Non-radial oscillations of stars in general relativity: a scattering problem.

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The problem of non-radial oscillations of stars can be formulated as a problem of resonant scattering of gravitational waves incident on the potential barrier generated by the spacetime curvature. This approach discloses some unespected correspondences between the theory of perturbations of stars and the theory of quantum mechanics. New relativistic effects are predicted, as the resonant behaviour of the axial modes in slowly rotating stars, due to the coupling with the polar modes induced by the Lense-Thirring effect.

1. Introduction

Non-radial oscillations of stars are manifested in a variety of astrophysical situations. For example, they are observed in the sun, and the corresponding frequencies, measured with very high accuracy, are used in modern heliosysmology to investigate the internal structure of the star. Moreover, non-radial pulsations are thought to be at the origin of the drifting subpulses and micropulses detected in some radio sources, and of the quasi-periodic variability seen in some X-ray burst sources and in a number of bright X-ray sources (McDermott, Van Horn & Hansen 1988). Due to their central role in astrophysics, oscillations of stars have been extensively studied both in the framework of the newtonian theory of gravity, and in general relativity. According to general relativity, a star vibrating into non radial modes emits gravitational waves, whereas gravitational waves do not exist in the newtonian theory. This difference is a substantial one, and it is the key point of a recent reformulation of the relativistic theory of stellar perturbations, whose main results we shall describe in this paper. This work has been developed in a series of papers, Chandrasekhar& Ferrari *a,b,c,d*, Chandrasekhar, Ferrari & Winston, 1991, Chandrasekhar& Ferrari *e*, to be referred to hereafter respectively as Paper I,II,III,IV,V and VI.

It is useful to clarify what are the specific questions to which one is addressed in formulating a theory of stellar oscillations. When a star is perturbed by some external agency, after a transient which depends on the cause of the perturbation, it will start to oscillate at some characteristic frequencies, that, as we have seen, appear to be coded in various radiative processes. Gravitational waves will also be emitted with these frequencies, and with some characteristic damping times which depend on the structure of the star. The determination of these characteristic frequencies is therefore one of the main objects of the theory. The new formulation of the problem of stellar oscillations presents several novelties with respect to the existing relativistic theory developed by Thorne and his collaborators (Thorne&Campolattaro 1967, Price & Thorne 1969, Thorne 1969). It leads to a different interpretation of the problem, which discloses some surprising and fascinating analogies with the theory of quantum mechanics. Moreover, it introduces a remarkable simplification of the problem, and allows a generalization of the theory to the case in which the star is slowly rotating. New phenomena, as the resonant behaviour of the axial modes, and the coupling between polar and axial modes induced by the Lense-Thirring effect, will emerge.

But in order to understand how the anticipated novelties are introduced by the new theory, we need to summarize and compare the newtonian theory and the previously formulated relativistic theory.

In the newtonian theory the equations that govern the adiabatic perturbations of a spherical star constitute a fourth-order linear differential system which couples the perturbation of the newtonian potential with the perturbations of the variables describing the fluid. All quantities are usually assumed to have a time dependence $\sim e^{i\sigma t}$, where σ is a constant frequency, an assumption which implies a Fourier decomposition of the modes of vibration. The system of equations must be integrated from the center to the surface of the star, with the boundary conditions that i) all physical quantities are regular at the origin, and ii) the perturbation of the pressure, δp, vanishes at the surface. These conditions are satisfied only for a specific set of *real* values of σ, $\{\sigma_n\}$, which are the frequencies of the *normal modes*. Thus the problem of finding the frequencies of the normal modes of a star in newtonian theory is *an eigenvalue problem*: one has to find the real values of σ such that the corresponding solution of the equations satisfies all the boundary conditions.

A relativistic theory of stellar perturbations can be constructed as a generalization of the newtonian theory. The resulting system of equations splits into two decoupled sets: the *polar modes*, (the even modes in Thorne's notation), that correspond to the tidal modes already present in the newtonian theory, and the axial modes (odd modes), whose effect is to induce a stationary rotation in the star, *but no pulsation in the fluid*. The axial modes do not have a counterpart in the newtonian theory, and since they do not induce any motion in the fluid, they have been disregarded as irrelevant in the literature. However, as we shall see in sections 4 and 9, under suitable circumstances they can exhibit very interesting properties.

Much more attention has been focused onto the polar modes, due to the fact that they do excite pulsations in the fluid. In the theory developed by Thorne and his collaborators it has been shown that the system of equations governing the polar perturbations can still be reduced, as in the newtonian case, to a fourth-order linear differential system *that couples the perturbations of the metric with the perturbations of the fluid.* (The reduction to a fourth order system has been accomplished by Lindblom& Detweiler 1983). This system describes the evolution of the perturbations inside the star. However, unlike the newtonian case, at this stage the description of the problem is not complete. The perturbations in the interior must be matched with the perturbations of the gravitational field in the exterior of the star, to properly take into account the emission of gravitational waves. In general relativity the frequencies of oscillation of a star are complex. The presence of an imaginary part derives from the fact that the mechanical energy of vibration is exponentially damped by the emission of gravitational waves. Consequently, the corresponding modes are called *quasi-normal* modes. They are defined as the solutions of the sistem of equations which govern the polar perturbations, both inside and outside the star, that satisfy the following boundary conditions: i) regularity of all functions at the center, (ii) $\delta p = 0$ at the surface, (iii) continuous matching of the interior and the exterior solution, and (iiii) at radial infinity the solution must reduce to a pure outgoing wave. In the approach we have described, the nature of the problem does not change substantially with respect to the newtonian theory: it is still an *eigenvalue problem associated to a system of equations which couples, in the interior of the star, the perturbations of the gravitational field with the perturbations of the fluid.*

The new relativistic theory of stellar perturbations has been constructed having as a guide the theory of perturbations of black holes rather than the newtonian theory. In order to describe the perturbed spacetime we have choosen the same gauge which has been used to study the perturbations of a Schwarzschild black hole (see *The mathematical theory of black holes*, Chandrasekhar 1983, this book will be referred in the sequel as *M.T.*).

This assumption, as remarked by Price & Ipser (Price & Ipser 1991), corresponds to an incomplete constraint on the coordinates. However this additional degree of freedom has no physical consequences because it is eliminated by the requirement that all perturbed quantities are well behaved at $r = 0$. Conversely, this choice is rich in consequences and implications. The first is that the resulting equations are particularly simple both for the polar and for the axial modes. A scrutiny of the structure of the equations for the polar modes immediately shows that it is possible to *decouple the equations describing the perturbations of the gravitational field from the equations describing the perturbations of the fluid.* As a consequence, the equations for the perturbed gravitational field can be solved with no reference to the motion that can be induced in the fluid. This is a relevant difference between our approach and the newtonian approach (or its previous relativistic generalization). In fact, due to this decoupling, the problem of finding the frequencies of the quasi-normal modes is transformed into a problem of *resonant scattering.* But in order to fully understand the physical content of the theory and its consequences, we now need to enter into the details of its mathematical formulation.

2. The equilibrium configuration

The metric for a static, spherically symmetric distribution of matter can be written in the standard form[1]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\mu_2}(dr)^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

Inside the star, the functions ν and μ_2 can be determined by solving Einstein's equations coupled to the equations of hydrostatic equilibrium. We shall assume that the star is

[1] We shall adopt the convenctions $G = c = 1$, $G_{ij} = 2T_{ij}$, and the Riemann tensor defined as in M.T. ch. 1

composed by a perfect fluid, whose energy-momentum tensor is given by

$$T^{\alpha\beta} = (p+\epsilon)u^\alpha u^\beta - pg^{\alpha\beta}, \tag{2}$$

where p and ϵ are respectively the pressure and the energy density, that are assumed to have an isotropical distribution, and u^α is the four-velocity of the fluid. By defining the mass contained inside a sphere of radius r as

$$m(r) = \int_0^r \varepsilon r^2 dr, \tag{3}$$

the relevant equations are

$$\begin{aligned} \nu_{,r} &= -\frac{p_{,r}}{p+\epsilon}, & (4)\\ \left[1-\frac{2m(r)}{r}\right] p_{,r} &= -(\varepsilon+p)\left[pr+\frac{m(r)}{r^2}\right], & (5)\\ \text{and} \qquad e^{2\mu_2} &= (1-\frac{2m(r)}{r})^{-1}. & (6) \end{aligned}$$

When the equation of state of the fluid is specified, eqs. (3) and (5) can be solved numerically and the distribution of pressure and energy-density through the star can be determined. Once ϵ and p are known, eq. (4) can be integrated

$$\nu = -\int_0^r \frac{p_{,r}}{(\epsilon+p)} dr + \nu_0. \tag{7}$$

The constant ν_0 is fixed by the condition that at the boundary of the star, $r = R$, the metric reduces to the Schwarzschild metric

$$(e^{2\nu})_{r=R} = (e^{-2\mu_2})_{r=R} = 1 - 2M/R, \tag{8}$$

where $M = m(R)$ is the total mass.

Outside the star the metric is the Schwarzschild metric in its standard form

$$ds^2 = \left(1-\frac{2M}{r}\right)(dt)^2 - \left(1-\frac{2M}{r}\right)^{-1}(dr)^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{9}$$

3. The perturbed spacetime

We shall restrict our analysis to the study of axisymmetric perturbations of a star. This assumption implies no loss of generality, since, due to the spherical symmetry of the background, non-axisymmetric modes can be deduced from axisymmetric perturbations by a suitable rotation of the polar axes (see M.T. §4.23). A line-element appropriate to describe an axially symmetric, time-dependent spacetimes is

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2. \tag{10}$$

In the following we shall project the equations onto an orthonormal tetrad

$$e^i_{(a)} e^j_{(b)} g_{ij} = \eta_{(a)(b)}, \tag{11}$$

where $\eta_{(a)(b)} = (1, -1, -1, -1)$.

When a star is perturbed, each element of fluid suffers an infinitesimal displacement from its equilibrium position, identified by the lagrangian displacement $\vec{\xi}$. Consequently, the metric and the thermodinamical variables change by an infinitesimal amount with respect to their unperturbed values (indicated by a bar)

$$\begin{array}{lll} \nu \longrightarrow \bar{\nu} + \delta\nu & \mu_2 \longrightarrow \bar{\mu_2} + \delta\mu_2 & \varepsilon \longrightarrow \bar{\varepsilon} + \delta\varepsilon \\ \psi \longrightarrow \bar{\psi} + \delta\psi & \mu_3 \longrightarrow \bar{\mu_3} + \delta\mu_3 & p \longrightarrow \bar{p} + \delta p, \end{array} \tag{12}$$

and

$$\omega \longrightarrow \delta\omega \quad , \qquad q_2 \longrightarrow \delta q_2 \quad , \qquad q_3 \longrightarrow \delta q_3. \tag{13}$$

It should be recalled that ω, q_2 and q_3 are zero in the unperturbed state. All perturbed quantities depend on t, r and θ. If we now write Einstein's equations, the hydrodynamical equations and the conservation of barion number (see Paper II §§4 and 11) we find that they decouple into two sets: the *polar modes*, involving the variables given in eqs.

(13) and the lagrangian displacement $\vec{\xi}$, and the *axial modes*, involving the off-diagonal perturbations of the metric (13). The same decoupling into axial and polar modes also occurs when a Schwarzschild black hole is perturbed. However, as we shall see in the following, polar and axial perturbations of stars behave differently.

4. The axial modes

The axial modes do not have a newtonian counterpart. They are purely gravitational modes, since they do not produce any motion in the fluid except for a stationary rotation. The equations for the axial modes are the following.

$$\delta R_{(1)(2)} = 2\delta T_{(1)(2)} \qquad \rightarrow \qquad (e^{3\psi+\nu-\mu_2-\mu_3} Q_{23})_{,\theta} + e^{3\psi-\nu-\mu_2+\mu_3} Q_{02,t} = 0, \tag{14}$$

$$\delta R_{(1)(3)} = 2\delta T_{(1)(3)} \qquad \rightarrow \qquad (e^{3\psi+\nu-\mu_2-\mu_3} Q_{23})_{,r} - e^{3\psi-\nu+\mu_2-\mu_3} Q_{03,t} = 0, \tag{15}$$

where:

$$Q_{23} = \delta q_{2,\theta} - \delta q_{3,r}, \qquad Q_{02} = \delta\omega_{,r} - \delta q_{2,t}, \qquad Q_{03} = \delta\omega_{,\theta} - \delta q_{3,t}\ .$$

Assuming that all perturbed quantities have a time dependence $e^{i\sigma t}$, eqs. (14) and (15) can easily be reduced to the following second order equation

$$(e^{-3\psi+\nu-\mu_2+\mu_3} X_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3} X_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3} X = 0, \tag{16}$$

where we have put

$$e^{+3\psi+\nu-\mu_2-\mu_3} Q_{23} = X. \tag{17}$$

Equation (16) can be separated by expanding the function X in terms of the Gegenbauer polynomials $C_n^\nu(\theta)$, defined by the equation

$$\left[\frac{d}{d\theta}\sin^{2\nu}\theta\frac{d}{d\theta} + n(n+2\nu)\sin^{2\nu}\theta\right] C_n^\nu(\theta) = 0\ . \tag{18}$$

By introducing a new radial variable r_* defined as

$$r_* = \int_0^r e^{-\nu+\mu_2} dr, \tag{19}$$

and putting

$$X = rZ(r)C_{\ell+2}^{-\frac{3}{2}}(\theta), \tag{20}$$

eq. (16) reduces to the following radial equation

$$\frac{d^2Z}{dr_*^2} + [\sigma^2 - U(r)]Z = 0, \tag{21}$$

where

$$U(r) = \frac{e^{2\nu}}{r^3}[\ell(\ell+1)r + r^3(\varepsilon - p) - 6m(r)] \ . \tag{22}$$

Outside the star ϵ and p are zero and eq. (22) reduces to the Regge-Wheeler potential (Regge & Wheeler 1957).

Thus *the axial modes are completely described by the Schroedinger equation (21), valid from the center of the star to radial infinity, with a potential barrier (22), that depends on how the energy-density and the pressure are distributed in the interior of the unperturbed star.*

Given a model of star, solution of the equilibrium equations, eq. (21) can be integrated numerically. The solution free of singularities at the origin has the expansion

$$Z \sim r^{l+1} + \frac{1}{2(2l+3)}\{(l+2)[\frac{1}{3}(2l-1)\epsilon_0 - p_0] - \sigma_0^2\}r^{l+3} + \dots\ , \tag{23}$$

where ϵ_0 and p_0 are the values of the energy-density and of the pressure at the center of the star, and $\sigma_0 = e^{2\nu_0}\sigma$. The asymptotic behaviour of the function Z when $r_* \to \infty$ is

$$\begin{aligned} Z \quad \rightarrow \quad & + \quad \{\alpha - \beta\frac{n+1}{\sigma r} - \frac{1}{2\sigma^2}[n(n+1)\alpha - 3M\sigma\beta]\frac{1}{r^2} + \dots\} \cos\sigma r_* \\ & - \quad \{\beta + \alpha\frac{n+1}{\sigma r} - \frac{1}{2\sigma^2}[n(n+1)\alpha + 3M\sigma\beta]\frac{1}{r^2} + \dots\} \sin\sigma r_*. \end{aligned} \tag{24}$$

α and β, which will play a relevant role in the following development of the problem, are functions of σ , and can be determined by matching the solution obtained by numerical integration of eq. (21), with the asymptotic behaviour (24).

Since the axial modes are described by the Schroedinger equation (21), the problem of studying the axial perturbations of a spherical star is a problem of *pure scattering* in a spherically symmetric, static potential. Therefore we can apply the methods developed in the framework of quantum mechanics in the context of the classical theory of relativity. We can assume that the star is perturbed by an incident gravitational wave of arbitrary frequency, and study the response of the star by evaluating how much of the incident wave will be transmitted or reflected by the potential barrier, in the same way in which the properties of a nucleus described by a Schroedinger equation are investigated by scattering waves of different energy on its potential barrier.

A relevant question which emerges at this point is whether the scattering is, in our context, resonant. If it is not resonant, the star will simply behave as a center of elastic scattering for incident gravitational radiation. Conversely, if it is resonant, the star will be able to emit gravitational waves with frequencies equal to the characteristic resonance frequencies. An extensive answer to this question will be given in section 9.

5. The polar modes

The polar modes couple the perturbations of the diagonal part of the metric (10) $(\delta\nu, \delta\psi, \delta\mu_2, \delta\mu_3)$, with the perturbations of the energy density $\delta\epsilon$, of the pressure δp, and the lagrangian displacement $\vec{\xi}$. In contraxt with the case of the axial modes, polar modes do excite the motion of the fluid that composes the star.

We shall assume that the perturbations take place adiabatically, i.e., that the changes in the pressure and in the energy-density arise without dissipation.

The equations which describe the polar perturbations are the Einstein equations, the hydrodynamical equations, and the conservation of barion number. Since we are mainly interested in showing the results of the new theory, we shall omit the esplicit derivation of these equations which can be found in Paper I and II. Here we only remark that the relevant equations can be separated by performing the following substitutions in terms of the Legendre polynomials, P_l, and their derivatives:

$$\delta\nu = N(r)P_\ell(\cos\vartheta) \qquad \delta\mu_2 = L(r)P_\ell(\cos\vartheta) \tag{25}$$

$$\delta\mu_3 = T(r)P_\ell + V(r)P_{\ell,\vartheta,\vartheta} \qquad \delta\psi = T(r)P_\ell + V(r)P_{\ell,\vartheta}\cot\vartheta\ ,$$

(cf. M.T. p. 147, eqs. (36)-(39) originally due to J.Friedman), and

$$\delta p = \Pi(r)P_\ell(\cos\vartheta) \qquad 2(\varepsilon + p)e^{\nu+\mu_2}\xi_2(r,\vartheta) = U(r)P_\ell$$

$$\delta\varepsilon = E(r)P_\ell(\cos\vartheta) \qquad 2(\varepsilon + p)e^{\nu+\mu_3}\xi_3(r,\vartheta) = W(r)P_{\ell,\vartheta}\ , \tag{26}$$

where ξ_2 and ξ_3 are respectively the r and θ tetrad-components of the lagrangian displacement. After the separation, we are left with a system of coupled equations involving the following variables: $N(r), L(r), T(r), V(r)$, which describe the radial part of the perturbation of the metric, and $U(r)$, $W(r)$, $\Pi(r)$, $E(r)$, which describe the radial part of the perturbation of the fluid. The resulting equations are:

$$\left[\frac{d}{dr}+\left(\frac{1}{r}-\nu_{,r}\right)\right](2T-kV)-\frac{2}{r}L \;=\; -\;U, \tag{27}$$

$$(T-V+N)_{,r}-\left(\frac{1}{r}-\nu_{,r}\right)N-\left(\frac{1}{r}+\nu_{,r}\right)L \;=\; 0, \tag{28}$$

$$\frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}N_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(2T-kV)_{,r}-\frac{2}{r}\left(\frac{1}{r}+2\nu_{,r}\right)L\right]+ \frac{1}{2}\left[-\frac{1}{r^2}(2nT+kN)+\sigma^2e^{-2\nu}(2T-kV)\right] \;=\; \Pi, \tag{29}$$

$$V_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)V_{,r}+\frac{e^{2\mu_2}}{r^2}(N+L)+\sigma^2e^{2\mu_2-2\nu}V \;=\; 0, \tag{30}$$

$$W = -(T-V+L), \tag{31}$$

$$\Pi = -\frac{1}{2}\sigma^2e^{-2\nu}W-(\varepsilon+p)N, \tag{32}$$

$$U = \frac{[\frac{1}{2}(\sigma^2e^{-2\nu}W)_{,r}+(Q+1)\nu_{,r}\frac{1}{2}(\sigma^2e^{-2\nu}W)+(\varepsilon_{,r}-Qp_{,r})N]}{\frac{1}{2}\left[\sigma^2e^{-2\nu}+\frac{e^{-2\mu_2}\nu_{,r}}{\varepsilon+p}(\varepsilon_{,r}-Qp_{,r})\right]}, \tag{33}$$

$$E = Q\Pi+\frac{e^{-2\mu_2}}{2(\varepsilon+p)}(\varepsilon_{,r}-Qp_{,r})U, \tag{34}$$

where

$$k=\ell(\ell+1), \qquad 2n=(\ell-1)(\ell+2)=k-2, \qquad Q=\frac{(\epsilon+p)}{\gamma p}, \tag{35}$$

and

$$\gamma=\frac{(\varepsilon+p)}{p}\left(\frac{\partial p}{\partial\varepsilon}\right)_{entropy=const}\;; \tag{36}$$

is the adiabatic exponent (defined in Paper I, equation (106)).

One can immediately recognize that eqs. (31)-(34) give the fluid variables as a combination of the metric perturbations T, V, L, and N. Therefore, if we replace the expressions of U and Π given by eqs. (33) and (32) on the right-hand side of eqs. (27) and (29), we are left with a system of equations which involves only the perturbations of the metric functions (T, V, L, N)!

It should be stressed that the decoupling of the equations governing the metric perturbations from the equations governing the hydrodynamical variables is possible in general, and requires no assumptions on the equation of state of the fluid.

