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M2–BRANE GAUGE THEORIES, KALUZA KLEIN SPECTRA and METRIC CONES on G/H

1. In the $'80.s$ Freund–Rubin solutions of D=11 on

$$AdS_4 \times (G/H)_7 \qquad \text{with } N_{G/H} \text{ SUSY}$$

2. Kaluza–Klein spectra of $Osp(N|4) \times G'$ multiplets, but no **singleton**!

3. $\exists$ classical M2–branes that interpolate between

$$AdS_4 \times (G/H)_7 \quad \text{and} \quad Mink_3 \times \mathcal{C}_8\,(G/H)$$

   $\mathcal{C}_8\,(G/H) =$ the metric cone over G/H

4. Since AdS/CFT search for corresponding $d = 3$ SCFT. Here **singleton** = fundamental fields of $d = 3$ gauge theory with N SUSY: gauge multiplets and chiral multiplets, if $N = 2$, in suitable flavor $G' \subset G$ representation

5. The Kaluza Klein spectrum must match the SCFT spectrum. Strategies: study singular cone properties and supermembrane action in its own background

Bibliography and Collaborators

RECENT WORK

M^{111} spect., hep-th 9903036; $Osp(N|4)$ supfiel., hep-th 9905134 — D. Fabbri, L. Gualtieri, P. Fré, P. Termonia

Brane gauge theory, *Work in pro.* — D. Fabbri, L. Gualtieri, P. Fré, A. Zaffaroni, C. Reina, A. Zampa, A. Tommasiello

OLD WORK of 1983-1985 :

A. Ceresole, R. D'Auria, P. Fré, L. Castellani, P. Van Nieuwnhuizen, & H. Nicolai, *in various combiantions*

(see hep-th 9903036 for complete set of references)

Mp–branes and the angular G/H–manifold

G/H **M2–branes** (*Ceresole et al 1998*) are solutions of the classical field equations of $D = 11$ SUGRA of the following form:

$$
\begin{aligned}
ds^2_{11} \;=\; & \left(1+\frac{k}{r^6}\right)^{-\frac{2}{3}} dx^\mu \, dx^\nu \eta_{\mu\nu} \\
+ \; & \left(1+\frac{k}{r^6}\right)^{\frac{1}{3}} \left(dr^2 + r^2 \, ds^2_{G/H}\right)
\end{aligned}
$$

where

$$
r \;\geq\; 0 \quad ; \quad ds^2_{\mathcal{C}(G/H)} \;\equiv\; dr^2 + r^2 \, ds^2_{G/H}
$$

describes the **metric cone** constructed over a 7–dimensional homogeneous space

$$
(G/H)_7
$$

The 8th coordinate r of the cone $\mathcal{C}(G/H)$ is also the **radial distance** from the brane in transverse space, $\mu = 0, \ldots, 2$.

The **isometry group** of the 11–dimensional metric is:

$$
\boxed{\mathcal{I}_{p-brane} \;=\; ISO(1,2) \otimes G}
$$

$\boxed{\text{Kaluza Klein and branes}}$

(1982-1984)

$D = 11$ SUGRA has **Freund–Rubin** vacua :

$$\begin{aligned} \mathcal{M}_{11} &= AdS_4 \times \left(\frac{G}{H}\right)_7 \\ \mathcal{M}_{11} &= AdS_7 \times \left(\frac{G}{H}\right)_4 \end{aligned}$$

Type IIB SUGRA in $D = 10$ has vacua:

$$\mathcal{M}_{10} = AdS_5 \times \left(\frac{G}{H}\right)_5$$

where **anti de Sitter space** *is*

$$AdS_D = \frac{SO(2, D-1)}{SO(1, D-1)}$$

and

$$\left(\frac{G}{H}\right)_n = n\text{–dimensional coset manifold}$$

AT THE SAME TIME:

$$\begin{array}{rcll} F_{\mu_1\mu_2\mu_3\mu_4} &=& e\,\epsilon_{\mu_1\mu_2\mu_3\mu_4} & \text{M2–brane} \\ {}^\star F_{\mu_1\dots\mu_7} &=& g\,\epsilon_{\mu_1\dots\mu_7} & \text{M2–brane} \\ F_{\mu_1\dots\mu_5} &=& g\,\epsilon_{\mu_1\dots\mu_5} & \text{D3–brane} \end{array}$$

