^{\Delta-4} + \frac{B(k)}{A(k)}\rho^{-\Delta}\right]$$

where ϕ_k is the value of the boundary source. We obtain poles iff $A(k) = 0$.

- Mass of the glueballs: $M^2 = -k^2$, where k is such that the eqs. of motion have NORMALIZABLE solutions ($A(k) = 0$).

- It is a Schrodinger-like problem: eigenvalues of a second order differential equation.

– No normalizable solutions for $k^2 \geq 0$

Proof: Assume $m^2 \geq 0$ (true for the lowest glueball $F^2, \Delta = 4, m^2 = 0$). On a solution of eqs. of motion, S reduces to $\int \phi \partial_n \phi$ (see eq. (3)), which is zero for a normalizable sol. Since S is a sum of positive definite terms, $\rightarrow \phi = 0$.

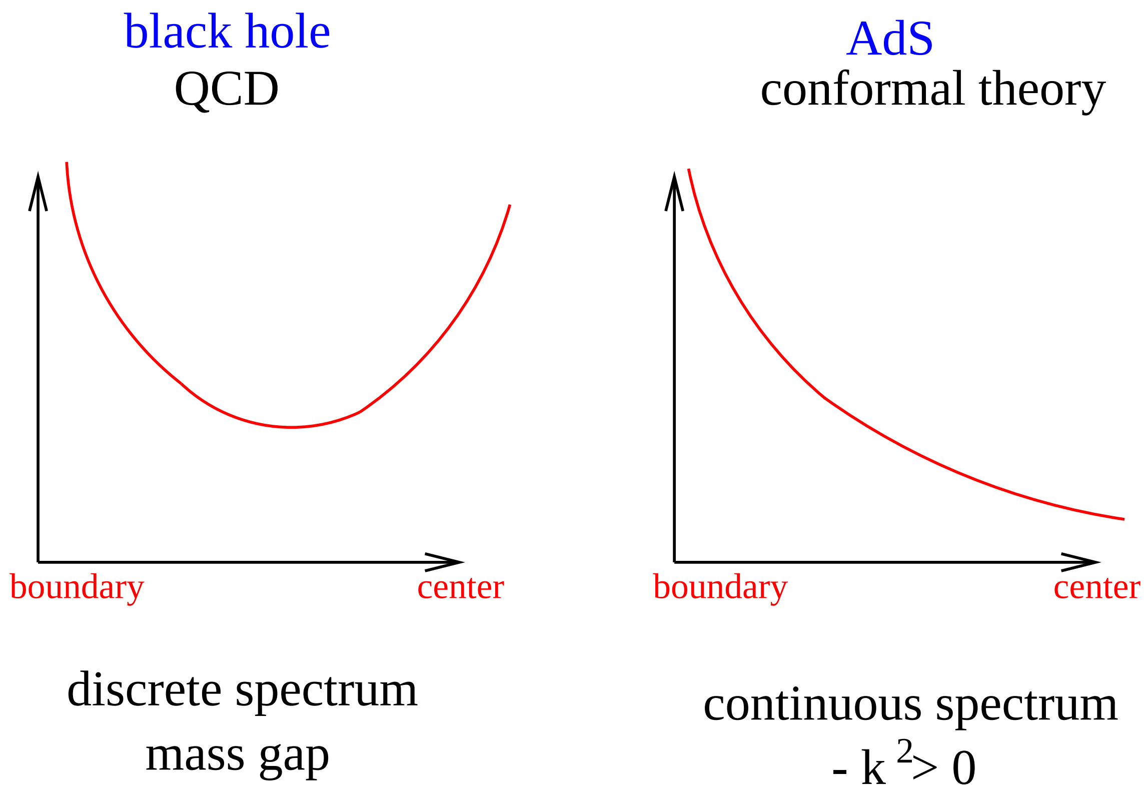
– Normalizable solutions may exist for $k^2 < 0$.

By redefinition of fields and coordinates ($\rho \rightarrow x(\rho), \phi \rightarrow \psi(x)$), we can always reduce the problem to a Schroedinger equation

$$-\psi'' + V(x)\psi = -k^2\psi, \qquad E = -k^2 = M^2$$

The eigenvalues give the glueball masses.

Exercise: verify that V has the form:



QCD$_4$ can be discussed in a similar way. We start with $AdS_7 \times S^4$ (dual to the 6d (0,2) exotic theory with tensionless strings) and compactify on a two-torus.

Validity of the approach:

We can think of R as a regulator for QCD$_3$. When embedded in N=4 SYM, the theory is finite. To get a well defined QCD$_3$ theory, we remove the cut-off ($R \to 0$) with a fine tuning of the coupling ($g_4(R) \to 0$). However, if we use supergravity, we are in the large g_4 regime. The KK modes on the circle have a mass comparable with the scale of QCD_3 and they do not decouple. We can think of this as a theory with an ultraviolet cut-off.

A good analogy is with lattice gauge theory. R corresponds to the lattice spacing. The continuum limit is obtained with a fine tuning $a \to 0, g(a) \to 0$. However we can study the lattice theory at strong coupling, far from the continuum limit. A standard computation at strong coupling (by Wilson) gives the area law. We just did an analogous computation with supergravity. Qualitative features of QCD still hold at strong coupling.

Result: What we discussed is not real QCD_3

- Supergravity is not valid for small x, where we can obtain QCD.

- Fermions and scalars do not decouple. *Glueballs* with SO(6) quantum numbers have the same mass of ordinary glueballs.

- Real QCD has Regge trajectories: glueballs with arbitrary spin. Supergravity have fields with maximum spin 2.

Alternative to the lattice? For real QCD, we need string theory on $AdS_5 \times S^5$ for small x. String theory contains fields with arbitrary spin. We need the exact 2d CFT describing $AdS_5 \times S^5$: problems with the RR-fields of the background. Few progresses.

REFERENCES

I did not included references (too many...).

Three basic ones:

Maldacena (hepth/9711200), Gubser-Klebanov-Polyakov (9802109), Witten (9802150).

And for: Wilson loop (Maldacena, 9803002), Finite temperature and QCD (Witten, 9803131)

An (almost) complet list of references can be found in the review Maldacena et al. (9905111), which covers all the topics in these notes.