We are therefore in a situation totally different from the newtonian case: *we can solve the equations for the perturbations of the metric independently on the motion which is induced in the fluid.*

Outside the star the variables related to the fluid, Π and U, vanish and the system of equations (27)-(30) can be reduced to a single Schroedinger equation (the Zerilli equation (Zerilli 1972*a,b*)) with an associated potential barrier

$$\left(\frac{d^2}{dr_*^2}+\sigma^2\right)Z=VZ, \tag{37}$$

where the function Z is defined as

$$Z=\frac{r}{nr+3M}\left(\frac{3M}{n}X-rL\right)\ , \tag{38}$$

and

$$V(r)=\frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3+3Mn^2r^2+9M^2nr+9M^3]\ . \tag{39}$$

The radial variable r_* is the 'tortoise' coordinate

$$r_*=r+2M\log(\frac{r}{2M}-1). \tag{40}$$

We now want to integrate the perturbation equations both inside and outside the star.

(a) The integration of the equations

In order to numerically integrate the *decoupled* system for (T,V,L,N) in the interior of the star (we assume that the aforementioned substitution for U and Π in eqs. (27) and (29) has been performed), we need to find the behaviour of these functions near $r=0$.

We can seek a power series solution of the type

$$(T, V, L, N) \sim (T_0, V_0, L_0, N_0) r^x + O(r^{x+2}), \tag{41}$$

where x is an exponent to be determined, but if we substitute these expressions into the equations we discover that the system is linearly dependent near the origin. This difficulty can be circumvented by introducing a suitably defined new variable. For the sake of simplicity, in the following we shall restrict our consideration to the case when the fluid obeys a *barotropic* equation of state, i.e. when the pressure is a unique function of the energy density, $p = p(\epsilon)$. In this case $Q = \frac{\epsilon_{,r}}{p_{,r}}$, and the equations considerably simplify. We shall replace the variable T by the new variable G defined as

$$\begin{aligned} G = \nu_{,r}[\frac{n+1}{n}X - T]_{,r} \quad &+ \quad \frac{1}{r^2}(e^{2\mu_2} - 1)[n(N+T) + N] + \frac{\nu_{,r}}{r}(N+L) - \\ &- \quad e^{2\mu_2}(\varepsilon + p)N + \frac{1}{2}\sigma^2 e^{2(\mu_2 - \nu)}[L - T + \frac{2n+1}{n}X] \,, \end{aligned} \tag{42}$$

and the variable V by $X = nV$.

The final set of equations we shall integrate is

$$X_{,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) X_{,r} + \frac{n}{r^2} e^{2\mu_2}(N+L) + \sigma^2 e^{2(\mu_2 - \nu)} X = 0, \tag{43}$$

$$(r^2 G), r = n\nu_{,r}(N-L) + \frac{n}{r}(e^{2\mu_2} - 1)(N+L) + r(\nu_{,r} - \mu_{2,r})X_{,r} + \sigma^2 e^{2(\mu_2 - \nu)} r X \ , \tag{44}$$

$$\begin{aligned} -\nu_{,r}N_{,r} \qquad &= -G + \nu_{,r}[X_{,r} + \nu_{,r}(N-L)] + + \frac{1}{r^2}(e^{2\mu_2} - 1)(N - rX_{,r} - r^2 G) \\ &-e^{2\mu_2}(\varepsilon + p)N + \frac{1}{2}\sigma^2 e^{2(\mu_2 - \nu)} \left\{ N + L + \frac{r^2}{n}G + \frac{1}{n}[rX_{,r} + (2n+1)X] \right\}, \end{aligned} \tag{45}$$

$$\begin{aligned} -L_{,r} = (N + 2X)_{,r} \qquad &+ \left(\frac{1}{r} - \nu_{,r}\right)(-N + 3L + 2X) + \\ &+ \left[\frac{2}{r} - (Q+1)\nu_{,r}\right] \left[N - L + \frac{r^2}{n}G + \frac{1}{n}(rX_{,r} + X)\right] \ . \end{aligned} \tag{46}$$

This is a fifth-order linear differential system. It has been shown (Price & Ipser 1991) that it can be reduced to a fourth order system, however we prefer not to use that reduction because our equations are considerably simpler.

The system of equations (43)-(46), which involves only the perturbations of the gravitational field, can now be integrated from the center to the surface of the star, in the following way. As before, we shall assume that, near the origin, the functions have the asymptotic expansion

$$(X, G, N, L) = (X_0, G_0, N_0, L_0) r^x + (X_2, G_2, N_2, L_2) r^{x+2} \;, \tag{47}$$

where both the exponent x and the coefficients of the expansion have to be determined by inserting eq. (47) into equations (43)-(46), and by setting to zero the coefficients of different powers of r. From the lower order terms we obtain a homogeneous algebraic system of four equations for the four coefficients (X_0, G_0, N_0, L_0)

$$\begin{aligned}
& x(x+1)X_0 + n(L_0 + N_0) = 0 \qquad (48)\\
& [(a-b)x + \sigma_0^2]X_0 - (x+2)G_0 + n(a+b)N_0 - n(a-b)L_0 = 0\\
& \left[(a-b)x + \frac{\sigma_0^2}{2n}(x+2n+1)\right] X_0 - G_0 + a(x-1)N_0 + \frac{1}{2}\sigma_0^2 N_0 + \frac{1}{2}\sigma_0^2 L_0 = 0\\
& 2\left[x\left(\frac{n+1}{n}\right) + 2\right] X_0 + (x+1)(N_0 + L_0) = 0,
\end{aligned}$$

where a and b are the coefficients of the expansion of the metric functions

$$e^{2\mu_2} \sim 1 + br^2 = 1 + (\frac{2}{3}\epsilon_0)r^2, \qquad e^{2\nu} \sim 1 + ar^2 = 1 + (p_0 + \frac{1}{3}\epsilon_0)r^2. \tag{49}$$

The system (48) admits a non-trivial solution only if the determinant is zero. This condition provides the indicial equation for the determination of x

$$na(x+1)(x-\ell)^2(x+\ell+1)^2 = 0 \;. \tag{50}$$

Surprisingly, we see that there are only two coincident values of x which correspond to regular solutions, i.e. $x = l$. That means that, although our original system is of order five, only two independent solutions are acceptable. This is a great simplification with respect to the old theory, where four independent solutions had to be integrated through the star and then matched in order to satisfy the boundary conditions. In selecting the admissible values of x, we eliminate the extra degree of freedom due to our incomplete gauge specification. A possible choice for the two independent solution is

$$1) \qquad L_0 = 0, \qquad N_0 = 1, \qquad X_0 = -\frac{n}{\ell(\ell+1)} N_0, \tag{51}$$
$$G_0 = +\frac{1}{2}\ell - 1) \left\{ a + b - \frac{1}{\ell(\ell+1)} [(a-b)\ell + \sigma_0^2] \right\} N_0 \ ,$$

$$2) \qquad N_0 = 0 \qquad L_0 = 1, \qquad X_0 = -\frac{n}{\ell(\ell+1)} L_0, \tag{52}$$
$$G_0 = -\frac{1}{2}\ell - 1) \left\{ a - b + \frac{1}{\ell(\ell+1)} [(a-b)\ell + \sigma_0^2] \right\} L_0 \ .$$

The coefficients (X_2, G_2, N_2, L_2) in the expansion (47), can be found by equating to zero the coefficients of the next power of x into the expanded equations.

We can now numerically integrate eqs. (43)-(46), with the initial conditions (51)-(52). It remains to be ascertained whether two independent solutions are sufficient to satisfy the boundary conditions required by the problem. As in the newtonian case, we need to impose that the perturbation of the pressure δp vanishes at the surface $r = R$, but in addition we need to impose that the interior solution joins continuously with the solution in the exterior of the star. In order to satisfy the continuity condition at $r = R$, eqs. (27)-(30), which are equivalent to eqs. (43)-(46), must reduce to those appropriate to the vacuum, and therefore it must be

$$\Pi = 0, \qquad \text{and} \qquad U = 0. \tag{53}$$

The vanishing of δp at the boundary is included in the first of eqs. (53), since Π is the radial part of δp (see eq. (26)). Since we are solving the barotropic case, from eqs. (32) and (33) it follows that

$$\Pi = -\frac{1}{2}\sigma^2 e^{-2\nu} W - (\varepsilon + p)N, \quad \text{and} \quad U = W_{,r} + (Q-1)\nu_{,r} W \ . \tag{54}$$

For a fluid star ϵ and p tend to zero at the boundary. Moreover

$$Q = \frac{\epsilon_{,r}}{p_{,r}}, \quad \to \quad \frac{Q_1}{(R-r)}, \quad \nu_{,r} \quad \to \quad \nu_1', \quad \text{and} \quad W \quad \to \quad (R-r)W_1 e^{\alpha(R-r)} \ , \tag{55}$$

where Q_1, ν_1', W_1 and α are constant. Since ϵ, p and W tend to zero, from equation (54) it follows that the first condition, $\Pi = 0$, is automatically satisfied by any independent solution! Conversely, from eqs. (55) it follows that U tends to a constant value

$$U \sim W_1 + \nu_1' Q_1 W_1 = \text{const} \ , \tag{56}$$

and we need to consider a linear combination of the two independent solutions in such a way that the remaining condition, $U = 0$, is satisfied at the boundary. Therefore the two degrees of freedom given by eq. (51) and (52) are precisely what we do need to match the interior and the exterior solution, and to satisfy the condition $\delta p = 0$.

Now the strategy of integration is clear: we integrate the two independent solutions of eqs. (43)-(46) for the metric perturbations, with the initial conditions (51) and (52). Then we linearly superimpose the two solutions in such a way that at the boundary $U = 0$. At this point we have the values of $X, L, X_{,r}$ and $L_{,r}$ at $r = R$, and we can construct the functions $Z(R)$ and $Z_{,r_*}(R)$ given by

$$Z(R) = \lim_{r\to R-0} \frac{r}{nr + 3M}\left(\frac{3M}{n}X - rL\right), \qquad Z_{,r_*}(R) = \left(1 - \frac{2M}{R}\right) \lim_{r\to R-0} Z_{,r}(r). \tag{57}$$

With these initial values, equation (37) can be integrated. The asymptotic behaviour of the function Z for large r is

$$Z \to \ + \ \left\{\alpha - \frac{n+1}{\sigma}\frac{\beta}{r} - \frac{1}{2\sigma^2}\left[n(n+1)\alpha + -\frac{3}{2}M\sigma\left(1 + \frac{2}{n}\right)\beta\right]\frac{1}{r^2} + ...\right\}\cos\sigma r_* \tag{58}$$

$$- \quad \left\{ \beta + \frac{n+1}{\sigma} \frac{\alpha}{r} - \frac{1}{2\sigma^2} \left[n(n+1)\beta + + \frac{3}{2} M\sigma \left(1 + \frac{2}{n}\right) \alpha \right] \frac{1}{r^2} + ... \right\} \sin \sigma r_* \,,$$

where, as in the axial case, α and β are functions of σ to be determined by matching the integrated solution with the asymptotic behaviour (58). The solution is now complete.

The consequences of the decoupling

In this section we have shown how to construct the solution for the polar modes by solving a system of equations that do not involve the variables which describe the perturbed fluid: they can be found, if required, from eqs. (31)-(34) in terms of the metric perturbations by simple algebraic relations. We therefore concentrate our attention on the perturbations of the gravitational field with no reference to the motion of the fluid, and, again, we can treat the problem as a scattering problem. This is a relevant result that does not have a counterpart in the newtonian theory.

A counterpart has to be found in the theory of perturbations of a Schwarzschild black hole. In that case, both the polar and the axial modes are governed by a Schroedinger equation, and the problem is manifestely a scattering problem: incident gravitational waves are scattered by the curvature of the spacetime. The analogy is immediate in the case of the stellar axial modes which, as we have seen, are also described by a unique Schroedinger equation. In that case the potential barrier is generated by the curvature of the spacetime produced by the particular distribution of energy density and pressure inside the star.

In the case of the polar modes we do not have a simple problem in potential scattering, as it was in the case of the axial modes. Here a Schroedinger equation holds only in the exterior of the star, and a much more complicated fifth-order system must be solved in the interior. However we can still imagine that the perturbation is originated by an incident polar gravitational wave, and that the incoming wave drives the fluid pulsations which emit the scattered component of the wave.

The consequences of this new viewpoint will be manifest in the next sections where we shall develop a very simple algorithm to find the frequencies of the quasi-normal modes, and a method to evaluate how the gravitational energy flows through the star and in the exterior. Another element of interest in this theory is the remarkable simplification of the problem: only two independent solutions are needed to find the complete solution and satisfy the boundary conditions.

6. An algorithm to find the frequencies of the quasi-normal modes

In Paper V we have developed a method to determine the complex characteristic frequencies of the quasi-normal modes, which is based on the scattering nature of the problem. We shall now formulate the theory in general, and then specify how it can be applied to the axial and the polar modes. Let us consider a Schroedinger equation

$$\frac{d^2 Z_c}{dr_*^2} + (\sigma^2 - V) Z_c = 0, \tag{59}$$

where V is a spherically symmetric, short-range potential barrier , i.e. $V < o(r_*^{-1})$ for $r_* \to \infty$. We want to find the complex values of the frequency such that the corresponding solution of equation (59), regular at $r_* = 0$, behaves as a pure outgoing wave at radial infinity, i.e.

$$Z_c \sim e^{-i\sigma_c r_*}, \qquad \text{when} \qquad r_* \to \infty, \tag{60}$$

where $\sigma_c = \sigma + i\sigma_i$ and $Z_c = Z + iZ_i$. By separating the real and the imaginary part in eq. (59), we find

$$\frac{d^2 Z}{dr_*^2} - VZ + (\sigma^2 - \sigma_i^2) Z - 2\sigma\sigma_i Z_i = 0, \tag{61}$$

$$\frac{d^2 Z_i}{dr_*^2} - VZ_i + (\sigma^2 - \sigma_i^2) Z_i - 2\sigma\sigma_i Z = 0. \tag{62}$$

We shall assume that $\sigma_i \ll \sigma$. In our context this condition implies that the decay time of the emission of gravitational waves, $\tau = \frac{1}{\sigma_i}$, is much longer than the real part of the frequency σ, a condition which is always satisfied for stars (only for black holes σ_i is comparable with σ). If we now put $Z_i = \sigma_i Y$, and neglect the terms of order $O(\sigma_i^2)$ in eqs. (61) and (62), they become

$$\frac{d^2 Z}{dr_*^2} + (\sigma^2 - V)Z = 0, \tag{63}$$

$$\frac{d^2 Y}{dr_*^2} + (\sigma^2 - V)Y + 2\sigma Z = 0 \,. \tag{64}$$

From eq. (64) it follows that

$$Y(r_*, \sigma) = \frac{\partial}{\partial \sigma} Z(r_*, \sigma) \,, \tag{65}$$

and consequently

$$Z_c(r_*, \sigma_c) = Z(r_*, \sigma) + i\sigma_i \left[\frac{\partial}{\partial \sigma} Z(r_*, \sigma) \right]. \tag{66}$$

Therefore when $\sigma_i << \sigma$, we can construct the complex solution Z_c corresponding to a complex value of the frequency σ_c, by integrating only equation (63) for the real part Z, and for real values of the frequency σ.

The asymptotic behaviour of Z_c

When $r_* \to \infty$, the potential V tends to zero and eq. (63) admits two linearly independent solutions Z_1 and Z_2 which have the following asymptotic behaviour

$$Z_1 \to \cos \sigma r_* + O(r_*^{-1}), \qquad Z_2 \to \sin \sigma r_* + O(r_*^{-1}) \,.$$

Thus the general real solution Z is

$$Z(r_*, \sigma) = \alpha(\sigma) Z_1(r_*, \sigma) - \beta(\sigma) Z_2(r_*, \sigma), \tag{67}$$

where $\alpha(\sigma)$ and $\beta(\sigma)$ are functions to be determined by matching eq. (67) with the integrated solution of eq. (63) for different initially assigned values of real σ. From eq. (65) and (66) it follows that the complete solution for Z_c, up to terms of order $O(\sigma_i^2)$ is

$$Z_c = Z + i\sigma_i \frac{\partial Z}{\partial \sigma} = \alpha(\sigma) Z_1 - \beta(\sigma) Z_2 - i\sigma_i [\alpha'(\sigma) Z_1 - \beta'(\sigma) Z_2 + \alpha(\sigma) Z_1' - \beta(\sigma) Z_2'] \;, \quad (68)$$

where the prime indicates differentiation with respect to σ. For sufficiently large values of r_* the behaviour of Z_c is

$$Z_c \to (\alpha + i\sigma_i\alpha' - i\sigma_i\beta r_*) \cos \sigma r_* - (\beta + i\sigma_i\beta' - i\sigma_i\alpha r_*) \sin \sigma r_*. \quad (69)$$

It is clear that the terms proportional to r_* would eventually diverge if $r_* \to \infty$. However, in the limit $\sigma_i << \sigma$, the asymptotic behaviour (67) that we use to determine α and β, is established long before these terms begin to dominate. Therefore, if the value of r_* where we start to match the integrated real solution Z with the asymptotic behaviour, is large enough that eq. (67) can be applied, but not so far that the exponential growth has taken over in eq. (69), the diverging terms can be neglected, and the asymptotic form of Z_c can be written as

$$\begin{aligned} Z_c \to & \frac{1}{2}[(\alpha - \sigma_i\beta') + i(\beta + \sigma_i\alpha')]e^{i\sigma r_*} + \frac{1}{2}[(\alpha + \sigma_i\beta') - i(\beta - \sigma_i\alpha')]e^{-i\sigma r_*} \\ & = I(\sigma)e^{+i\sigma r_*} + O(\sigma)e^{-i\sigma r_*}. \end{aligned} \quad (70)$$

(That such value of r_* does indeed exist has been shown be a direct verification in Paper V). We now impose the outgoing wave condition, by setting to zero the coefficient of the ingoing wave, $I(\sigma)$, in eq. (70)

$$\alpha - \sigma_i\beta' = 0, \qquad \text{and} \qquad \beta + \sigma_i\alpha' = 0 \;. \quad (71)$$

Eliminating σ_i we finally find

$$\alpha\alpha' + \beta\beta' = 0 \;. \quad (72)$$

This equation says that if there exists a value of real σ, say $\sigma = \sigma_0$, where the function $(\alpha^2 + \beta^2)$ has a minimum, then the solution Z_c at infinity will represent a pure outgoing wave. Therefore σ_0 is the real part of the complex characteristic frequency belonging to a quasi-normal mode. The imaginary part can be obtained from eqs. (71) evaluated at $\sigma = \sigma_0$

$$\sigma_i = i \left.\frac{\alpha}{\beta'}\right|_{(\sigma=\sigma_0)} = -\left.\frac{\beta}{\alpha'}\right|_{(\sigma=\sigma_0)} . \tag{73}$$

Equation (72) suggests an alternative method to find σ_i. Since the function $(\alpha^2 + \beta^2)$ has a minimum when $\sigma = \sigma_0$, in the region $\sigma \sim \sigma_0$ it can be approximated by a parabola

$$\alpha^2 + \beta^2 = const \left[(\sigma - \sigma_0)^2 + \sigma_i^2 \right] \tag{74}$$

and σ_i can be determined by matching the values of $(\alpha^2 + \beta^2)$ obtained by numerical integration, with eq. (74).

The application of the algorithm we have described to the axial modes is straightforward. We integrate the Schroedinger equation (21) with the initial conditions (23) for different values of *real* σ. For sufficiently large r_*, we match the integrated solution with the asymptotic behaviour of Z given in eq. (24) and determine the values of α and β. Then we find the values of $\sigma = \sigma_0$ where the resonance curve $(\alpha^2 + \beta^2)$ has a minimum (if they exist): σ_0 will be the real part of the eigenfrequency. The imaginary part will be found from eq. (73), or alternatively, by fitting the resonance curve with the parabola (74). The same procedure can be applied in the case of the polar modes. The difference with respect to the axial case is that inside the star we need to integrate the system of equations (43)-(46) in the manner described in section 5. The purpose is to find the initial values for the function Z at the boundary of the star, which are needed to integrate the Schroedinger equation (37) outside the star. At sufficiently large values of r_*, the integrated solution will be matched with the asymptotic behaviour (58), and α and β will be determined. We shall then proceed as in the axial case.

To conclude this section we would like to stress the basic difference that exist between the newtonian and the relativistic approach to the problem of finding the frequencies of the normal (quasi-normal in the relativistic case) modes. In the newtonian theory one has to solve an eigenvalue problem associated to a system of equations *which couple the perturbations of the fluid with the perturbations of the gravitational field*. In the relativistic theory we solve a problem of *resonant scattering of gravitational waves by a potential barrier*. The implications of the analogy with resonant scattering in quantum mechanics will be further discussed in the next sections.