Near the horizon ($r \to 0$)

the exact metric is approximated by:

$$M_p^{hor} = AdS_{p+2} \times (G/H)_{D-p-2}$$

with isometry group:

$$\mathcal{I}_p^{hor} = SO(2, p+1) \times G$$

Near the spatial–infinity ($r \to \infty$)

the exact metric is approximated by :

$$M_p^{\infty} = \mathrm{Mink}_{p+1} \times \mathcal{C}\,(G/H)$$

The cone $\mathcal{C}\,(G/H)$ is a flat manifold with a singularity at $r = 0$. G/H is **Sasakian** *iff* $\mathcal{C}\,(G/H)$ is **Kähler**. It is **Sasakian Einstein** if $\mathcal{C}\,(G/H)$ is **Ricci flat Kähler**, i.e. a singular Calabi Yau fold. G/H must be **Sasakian Einstein** for $N = 2$ SUSY in $D = 4$.

$\boxed{\textbf{Magic Observation}}$

(Claus, Kallosh and Van Proeyen (November 1997)

For $p = 2$ and $p = 5$ $\mathcal{I}_p^{hor}$ is the bosonic sector of a algebra $\mathcal{SC}_p$ interpreted as **superconformal algebra** on the p–brane world–volume.

$$\mathcal{I}_2^{hor} = SO(2,3) \times SO(8) \quad ; \qquad \mathcal{SC}_2 = Osp(8|4)$$

$$\mathcal{I}_5^{hor} = SO(2,6) \times SO(5) \quad ; \quad \mathcal{SC}_5 = Osp(2,6|4)$$

$\boxed{\textbf{IN GENERAL}}$:

G/H **M–branes** are interpolating solitons between Minkowski$_{p+1}$ $\times$ **Cone**(G/H) and AdS_{p+2} $\times$ G/H , with $N_{G/H} < N_{max}$ SUSY.

Superconformal theory on the boundary of AdS_{p+2} with $N_{G/H}$ SUSY.

At the horizon: $N_{\frac{G}{H}}$–extended supergravity in AdS_{p+2}, $\times\ G/H$

$N_{\frac{G}{H}}$ Killing spinors, i.e. solutions of:

$$\left[\mathcal{D}_m^{G/H} + e\Gamma_m\right]\eta = 0$$

For all Freund Rubin manifolds $N_{\frac{G}{H}}$ was determined in the eighties

Each η yields a Killing spinor for the *corresponding soliton*.

New Killing spinor is restricted, by a projection operator that halves its components:

Hence in G/H M–branes number of preserved SUSY is

$$\begin{aligned} \#\text{SUSY in M2} &= \frac{1}{2} \times \frac{32}{8} \times N_{\frac{G}{H}} = 2\,N_{\frac{G}{H}} \\ \#\text{SUSY in M5} &= \frac{1}{2} \times \frac{32}{4} \times N_{\frac{G}{H}} = 4\,N_{\frac{G}{H}} \end{aligned}$$

The Role of Holonomy

Holonomy is the integrability condition for the Killing spinor equation. On the internal manifold G/H we obtain:

$$(R^{mn}{}_{rs} - \delta^{mn}_{rs})\,\Gamma_{mn}\,\eta = 0$$

The left hand side of this equation defines combination of $SO(7)$ generators spanning a subalgebra Hol.