7. Some further analogies between oscillations of stars and resonant scattering in quantum mechanics

There is clearly a strong resemblance between eq. (74) and the Breit-Wigner formula

$$\text{cross-section} = \frac{const}{(E-E_0)^2 + \frac{1}{4}\Gamma^2} \tag{75}$$

(see for example Landau & Lifschitz 1977, pp.603-611) used in atomic and nuclear physics, and it is interesting to clarify this analogy. In the context of quantum mechanics, resonant scattering occurs when a system is in a quasi-stationary state that decays, as for example a radioactive nucleus which emits an α-particle with energy E_0 and lifetime $\tau = \frac{\hbar}{\Gamma}$. The Schroedinger equation appropriate to that problem is

$$\frac{d^2 Z}{dr_*^2} + (E - V)Z = 0, \tag{76}$$

and one assumes that Z is *an analytic function of the complex energy* E. (In the notation of this paper $\sigma = const\sqrt{E}$, and the constant of proportionality is real and positive.) The complex plane is cut along the positive real E-axis in order to make Z a single valued function. The asymptotic solution for large values of r_* is of the form

$$Z(E) \sim I(E)e^{+i\sqrt{E}r_*} + O(E)e^{-i\sqrt{E}r_*}, \tag{77}$$

and if E is real and positive $O(E) = I^*(E)$, and $Z(E)$ is real. The scattering amplitude follows in the usual way

$$S_l = e^{2i\delta_l} = (-1)^{l+1}(I^*/I), \tag{78}$$

where l is the angular momentum associated to the order of the Legendre polynomial, and δ_l is the phase-shift. A quasi-stationary state corresponds to a zero of the function $I(E)$ (or to a pole of the scattering amplitude S_l), where the corresponding asymptotic wave function (77) reduces to a pure outgoing wave. In order to obtain the Breit-Wigner formula, one *postulates* the existence of a pole lying close to the positive real axis, at some *complex energy* $E = E_0 - \frac{1}{2}i\Gamma$, and by expanding $I(E)$ in the vicinity of the zero

$$I(E) \sim const(E - E_0 + \frac{1}{2}i\Gamma), \tag{79}$$

the cross-section (75) immediately follows.

Let us now see what is the connection between this approach and the algorithm developed in section 6. We have shown that if the function $(\alpha^2 + \beta^2)$ has a minimum for a value of *real* E (real σ), then the amplitude of the ingoing part of the asymptotic wavefunction $I(E)$ is zero, provided $\Gamma << E$ ($\sigma_i << \sigma$). Therefore, for such value $E = E_0$, $|I(E)|^2$ must also have a minimum and

$$II^{*\prime} + I^*I' = 0 \qquad \text{or} \qquad I'/I = -(I^{*\prime}/I^*), \tag{80}$$

where the prime indicates differentiation with respect to E. Thus, apart from the trivial case $I' = 0$, (I'/I) is imaginary at E_0, say $-2i/\Gamma$. Since the logarithmic derivative is purely imaginary at E_0, we may analytically continue the function I in the complex plane and expand in the vicinity of E_0

$$I(E) \sim I(E_0)[1 + (I'/I)_{E=E_0}(E - E_0)] \sim I(E_0)[1 + 2i(E - E_0)/\Gamma]. \tag{81}$$

A comparison with eq. (79) shows that the Breit Wigner formula can now be derived by the usual procedure. Thus our approach also leads to the Breit-Wigner formula, but we have focused the attention on the amplitude of the standing wave prevailing at infinity

$$A(\sigma) = < 2Z^2 >^{\frac{1}{2}}_{av} = \alpha + i\beta, \tag{82}$$

rather than on the amplitude of the ingoing part of the wave $I(\sigma)$. It should be stressed however that, while in quantum mechanics the existence of a resonance is *postulated*, and the values of E_0 and Γ are known from experiments, in our context we provide a method to evaluate both σ_0 and σ_i.

The analogies between the theory of oscillations of stars and quantum mechanics do not end here. We shall see in the next section that a suitable generalization of the Regge theory allows to define the flow of gravitational energy through the star.

8. The flow of gravitational energy, an application of the Regge theory

The Regge theory (Alfaro & Regge 1963) is applicable to the problem of potential scattering when the wave equation is separable, and the wave function can be written in terms of a radial function and a Legendre polynomial $P_l(\cos\theta)$. In that case the radial wave equation can be written by separating explicitely the 'centrifugal' part of the potential barrier

$$\frac{d^2Z}{dr^2} + \left[\sigma^2 - \frac{l(l+1)}{r^2} - U(r)\right] Z = 0, \tag{83}$$

and $U(r)$ is a short range, central potential. The amplitude of the standing wave at infinity is now considered as a function of the frequency and of the angular momentum l

$$A(\sigma, l) = \alpha(\sigma, l) + i\beta(\sigma, l), \tag{84}$$

and it is assumed to be an analytic function in the variables σ and l, which are both assumed to be *complex.* Further, to any given pole $(\sigma_0 + i\sigma_i; l_0)$, corresponding to a fixed integral value of the angular momentum l_0, there exists a Regge pole in the complex l-plane, $(\sigma_0; l_0 + il_i)$, belonging to the same quasi-stationary state. Consequently, in the neighbourhood of (σ_0, l_0), the amplitude A can be analytically continued either in the complex σ-plane

$$A(\sigma) \sim \left[\frac{\partial A(\sigma)}{\partial \sigma}\right]_{\sigma=\sigma_0} [\sigma - (\sigma_0 + i\sigma_i)] \tag{85}$$

and

$$|A(\sigma)|^2 = \alpha^2 + \beta^2 \sim \left[\frac{\partial A(\sigma)}{\partial \sigma}\right]^2_{\sigma=\sigma_0} [(\sigma - \sigma_0)^2 + \sigma_i^2]. \tag{86}$$

or in the complex l-plane

$$A(\sigma) \sim \left[\frac{\partial A(\sigma_0; l)}{\partial l}\right]_{l=l_0} [l - (l_0 + il_i)], \tag{87}$$

and

$$(\alpha^2 + \beta^2) \sim \left[\frac{\partial A(\sigma_0; l)}{\partial l}\right]^2_{l=l_0} [(l - l_0)^2 + l_i^2]. \tag{88}$$

It is now clear that we can generalize the algorithm developed in section 6 to find the resonance in the complex σ-plane, to determine the corresponding resonances in the complex l-plane. We shall assume that $\sigma = \sigma_0$ is known and fixed, and that the angular momentum is complex

$$l_c = l + il_i. \tag{89}$$

If we assume that $|l_i| << l$, in analogy with eq. (66) the corresponding complex solution $Z_c = Z + iZ_i$, where now $Z = Z(r; \sigma_0, l)$, can be written as

$$Z_c(r; \sigma_0, l + il_i) = Z(r; \sigma_0, l) + il_i \left[\frac{\partial}{\partial l} Z(r; \sigma_0, l)\right], \tag{90}$$

and the complete complex solution Z_c can be derived from the only knowledge of the real solution $Z(r; \sigma_0, l)$. The procedure to find l_0 and l_i is therefore the same as that described in section 6 (eqs. (72), (73) and (74)), with the only difference that now the square amplitude of the standing wave at infinity $(\alpha^2+\beta^2)$ has to be considered a function of real l.

Once l_0 and l_i are known, they can be substituted explicitely into eq. (83), that becomes

$$\frac{d^2 Z_c}{dr^2} + \left[\sigma^2 - \frac{l_0(l_0+1)}{r^2} - U(r)\right] Z_c = i l_i \frac{(2l_0+1)}{r^2} Z_c + O(l_i^2). \tag{91}$$

Multiplying equation (91) by Z_c^* and subtracting from the resulting equation its complex conjugate (complex conjugation is taken with respect to l_c), we find that

$$\frac{d}{dr}[Z_c, Z_c^*]_r = 2 i l_i \frac{(2l_0+1)}{r^2} |Z_c|^2, \tag{92}$$

where

$$[Z_c, Z_c^*]_r = Z_{c,r} Z_c^* - Z_{c,r}^* Z_c \tag{93}$$

is the wronskian. Since Z_i is of order l_i (cfr. eq.(90)), up to terms of order $\mathrm{O}(l_i^2)$ $|Z_c|^2 = Z^2$, and from equation (92) it follows that

$$[Z_c, Z_c^*]_r = 2 i l_i (2l_0+1) \int_0^r \frac{dr}{r^2} Z^2. \tag{94}$$

The integral on the right-hand side converges for $r \to \infty$ and it is positive definite. In quantum mechanics the non constancy of the wronskian exhibited in eq. (94) is interpreted as the emission of a new particle in the field volume. (see for example Landau & Lisfshitz 1977, bottom of page 588). The knowledge of the pole (l_0, l_i) is therefore essential to evaluate eq. (94).

The theory now described can be immediately applied to the axial modes, provided they are resonant. The fact that the radial wave equation (21) is obtained by expanding the wave-function in Gegenbauer polynomials $C_{l+2}^{-\frac{3}{2}}$, instead of Legendre polynomial $P_l(\cos\theta)$, does not affect any conclusion we have reached so far. The radial equation (21) can be rewritten in a form analogous to eq. (83):

$$\frac{d^2 Z_c}{dr_*^2} + \left[\sigma^2 - \frac{e^{2\nu}}{r^2} l(l+1) - U(r)\right] Z_c = 0, \tag{95}$$

where

$$U(r) = e^{2\nu}\left[(\epsilon - p) - \frac{6M}{r^3}\right]. \tag{96}$$

If we now assume that $\sigma = \sigma_0$ is the real part of the frequency of a quasi-normal mode previously determined, and $l = l_0 + il_i$, is the corresponding pole in the complex l-plane, equation (95) can be written in a form equivalent to eq. (91)

$$\frac{d^2 Z_c}{dr_*^2} + \left[\sigma_0^2 - \frac{e^{2\nu}}{r^2} l_0(l_0+1) - U(r)\right] Z_c = il_i(2l_0+1)\frac{e^{2\nu}}{r^2} Z_c, \tag{97}$$

where $Z_c = Z_c(r_*, \sigma_0, l_0 + il_i)$. Multiplying by Z_c^* and subtracting from the complex conjugate equation we find

$$[Z_c, Z_c^*]_{r_*} = 2il_i(2l_0+1)\int_0^{r_*} \frac{e^{2\nu}}{r^2} Z^2 dr_*. \tag{98}$$

In analogy with the interpretation of equation (94) given in the context of quantum mechanics, we can interpret the right-hand side of eq. (98) as the a measure of gravitational energy which crosses a sphere of radius r_*.

It should be stressed that in order to define the flow of energy through and outside the star *we do need* to use the Regge theory. One may ask why didn't we try to evaluate the flux by assuming l real, σ complex, and operating on eq. (95) with the function Z_c^* complex conjugate to Z_c with respect to σ. The result in that case would be

$$\frac{d}{dr_*}[Z_c, Z_c^*]_{r_*} = -4i\sigma_0\sigma_i |Z_c|^2, \tag{99}$$

where now $Z_c = Z_c(r_0, \sigma_0 + i\sigma_i, l_0)$. Consequently

$$[Z_c, Z_c^*]_{r_*} = -4i\sigma_0\sigma_i \int_0^{r_*} |Z_c|^2 dr_*. \tag{100}$$

But in this case, if we require that the solution is exponentially damped in time, i.e. $\sigma_i > 0$, the asymptotic behaviour of $|Z_c|^2$ would be

$$|Z_c|^2 \sim e^{-2\sigma_i t} e^{2\sigma_i r_*}, \tag{101}$$

and the integral would explode. *The application of the Regge theory is therefore essential to circumvent the obstacle of the divergent integral.* The question now is whether this theory can be applied to the polar modes.

The resonant scattering of polar gravitational waves is not a conventional potential scattering. Inside the star we have to integrate a fifht-order differential system whose solution must be properly matched with the solution of the Schroedinger equation which governs the perturbations of the gravitational field outside the star. Thus the Regge theory cannot be applied in its standard form. However a generalization is possible. In Part II of Paper I (eqs. (132)-(134)) it was shown that the polar perturbations allow a conservation equation of the form

$$E_{2,2} + E_{3,3} = 0, \tag{102}$$

where $\mathbf{E}$ is a vector we shall define, $x^2 = r$ and $x^3 = \mu = \cos\theta$. By Gauss's theorem, it follows that, if C_1 and C_2 are any two closed contours, one inside the other, in the (x^2, x^3)-plane

$$\int_{C_1} (E_2 dx^3 - E_3 dx^2) = \int_{C_2} (E_2 dx^3 - E_3 dx^2), \tag{103}$$

provided $\mathbf{E}$ is not singular inside the area included between C_1 and C_2. If we now assume that the closed contour is a circle of radius r eq. (103) becomes

$$< E_2 >= \int_0^\pi r^2 E_2 \sin\theta d\theta = const, \tag{104}$$

which expresses the conservation of the flux of the vector $\mathbf{E}$ across a spherical surface of radius r surrounding the star. We shall now write explicitely E_2, which is the only component of **E** relevant to our problem,

$$E_2 = r^2 e^{\nu-\mu_2} \sin\theta \{[\delta\mu_3, \delta\mu_3^*]_2 + [\delta\psi, \delta\psi^*]_2 - [\delta\nu_{,2}\delta(\psi+\mu_3)^* - c.c] + \tag{105}$$
$$+ [\delta\mu_2\delta(\psi+\mu_3)^*_{,2} - c.c] + \Big[2[(\epsilon+p)\delta(\psi+\mu_3-\mu_2)^* - \delta p]e^{\nu+\mu_2}\xi_2 - c.c\Big]\}.$$

Separating the variables as in eqs. (25), (26), after some reduction we find

$$\frac{1}{4}(2l+1) < E_2 >= -(n+1)\int_0^r e^{\nu+\mu_2}[(N+L)X^* - (N+L)^*X]dr \tag{106}$$
$$+ e^{\nu+\mu_2}\Big\{(n+1)r[(N+X)F^* - (N+X)^*F] + r^3(\Pi F^* - \Pi^* F)\Big\}$$
$$+ r^2 e^{\nu-\mu_2}\left\{\frac{1}{2}r\nu_{,r}(UF^* - U^*F) + \frac{1}{2(\epsilon+p)}(\Pi U^* - \Pi^* U)\right\},$$

where we have defined $F = L+X+W$. Equation (106) has been formally derived from the equations describing the polar perturbations, by considering a solution $(\delta\psi, \delta\mu_2, \delta\mu_3, \delta\nu, \xi_2)$, and the complex conjugate solution of the same equations, for *real* σ *and real* l. Under these conditions, according to eq. (104), $< E_2 >$ has the meaning of a conserved quantity. But the boundary conditions of the problem discussed in section 5 do not allow a physically meaningful complex conjugate solution for real σs and real ls. Therefore equation (106) cannot be used as it stands. However, we can assume, as we did for the axial modes, that the polar perturbations are described by functions that are analytic in the complex l-plane, and we can extend all perturbations as in eq. (88). For example, we can assume

$$N_c = N(r;\sigma_0, l_0) + il_i\left[\frac{\partial}{\partial l}N(r;\sigma_0, l)\right], \quad X_c = X(r;\sigma_0, l_0) + il_i\left[\frac{\partial}{\partial l}X(r;\sigma_0, l)\right], \tag{107}$$

and similarly for the other functions, where $N(r;\sigma_0, l)$, $X(r;\sigma_0, l)$, etc. are solutions of eqs. (43)-(46) corresponding to real $\sigma = \sigma_0$, and real l. In Paper VI we have shown that this extension is indeed possible, and that *to any pole* (σ_0, σ_i) *there exists a corresponding pole* (l_0, l_i) *belonging to the same quasi-normal mode.* Under these premises, the analytic extension of $< E_2 >$ in the complex l-plane can also be performed, and the right-hand side of equation (106) can be evaluated in terms of the extended solution (107) to give the flux of gravitational energy through the star.

9. Some consequences of the new relativistic theory

One of the major novelties introduced by the relativistic theory that we have described in the preceeding sections is that both the polar and the axial perturbations of a spherical star can be studied as a problem of scattering of incident gravitational waves by the curvature of the spacetime. However the two classes of modes differ in one important respect: the incidence of polar gravitational waves induces oscillations in the fluid, the incidence of axial gravitational waves does not. It is known from the newtonian theory that the polar modes are resonant, and frequencies of oscillation have been measured in several astrophysical contexts. Thus stars are expected to emit polar gravitational waves with these characteristic frequencies. The question now is whether the axial modes can be resonant, and, in that case, what are the frequencies of the emitted gravitational axial waves. We shall answer this question in two different context: *(a)* very compact stars, and *(b)* slowly rotating stars.

The phenomena which we are going to describe do not have any counterpart in the newtonian theory since they derive from purely general relativistic effects.

(a) The resonant behaviour of the axial modes

In order to ascertain whether the axial modes can be resonant, in Paper IV we have applied the method to find the complex eigenfrequencies developed in section 6 to a model of star with uniform energy density distribution. This model, although clearly unrealistic, presents several advantages. The equilibrium configuration is known as an exact solution of Einstein's equations (the Schwarzschild solution). Moreover, this assumption enables us to study the axial modes in a regime where the effects of general relativity are as strong as they can ever become under conditions of hydrostatic equilibrium. The unperturbed configuration (c.f.r. S.Chandrasekhar & J.C. Miller, 1974), is

$$\begin{aligned} &\epsilon = constant, \qquad && m(r) = \frac{\epsilon r^3}{3}, \qquad p = \frac{\epsilon(y-y_1)}{(3y_1-y)}, \\ &e^{2\nu} = \tfrac{(3y_1-y)^2}{4}, && e^{-2\mu_2} = (1 - \frac{2\epsilon r^2}{3}), \\ &y = (1-\tfrac{2\epsilon r^2}{3})^{\frac{1}{2}}, && y_1 = (1 - \frac{2\epsilon {r_R}^2}{3})^{\frac{1}{2}}. \end{aligned} \tag{108}$$

At the boundary of the star $r = R$,

$$e^{2\nu}_{(r=R)} = e^{-2\mu_2}_{(r=R)} = 1 - 2M/R, \tag{109}$$

and the metric exterior to the star reduces to the Schwarzschild metric.

Homogeneous stars can exist only if their radius R exceeds 9/8 times the Schwarzschild radius R_s, or $R/M > 2.25$. The models we shall consider in the following will be labelled by the parameter (R/M). For values of $(R/M) > 2.6$, we find that the axial modes are not resonant. The reason can be understood by plotting the potential barrier (22), computed for the model of star described in eqs. (108), as a function of r/M for different values of (R/M), as shown in fig. 1. It is known from atomic physics, that scattering by a potential barrier will exhibit resonances if the potential has a minimun followed by a maximum, and if the potential well is sufficiently deep to ensure the occurrence of quasi-stationary states. In our present context we see that only when $(R/M) < 2.6$, namely when the star becomes very compact, this condition is satisfied, and the axial modes *do become resonant.* In Table 1 it is also shown that the imaginary part of frequency dramatically tends to zero as we approch the limit $(R/M) = 2.25$. Therefore, the more compact is the star, the longer will be the time needed to damp the axial oscillation.

It is interesting to note that the axial quasi-normal modes that we have found for

Table 1: *The $l = 2$ axial resonances for homogeneous star with $\epsilon = 1$*

(M and σ are measured in the units $\epsilon^{\frac{1}{2}}$ and $\epsilon^{-\frac{1}{2}}$)

$(\frac{R}{M})$	M	σ_0	σ_i
2.26	0.509798	0.213863874	$0.23 \cdot 10^{-8}$
2.28	0.503105	0.3689962	$0.12 \cdot 10^{-5}$
2.30	0.496557	0.473525	$0.26 \cdot 10^{-4}$
2.40	0.465848	0.7767	$0.92 \cdot 10^{-2}$.

homogeneous stellar models with radii approaching the limiting radius, are not related to the Schwarzschild quasi-normal modes. We might have expected that, when the star tends to the limiting configuration, the frequencies of the quasi-normal modes would tend to those of a Schwarzschild black hole of the same mass. But, as on e can see from Table 1, this is not the case. For example, for a star with $R/M = 2.26$ we find $\tau \sim 4 \times 10^8$, while a Schwarzschild black hole of the same mass would have $\tau = 5.73$! The reason is that the nature of the scattering in the two cases (a compact star and a black hole) is different, and different are the boundary conditions associated to the problem. In the case of a star, we require that at $r = 0$ the solution is free of singularity, and that at $r = R$ the metric functions and their derivatives are continuous, *with no restrictions on the direction of the flow of radiation.* In contraxt, in the case of a black hole the only boundary condition is that at the horizon there cannot be an *outward directed wave*, and only *inward radiation* can be present. Consequently, a black hole will be characterized by a reflection and an absorbtion coefficient, while a star will behave as a center of *elastic* scattering for incident radiation. The progressive increasing of the damping time τ as the star tends to the limiting configuration means that the lowest quasi-stationary state is effectively trapped, and the star cannot radiate in that resonance frequency. In conclusion, we have shown that in extremely compact stars axial modes can become resonant. Since neutron stars are likely to have radii in the range $4 < R/M < 6$, resonant scattering of axial gravitational waves by neutron stars is not to be expected. However it is possible that these modes may be excited as transients during the gravitational collapse.

(b) The coupling of the axial and polar modes in slowly rotating stars.

The theory of non-radial oscillations of stars has been developed by assuming that the unperturbed star is static and spherically symmetric. However, all celestial objects are known to be rotating, and a generalization of the theory is needed to describe realistic situations. In Paper III we have considered the case of a star that rotates with an agular velocity Ω so slow that the distortion of its figure from spherical symmetry is of order Ω^2, and can be ignored. For compact objects, small angular velocity means

$$\Omega R << 1, \tag{110}$$

a condition which is satisfied by most realistic neutron star models. We have restricted our analysis to the axial modes of slowly rotating stars.

The metric for the unperturbed spacetime is (Hartle 1967, Chandrasekhar & Miller 1974)

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \tag{111}$$

where ν, ψ, μ_2, μ_3 differ from those of a spherical non-rotating star by quantities of order Ω^2, and ω (that is zero in the non-rotating case) is now of order Ω. The equations governing ν, ψ, μ_2, μ_3 to order zero in Ω are given in section 4. The equation determining ω is

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} - (\mu_2 + \nu)_{,r}\left(\varpi_{,r} + \frac{4}{r}\varpi\right) = 0, \tag{112}$$

where we have defined

$$\varpi = \Omega - \omega. \tag{113}$$

In the vacuum outside the star, $\mu_2 + \nu = 0$ and the solution of eq. (112) can be written as

$$\varpi = \Omega - 2Jr^{-3}, \tag{114}$$

where J is the angular momentum of the star. Both inside and outside the star ϖ is a function of r only, and the continuity of ϖ at the boundary requires that $(\varpi)_{r=R} = 6JR^{-4}$. It should be noted that the function ϖ is responsible for the dragging of inertial frames predicted by the Lense-Thirring effect.

The equations governing the perturbations of a slowly rotating star can be derived by assuming that the metric appropriate to describe the phenomenon has the same form as eq. (10). We retain the hypothesis of axisymmetric perturbations because the distortion of the unperturbed configuration from spherical symmetry due to the rotation is only of order Ω^2. However, there will be relevant changes with respect to the equations that we have derived in section 4 for the non-rotating case, since now the *unperturbed* fluid is in slow rotation with a velocity

$$v^{(\alpha)} = 0, \qquad (\alpha = 2, 3), \qquad v^{(1)} = V = e^{\psi-\nu}(\Omega - \omega) = e^{\psi-\nu}\varpi, \tag{115}$$

where $v^i = x^i_{,t}$, and $v^{(i)}$ are the tetrad components. The basic equation appropriate to describe the axial modes in the present context is

$$\begin{aligned} &(e^{-3\psi+\nu-\mu_2+\mu_3} X_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3} X_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3} X \\ &= \varpi_{,r}(3\delta\psi - \delta\nu - \delta\mu_2 + \delta\mu_3)_{,\theta} - 4[(\epsilon + p)e^{\nu+\mu_2}\xi_2\varpi]_{,\theta} + 4[(\epsilon + p)e^{\nu+\mu_3}\xi_3\varpi]_{,r}. \end{aligned} \tag{116}$$

where we have made the assumption that all perturbed quantities have the *same* time-dependence $e^{i\sigma t}$, and that X is the same function defined in eq. (17). Equation (116) should be compared with eq. (16) valid in the non-rotating case. The difference is that on the right-hand side of eq. (116) in place of zero we have a combination of the perturbations $(\delta\psi, \delta\nu, \delta\mu_2, \delta\mu_3, \xi_2, \xi_3)$, that describe the *polar* modes, multiplied by ϖ and $\varpi_{,r}$.

Thus, if a star is slowly rotating the polar and the axial modes are no longer independent: they couple through the 'coupling function' ϖ that is responsible for the dragging of inertial frames.

In order to further clarify the nature of this coupling, we may expand all perturbed quantities in terms of Ω, say ($X = X^0 + \Omega X^1 + ...,\quad \delta\psi = \delta\psi^0 + \Omega\delta\psi^1 + ...,\quad etc.$). Let us consider eq. (116) at lower order in Ω. Since ϖ is of order Ω, we shall substitute to $(\delta\psi, \delta\nu, \delta\mu_2, \delta\mu_3, \xi_2, \xi_3)$, their zero order terms in Ω, i.e. $(\delta\psi^0, \delta\nu^0, \delta\mu_2^0, \delta\mu_3^0, \xi_2^0, \xi_3^0)$. Consequently, the axial perturbations X on the left-hand side of eq. (116) will be of order one in Ω (X^1):

$$(e^{-3\psi+\nu-\mu_2+\mu_3} X^1_{,r})_{,r} + (e^{-3\psi+\nu+\mu_2-\mu_3} X^1_{,\theta})_{,\theta} + \sigma^2 e^{-3\psi-\nu+\mu_2+\mu_3} X^1 \tag{117}$$
$$= \varpi, r(3\delta\psi^0 - \delta\nu^0 - \delta\mu_2^0 + \delta\mu_3^0)_{,\theta} - 4[(\epsilon + p)e^{\nu+\mu_2} \xi_2^0 \varpi]_{,\theta} + 4[(\epsilon + p)e^{\nu+\mu_3} \xi_3^0 \varpi]_{,r}.$$

In a similar manner, the zero-order (with respect to Ω) axial perturbations X^0 will be the source for the first order polar modes, $(\delta\psi^1, \delta\nu^1, \delta\mu_2^1, \delta\mu_3^1, \xi_2^1, \xi_3^1)$, of a slowly rotating star, a case that we are not going to treat in the present paper.

Since the left-hand side of eq. (117) is the same as eq. (16), we can expand X^1 in terms of Gegenbauer polynomials (see eq. (20)). It should be stressed that $(\delta\psi^0, \delta\nu^0, \delta\mu_2^0, \delta\mu_3^0, \xi_2^0, \xi_3^0)$ are the solution of the polar equations to order zero in Ω, namely the solution appropriate to a non-rotating star that we have discussed in section 5. Therefore, the 'source term' on the right-hand side can be separated in terms of Legendre polynomials as indicated in eqs. (25)-(26). By introducing the variable r_* defined in eq. (19), and the function $Z^1 = X^1/r$, we find that eq. (117) reduces to

$$\sum_{l=2}^{\infty} \left\{ \frac{d^2 Z_l^1}{dr_*^2} + \sigma^2 Z_l^1 - \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)] Z_l^1 \right\} C_{l+2}^{-\frac{3}{2}}(\mu) \tag{118}$$
$$= 6\frac{e^{2\nu}}{r^3} J(1-\mu^2)^2 \sum_{l=2}^{\infty} S_l^0(r, \mu),$$

where

$$S_l^0 = \varpi_{,r}[(2W_l^0 + N_l^0 + 5L_l^0 + 2nV_l^0 P_{l,\mu} + 2\mu V_l^0 P_{l,\mu,\mu}] + 2\varpi W_l^0(Q-1)\nu_{,r} P_{l,\mu}, \qquad (119)$$

and Q has been defined in eq. (35). Eq. (118) is valid from the center of the star up to radial infinity, remembering that outside the star, ϵ, p and W are zero. In order to eliminate the angular dependence in eqs. (118), we multiply by $C_{m+2}^{-\frac{3}{2}}$ and integrate over the range $\mu = \cos\theta = (-1, 1)$. Since $C_{m+2}^{-\frac{3}{2}}$, $P_{l,\mu}$ and $\mu P_{l,\mu,\mu}$ are of opposite parities, it follows that the polar modes belonging to *even* l can couple only with the axial modes belonging to *odd* l, and conversely, and it must be

$$l = m + 1, \qquad \text{or} \qquad l = m - 1. \qquad (120)$$

Moreover, a *propensity rule* is true. Due to the behaviour of the source term S_l^0 near the origin (for details, see Paper III, eqs. (61)-(63)), the transition $l \rightarrow l+1$ is strongly favoured over the transition $l \rightarrow l-1$. It is interesting to note that these 'coupling rules' are known in atomic theory: the first is the Laporte rule, while the propensity rule has been formulated (Fano,1985) in the context of light absorption. Once again, we are dealing with a phenomenon in general relativity that has a counterpart in the theory of quantum mechanics.

The problem which we have formulated is essentially a two-channel problem, the two channels being the axial and the polar modes, and it is clear that a whole range of problems with different initial conditions can be formulated. We have seen that in general the axial modes of a non-rotating star *are not resonant*, unless the star is extremely compact. Conversely, the polar modes are *always* resonant. In a slowly rotating star the axial and the polar modes couple in the manner that we have now described, and it is interesting to ask whether, due to this coupling, the axial modes may exhibit resonances. To answer this question we consider the following situation. Suppose that a polar gravitational wave of frequency σ excites the star in its quadrupole polar mode $l = 2$. If the star is slowly rotating, the polar perturbation of order zero in Ω, (the same as if the star were non-rotating), will act as a source for the axial perturbation with $m = 3$, according to the Laporte and the propensity rule, as shown in eqs. (118) and (119). We can solve eq. (118) and find the values of σ for which the solution at infinity reduces to a pure outgoing wave. All the methods developed in the previous sections can now be applied, since at infinity the right-hand side of eq. (118) goes to zero at least as fast as r^{-3}, and the wave equation tends to a homogeneous Schroedinger equation. As an example, in Paper III we have applied this procedure to a politropic model of star, with a politropic index $n = 1.5$, for different values of the angular velocity Ω. For this star the axial modes were not resonant in the non-rotating case. We have found that when the star does rotate *the axial modes become resonant. Their resonances are different from that of the polar modes, and in particular, the damping times are considerably longer (hundred times longer in the example we have considered).* Thus, in a slowly rotating star, the axial modes are resonant even if the star is not extremely compact, and this resonant behaviour is a consequence of the coupling between the polar and the axial modes, that is induced by the dragging of inertial frames.

10. Concluding remarks

The idea that certain types of variable stars owe their variability to periodic oscillations, originally due to Shapley (1914), received a first mathematical formulation in 1919 (Eddington 1919*a,b*). Since then, stellar pulsations have been studied both in the framework of the newtonian theory, and in general relativity, and one might think that nothing new can be said on the subject. However, if the search is focused on those phenomena that are of pure relativistic origin, some new interesting effects emerge which disclose the original content of the theory of general relativity.

A first result of this approach is a totally new interpretation of the phenomenon of non-radial oscillations of stars: we have shown that it can be studied as a problem of pure scattering of gravitational waves by the curvature of the spacetime. This interpretation is straightforward for the axial modes, since they are governed by a single Schroedinger equation with a potential barrier depending on the particular distribution of energy density and pressure inside the star. In the case of the polar modes, the scattering nature of the problem emerges as a consequence of the decoupling of the equations that govern the perturbations of the gravitational field from those that describe the perturbations of the fluid.

Moreover, we have shown that, although the axial modes do not produce a pulsating motion in the fluid, they can exhibit a resonant behaviour, either if the star is non-rotating but compact enough, or if the star is slowly rotating. In this case the resonances are induced by a coupling between the polar and the axial modes due to the dragging of inertial frames.

These effects are new. They could not have been anticipated by the newtonian theory of gravity, and they were obscured in the existing relativistic treatment of the problem.

An interesting possibility follows from these results. When a Schwarzschild black hole is perturbed, both the axial and the polar modes are resonant, and they *have exactely the same resonances.* Conversely, when a star is perturbed the spectrum of the axial and the polar modes *is different.* Thus, there is a clear signature in the spectrum of the quasi-normal modes which allows to distinguish whether the emitting source is a star or a black hole. An unanbiguous identification of black holes will therefore be possible when axial and polar gravitational waves will be detected.

But perhaps one of the most interesting consequences of our approach is that it discloses analogies and correspondences between the theory of general relativity and the theory of quantum mechanics. The fact that we can evaluate the frequencies of the quasi-normal modes, and compute the flux of gravitational radiation by generalizing the Breit-Wigner and the Regge theory, or the existence of a Laporte, selection and propensity rule which govern the coupling between the axial and polar modes of a slowly rotating star, provide an example of a close interconnection between the two theories, that has remained veiled for more than fifthy years.

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FIGURE CAPTIONS

fig. 1

The potential V for $l = 2$, computed for a model of homogeneous star and for different values of the ratio R/M. The discontinuity at $r = R$ is due to the discontinuity of ϵ. The dashed lines are the values of $(\sigma_0 M)^2$ corresponding to the quasi-stationary states.

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THE QUASI-NORMAL MODES OF STARS AND BLACK HOLES

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Non-radial oscillations of stars excited by external perturbations, are associated to the emission of gravitational waves. The characteristic eigenfrequencies of these oscillations, computed by using the relativistic theory of stellar perturbations, will be compared with those of black holes.

1 Introduction

The study of stellar oscillations started at the beginning of this century, when Shapley[1] (1914) and Eddington[2] (1918) suggested that the variability observed in some stars is due to periodic pulsations. The subsequent study of this phenomenon, carried out in the framework of the newtonian theory of gravity, has been a powerful tool in the investigation of stellar structure. In General Relativity, the interest in the theory of stellar pulsations is enhanced by the fact that a pulsating star emits gravitational waves with frequencies and damping times each belonging to characteristic "quasi-normal" modes. Since the fluid composing the star and the gravitational field are coupled, the emitted radiation carries information on the structure of the star, and also on the manner in which the gravitational field couples to matter. Conversely, for black holes the quasi-normal modes are purely gravitational, and the corresponding eigenfrequencies depend only on the parameters that identify the spacetime geometry: mass, charge and angular momentum. In sections 2, 3 and 4 of this lecture, I shall introduce the basic equations of the theory of stellar perturbations which has been developed in collaboration with S. Chandrasekhar[3,9], under the assumption of no rotation. In section 5 the characteristics of the spectrum of the quasi-normal modes of stars and black holes, and the information it gives on the nature and the structure of the source will be discussed.

2 The perturbed spacetime

As a consequence of a perturbation, all metric functions change by an infinitesimal amount with respect to their unperturbed values, and, if we are dealing with a star, each element of fluid suffers an infinitesimal displacement from its equilibrium position, identified by the lagrangian displacement $\vec{\xi}$. Consequently, the thermodynamical variables ϵ and p, respectively the energy-density and the pressure, also change by an infinitesimal amount. Our analysis will presently be restricted to the study of adiabatic, axisymmetric perturbations of stars composed by a perfect fluid, and we shall assume that all perturbed quantities have a time dependence $e^{i\sigma t}$. The perturbed quantities are determined by solving Einstein's equations coupled to the hydrodynamical equations for a star, while for a black hole only Einstein's equations for the metric perturbations need to be considered. In order to separate the variables, all tensors can be expanded in tensorial spherical harmonics, and the azimuthal number m can be set to zero (axisymmetic perturbations). These harmonics belong to two different classes depending on the way they transform under the parity transformation $\theta \to \pi - \theta$ and $\varphi \to \pi + \varphi$. In particular those that transform like $(-1)^{(\ell+1)}$ are said to be *axial*, and those that transform like $(-1)^{(\ell)}$ are said to be *polar*. Consequently, the perturbed equations split into two distinct sets the *axial* and the *polar*, each belonging to different parities. If we choose the following line-element, appropriate to describe an axially symmetric, time-dependent spacetimes, [a]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \quad (1)$$

we find that the *axial* equations involve the perturbations of the off-diagonal components of the metric, i.e. $\{\delta\omega, \delta q_2$ and $\delta q_3\}$, and that the *polar* equations involve the diagonal part of the metric $\{\delta\nu, \delta\mu_2, \delta\psi, \delta\mu_3\}$, coupled to the thermodynamical variables $\{\delta\epsilon, \delta p, \vec{\xi}\}$ in the case of stars.

3 A Schroedinger equation for the axial perturbations

The equations for the axial perturbations can be considerably simplified by introducing, after separating the variables, a new function $Z_\ell(r)$, constructed from the radial part of the axial metric components, and which satisfies the

[a] It may be noted that with our choice of the gauge the number of free functions is seven. The extra degree of freedom which we allow will be eliminated by imposing boundary conditions suitable to the problem on hand.

following Schroedinger-like equation

$$\frac{d^2 Z_\ell^{ax}}{dr_*^2} + [\sigma^2 - V_\ell(r)] Z_\ell^{ax} = 0, \tag{2}$$

where $r_* = \int_0^r e^{-\nu+\mu_2} dr$. For a black hole[10]

$$V_{\ell BH}(r) = \frac{e^{2\nu}}{r^3}[l(l+1)r - 6Mr], \qquad \text{and} \qquad e^{2\nu} = 1 - \frac{2M}{r}, \tag{3}$$

and for a star[4]

$$V_{\ell Star}(r) = \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)], \qquad \nu_{,r} = -\frac{p_{,r}}{\epsilon + p}. \tag{4}$$

Outside the star ϵ and p are zero and eq. (4) reduces to eq. (3), also known as the Regge-Wheeler potential.

Thus the axial perturbations of black holes and stars are fully described by a Schroedinger-like equation with a potential barrier that depends, respectively, on the black hole mass, and on how the energy-density and the pressure are distributed inside the star in its equilibrium configuration. It should be stressed that the axial perturbations of stars are not coupled to any fluid pulsation: *they are pure gravitational perturbations, and do not have a newtonian counterpart.*

4 The polar perturbations

The expansion in tensorial spherical harmonics (with $m = 0$) shows that the polar metric functions and the thermodynamical variables have the following angular dependence

$$\begin{aligned}
\delta\nu &= N_\ell(r) P_l(\cos\theta) e^{i\sigma t} & \delta\mu_2 &= L_\ell(r) P_l(\cos\theta) e^{i\sigma t} \\
\delta\mu_3 &= [T_\ell(r) P_l + V_\ell(r) P_{l,\theta,\theta}] e^{i\sigma t} & \delta\psi &= [T_\ell(r) P_l + V_\ell(r) P_{l,\theta} \cot\theta] e^{i\sigma t}, \\
\delta p &= \Pi_\ell(r) P_l(\cos\theta) e^{i\sigma t} & 2(\epsilon + p) e^{\nu+\mu_2} \xi_r(r,\theta) e^{i\sigma t} &= U_\ell(r) P_l e^{i\sigma t} \\
\delta\epsilon &= E_\ell(r) P_l(\cos\theta) e^{i\sigma t} & 2(\epsilon + p) e^{\nu+\mu_3} \xi_\theta(r,\theta) e^{i\sigma t} &= W_\ell(r) P_{l,\theta} e^{i\sigma t},
\end{aligned} \tag{5}$$

where $P_l(\cos\theta)$ are the Legendre polynomials. After separating the variables the relevant Einstein's equations become

$$\begin{cases} (T_\ell - V_\ell + N_\ell)_{,r} - \left(\frac{1}{r} - \nu_{,r}\right) N_\ell - \left(\frac{1}{r} + \nu_{,r}\right) L_\ell = 0, \\ V_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) V_{\ell,r} + \frac{e^{2\mu_2}}{r^2}(N_\ell + L_\ell) + \sigma^2 e^{2\mu_2 - 2\nu} V_\ell = 0, \end{cases} \tag{6}$$

$$\begin{cases} -(T_\ell - V_\ell + L_\ell) = W_\ell & (= 0 \quad \text{for B.H.}), \\ \left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2T_\ell - kV_\ell) - \frac{2}{r}L_\ell = -U_\ell & (= 0 \quad \text{for B.H.}), \\ \frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}N_{\ell,r} + \left(\frac{1}{r} + \nu_{,r}\right)(2T_\ell - kV_\ell)_{,r} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,r}\right)L_\ell\right] + & \\ \frac{1}{2}\left[-\frac{1}{r^2}(2nT_\ell + kN_\ell) + \sigma^2 e^{-2\nu}(2T_\ell - kV_\ell)\right] = \Pi_\ell & (= 0 \quad \text{for B.H.}), \end{cases} \tag{7}$$

where $k = l(l+1)$, and $2n = (l-1)(l+2)$. After some manipulation, the hydrodynamical equations and the conservation of barion number give the following expression for the hydrodynamical quantities

$$\Pi_\ell = -\frac{1}{2}\sigma^2 e^{-2\nu} W_\ell - (\epsilon + p)N_\ell, \qquad E_\ell = Q\Pi_\ell + \frac{e^{-2\mu_2}}{2(\epsilon + p)}(\epsilon_{,r} - Qp_{,r})U_\ell, \tag{8}$$

$$U_\ell = \frac{[(\sigma^2 e^{-2\nu}W_\ell)_{,r} + (Q+1)\nu_{,r}(\sigma^2 e^{-2\nu}W_\ell) + 2(\epsilon_{,r} - Qp_{,r})N_\ell](\epsilon + p)}{[\sigma^2 e^{-2\nu}(\epsilon + p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})]}, \tag{9}$$

where

$$Q = \frac{(\epsilon + p)}{\gamma p}, \qquad \gamma = \frac{(\epsilon + p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const} \tag{10}$$

and γ is the adiabatic exponent (defined in ref. [3], equation (106)).

For a black hole, a suitable reduction of eqs. (6) and (7), with W_ℓ, U_ℓ, Π_ℓ set equal zero, shows that the new function

$$Z_\ell^{pol}(r) = \frac{r}{nr + 3M}\left(3MV_\ell(r) - rL_\ell(r)\right), \tag{11}$$

satisfies the following wave equation

$$\frac{d^2 Z_\ell^{pol}(r)}{dr_*^2} + [\sigma^2 - V_{BH}]Z_\ell^{pol}(r) = 0, \tag{12}$$

where

$$V_{BH}(r) = \frac{2(r - 2M)}{r^4(nr + 3M)^2}[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3]. \tag{13}$$

Thus, as for the axial perturbations, the equations for the polar perturbations of a Schwarzschild black hole reduce to a single Schroedinger-like equation, but with a different potential barrier. Equation (12) with the potential (13) is known as the Zerilli equation[11], and it will also governe the metric perturbations in the exterior of a non-rotating star. The functions Z_ℓ^{ax} and Z_ℓ^{pol} contain all information on the gravitational waves emerging at infinity. In fact, it has been shown that the imaginary and the real part of the Weyl scalar Ψ_0, which represents the outgoing part of the radiative field (cfr. [12] eqs. 345 and 353), can be expressed in terms of Z_ℓ^{ax} and Z_ℓ^{pol}, respectively.