Holonomy determines how many Killing spinors we can have. It is the number of **singlets** when we decompose the spinorial $\mathbf{8}$ of $SO(7)$. We have:

$$\begin{array}{rcccc}
Hol = G_2 & \rightarrow & N_{\frac{G}{H}} = 1 & & \mathbf{8} = \mathbf{7} + \mathbf{1} \\
Hol = SU(3) & \rightarrow & N_{\frac{G}{H}} = 2 & & \mathbf{8} = \mathbf{3} + \bar{\mathbf{3}} + 2 \times \mathbf{1} \\
Hol = SU(2) & \rightarrow & N_{\frac{G}{H}} = 3,4 & & \mathbf{8} = \mathbf{2} + \bar{\mathbf{2}} + 4 \times \mathbf{1} \\
Hol = SO(7)) & \rightarrow & N_{\frac{G}{H}} = 0 & & \mathbf{8} = \mathbf{8}
\end{array}$$

Note this **Holonomy** of $SO(8)$ Lie algebra valued connection, **not usual holonomy !**

G/H M–branes with $N_{G/H} > 0$ are BPS states with isometry:

$$\mathcal{I}_{G/H-p-brane} = ISO(1,p) \otimes G$$

$\boxed{\textbf{OLD KK RESULT}}$: If $N_{G/H}$ Killing spinors, then necessarily (*Fré & D'Auria 1983*)

For $(G/H)_7$: $G = G' \otimes SO\left(N_{G/H}\right)$

For $(G/H)_4$: $G = G' \otimes Usp\left(N_{G/H}\right)$

The factor $SO\left(N_{G/H}\right)$ with $SO(2,3)$ of AdS_4 and the supercharges yields

$$Osp\left(N_{G/H}|4\right)$$

Similarly $Usp\left(N_{G/H}\right)$ with $SO(2,6)$ of AdS_7 and the supercharges yields

$$Osp\left(2,6|N_{G/H}\right)$$

Hence full superconformal symmetry is:

$$\mathcal{SC}_2^{G/H} = Osp\left(N_{G/H}|4\right) \times G'$$

$$\mathcal{SC}_5^{G/H} = Osp\left(2,6|N_{G/H}\right) \times G'$$

Two Interpretations

IN KALUZA KLEIN THEORY

$$G' \;=\; \textbf{gauge group}$$

IN CONFORMAL THEORY

$$G' \;=\; \textbf{flavour group}$$

Indeed in the **bulk** = (supergravity) we have massless gauge multiplets of G' group.

In the brane gauge theory G' must be a global symmetry group acting on the **microscopic singleton fields** living on the brane world volume.

$\boxed{\text{The general form of the brane conformal theory}}$

$\mathcal{N} = N_{G/H}$ supersymmetric gauge theory of $U(N) \times U(N)$ in $d = p + 1$ with gauge and matter multiplets and **Conformal fixed point in the** $N \to \infty$ **limit. In our case** $d = 3$

Supersymmetric Freund Rubin Cosets

G/H	G/H	$N_{G/H}$
S^7	$\frac{SO(8)}{SO(7)}$	8
squashed S^7	$\frac{SO(5)\times SO(3)}{SO(3)\times SO(3)}$	1
M^{ppr}	$\frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)^2}$	2
N^{010}	$\frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$	3
N^{pqr}	$\frac{SU(3)\times U(1)}{U(1)^2}$	1
Q^{ppp}	$\frac{SU(2)^3}{U(1)^3}$	2
B^7_{irred}	$\frac{SO(5)}{SO(3)_{max}}$	1
$V_{5,2}$	$\frac{SO(5))\times U(1)}{SO(3)\times U(1)}$	2

General Features of the $N = 2$ cosets

We have:

$$\frac{G}{H} = \text{circle bundle over } \frac{G'}{H}$$

The 6–manifold G'/H is a 3–dimensional complex projective algebraic variety (*work in progress with C. Reina*). The structure of the singular $C.Y.$ 4–fold corresponding to the metric cone on G/H is determined by the grade of this circle bundle.