It is now interesting to see how eqs. (6)-(7) and the hydrodynamical equations (8,9) can be reduced if the perturbed object is a star. One may try to operate on these equations in a way similar to that used to find equation (12), hoping to find again a Schroedinger-like equation, possibly with some source in terms of the fluid variables. Unfortunately this is not possible, since the Schroedinger equation for black holes arises by virtue of the equilibrium equations, that are very different in the case of a star. In addition, this fact was to be expected, as already in newtonian theory the equations for the polar perturbations are described by a fourth order linear differential system. However a remarkable simplification is still possible. The first of eqs. (7) and eqs. (8,9) show that the fluid variables $[W_\ell, U_\ell, E_\ell, \Pi_\ell]$ can be expressed as a combination of the metric perturbations $[T_\ell, V_\ell, L_\ell, N_\ell]$ and their first derivatives. Therefore, after their direct substitution on the right hand side of the last three eqs. (7) we obtain a set of new equations which involves only the perturbations of the metric functions $[T_\ell, V_\ell, L_\ell, N_\ell]$. The final set is

$$\left\{\begin{array}{l} X_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) X_{\ell,r} + \frac{n}{r^2} e^{2\mu_2}(N_\ell + L_\ell) + \sigma^2 e^{2(\mu_2-\nu)} X_\ell = 0, \\ (r^2 G)_{\ell,r} = n\nu_{,r}(N_\ell - L_\ell) + \frac{n}{r}(e^{2\mu_2} - 1)(N_\ell + L_\ell) + r(\nu_{,r} - \mu_{2,r})X_{\ell,r} + \sigma^2 e^{2(\mu_2-\nu)} r X_\ell, \\ -\nu_{,r} N_{\ell,r} = -G_\ell + \nu_{,r}[X_{\ell,r} + \nu_{,r}(N_\ell - L_\ell)] + \frac{1}{r^2}(e^{2\mu_2} - 1)(N_\ell - rX_{\ell,r} - r^2 G_\ell) \\ -e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\sigma^2 e^{2(\mu_2-\nu)} \left\{ N_\ell + L_\ell + \frac{r^2}{n} G_\ell + \frac{1}{n}[rX_{\ell,r} + (2n+1)X_\ell] \right\}, \\ L_{\ell,r}(1-D) + L_\ell\left[\left(\frac{2}{r} - \nu_{,r}\right) - \left(\frac{1}{r} + \nu_{,r}\right) D\right] + X_{\ell,r} + X_\ell\left(\frac{1}{r} - \nu_{,r}\right) + DN_{\ell,r} + \\ +N_\ell\left(D\nu_{,r} - \frac{D}{r} - F\right) + \left(\frac{1}{r} + E\nu_{,r}\right)\left[N_\ell - L_\ell + \frac{r^2}{n} G_\ell + \frac{1}{n}\left(rX_{\ell,r} + X_\ell\right)\right] = 0, \end{array}\right. \tag{14}$$

where

$$\left\{\begin{array}{l} A = \frac{1}{2}\sigma^2 e^{-2\nu}, \qquad B = \frac{e^{-2\mu_2}\nu_{,r}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r}), \\ D = 1 - \frac{A}{2(A+B)} = 1 - \frac{\sigma^2 e^{-2\nu}(\epsilon+p)}{\sigma^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \\ E = D(Q-1) - Q, \\ F = \frac{\epsilon_{,r} - Qp_{,r}}{2(A+B)} = \frac{2[\epsilon_{,r} - Qp_{,r}](\epsilon+p)}{2\sigma^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \end{array}\right. \tag{15}$$

and V_ℓ and T_ℓ have been replaced by X_ℓ and G_ℓ defined as

$$\left\{\begin{array}{l} X_\ell = nV_\ell \\ G_\ell = \nu_{,r}[\frac{n+1}{n}X_\ell - T_\ell]_{,r} + \frac{1}{r^2}(e^{2\mu_2} - 1)[n(N_\ell + T_\ell) + N_\ell] \\ +\frac{\nu_{,r}}{r}(N_\ell + L_\ell) - e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\sigma^2 e^{2(\mu_2-\nu)}[L_\ell - T_\ell + \frac{2n+1}{n}X_\ell]. \end{array}\right. \tag{16}$$

Equations (14) describe the perturbations of the gravitational field in the interior of the star, with no reference to the motion of the fluid. Once these equations have been solved, the fluid variables can be obtained in terms of the metric functions from the first of eqs. (7) and eqs. (8,9). This fact is remarkable: it shows that all the information on the dynamical evolution of a physical system is encoded in the gravitational field, a result which expresses the physical content of Einstein's theory of gravity. Moreover, it should be stressed that the decoupling of the equations governing the metric perturbations from the equations governing the hydrodynamical variables is possible in general, and *requires no assumptions on the equation of state of the fluid.* Thus, if we are interested exclusively in the study of the emitted gravitational radiation, we can solve the system (14) and disregard the fluid behaviour.

Equations (14) have to be integrated for each value of the frequency from $r = 0$, up to the boundary of the star. There the spacetime becomes a vacuum sperically symmetric spacetime, and the perturbed metric functions match continuously with the metric functions that describe the polar perturbations of a Schwarzschild black hole, i.e. eqs (11,12,13). Thus the boundary conditions appropriate to the problem are

$i)$ all functions are regular at $r = 0$, (17)

$ii)$ $\delta p = 0$ at the boundary of the star

$iii)$ all functions and their first derivatives are continuous at the boundary of the star.

5 The characteristic frequencies of the quasi-normal modes

The concept of quasi-normal modes plays a central role in the theory of perturbations of stars and black holes. In newtonian theory the oscillations of a perturbed star can be decomposed into normal modes, i.e. solutions of the perturbed equations that satisfy the boundary conditions (17) i), ii), and that correspond to a discrete set of real eigenfrequencies. Their relativistic generalization are the quasi-normal modes, and in this case the characteristic frequencies are complex, since the imaginary part is the inverse of the damping time associated to the emission of gravitational waves. Although the completeness of the quasi-normal modes has never been proved, numerical simulations show that an initial perturbation will, during the very last stages, decay as a superposition of these pure modes, and that a large fraction of the radiation will be emitted at the corresponding frequencies. The boundary conditions that identify the quasi-normal modes of a star are that, in addition to (17), at radial infinity only pure outgoing waves must prevail. The role of the equations in the interior of the star is that of providing the initial conditions for the integration of the Zerilli or the Regge-Wheeler equation in the exterior. Since a polar perturbation excites the fluid motion, the amount of energy which leaks out of

Table 1: The complex characteristic frequencies of the quasi-normal modes of a Schwarzschild black hole.

	$M\sigma_0 + iM\sigma_i$		$M\sigma_0 + iM\sigma_i$
$\ell = 2$	0.3737+i0.0890	$\ell = 3$	0.5994+i0.0927
	0.3467+i0.2739		0.5826+i0.2813
	0.3011+i0.4783		0.5517+i0.4791
	0.2515+i0.7051		0.5120+i0.6903

the star in the form of gravitational waves depends on the exchange of energy between the fluid and the gravitational field. Conversely, an axial perturbation does not excite any fluid motion, and the boundary conditions depend only on the shape of the potential of the wave-equation, i.e. on how the energy-density and the pressure are distributed in the equilibrium configuration. Thus, the eigenfrequencies of the axial quasi-normal modes carry information essentially on the structure of the star, and the polar, in addition, elucidate the manner in which the fluid and the gravitational field couple at supernuclear regimes.

For a black hole, the quasi-normal modes are defined to be solutions of the wave-equations that satisfy the boundary conditions of a *pure outgoing wave at infinity* and of a *pure ingoing wave at the horizon* (no radiation can emerge from the horizon). The corresponding frequencies are characteristic of many different processes involving the dynamical perturbations of black holes, and are the same both for the polar and for the axial perturbations, i.e. *the two potential barriers (3) and (13) are isospectral.* In 1975 Chandrasekhar and Detweiler[13] computed the first few eigenfrequencies of a Schwarzschild black hole, and subsequently Leaver [14] determined the next values with very high accuracy. He showed that, for a given ℓ, $M\sigma_0$ decreases with the order of the mode, and approaches a non-zero constant value, while $M\sigma_i$ increases, i.e. the damping time decreases. In Table 1 we show the first four values, respectively for $\ell = 2$ and $\ell = 3$. For example, remembering that $1M_\odot = 1.48 \cdot 10^5 cm$ and assuming that the black hole mass is $M = nM_\odot$, the conversion to physical unities gives the following values of the frequency and damping time

$$\nu_0 = \frac{c}{2\pi n \cdot M_\odot (M\sigma_0)} = \frac{32.26}{n}(M\sigma_0) kHz, \quad \tau = \frac{nM_\odot}{(M\sigma_i)c} = \frac{n \cdot 0.4937 \cdot 10^{-5}}{(M\sigma_i)} s. \tag{18}$$

In order to compare the frequencies at which black holes and stars emit gravitational waves, we shall first consider, as an example, the polar perturba-

Table 2: Parameters of the three models of polytropic stars used to compute the polar eigenfrequencies

ρ in gr/cm^3	$\frac{M}{M_\odot}$	R in km	$\frac{2M}{R}$
$3 \cdot 10^{15}$	1.266	8.861	0.422
$6 \cdot 10^{15}$	1.35	7.413	0.538
10^{16}	1.3	6.465	0.594

Table 3: The characteristic frequencies and damping times of the $\ell = 2$ s and **w** polar modes of polytropic stars, compared with the first three eigenfrequencies of a Schwarzschild black hole with the same mass

	s-modes		**w**-modes		black hole	
$\frac{2M}{R}$	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s
0.422	3.0366	0.076	13.1556	$2.42 \cdot 10^{-5}$	9.5226	$7.02 \cdot 10^{-5}$
	6.7384	5.642	22.3438	$1.83 \cdot 10^{-5}$	8.8346	$2.28 \cdot 10^{-5}$
	10.1980	0.077	31.2207	$1.26 \cdot 10^{-5}$	7.6726	$1.31 \cdot 10^{-5}$
0.538	3.9166	0.060	12.4960	$3.65 \cdot 10^{-5}$	8.9300	$7.49 \cdot 10^{-5}$
	7.9610	0.623	19.4390	$2.30 \cdot 10^{-5}$	8.2848	$2.43 \cdot 10^{-5}$
	11.8669	0.035	26.3559	$1.94 \cdot 10^{-5}$	7.1952	$1.39 \cdot 10^{-5}$
0.594	4.5310	0.061	10.8420	$6.20 \cdot 10^{-5}$	9.2735	$7.21 \cdot 10^{-5}$
	8.7109	0.151	16.9960	$3.27 \cdot 10^{-5}$	8.6035	$2.34 \cdot 10^{-5}$
	12.7429	0.035	22.5540	$2.59 \cdot 10^{-5}$	7.4719	$1.34 \cdot 10^{-5}$

tions of three models of star with a polytropic equation of state

$$p = K\rho^{1+\frac{1}{m}}, \qquad m = 1, \qquad K = 100 \quad km, \tag{19}$$

identified by different values of the central density. The corresponding mass, radius and surface gravity are given in Table 2. The polar quasi-normal modes of a star belong essentially to two different classes

i) slowly-damped modes, or **s**-modes,

ii) highly-damped modes, or **w**-modes,

and the values of the first three eigenfrequencies of the $\ell = 2$ **s**-[15] and **w**-modes[16] are shown in Table 3, compared with the polar eigenfrequencies of a Schwarzschild black hole having the same mass.

The damping time τ indicates how fast the energy is dissipated in the form of gravitational waves, and since the τ's associated to the **w**-modes are of the same order of magnitude both for stars and black holes, (note also that they both decrease with the order of mode), it is natural to interpret the **w**-modes as being essentially modes of the gravitational field. However, since the boundary conditions to be imposed at the surface of the star and at the black hole horizon are different, the real part of the eigenfrequency, ν_0, will, in general, be different: higher for a star than for a black hole with the same mass, and increasing with the order of mode rather than decreasing. The **s**-polar modes have a different physical origin. They are essentially fluid pulsations whose energy is dissipated in the form of gravitational radiation at a rate which depends on how strong is the coupling between the fluid and the gravitational field. Thus, the values of the damping times are considerably longer that those of the **w**-modes. The frequency of the fundamental mode is smaller than that of a black hole with the same mass, and increases with the compactness of the star, because the time scale of these processes is related to the speed of acoustic waves in the fluid.

Let us now consider the axial perturbations. Since they do not excite any motion in the fluid, one may expect that slowly damped axial quasi-normal modes should not exist. However, this is not the case for the following reason. The slowly damped quasi-normal modes associated to the Schroedinger-like equation (2) with the potential barrier (4), are the equivalent of the quasi-stationary states that one encounters in quantum mechanics in the study of the emission of α-particles by a radioactive nucleus, also described by a Schroedinger equation. In that case σ^2 is replaced by the energy E and the potential barrier is suitable for the problem on hand. The boundary conditions for the two problems are the same: regularity of the wave function at the center, and pure outgoing waves emerging at infinity. In a quasi-stationary state E is allowed to be complex: $\Re E$ is the energy of the α-particle, and $\Im E$ is the inverse of the mean life-time (Γ) of the particle (the inverse of the damping time in our context). It is known from atomic physics that a quasi-stationary state will exist if the potential barrier has a minimum followed by a maximum, and if the potential well is sufficiently deep. For a star, the potential barrier should be considered in two regions: the interior $r < r_1$, where it depends on ϵ and p, and the exterior $r > r_1$, where it reduces to the barrier of a Schwarzschild black hole which has a maximum at $r = 3M$. If the radius of the star is smaller than $3M$ and if the star is very compact, the potential well in the interior may be deep enough to allow the existence of one or more quasi-normal **s**-mode. This conjecture can easily be proved, and in Table 4 we show the eigenfrequencies of the first four **s**-[6] and **w**-modes[17] computed for the very simple models of homogenous stars with decreasing values of the ratio R/M, i.e. increasing compactness. It emerges that if $R/M > 2.4$ the depth

Table 4: The characteristic frequencies and damping times of the first four $\ell = 2$, s and w axial modes of homogenoeus stars, with $M = 1.35 M_\odot$, and different values of R/M. The data are compared with the eigenfrequencies of a black hole with the same mass.

	s-modes		**w**-modes		black hole	
$\frac{R}{M}$	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s	ν_0 in kHz	τ in s
2.4	8.6293	$1.52 \cdot 10^{-3}$	11.1738	$1.70 \cdot 10^{-4}$	8.9300	$7.49 \cdot 10^{-5}$
	–	–	14.2757	$8.03 \cdot 10^{-5}$	8.2848	$2.43 \cdot 10^{-5}$
	–	–	18.2232	$5.70 \cdot 10^{-5}$	7.1952	$1.39 \cdot 10^{-5}$
	–	–	22.6669	$4.88 \cdot 10^{-5}$	6.0099	$0.95 \cdot 10^{-5}$
2.3	5.6153	0.54	11.1084	$3.02 \cdot 10^{-4}$		
	7.5566	$1.16 \cdot 10^{-2}$	13.0403	$1.73 \cdot 10^{-4}$		
	9.3319	$1.02 \cdot 10^{-3}$	15.1512	$1.28 \cdot 10^{-4}$		
	–	–	17.4412	$1.06 \cdot 10^{-4}$		
2.28	4.4333	10.8	10.4128	$5.45 \cdot 10^{-4}$		
	6.0168	$2.50 \cdot 10^{-1}$	11.9074	$2.91 \cdot 10^{-4}$		
	7.5462	$1.44 \cdot 10^{-2}$	13.4813	$2.07 \cdot 10^{-4}$		
	8.9891	$1.83 \cdot 10^{-3}$	15.1428	$1.67 \cdot 10^{-4}$		
2.26	2.6041	$5.38 \cdot 10^{3}$	10.7852	$7.60 \cdot 10^{-4}$		
	3.5427	$1.69 \cdot 10^{2}$	11.6922	$5.34 \cdot 10^{-4}$		
	4.4802	$1.22 \cdot 10^{1}$	12.6138	$4.22 \cdot 10^{-4}$		
	5.4127	$1.37 \cdot 10^{-1}$	13.5512	$3.56 \cdot 10^{-4}$		

of the potential well in the interior is not sufficient to allow the existence of an **s**-mode, and only the **w**-modes survive. However, if $R/M < 2.4$, the **s**-modes appear, and their number is finite and increases with the compactness of the star, as well as the damping times.

The spectrum of the quasi-normal modes, whose main properties we have described, gives important information on the nature of the perturbed source: 1) If the axial and the polar spectra coincide, the source is a black hole. *This is a very strong signature.* In a suitably choosen TT-gauge the axial and the polar part of the metric tensor are respectively

$$h_{\mu\nu}^{ax} = \begin{pmatrix} (t) & (r) & (\varphi) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{\vartheta\vartheta}^{ax} & h_{\vartheta\varphi}^{ax} \\ 0 & 0 & h_{\varphi\vartheta}^{ax} & h_{\varphi\varphi}^{ax} \end{pmatrix}, \qquad h_{\mu\nu}^{pol} = \begin{pmatrix} (t) & (r) & (\varphi) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & h_{rr}^{pol} & h_{r\vartheta}^{pol} & 0 \\ 0 & h_{\vartheta r}^{pol} & h_{\vartheta\vartheta}^{pol} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

thus, the detection of these two components of the emitted radiation will provide a direct evidence of the existence of black holes.
2) If the source is a star, the presence of the **s**-modes in the axial spectrum indicates that the star has a very compact core, while their number is directly related to the value of the ratio R/M. The question whether stars with a core compact enough to allow the existence of axial **s**-modes can exist in nature is open, and it will probably receive an answer when axial gravitational waves will be observed.

Can the quasi-normal modes be excited? In the case of black holes we know they can in a variety of situations, for example when a gravitational wave-packet is scattered on the potential barrier, or when a mass $m_0 << M$ is captured by the black hole. In this case, the integration of the Zerilli and the Regge-Wheeler equations with the source term given by the stress-energy tensor of the infalling mass allows to compute the waveform and the energy emitted in these processes. It has been shown (see ref. [18] for an extensive bibliography on the subject) that the burst of gravitational waves ends in a ringing tail emitted when the particle coaleshes into the black hole ($2 < \frac{r}{M} < 4.5$). This part of the signal can be fitted with a linear superposition of quasi-normal modes. For a particle falling radially the total radiated energy is $\Delta E \sim 0.01 \left(\frac{m_0^2}{M}\right)$, which can be increased by up to a factor of 50 if the particle has an initial angular momentum.

In the case of a star it has been shown (see K. Kokkotas' paper in this volume) that both the **s**- and the **w**-axial modes can be excited if a gravitational wave-packet is scattered by the potential barrier, but much remains to be done in more realistic situations like the capture of infalling masses. For a star, these kind of calculations are complicated by the fact that we do not know how the mass m_0 interacts with the fluid composing the star after it crosses the surface. A preliminary integration of the axial[19] and the polar equations[15] with a source due to an infalling mass, and performed by truncating the integration when m_0 reaches the surface of the star, shows that indeed both the **s**- and the **w**-axial modes are excited, and that a considerable fraction of the emitted energy goes into the **w**-modes. Further work on this subject is in progress.

I would like to conclude this lecture by stressing an interesting aspect of the theory of perturbations: although it is based on the simplifying assumption that the perturbations of the physical quantities are small with respect to their unperturbed values, nevertheless, the results that one obtains by using this assumption are, to some extent, general. For example, in 1985 Stark and Piran[20,21] computed the energy spectrum emitted when an axisymmetric distribution of rotating polytropic fluid collapses to form a black hole, and they showed that it is very similar to that one obtains by integrating the Zerilli or the Regge-Wheeler equations when a mass falls in. In particular, the relevant contribution to the emitted energy is given at those frequencies at which the newborn black hole oscillate, namely at the frequencies of the quasi-normal modes.

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ASTROPHYSICAL SOURCES OF GRAVITATIONAL WAVES

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The spectral properties of the gravitational signals emitted by stars and black holes in different astrophysical processes are reviewed.

1 Introduction

Stars and black holes emit gravitational waves in a variety of astrophysical situations. Depending on the features of the signals, these sources can be classified essentially in three cathegories: i) Sources of continuous radiation, such as binary systems or rotating stars. ii) "Impulsive" sources. These include the gravitational collapse of massive stars, the coalescence of compact stars or black holes, or perturbation processes excited, for example, by the capture or the scattering of masses by an already formed compact object. In these cases gravitational waves are emitted as a burst. iii) Stochastic sources. The cumulative effect of the radiation emitted in gravitational collapses occurring in galaxies, now and in the past, should present the characteristics of a stochastic background, the spectral energy density of which would contain information on the process of galaxy and star formation.

In this lecture I shall discuss these issues, with particular reference to impulsive and stochastic sources, and I shall show what kind of information on the generating processes the energy spectrum of the emitted gravitational radiation may contain.

In view of a possible detection of gravitational waves, the knowledge of the frequencies at which the radiation will be emitted is crucial. If the source of a burst of gravitational waves is a star or a black hole, these frequencies are associated to proper modes of vibration, said *quasi-normal modes*, because they are damped by the emission of waves. These modes are central to the theory of gravitational waves because they play an important role in several dynamical processes. For example, they are excited when an external agent, such as an infalling mass, perturbs the spacetime generated by a compact object; or during the last phases of a gravitational collapse and of the coalescence of stars or black holes, when the newborn object oscillates until its residual mechanical energy is radiated away in gravitational waves. Numerical studies have shown that in this stages the dominant contribution to the emitted radiation is due to the quasi-normal modes. The eigenfrequencies of the quasi-normal modes can be computed by studying the source-free perturbations of the equilibrium configuration, and by solving the perturbed equations with boundary conditions appropriate to the nature of the source. I shall describe this approach in next section.

2 The quasi-normal modes of compact objects.

The equations describing the perturbations of black holes and stars are obtained by writing the Einstein equations, plus the equations of hydrodynamics for stars, under the assumption that the metric functions and the fluid variables undergo small changes with respect to their equilibrium values. By retaining only the first order terms, one obtains a set of linear equations describing the perturbed configuration. If the black hole or the star are static and spherically symmetric, the perturbed equations split in two classes depending on the behaviour of the angular part of the perturbation under the transformation $\theta \rightarrow \pi - \theta$ and $\varphi \rightarrow \pi + \varphi$. In particular those that transform like $(-1)^{(\ell+1)}$ are said to be $AXIAL$, and those that transform like $(-1)^{(\ell)}$ are said to be $POLAR$. In a suitably choosen TT-gauge the axial and the polar asymptotic components of the metric tensor are respectively

$$h_{\mu\nu}^{ax} = \begin{pmatrix} (t) & (r) & (\varphi) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{\vartheta\vartheta}^{ax} & h_{\vartheta\varphi}^{ax} \\ 0 & 0 & h_{\varphi\vartheta}^{ax} & h_{\varphi\varphi}^{ax} \end{pmatrix}, \qquad h_{\mu\nu}^{pol} = \begin{pmatrix} (t) & (r) & (\varphi) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & h_{rr}^{pol} & h_{r\vartheta}^{pol} & 0 \\ 0 & h_{\vartheta r}^{pol} & h_{\vartheta\vartheta}^{pol} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

The quasi-normal modes are solutions of the perturbed equations belonging to complex eigenfrequencies $\sigma = \sigma_0 + i\sigma$ (the imaginary part is the inverse of the damping time), and satisfying the boundary conditions of a pure outgoing wave at infinity. This condition identifies physically acceptable modes, i.e. those that damp the oscillations. In addition, for a black hole one must require that the solution at the horizon reduces to a pure ingoing wave, corresponding to the requirement that nothing can escape from the horizon. Conversely, for a star all perturbed functions must have a regular behaviour at $r = 0$, and match continuously with the exterior perturbation at the surface .

2.1 The quasi-normal modes of black holes

In 1975 S. Chandrasekhar and S. Detweiler [1] computed the frequencies and the damping times of the quasi-normal modes of a Schwarzschild black hole. Those of a rotating black hole were first determined by Detweiler [2 3 4], and subsequently by Leaver [5], Seidel and Iyer [6] and Kokkotas [7]. The eigenfrequencies depend on the parameters that identify the spacetime geometry, i.e. the mass, the angular momentum and the charge. In particular the frequency of oscillation of a black hole is directly proportional to its mass M, while the damping time scales as the inverse of M. For example, for the fundamental $\ell = 2$-mode of a Schwarzschild black hole of mass $M = nM_\odot$

$$\nu_0 = \frac{12.1}{n} kHz, \quad \tau = n \cdot 5.5 \cdot 10^{-5} s. \tag{2}$$

In ref. 1 S. Chandrasekhar and S. Detweiler also showed that the transmission and the reflection coefficients for the axial and the polar perturbations of a Schwarzschild black hole are equal, i.e. the polar and the axial perturbations are isospectral. *This is a definite signature that gravitational waves carry on the nature of the emitting source.* In fact for stars the situation is much different.

2.2 The **polar** *quasi-normal modes of a non-rotating star*

We shall consider in the following adiabatic perturbations of stars composed by a perfect fluid. Let us analyze firstly the polar perturbations, which also exist in newtonian theory. Inside the star they are described by a set of linear equations that couple the perturbations of the fluid and the metric variables. However, it is possible to rearrange these equations and derive a set of equations that describe exclusively the metric perturbations. The thermodynamical variables can be obtained in terms of these by simple algebraic relations [8]. This decoupling is possible in general, and requires no assumption on the equation of state of the fluid.

In newtonian theory, the classification of the polar modes is based on the behaviour of the perturbed fluid, thus, it is interesting to see whether this classification survives in the relativistic approach which, conversely, focuses on the gravitational field.

An inspection of the newtonian hydrodynamical equations shows that when a star is perturbed each element of fluid moves under the competing action of two restoring forces

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\delta\rho}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla\delta p + \delta\mathbf{f}\,, \qquad \delta\mathbf{f} = -\nabla\delta\phi, \tag{3}$$

where $\delta\phi$ is the variation of the gravitational potential. Thus, the modes of oscillations are classified according to the restoring force that is prevailing: the **g**-modes, or gravity modes, when the force is due to the eulerian change in the density $\delta\rho$, and the **p**-modes, when it is due to a change in pressure. (This classification scheme was introduced by Cowling, 1942 [9]). The two classes of modes occupy well defined regions of the spectrum, and they are separated by the **f**-mode that is the generalization of the only possible mode of oscillation of an incompressible homogeneous sphere [10]. The characteristic of this mode is that the corresponding eigenfunction has no nodes inside the star.

The relativistic approach is completely different from the newtonian approach. As mentioned earlier, the equations used to find the oscillation frequencies involve only the perturbations of the gravitational field, and the algorithm used to find them is the following. By integrating the perturbed equations for different values of the real frequency σ, one constructs the function $[\alpha^2(\sigma)+\beta^2(\sigma)]$, that is the squared amplitude of the stationary wave prevailing at infinity. It can be shown[11] that, under the hypothesis that the imaginary part of an eigenfrequency is much smaller than the real part, the values of the frequency where $[\alpha^2+\beta^2]$ has a minimum correspond to the real part of the frequencies of the quasi-normal modes. The imaginary part, i.e. the inverse of the damping time, can be obtained from the width of the parabola which fits the curve near each minimum.

For example, let us consider a star with a non-barotropic polytropic equation of state with $n=3$, $\gamma=5/3$, $\epsilon_0/p_0 = 5.35\cdot 10^5$ (the non-barotropic character of the equation of state is clear when we note that the chosen value of the adiabatic exponent γ is different from 4/3). The ratio between the central energy density ϵ_0 and pressure p_0 has been chosen to coincide with that at the centre of the sun [12]. In figure 1 we show a graph of the resonance curve $log(\alpha^2+\beta^2)$ as a function of the frequency. Although the identification of the different classes of modes can be traced back to the hydrodynamical equations that generalize eqs. (3) in the relativistic case, it is impressive to see how the distinction between the **g**-modes and the **p**-modes, separated by the **f**-mode, graphically emerges from this plot, which is based on the behaviour of the metric functions at infinity. Thus, in the relativistic approach the information on the different kind of fluid modes is coded in the gravitational field. Typical values of the frequency of the **f**-mode for neutron stars are $\nu_f \approx 1-2kHz$, and for the damping times $\tau_f \approx 0.1-0.5s$.

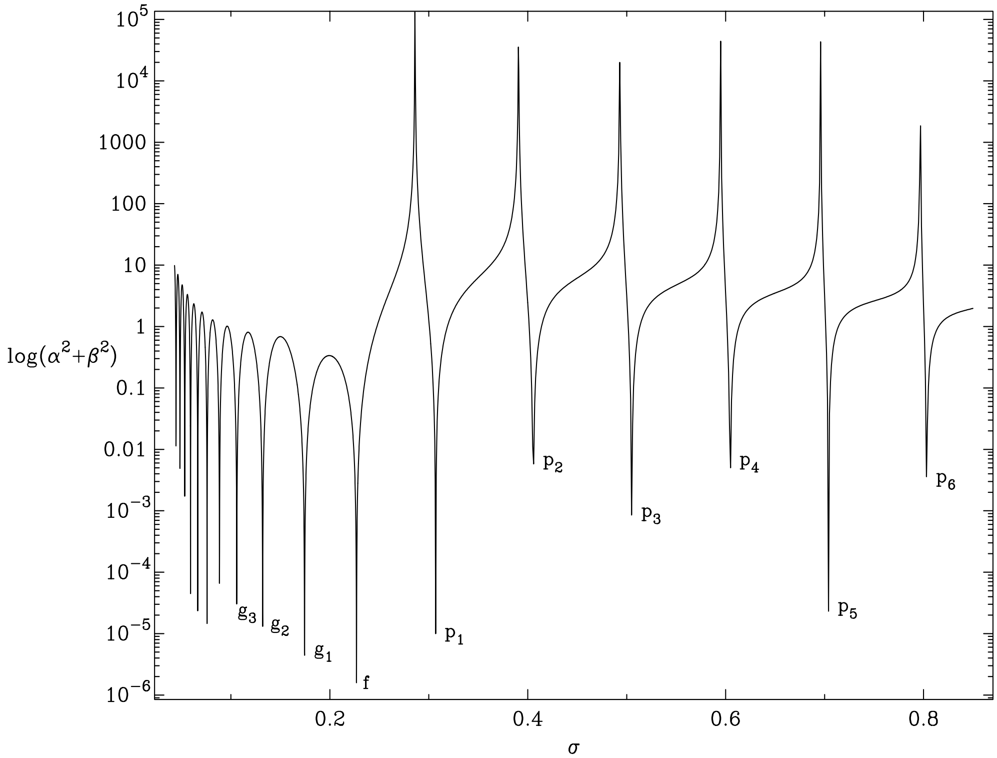

Figure 1: The resonance curve $\log(\alpha^2+\beta^2)$, is plotted versus the real frequency σ, for $\ell = 2$. σ is measured in unities of $\epsilon_0^{-1/2}$, where ϵ_0 is the energy density at the centre of the star.

From the knowledge of the eigenfrequencies of the polar quasi-normal modes one can derive interesting information. N. Andersson and K. Kokkotas [14] have determined the frequency and the damping time of the **f**-mode for several equations of state proposed in the literature for neutron stars. They fit the data with the following relations

$$\nu_f = 0.39 + 44.45\sqrt{\left(\frac{M}{R^3}\right)} \qquad \tau_f = 0.1 - \left(\frac{M}{R}\right) + 2.69\left(\frac{M}{R}\right)^2, \qquad (4)$$

where M and R are expressed in km, ν_f in kHz and τ_f in ms. These two relations provide an estimate both for M and R, good within 5% if compared with the true values.

In addition to the **g**-,**f**-,**p**-modes, that exist also in newtonian theory, in general relativity there exists a new family of modes that are essentially spacetime modes, since the corresponding motion of the fluid is negligible[13]. They are called **w**-modes, and are characterized by high frequencies $\nu_w \approx 8{-}12kHz$, and short damping times $\tau_w \approx 0.02 - 0.1ms$.

A further relation is provided by N. Andersson and K. Kokkotas[14] for the damping time of the lowest **w**-mode computed for the same neutron stars models

$$\frac{1}{\tau_{w_0}} = 0.104 - 0.063 \left(\frac{M}{R}\right). \qquad (5)$$

2.3 The **axial** *quasi-normal modes of a non-rotating star*

The radial evolution of the axial perturbations of stars is described by a Schroedinger-like equation with a potential barrier that depends on the distribution of energy and pressure in the interior of the star in the equilibrium configuration [8]. *The axial perturbations are not coupled to the oscillations of the fluid, and do not have a newtonian counterpart.* Consequently, the axial quasi-normal modes are pure spacetime modes, and they belong to two cathegories:
w-modes - highly damped and with properties similar to the polar **w**-modes,
s-modes - slowly damped[15] and related to the shape of the potential barrier. These modes appear if the star is extremely compact. For example, the potential well in the interior of homogeneous stars becomes deeper as the value of (R/M) decreases and the star shrinks. When the ratio (R/M) is sufficiently small, $(R \lesssim 3M,)$ the potential barrier outside the star has a maximum, and the potential well in the interior may become deep enough to allow for the existence of one or more quasi-stationary states, i.e. of quasi normal modes.

It is interesting to see explicitely to what extent the **s**-modes are slowly damped compared to the **w**-modes. As an illustrative example, in table 1 we show the characteristic frequencies and damping times of the first $\ell = 2$, **s**- and **w**-axial modes of homogenoeus stars, with $M = 1.35 M_\odot$, and different values of R/M. It should be stressed that the modes that one finds when the radius of the star approaches the limiting value $R/(2M) = 9/8$, are not related to the quasi-normal modes of a Schwarzschild black hole, because the boundary conditions are different. Moreover, the progressive increasing of the damping time for these modes means that they are more effectively trapped by the curvature of the star.

3 The excitation of the quasi-normal modes

It is now interesting to ask whether the quasi-normal modes can be excited in some astrophysical situations. For example one can compute the energy spectrum of the gravitational radiation emitted when a mass $m_0 << M$ is captured by a star or by a black hole of mass M, and compare the results[16]. The difficulty in the case of a star is that we do not know how the mass m_0 interacts with the fluid in the interior, and therefore the integration of the equations must be stopped when m_0 reaches the surface of the star.

Table 1: *The characteristic frequencies and damping times of the first* $\ell = 2$, **s**-*and* **w**-*axial modes of homogenoeus stars, with* $M = 1.35 M_{\odot}$, *and different values of* R/M.

	s-modes		**w**-modes	
$\frac{R}{M}$	ν_s in kHz	τ_s in s	ν_w in kHz	τ_w in s
2.4	8.6293	$1.52 \cdot 10^{-3}$	11.1738	$1.70 \cdot 10^{-4}$
2.3	5.6153	0.54	11.1084	$3.02 \cdot 10^{-4}$
2.28	4.4333	10.8	10.4128	$5.45 \cdot 10^{-4}$
2.26	2.6041	$5.38 \cdot 10^{3}$	10.7852	$7.60 \cdot 10^{-4}$

As a consequence of this truncation, the computed energy spectrum may be quite distorted with respect to the true spectrum, but still it will provide an indication on whether the modes are excited or not. In figure 2 the energy spectrum emitted in the axial perturbations when a mass m_0 is captured by a Schwarzschild black hole or by a star are plotted versus the frequency. m_0 starts its flight from radial infinity with a given angular momentum $\bar{L} = L/M$, and the mass of the black hole and of the star are chosen in such a way that the frequencies of their lowest quasi-normal mode, if expressed in physical units, coincide. This means that, for example, if the mass of the star is $M_S = 1.8 M_{\odot}$, that of the black hole will be $M_{BH} = 2.07 M_{\odot}$. The star is supposed to be homogeneous, with energy density ϵ, and with $R/M = 2.3$. This star possesses only two **s**-modes and several **w**-modes. Figure 2 shows that the energy spectra emitted by a black hole and by a star are morphologically very different and contain a clear signature on the nature of the source. For a black hole, figure 2a, there is only one peak at approximately the frequency of the lowest quasi-normal mode. The reason why the contribution of the different modes is not distinguishable is that, being the damping time of each mode very short, the width associated to each peak is large and its contribution cannot be isolated from the others, so that the result is the envelope. In the case of a star, figure 2b gives a clear indication that both **s**-modes are excited, though we cannot say anything definite about the relative height of the two peaks because of the truncation of the signal, as explained before. The two peaks are so well resolved because the damping times of the corresponding modes are quite large. A zoom of the spectrum at higher frequencies given in figure 3c, shows that also the **w**-modes of this star are excited.

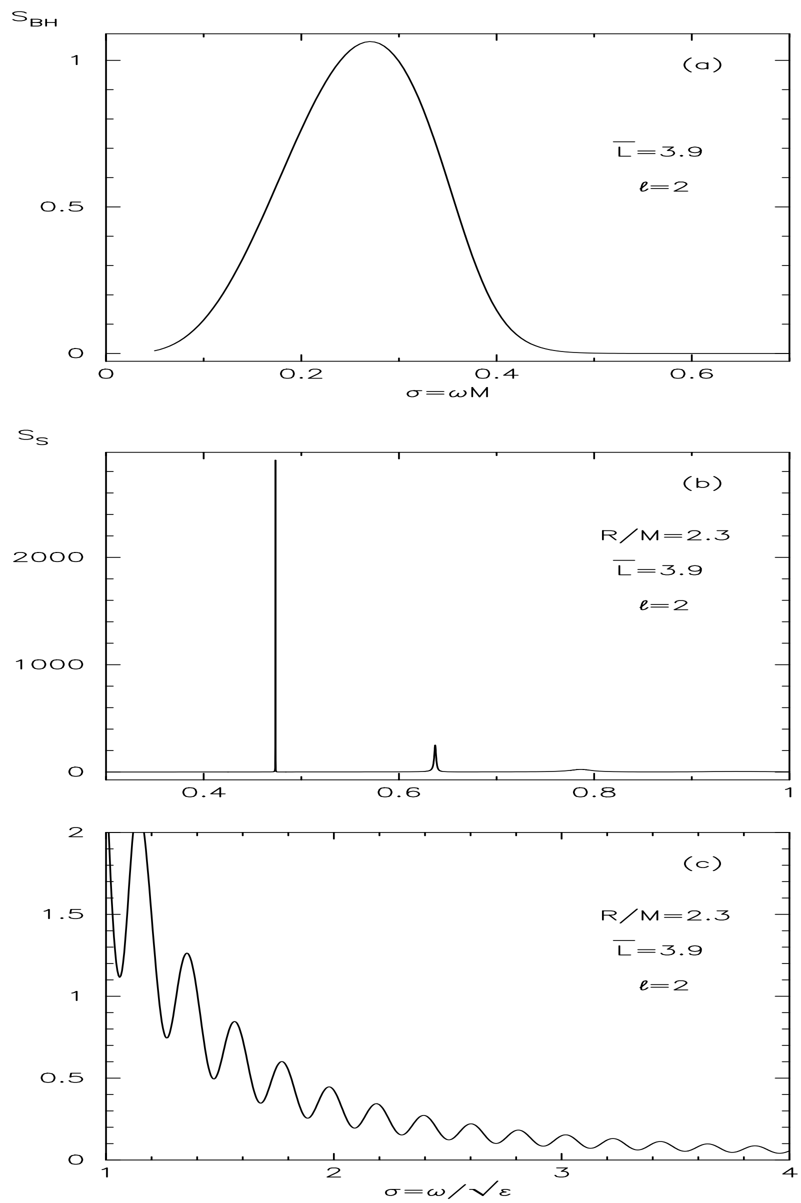

Figure 2: *The energy spectra emitted by a black hole (a) and by a homogeneous star (b,c) are plotted versus a normalized frequency.* $\left(\frac{dE}{d\omega}\right)_{BH} = M_{BH} m_0^2 S_{BH}$, *and* $\left(\frac{dE}{d\omega}\right)_{S} = S_S m_0^2 \epsilon^{-1/2}$.

4 A stochastic background

The evolution of sufficiently massive stars leads to gravitational collapse and to the emission of bursts of gravitational radiation. If the rate of star formation is sufficiently high, the cumulative effect of these processes produces a stochastic background of gravitational waves, the spectral features of which depend on how the process of galaxy and star formation took place.

A possible scenario for galaxies and stars formation, proposed by A.Di Fazio[17] and A. Di Fazio and Yu. Izotov[18], is the following. After radiation decoupled from matter, gravitational instabilities caused the formation of self-gravitating gaseous clouds, with primordial chemical composition. Being unstable, they collapsed and underwent fragmentation. This process recurred inside the newly formed structures originating generations of smaller fragments, up to when the first protostars were formed. The subsequent evolution of these protostars produced an intense burst of gravitational collapses, the biggest in the history of the universe, since at that time the gas available to form stars was much more than it is today. The resulting normalized mass distribution functions for galaxies, $\Psi_G(M)$, and for stars, $\Psi_S(m)$, can be modeled as follows

$$\psi_G(M) = \frac{M^{-1.87}\sqrt{1-\left(\frac{M_{min}}{M}\right)^{\frac{2}{3}}}}{\int_{\Delta M} M^{-1.87}\sqrt{1-\left(\frac{M_{min}}{M}\right)^{\frac{2}{3}}}\,dM}, \qquad \int_{\Delta M}\psi_G(M)dM = 1. \tag{6}$$

where ΔM is the protogalaxy mass range $[8\cdot 10^8 M_\odot < M < 5\cdot 10^{12} M_\odot]$, and

$$\psi_S(m) = \frac{m^{-1.77}\sqrt{1-\left(\frac{m_{min}}{m}\right)^{\frac{2}{3}}}}{\int_{\Delta m} m^{-1.77}\sqrt{1-\left(\frac{m_{min}}{m}\right)^{\frac{2}{3}}}\,dm}, \qquad [4M_\odot < m < 100 M_\odot]. \tag{7}$$

In the framework of this scenario we have computed[19] the rate of gravitational collapses associated to the first big burst of star formation, the expected spectral energy density and the strain amplitude. In our calculation we have made the simplifying assumption that all galaxies were formed at some redshift z_{GF}. The rate of gravitational collapses can be obtained by integrating the following expression

$$d\Re = \frac{dN_S}{(1+z_S)\Delta t_{burst}}, \tag{8}$$

over the allowed range of masses for galaxies and stars. In eq. (8) Δt_{burst} is the time interval during which the first burst of primordial collapses occurred,

z_S is the redshift of star formation which is related to z_{GF}, and dN_S is the number of protostars, with mass in the range $[m, m+dm]$, which form in primordial galaxies i.e.

$$dN_S = N_0 \Psi_G(M) dM \cdot N_S(M) \psi_S(m) dm, \tag{9}$$

where N_0 and $N_S(M)$ are respectively the total number of protogalaxies and the number of stars in each protogalaxy. We have found that the rate of gravitational collapses that led to black hole formation is $\Re \gtrsim 10^5$ events per second, depending on the values of the parameters present in our calculation, i.e.