$\boxed{\textbf{EXAMPLE: The } M^{111} \textbf{ space}}$

$$M^{pqr} = \frac{\mathcal{G}_s}{H} \equiv \frac{SU(3)^c \times SU(2)^w \times U(1)^Y}{SU(2)^c \times U(1)' \times U(1)''} \qquad (1)$$

M^{pqr} manifolds, introduced by Witten in 1981, were shown to be solutions of $D = 11$ supergravity by Castellani D'Auria and Fré in 1982, who also found N=2 SUSY

$$p = q = \text{odd}$$

The M^{111} space

$$\begin{cases} \mathbb{G} = \mathbb{H} \oplus \mathbb{K} \\ \mathbb{G} = SU(3) \times SU(2) \times U(1) \\ \mathbb{H} = SU(2)^c \times U(1)' \times U(1)'' \end{cases}$$

$SU(2)^c$ is the isospin $SU(2)$ subgroup of $SU(3)$ so that triplet representation branches as follows:

$$\mathbf{3} \stackrel{SU(2)^c}{\longrightarrow} \mathbf{2} \oplus \mathbf{1}.$$

The $U(1)'$ and $U(1)''$ factors of H are generated by Z' and Z'', *i.e.* two independent linear combinations of the three remaining abelian generators of $SU(3) \times SU(2) \times U(1)$ commuting with $SU(2)^c$.

Using the Gell–Mann matrices for $SU(3)$, Pauli matrices for $SU(2)$ the list is $\frac{\sqrt{3}}{2} i\lambda_8$, $\frac{1}{2} i\sigma_{\ddot{3}}$, and $iY_{\dddot{3}}$, Z' and Z'' are a basis for the orthogonal complement to the $U(1)$ sitting in $\mathbb{K}$ rather than $\mathbb{H}$:

$$\boxed{Z = p\frac{\sqrt{3}}{2} i\lambda_8 + \frac{1}{2} qi\sigma_{\ddot{3}} + riY_{\dddot{3}}}$$

$$\text{with} \quad p = q = r = 1$$

The $U(1)$–bundle structure of the space M^{111}, when extended to a line–bundle yields the metric cone which, therefore, is identified with:

$$\mathcal{C}(M^{111}) \;=\; \mathcal{O}_{\mathbf{P^2}}(-3) \times \mathcal{O}_{\mathbf{P^1}}(-2) \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$$

If U^i (i=1,2,3) are the homogeneous coordinates of $\mathbb{P}_2$. and V^A (Ai=1,2) those of $\mathbb{P}_1$ we have the immersion

$$\mathbb{P}_2 \times \mathbb{P}_1 \hookrightarrow \mathbb{P}_{29}$$

writing the 30 homogeneous coordinates X of $\mathbb{P}_{29}$ as:

$$X^{ijk|AB} \;=\; U^i U^j U^k V^A V^B$$
$$(i,j,k = 1,2,3\,;\; A,B = 1,2.)$$

Namely the coordinates X are assigned to the following representation of $SU(3) \times SU(2)$:

$$X^{ijk|AB} \mapsto \mathbf{(10,2)} \equiv \boxed{}\boxed{}\boxed{} \otimes \boxed{\times}\boxed{\times}$$

Following a general method this embedding is reformulated by writing $\mathbf{325}$ quadric equations. This defines a 4–fold in $\mathbb{C}^{30}$ $=$ **algebraic description of metric cone.**

These equations $\longrightarrow$ superpotential of the $SCFT_3$ (work in progress) (Example given by Witten Klebanov for the case of $SCFT_4$ from D3-brane on T^{11})

Nature of the embedding equations

There are

$$\frac{30 \times 31}{2} = \mathbf{465}$$

quadric monomials X^2, spanning the tensor product:

$$\left(\left[\begin{ytableau} \; & \; & \; \end{ytableau} \otimes \begin{ytableau} \times & \times \end{ytableau} \right] \otimes \left[\begin{ytableau} \; & \; & \; \end{ytableau} \otimes \begin{ytableau} \times & \times \end{ytableau} \right] \right)_{symm}$$