- The value of the Hubble constant, which we write as $H_0 = 50 \frac{km}{sMpc} \cdot h$.
- The fraction of barion mass which goes into galaxies, $0.5 \lesssim \eta_G \lesssim 1$.
- The uncertainty on the value of the time interval a star stays in the main sequence before collapsing or exploding as a supernova, $\tau_{MS} \sim [2-3] My$ (for stars with masses in the range $25 M_\odot \lesssim m \lesssim 100 M_\odot$.)
- The redshift at which galaxy formation occurred, which we assume to be $4 \lesssim z_{GF} \lesssim 8$.

Since the sources are isotropically distributed, and due to the high rate and to the short duration of the signal generated in each event (typically a few milliseconds at the emission), the assumption that the gravitational radiation produced in these processes has the character of an isotropic, continuous stochastic background, is justified. We have considered only the collapses of those stars that gave birth to a black hole because, in this case, the energy spectra available in the literature present, quite independently from the initial conditions, some common features that can easily be modeled. As a model for a single event we have used the energy spectrum computed by R.F.Stark and T.Piran[21], who integrated by a fully relativistic computer code the equations governing the evolution of a rigidly rotating, axisymmetric polytropic configuration, with adiabatic index $\gamma = 2$. The collapse was ignited by a pressure reduction to a chosen fraction f_p of its equilibrium central pressure. The efficiency of the process was always $\frac{\Delta E_{GW}}{mc^2} < 7 \cdot 10^{-4}$.

The spectral energy-density of the stochastic background is given by

$$l_{tot}(a, \nu) = \frac{dE}{dtdSd\nu} = \int f(a, m, \nu) \cdot d\Re, \tag{10}$$

where $f(a, m, \nu)$ is the energy spectrum of the single event. It is related to the strain amplitude $\sqrt{S_h(\nu)}$ (expressed in $\frac{1}{\sqrt{Hz}}$) by the equation

$$l_{tot}(a, \nu) = \frac{\pi}{2} \frac{c^3}{G} \nu^2 S_h(\nu) \left[\frac{ergs}{cm^2 Hzs} \right]. \tag{11}$$

In Fig. 3 we plot the spectral amplitude $\sqrt{S_h(a,\nu)}$ as a function of the frequency of observation ν, for different values of the redshift of galaxy formation, and the function $\Omega_G(a,\nu) = \frac{4}{3}\frac{\pi^2}{H_0^2}\nu_o^3 S_h(a,\nu)$, related to it. In that picture all formed black holes have been assumed to have the same, and quite low, angular momentum $a = 0.5$. (The maximum value, reported in ref. [21], is $a_{crit} = 1.2 \pm 0.2$). Depending on the value of the redshift of galaxy formation, $\sqrt{S_h(a,\nu)}$ reaches its maximum values respectively in the following regions

$$\begin{aligned} z_{GF} &= 4 \quad 240Hz < \nu < 370Hz \\ z_{GF} &= 6 \quad 195Hz < \nu < 295Hz \\ z_{GF} &= 8 \quad 165Hz < \nu < 255Hz \end{aligned}$$

These data do not significantly change if we change the value of the Hubble constant. For example if we assume $H_0 = 33\frac{km}{sMpc}$ for $z_{GF} = 6$ we find that $175Hz < \nu < 270Hz$. The amplitude of $\sqrt{S_h(a,\nu)}$ scales as $\sqrt{H_0}$.

5 Concluding Remarks

Many are the problems related to the emission of gravitational waves from astrophysical sources that remain to be investigated and clarified. For example, our knowledge on the information that the energy spectrum emitted by compact sources carries on the internal structure of the source is still very limited. Furthermore, we have indications on how the rotation of a star affects its emission [22] if the rotation is slow, but much remains to be understood in the case of fast rotation. Lastly, we have a very poor understanding of the gravitational collapse, for which a fully relativistic numerical approach seems to be unavoidable. However, apart from the difficulties of modeling the physics of such catastrophic events, the computer codes designed to study the problem suffer of several problems related to the strongly non-linear regime in which they are forced to operate. Thus, there is a strong need to support the non-linear numerical approach with other techniques and semi-analytical methods that will be of great help in testing and interpreting the numerical results. In this respect, the theory of perturbations is far from being cut out of the future.

The preliminary results presented in this paper on the stochastic background of gravitational waves are only a first step towards the comprehension of a phenomenon of extreme complexity, since it is intimately related to the theory of galaxy and star formation, which is subject of debate among cosmologists. We plan to repeat our calculations in the framework of alternative theories, in order to predict from the spectral properties of the resulting grav-

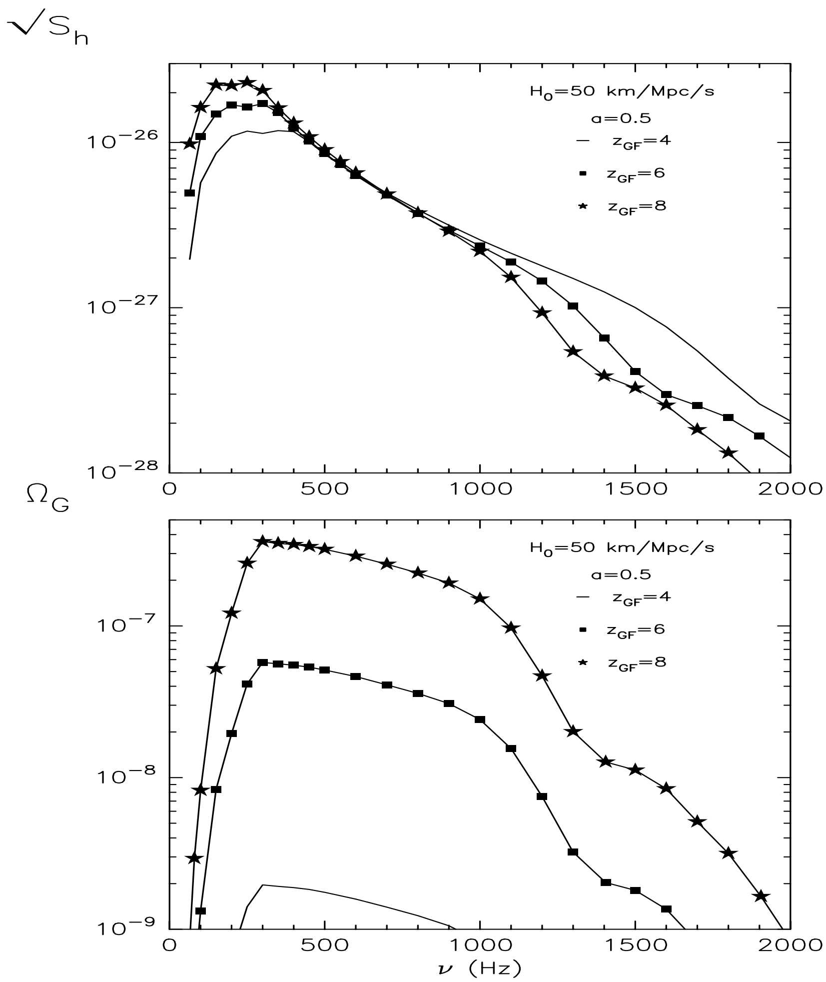

Figure 3: *The spectral amplitude* $\sqrt{S_h(a,\nu)}$, *and the spectral energy density expressed in units of the critical density,* $\Omega_G(a,\nu)$, *are plotted versus the frequency of observation, for* $a = 0.5$ *and for three different values of the redshift of galaxy formation* z_{GF}. *In these calculations we have assumed* $h = 1,\quad \eta_G = 0.8,\quad \tau_{MS} = 3 Myears.$ $\sqrt{S_h(a,\nu)}$ *is proportional to* $\left(\frac{h\eta_G}{\beta}\right)^{1/2}$, *and* $\Omega_G(a,\nu) \sim \frac{\eta_G}{h\beta}$.

itational background, those features that may discriminate among different galaxy formation scenarios.

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Gravitational waves, stars and black holes

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I had the priviledge of collaborating with professor Chandrasekhar for twelve years during which we explored the General Theory of Relativity and developed a new formulation of the theory of stellar perturbations, the startling complexity and richness of which I will try to describe in this lecture.

In order to understand the basic ideas underlying our approach, we need to frame the problem in an hystorical perspective, and start describing some major results of the theory of perturbations of a Schwarzschild black hole, which is beautifully illustrated in Chandra's book *The mathematical theory of black holes"* [1].

In 1957 T. Regge and J.A. Wheeler [2] derived the equations governing the perturbations of a static, spherically symmetric black hole. The separation of variables was accomplished by expanding the perturbed metric tensor in tensorial spherical harmonics, and since these harmonics have a different behaviour under the angular transformation $\theta \to \pi - \theta, \quad \varphi \to \pi + \varphi$, the separated equations split in two sets: the *polar* or *even*, belonging to the parity $(-1)^\ell$, and the *axial* or *odd*, belonging to the parity $(-1)^{(\ell+1)}$. Regge and Wheeler reduced the equations describing the *axial* perturbations to a single Schroedinger-like equation

$$\frac{d^2 Z_\ell^-}{dr_*^2} + \left[\omega^2 - V_\ell(r)\right] Z_\ell^- = 0, \tag{1}$$

$$V_\ell^-(r) = \frac{1}{r^3}\left(1 - \frac{2M}{r}\right)[\ell(\ell+1)r - 6M],$$

where $r_* = r + 2M\log(\frac{r}{2M} - 1)$, M is the black hole mass, ω is the frequency and the perturbed functions have been Fourier-expanded. The theory of perturbations of black holes was born.

Due to the analytical complexity of the polar equations, only much later, in 1970, F. Zerilli [3] was able to derive also for the *polar* perturbations a single Schroedinger-like equation, but with a different potential barrier

$$\frac{d^2 Z_\ell^+}{dr_*^2} + [\omega^2 - V_\ell^+(r)] Z_\ell^+ = 0, \tag{2}$$

$$V_\ell^+(r) = \frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3],$$
$$n = \frac{1}{2}(\ell+1)(\ell-2).$$

Equations (1) and (2) show that the curvature generated by a point-like mass appears in the perturbed equations as a potential barrier which extends throughout spacetime. Consequently, the response of a black hole to a generic perturbation can be studied by investigating the manner in which a gravitational wave incident on the black hole is transmitted, absorbed and reflected by this barrier, a phenomenon with which we are familiar in elementary quantum theory.

1. The quasi-normal modes of a black hole

In 1970 Vishveshwara [4] had pointed out that the equations governing the perturbations of a Schwarzschild black hole should allow complex frequency solutions behaving at radial infinity as pure outgoing waves. W.H. Press [5] confirmed this idea by numerically integrating the equations, and by showing that an arbitrary initial perturbation ends in a ringing tail, which indicates that black holes possess some proper modes of vibration.

Since the oscillations must be damped by the emission of gravitational waves, these modes were called *quasi-normal modes*, and they were defined to be solutions of the perturbed equations belonging to complex eigenfrequencies $\omega = \omega_0 + i\omega_i$, and satisfying the boundary conditions of a pure outgoing wave at infinity and of a pure ingoing wave at the horizon. The first condition identifies physically acceptable modes, i.e. those that damp the star (provided $\omega_i > 0$). The latter is the requirement that nothing can escape from the horizon. It should be noted that in scattering theory these boundary conditions associated to a Schroedinger equation with a one-dimensional potential barrier identify the singularities of the scattering amplitude.

In 1975 S. Chandrasekhar and S. Detweiler [6] computed the complex eigenfrequencies of the quasi-normal modes of a Schwarzschild black hole. The first few values for $\ell = 2$ and $\ell = 3$ are given in Table 1.

The real part of the frequency is inversally proportional to the mass, while the damping is proportional to it. If the black hole mass is $M = nM_\odot$, the oscillation frequency and the damping of the modes can be computed by the

	$M\omega + iM\omega_i$		$M\omega + iM\omega_i$
$\ell = 2$	0.3737+i0.0890	$\ell = 3$	0.5994+i0.0927
	0.3467+i0.2739		0.5826+i0.2813
	0.3011+i0.4783		0.5517+i0.4791
	0.2515+i0.7051		0.5120+i0.6903

Table 1: *The complex characteristic frequencies of the quasi-normal modes of a Schwarzschild black hole.*

following formulae

$$\nu_0 = \frac{c}{2\pi n \cdot M_\odot (M\omega)} = \frac{32.26}{n} (M\omega) kHz, \quad \tau = \frac{nM_\odot}{(M\omega)c} = \frac{n \cdot 0.4937 \cdot 10^{-5}}{(M\omega)} s. \tag{3}$$

For example, the lowest $\ell = 2$ quasi-nomal mode of a black hole of one solar mass, and of a supermassive black hole of $10^6 M_\odot$ belong, respectively, to the following frequencies

$$\begin{array}{lll} M = 1 M_\odot, & \nu_0 = 12.06 \ kHz, & \tau = 5.55 \cdot 10^{-5} s \\ M = 10^6 M_\odot, & \nu_0 = 1,21 \cdot 10^{-2} Hz, & \tau = 55.5 s. \end{array} \tag{4}$$

The frequencies of oscillation of a black hole depend exclusively on the parameters that identify the spacetime geometry: the mass, and the angular momentum or the charge if the black hole is rotating or charged.

In ref. [6] S. Chandrasekhar and S. Detweiler also showed that the transmission and the reflection coefficients associated respectively to the polar and to the axial potential barriers are equal. This equality can be explained in terms of a transformation theory which clarifies the relations that exist between potential barriers admitting the same reflection and absorption coefficients (this theory is extensively illustrated in ref. [1]). [1] However the physical reason why this happens is still unclear:

[1]The equality of the transmission and reflection coefficients can also be justified by the following considerations. The perturbations of a Schwarzschild black hole can be described in terms of the Bardeen-Press equation [7] written for the Weyl scalars Ψ_0 and Ψ_4, which represent the ingoing and outgoing radiative part of the gravitational field. The

"In spite of $V^{(+)}$ and $V^{(-)}$ appearing so very different, they are *isospectral* in the sense that the reflection and absorption coefficient for incident polar and axial gravitational waves are identically the same for all frequencies. In tracing the origin of this identity, one is led to a 'transformation theory' whose significance remains illusive"
(From S. Chandrasekhar "The series Paintings of Claude Monet and the Landscape of General Relativity" 1992 [8]).

Numerical integration of the wave equations (1) and (2) with different sources (see ref. [9] for an extensive bibliography) have shown that the gravitational signal emitted as a consequence of a generic perturbation will, during the last stages, decay as a superposition of the quasi-normal modes. In addition, a newborn black hole generated either by the gravitational collapse of a massive star or by the coalescence of two compact objects, will oscillate and emit gravitational waves until its residual mechanical energy is radiated away, and again the dominant contribution is expected to be due to the quasi-normal modes. Being the axial and polar perturbations isospectral, the gravitational radiation emitted in these processes will carry a definite signature on the nature of the emitting source; in fact, as we shall later discuss, the axial and polar perturbations of a star **are not** isospectral [10].

2. A conservation law for the scattering of gravitational waves by a black hole

One of the major problems in General Relativity is that an energy conservation law governing the scattering of gravitational waves by black holes does not exist in the framework of the exact non linear theory. However, such law can be derived in perturbation theory both for Schwarzschild , Kerr and Reissner-Nordstrom black holes. We shall now derive the conservation law for a Schwarzschild black hole, by following the procedure adopted in ref. [1].

Due to the short-range character of the potential barriers of eqs. (1) and (2), the asymptotic behaviour of the solution Z at $r_* = \pm\infty$ is, in

Bardeen-Press equation admits solutions which satisfy the boundary conditions of the quasi-normal modes, and since both Ψ_0 and Ψ_4 can be expressed as a combination of the Regge-Wheeler and of the Zerilli functions and their first derivatives [1], the axial and the polar perturbations must be isospectral.

general, a superposition of outgoing and ingoing waves

$$Z_{out} \sim e^{-i\omega r_*}, \qquad \text{and} \qquad Z_{in} \sim e^{+i\omega r_*}. \tag{5}$$

Consider two solutions of the wave equations, say Z_1 and Z_2, satisfying respectively the following boundary conditions

$$\begin{array}{lll} r_* \to +\infty & Z_1 \to e^{-i\omega r_*}, & \text{pure outgoing wave} \\ r_* \to -\infty & Z_2 \to e^{+i\omega r_*} & \text{pure ingoing wave.} \end{array} \tag{6}$$

The pairs (Z_1, Z_1^*) and (Z_2, Z_2^*), where the $*$ indicates complex conjugation and ω is assumed to be real, will be pairs of independent solutions of the wave equation, since their Wronskians are different from zero. In fact, by a direct evaluation, for example, at $\pm\infty$, one finds [2]

$$r_* \to +\infty, \quad [Z_1, Z_1^*]_{r_*} = -2i\omega, \qquad\qquad r_* \to -\infty, \quad [Z_2, Z_2^*]_{r_*} = +2i\omega. \tag{7}$$

Therefore, we can write Z_1 as a linear combination of (Z_2, Z_2^*) and viceversa:

$$\begin{aligned} Z_1 &= A(\omega)Z_2 + B(\omega)Z_2^*, \\ Z_2 &= C(\omega)Z_1 + D(\omega)Z_1^*. \end{aligned} \tag{8}$$

We now divide Z_2 by $D(\omega)$ and define

$$Z_R = \frac{Z_2}{D(\omega)} = R_1(\omega)Z_1 + Z_1^*, \tag{9}$$

where $R_1(\omega) = \frac{C(\omega)}{D(\omega)}$, and similarly

$$Z_L = \frac{Z_1}{B(\omega)} = R_2(\omega)Z_2 + Z_2^*, \tag{10}$$

where $R_2(\omega) = \frac{A(\omega)}{B(\omega)}$. Z_R and Z_L have the following asymptotic behaviour

$$Z_R \to \begin{cases} T_1(\omega)e^{+i\omega r_*} & r_* \to -\infty \\ e^{+i\omega r_*} + R_1(\omega)e^{-i\omega r_*} & r_* \to +\infty \end{cases} \tag{11}$$

[2] $[A, B]_r = A_{,r} \cdot B - A \cdot B_{,r}$

$$Z_L \to \begin{cases} e^{-i\omega r_*} + R_2(\omega) e^{+i\omega r_*} & r_* \to -\infty \\ T_2(\omega) e^{-i\omega r_*} & r_* \to +\infty \end{cases}$$

where we have set $T_1(\omega) = \frac{1}{D(\omega)}$, and $T_2(\omega) = \frac{1}{B(\omega)}$.

Thus, Z_R represents a wave of unitary amplitude incident on the potential barrier from $+\infty$ which gives rise to a reflected wave of amplitude $R_1(\omega)$ and to a transmitted wave of amplitude $T_1(\omega)$. Conversely, Z_L is a unitary wave incident from $-\infty$ which is partially reflected ($R_2(\omega)$) and partially transmitted ($T_2(\omega)$). Furthermore, by computing the Wronskian of the two solutions at $\pm\infty$ it is easy to verify that

$$\begin{aligned} [Z_L, Z_R]_{r_*} &= -2i\omega T_2(\omega) \qquad && r_* \to +\infty \\ [Z_L, Z_R]_{r_*} &= -2i\omega T_1(\omega) \qquad && r_* \to -\infty, \end{aligned} \tag{12}$$

and since the Wronskian is constant, it follows that

$$T_1(\omega) = T_2(\omega) = T(\omega). \tag{13}$$

Similarly

$$\begin{aligned} [Z_L, Z_L^*]_{r_*} &= 2i\omega(|R_2(\omega)|^2 - 1) \qquad && r_* \to -\infty \\ [Z_L, Z_L^*]_{r_*} &= -2i\omega |T_2(\omega)|^2 \qquad && r_* \to +\infty, \end{aligned} \tag{14}$$

and consequently

$$|R_2|^2 + |T_2|^2 = 1. \tag{15}$$

By a similar procedure applied to Z_R we easily find

$$|R_1|^2 + |T_1|^2 = 1. \tag{16}$$

This means that R_1 and R_2 can differ only by a phase factor and that

$$|R|^2 + |T|^2 = 1 \tag{17}$$

holds in general. This equation establishes the symmetry and the unitarity of the S-matrix, and it expresses the conservation of energy because it says that if a wave of unitary amplitude is incident on one side of the potential barrier, it splits into a reflected and a transmitted wave such that the sum of the square of their amplitudes is still one.

The existence of conservation laws for the scattering of gravitational waves by a black hole raised an interesting question: is it possible to establish a similar conservation law for the polar perturbations of a static, spherically symmetric spacetime generated either by an electromagnetic source or by a non rotating star? That such law should exist was known on a theoretical ground: A. Ashtekar, J. Friedmann, R.Sorkin and R. Wald had told us that the existence of a *conserved simplectic current* can in principle be inferred for any field theory derived from a suitably defined Lagrangian action. However, Chandra wanted to derive the conserved current by using a procedure similar to that used for Schwarzschild black holes. In that case, the central point of the derivation was to show that the Wronskian of two independent solutions of the wave equations is a constant. Conversely, the equations for the polar perturbations of a star are a fourth order linear differential system: what would be the role played by a Wronskian in this context? The solution of the problem required a considerable amount of hard work on the equations, but at the end the result was rewarding: we found that there exists a vector $\vec{\mathbf{E}}$ which satisfies the following equation [11]

$$\frac{\partial}{\partial x^\alpha}E^\alpha = 0, \qquad \alpha = (x^2 = r, x^3 = \vartheta). \tag{18}$$

The vanishing of the ordinary divergence implies that, by Gauss's theorem, the flux of $\vec{\mathbf{E}}$ across a closed surface surrounding the star is a constant.