On the locus, defined by the explicit embedding only $28 \times 5 = \mathbf{140}$ of these components are independent. These components fill the representation of highest weight:

$$(\mathbf{28}, \mathbf{5}) \equiv \begin{ytableau} \; & \; & \; & \; & \; & \; \end{ytableau} \times \begin{ytableau} \times & \times & \times & \times \end{ytableau}$$

The complete decomposition into **irreps** of the tensor product is the following:

$$\begin{aligned} \mathbf{465} &= \underbrace{(\mathbf{28}, \mathbf{5})}_{140} \oplus \underbrace{\sum_{i=1}^{9} \Lambda_i}_{325} \\ \sum_{i=1}^{9} \Lambda_i &= (\mathbf{28}, \mathbf{1}) \oplus (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{9}, \mathbf{1}) \oplus (\mathbf{8}, \mathbf{1}) \\ &\quad \oplus (\mathbf{10}, \mathbf{5}) \oplus (\mathbf{9}, \mathbf{5}) \oplus (\mathbf{8}, \mathbf{5}) \\ &\quad \oplus (\mathbf{35}, \mathbf{3}) \oplus (\overline{\mathbf{10}}, \mathbf{3}) \end{aligned}$$

The 325 quadric equations are

$$\boxed{\Lambda_i = 0}$$

Another example : Q^{111} **(***D'Auria, Fré, van Nieuwenhuizen, 1984***)**

$$Q^{111} = \frac{SU(2) \times SU(2) \times SU(2) \times U(1)}{U(1) \times U(1)' \times U(1)'' \times U(1)'''}$$

Here the line bundle structure of the cone is:

$$\begin{aligned} \mathcal{C}(Q^{111}) &= \mathcal{O}_{\mathbf{P2}}(-1) \times \mathcal{O}_{\mathbf{P1}}(-1) \times \mathcal{O}_{\mathbf{P1}}(-1) \\ &\quad \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \end{aligned}$$

and we have the embedding:

$$\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \hookrightarrow \mathbb{P}_7$$

where the 8 homogeneous coordinates X of $\mathbb{P}_7$ are threelinear in the homogeneous coordinates of the three $\mathbb{P}_1$ A_i, B_j, C_k:

$$X^{ijk|AB} = A^i B^j C^k \quad (i, j, k = 1, 2)$$

namely they are assigned to the following irrep of $SU(2) \times SU(2) \times SU(2)$:

$$X^{ijk} \mapsto (\mathbf{2}, \mathbf{2}, \mathbf{2}) \equiv \boxed{} \otimes \boxed{\times} \otimes \boxed{\cdot}$$

In angular momentum notation we have:

$$X^{ijk} = \left(j_1 = \tfrac{1}{2}, j_2 = \tfrac{1}{2}, j_3 = \tfrac{1}{2}\right)$$

and it is easy to find struture of embedding equations $\Rightarrow$ **next slide**

Here we have:

$$\left[\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)\times\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)\right]_{sym} = \underbrace{(1,1,1)}_{27}$$
$$\oplus \underbrace{(1,0,0)+(0,1,0)+(0,0,1)}_{9}$$

There are 9 embedding equations Λ_i given by the vanishing of the irreducible representations not of highest weight:

$$\begin{aligned}
0 &= \left(\epsilon\sigma^A\right)_{ij} X^{i\ell p} X^{jmq}\, \epsilon_{\ell m}\, \epsilon_{pq} \\
0 &= \left(\epsilon\sigma^A\right)_{\ell m} X^{i\ell p} X^{jmq}\, \epsilon_{ij}\, \epsilon_{pq} \\
0 &= \left(\epsilon\sigma^A\right)_{pq} X^{i\ell p} X^{jmq}\, \epsilon_{ij}\, \epsilon_{\ell m}
\end{aligned}$$

Hence the metric cone $\mathcal{C}(G/H)$ defines the classical commutative ring of polynomials in the homogeneous coordinates of embedding projective space $\mathbf{P}^{\cdots}$, modulo the ideal generated by the quadric equations:

$$\boxed{\mathcal{R}_{class} = \frac{\mathbb{C}[X]}{\Lambda_i}}$$

This is the ring of

$\boxed{\textbf{Kaluza Klein chiral supermultiplets !}}$

$\boxed{\text{PROBLEM:}}$ **Determine IR gauge theory on the boundary.** From stability group of the horizon G/H we guess gauge group and representation assignments of matter multiplets (*Work in progress: Zaffaroni, Reina, Fré, Fabbri, Gualtieri, Zampa and Tommasiello*)

$\boxed{Q^{111}}$ is given by the locus:

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = |C_1|^2 + |C_2|^2$$

modded by the identifications with respect to $U(1) \times U(1)$:

$$\begin{aligned}(A, B, C) &\rightarrow (e^{i\alpha} A, B, e^{-i\alpha} C)\\(A, B, C) &\rightarrow (A, e^{i\alpha} B, e^{-i\beta} C)\end{aligned}$$

$U(1) \times U(1)$ is the gauge group for a single brane. For N branes we have:

$$\begin{aligned}\mathcal{G}_{gauge} &= U(N) \times U(N)\\A &\rightarrow (N, 1) \quad \text{(chiral multiplet)}\\B &\rightarrow (1, N) \quad \text{(chiral multiplet)}\\C &\rightarrow (\bar{N}, \bar{N}) \quad \text{(chiral multiplet)}\end{aligned}$$

OPEN PROBLEM: we have to find a flavor invariant superpotential $W(A, B, C)$ such that modulo its derivatives we reconstruct as ring chiral operators the classical homogeneous coordinate ring. This is necesssary to match Kaluza Klein spectrum. (*Work in progress*)

$Osp(2|4)$ supermultiplets

The even subalgebra is a direct sum:

$$G_{\text{even}} = Sp(4, \mathbb{R}) \oplus SO(2) \subset Osp(2|4)$$

where $Sp(4, \mathbb{R}) \sim SO(2,3)$ is the isometry of AdS_4 while $SO(2)$ is R–symmetry. The maximally compact subalgebra of G_{even} is

$$G_{\text{compact}} = SO(2)_E \oplus SO(3)_S \oplus SO(2)_R \subset G_{\text{even}}.$$

The generator of $SO(2)_E$ is the hamiltonian. A (UIR) of $Osp(2|4)$ is composed of a *finite* number of UIR of G_{even}, each of them being a particle state, characterized by a spin "s", a mass "m" and a hypercharge "y". Each UIR of G_{even} is an *infinite* tower of finite dimensional UIR.s of the compact subalgebra G_{compact}.

We have a Clifford vacuum, that has energy E_0, spin s_0 and hypercharge y_0:

$$|E_0, s_0, y_0 >$$

and there is a unitarity bound:

$$E_0 \geq |y_0| + s_0 + 1.$$

Our programme

- Construction of the three $\mathcal{N}=2$ multiplet spectra:

    1. $Osp(2|4)\times SU(3)\times SU(2)$ for M^{111}

    2. $Osp(2|4)\times SO(5)$ for $V_{5,2}$

    3. $Osp(2|4)\times SU(2)^3$ for Q^{111}

- Construction of the single $\mathcal{N}=3$ on N^{010}. Here we have $Osp(3|4)\times SU(3)$ multiplets.

- Search of the appropriate Kählerian conifold that yields SCFT for each case.

- Test of the AdS/CFT conjecture at the level of long and the short multiplets.

- Search of effective low energy $\mathcal{N}=2$ lagrangians, *i.e.* of the Special Kähler manifold for each $\mathcal{N}=2$ case.