In order to write explicitely the components of the vector $\vec{\mathbf{E}}$ (I shall omit the details of its derivation) we write the metric of a generic static, spherically symmetric spacetime in the following form

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}d\theta^2 + e^{2\psi}d\varphi^2, \tag{19}$$

where the metric functions depend only on r and ϑ. We shall restrict to the case when this metric represents the spacetime generated by an unperturbed star composed by a perfect fluid, though in ref. [11] we derived a similar conservation law also for charged solutions of Einstein's equations. The axisymmetric perturbations of the spacetime (19) can be described by the line-element

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - q_2dr^2 - q_3d\theta - \omega dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2. \tag{20}$$

It should be noted that the number of unknown functions in eq. (20) is seven, one more than needed. However, this extra degree of freedom disappears when the boundary conditions of the problem are fixed. As a consequence of a generic perturbation, the metric functions will experience small changes with respect to their unperturbed values, which we assume to be known

$$\begin{aligned} \nu &\longrightarrow \nu + \delta\nu, & \mu_2 &\longrightarrow \mu_2 + \delta\mu_2, \\ \psi &\longrightarrow \psi + \delta\psi, & \mu_3 &\longrightarrow \mu_3 + \delta\mu_3, \\ \omega &\longrightarrow \delta\omega, & q_2 &\longrightarrow \delta q_2 \quad , \qquad q_3 \longrightarrow \delta q_3. \end{aligned} \tag{21}$$

Since each element of fluid in the interior of the star undergoes an infinitesimal displacement from its equilibrium position, identified by the lagrangian displacement $\vec{\xi}$, the energy density and the pressure will change by an infinitesimal amount

$$\epsilon \longrightarrow \epsilon + \delta\epsilon, \qquad\qquad p \longrightarrow p + \delta p. \tag{22}$$

Under the assumption of axisymmetric perturbation, all perturbed quantities depend on t, r and θ. If we now write Einstein's equations supplemented by the hydrodynamical equations and the conservation of barion number, expand all tensors in tensorial spherical harmonics and Fourier-expand the time dependent quantities, we find that, as for black holes, the equations decouple into two sets, the *polar* and the *axial*, but with a major difference: the *polar* perturbations involve the same metric variables $(\delta\nu, \delta\mu_2, \delta\psi, \delta\mu_3)$ as for black holes, but now they are coupled to the thermodynamical variables

$$\begin{cases} \delta\nu \\ \delta\mu_2 \\ \delta\psi \\ \delta\mu_3 \end{cases} \longrightarrow \begin{cases} \delta\epsilon \\ \delta p \\ \xi_r \\ \xi_\vartheta \end{cases}. \tag{23}$$

Conversely the *axial* perturbations $[\delta\omega, \delta q_2, \delta q_3]$ do not induce motion in the fluid except for a stationary rotation. However, we shall see that the fluid plays a role, though different from that played in the polar case. In terms of the perturbed metric and fluid variables the E_2-component of the polar vector $\vec{\mathbf{E}}$ is

$$E_2 = r^2 e^{\nu-\mu_2} \sin\theta \{ [\delta\mu_3, \delta\mu_3^*]_2 + [\delta\psi, \delta\psi^*]_2 - [\delta\nu_{,2} \delta(\psi+\mu_3)^* - c.c] + \tag{24}$$
$$+[\delta\mu_2 \delta(\psi+\mu_3)^*_{,2} - c.c] + \Big[2[(\epsilon+p)\delta(\psi+\mu_3-\mu_2)^* - \delta p] e^{\nu+\mu_2} \xi_2 - c.c \Big] \},$$

and the E_3-component can be obtained by interchanging 2 with 3.

Equation (24) includes, as expected, Wronskians of the polar functions $[\delta\mu_3, \delta\mu_3^*]_2$ and $[\delta\psi, \delta\psi^*]_2$, and it reduces to the Wronskian of the solutions of the Zerilli equation as indicated in section 2, when the source terms ϵ and p are zero. We derived a similar expression for $\vec{\mathbf{E}}$ when the source is an electromagnetic field. G. Burnett and R. Wald [12] subsequently showed that in the Einstein-Maxwell case our conservation law can be obtained by constructing a simplectic current associated to the perturbed equations derived from a Lagrangian variational principle.

The conserved current $\vec{\mathbf{E}}$ represents the flux of gravitational energy which develops through the stars and propagates outside. Indeed, it can also be derived from the second variation of the Einstein pseudo-tensor $t_E^{\mu\nu}$ [13], [14]. The reason for choosing the Einstein pseudo-tensor is that among the infinite number of pseudo-tensors that can be defined for the gravitational field, all differing by a divergenceless term, $t_E^{\mu\nu}$ is the only one the second variation of which retains the divergence-free property, provided only the equations governing the static spacetime and its linear perturbations are satisfied. This property derives from the fact that the Einstein pseudo-tensor is a Noether operator for the gravitational field.

In addition, Raphael Sorkin pointed out that the contribution of the source should be introduced not by adding the second variation of the stress-energy tensor of the source $T^{\mu\nu}$, but through a suitably defined Noether operator, the form of which he derived for the electromagnetic case. This operator does not coincide with $T^{\mu\nu}$, but it gives the same conserved quantities. It should be mentioned that the Noether operator to be added to the Einstein pseudo-tensor when the source is a fluid has been derived only much recently by Vivek Iyer [15].

The existence of a conservation law for a spacetime with a perfect fluid source suggested to Chandra that the non-radial oscillations of stars should be reformulated as a problem in scattering theory.

"In general relativity, any distribution of matter (or more generally energy of any sort) induces a curvature of the spacetime – a potential well. Matter implies gravity and gravity implies matter. Therefore, instead of picturing the non-radial oscillations of a star as caused by some unspecified external perturbation, we can picture them as caused by incident gravitational radiation. Viewed in this manner, the reflection and absorption of incident gravitational waves by black holes and the non-radial oscillations of stars, become different aspects of the same basic theory. But how different – as we shall see!"

After completing the first paper on the flux integral, Chandra and I started to work on the perturbed equations, and reduced them to an interesting form [16], fairly different from that obtained by Thorne and his collaborators, who first developed the theory of stellar perturbations in general relativity in 1967 [17].

3. The polar equations

If one expands the perturbed metric tensor and the stress-energy tensor of the fluid in tensorial spherical harmonics, under the hypothesis of axisymmetric perturbations the polar metric functions and the thermodynamical variables turn out to have the following angular dependence

$$\begin{aligned} \delta\nu &= N_\ell(r)P_l(\cos\theta)e^{i\omega t} & \delta\mu_2 &= L_\ell(r)P_l(\cos\theta)e^{i\omega t} \\ \delta\mu_3 &= [T_\ell(r)P_l + V_\ell(r)P_{l,\theta,\theta}]e^{i\omega t} & \delta\psi &= [T_\ell(r)P_l + V_\ell(r)P_{l,\theta}\cot\theta]e^{i\omega t}, \\ \delta p &= \Pi_\ell(r)P_l(\cos\theta)e^{i\omega t} & 2(\epsilon+p)e^{\nu+\mu_2}\xi_r(r,\theta)e^{i\omega t} &= U_\ell(r)P_l e^{i\omega t} \\ \delta\epsilon &= E_\ell(r)P_l(\cos\theta)e^{i\omega t} & 2(\epsilon+p)e^{\nu+\mu_3}\xi_\theta(r,\theta)e^{i\omega t} &= W_\ell(r)P_{l,\theta}e^{i\omega t}, \end{aligned} \tag{25}$$

where $P_l(\cos\theta)$ are the Legendre polynomials. After separating the variables the relevant Einstein's equations become

$$\begin{aligned} &a) \quad (T_\ell - V_\ell + L_\ell) = -W_\ell \\ &b) \quad \left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2T_\ell - kV_\ell) - \frac{2}{r}L_\ell = -U_\ell \\ &c) \quad \frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}N_{\ell,r} + \left(\frac{1}{r} + \nu_{,r}\right)(2T_\ell - kV_\ell)_{,r} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,r}\right)L_\ell\right] + \\ &\qquad \frac{1}{2}\left[-\frac{1}{r^2}(2nT_\ell + kN_\ell) + \omega^2 e^{-2\nu}(2T_\ell - kV_\ell)\right] = \Pi_\ell \\ &d) \quad (T_\ell - V_\ell + N_\ell)_{,r} - \left(\frac{1}{r} - \nu_{,r}\right)N_\ell - \left(\frac{1}{r} + \nu_{,r}\right)L_\ell = 0, \\ &e) \quad V_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)V_{\ell,r} + \frac{e^{2\mu_2}}{r^2}(N_\ell + L_\ell) + \omega^2 e^{2\mu_2 - 2\nu}V_\ell = 0, \end{aligned} \tag{26}$$

where $k = l(l+1)$, and $2n = (l-1)(l+2)$. After some reduction, the hydrodynamical equations and the conservation of barion number provide the following expressions for the fluid variable [3]

$$\Pi_\ell = -\frac{1}{2}\omega^2 e^{-2\nu} W_\ell - (\epsilon+p)N_\ell, \qquad E_\ell = Q\Pi_\ell + \frac{e^{-2\mu_2}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r})U_\ell, \tag{27}$$

$$U_\ell = \frac{[(\omega^2 e^{-2\nu}W_\ell)_{,r} + (Q+1)\nu_{,r}(\omega^2 e^{-2\nu}W_\ell) + 2(\epsilon_{,r} - Qp_{,r})N_\ell](\epsilon+p)}{[\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})]}, \tag{28}$$

where

$$Q = \frac{(\epsilon+p)}{\gamma p}, \qquad \gamma = \frac{(\epsilon+p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const} \tag{29}$$

and γ is the adiabatic exponent.

Outside the star, the source vanishes and the polar equations can be reduced to the Zerilli equation (2), with the following identification

$$Z^+_\ell(r) = \frac{r}{nr+3M}\left(3MV_\ell(r) - rL_\ell(r)\right). \tag{30}$$

A remarkable simplification of eqs. (26) is possible. Equation 26a) and eqs. (27) show that the fluid variables $[W_\ell, U_\ell, E_\ell, \Pi_\ell]$ can be expressed as a combination of the metric perturbations $[T_\ell, V_\ell, L_\ell, N_\ell]$ and their first derivatives. Therefore, after their direct substitution on the right hand side of the last four eqs. (26) a set of new equations which involves exclusively the perturbations of the metric functions $[T_\ell, V_\ell, L_\ell, N_\ell]$ can be derived. The final set is

$$\begin{cases} X_{\ell,r,r} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) X_{\ell,r} + \frac{n}{r^2} e^{2\mu_2}(N_\ell + L_\ell) + \omega^2 e^{2(\mu_2-\nu)} X_\ell = 0, \\ (r^2 G_\ell)_{,r} = n\nu_{,r}(N_\ell - L_\ell) + \frac{n}{r}(e^{2\mu_2} - 1)(N_\ell + L_\ell) + r(\nu_{,r} - \mu_{2,r})X_{\ell,r} + \omega^2 e^{2(\mu_2-\nu)} rX_\ell, \\ -\nu_{,r}N_{\ell,r} = -G_\ell + \nu_{,r}[X_{\ell,r} + \nu_{,r}(N_\ell - L_\ell)] + \frac{1}{r^2}(e^{2\mu_2} - 1)(N_\ell - rX_{\ell,r} - r^2 G_\ell) \\ -e^{2\mu_2}(\epsilon+p)N_\ell + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}\left\{N_\ell + L_\ell + \frac{r^2}{n}G_\ell + \frac{1}{n}[rX_{\ell,r} + (2n+1)X_\ell]\right\}, \\ L_{\ell,r}(1-D) + L_\ell\left[\left(\frac{2}{r} - \nu_{,r}\right) - \left(\frac{1}{r} + \nu_{,r}\right)D\right] + X_{\ell,r} + X_\ell\left(\frac{1}{r} - \nu_{,r}\right) + DN_{\ell,r} + \\ +N_\ell\left(D\nu_{,r} - \frac{D}{r} - F\right) + \left(\frac{1}{r} + E\nu_{,r}\right)\left[N_\ell - L_\ell + \frac{r^2}{n}G_\ell + \frac{1}{n}(rX_{\ell,r} + X_\ell)\right] = 0, \end{cases} \tag{31}$$

where

$$\begin{cases} A = \frac{1}{2}\omega^2 e^{-2\nu}, \qquad B = \frac{e^{-2\mu_2}\nu_{,r}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r}), \\ D = 1 - \frac{A}{2(A+B)} = 1 - \frac{\omega^2 e^{-2\nu}(\epsilon+p)}{\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \\ E = D(Q-1) - Q, \\ F = \frac{\epsilon_{,r} - Qp_{,r}}{2(A+B)} = \frac{2[\epsilon_{,r} - Qp_{,r}](\epsilon+p)}{2\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \end{cases} \tag{32}$$

[3] We restrict our analysis to adiabatic perturbations of fluid stars.

and V_ℓ and T_ℓ have been replaced by X_ℓ and G_ℓ defined as

$$\begin{cases} X_\ell = nV_\ell \\ G_\ell = \nu_{,r}[\frac{n+1}{n}X_\ell - T_\ell]_{,r} + \frac{1}{r^2}(e^{2\mu_2} - 1)[n(N_\ell + T_\ell) + N_\ell] \\ +\frac{\nu_{,r}}{r}(N_\ell + L_\ell) - e^{2\mu_2}(\epsilon + p)N_\ell + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}[L_\ell - T_\ell + \frac{2n+1}{n}X_\ell]. \end{cases} \tag{33}$$

Equations (31) describe the perturbations of the gravitational field in the interior of the star, with no reference to the motion of the fluid.
Once these equations have been solved, the fluid variables can be obtained in terms of the metric functions from eqs. 26a) and eqs. (27). This fact is remarkable: it shows that all the information on the dynamical evolution of a perturbed star is encoded in the gravitational field, a result which expresses the physical content of Einstein's theory of gravity. Moreover, it should be stressed that the decoupling of the equations governing the metric perturbations from those governing the hydrodynamical variables is possible in general, and requires no assumptions on the equation of state of the fluid. Thus, if one is interested exclusively in the study of the emitted gravitational radiation, one can solve the system (31) and disregard the fluid behaviour.

Equations (31) have to be integrated for each value of the frequency from $r = 0$, where all functions must be regular, up to the boundary of the star. There, the spacetime becomes vacuum and sperically symmetric, and the perturbed metric functions and their first derivatives must be matched continuously with the functions that describe the polar perturbations of a Schwarzschild black hole (for a detailed discussion of the boundary conditions see refs. [16] and [18]).

It was subsequently shown by J.R.Ipser and R.H.Price [19] that the equations describing the polar gravitational perturbations decoupled from the fluid variables can be reduced to a fourth-order system.

4. A Schroedinger equation for the axial perturbations

The equations for the axial perturbations are much simpler than the polar ones. Their radial behaviour is completely described by a function $Z_\ell(r)$, which satisfies the following Schroedinger-like equation

$$\frac{d^2 Z_\ell}{dr_*^2} + [\omega^2 - V_\ell(r)]Z_\ell = 0, \tag{34}$$

where $r_* = \int_0^r e^{-\nu+\mu_2} dr$, and

$$V_\ell(r) = \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)], \qquad \nu_{,r} = -\frac{p_{,r}}{\epsilon + p}. \qquad (35)$$

Outside the star ϵ and p vanish and eq. (35) reduces to the Regge-Wheeler potential barrier (1). It should be stressed that the potential depends on how the energy-density and the pressure are distributed inside the star in its equilibrium configuration.

Since an axial gravitational wave incident on a star does not induce fluid motion, for a long time these perturbations have been considered as trivial. But this is not true if we adopt the scattering approach: the absence of fluid motion simply means that an incident axial wave experiences a potential scattering as it does in the case of a Schwarzschild black hole. There is however an important difference. The Schwarzschild potential vanishes at the black hole horizon, and it has a maximum at $r_{max} \sim 3M$. Conversely, due to the centrifugal contribution $\frac{\ell(\ell+1)}{r^2}$ the potential barrier of a perturbed star tends to infinity at $r = 0$. In addition, for a Schwarzschild black hole the Schroedinger-like equation describes a problem of scattering by a one-dimensional potential barrier, whereas in the case of a star it describes the scattering by a central potential.

Being the axial perturbations described by a Schroedinger equation, the axial component of the energy flux can be derived from the Wronskians of independent solutions, as in the black hole case. However, due to the different boundary conditions, the evaluation of this flux requires the application of the Regge theory of potential scattering in a central field. This theory can be generalized to be applicable also to the polar perturbations, and to explicitely compute the energy flux associated to the vector $\vec{\mathbf{E}}$ [20].

5. The quasi normal modes of a star

In our approach the non-radial oscillations of stars are thought to be induced by the incidence of polar or axial gravitational waves on the spacetime curvature generated by the star. In this view, a resonant scattering occurs when the star is in a quasi-stationary state that decays, i.e. when it oscillates in a quasi-normal mode.

The quasi-normal modes are solutions of the axial and polar equations that satisfy the following boundary conditions. As in the black hole case, at radial infinity only pure outgoing waves must prevail, whereas the pure ingoing wave condition at the black hole horizon is replaced by the requirement that all perturbed functions have a regular behaviour at $r = 0$. Furthermore, they must match continuously with the exterior perturbation at the surface of the star. Both the polar and the axial quasi-normal modes satisfy the same boundary conditions, but the underlying scattering problem is much different in the two cases. In fact, since a polar perturbation excites the fluid motion, the amount of radiation which leaks out of the star depends on the exchange of energy between the fluid and the gravitational field, whereas the scattering of axial gravitational waves is a pure scattering by a spherically symmetric, static potential.

In studying the theory of stellar perturbations in the framework of General Relativity, one encounters new phenomena that do not have a newtonian counterpart. A first example is the existence of new families of modes of vibration, which are modes of the radiative field. They appear because the spacetime is not symply a medium in which gravitational waves propagate: it has its own dynamics and spectrum, as it is clearly shown by the existence of the quasi-normal modes of black holes. Spacetime modes exist also for stars but, due to the different boundary conditions, their spectrum will be much different from that of black holes. One of these new families are the highly damped polar and axial **w**-modes, discovered by K.Kokkotas and B.Schutz [21]. Actually it was later shown that there exist two families of such modes [22], but we shall not go into such a detail in the present context. The **w**-modes are modes of vibration in which the motion of the fluid is barely excited, if not excited at all as in the axial case. In an article appeared in Physics World in 1991, Bernard Schutz makes an interesting analogy that vividly illustrates the nature of these modes [23]:

"Consider a violin played in an infinitely large room. The air by itself does not have conventional outgoing-wave modes: any sound waves are coming in from somewhere and going out somewhere else. But put a violin string in the room, and there appears a family of modes with purely outgoing sound waves that exchange a small amount of energy with the string, and die away very fast. These modes are strongly damped, and the weaker the coupling of the string to the air, the faster they damp away, so that in the limit of a vacuum around the string, they go away entirely."

Typical values of the lowest **w**-mode range between $\approx 8 - 12kHz$, (the

$\frac{R}{M}$	ν_0 in kHz	τ in s	$\frac{R}{M}$	ν_0 in kHz	τ in s
2.4	8.6293	$1.52 \cdot 10^{-3}$	2.28	4.4333	10.8
	-	-		6.0168	$2.50 \cdot 10^{-1}$
	-	-		7.5462	$1.44 \cdot 10^{-2}$
	-	-		8.9891	$1.83 \cdot 10^{-3}$
2.3	5.6153	0.54	2.26	2.6041	$5.38 \cdot 10^{3}$
	7.5566	$1.16 \cdot 10^{-2}$		3.5427	$1.69 \cdot 10^{2}$
	9.3319	$1.02 \cdot 10^{-3}$		4.4802	$1.22 \cdot 10^{1}$
	-	-		5.4127	$1.37 \cdot 10^{-1}$

Table 2: *The characteristic frequencies and damping times of the* $\ell = 2$ **s**-*modes of homogenoeus stars, with* $M = 1.35 M_\odot$ *and increasing compactness.*

frequency of the **w**-modes increases with the order of the mode), and the corresponding damping times are $\approx 0.02 - 0.1 ms$.

Chandra and I brought to light a further family of spacetime modes [24]. Contrary to the **w**-modes they are slowly damped, and therefore I shall call them the **s**-modes. They exist only for the axial perturbations and their appearence is related to the depth of the potential well inside the star, as the following illustrative example shows. Let us compare the shape of the axial potential barriers generated by homogeneous stars of increasing compactness, i.e. of decreasing ratio $\frac{R}{M}$. It should be reminded that homogeneous stars can exist only if their radius R exceeds $\frac{9}{8} R_s$, or equivalently, if $\frac{R}{M} > 2.25$. In figure 1 it is shown how the potential well inside the star becomes deeper as the value of $\frac{R}{M}$ decreases and the star shrinks. In the exterior the potential coincides with the Regge-Wheeler potential that has a maximum at $r \approx 3M$. When $(R/M) < 2.6$ the potential well in the interior becomes deep enough to allow the existence of one or more quasi-normal modes. In table 2 the characteristic frequencies and damping times of the $\ell = 2$ **s**-modes of homogenoeus stars with $M = 1.35 M_\odot$ and different values of R/M are listed.

It should be stressed that the modes that one finds when the radius of the star approaches the limiting value, are not related to the quasi normal modes of a Schwarzschild black hole, because both the boundary conditions and the underlying scattering process are different. Moreover, the progressive increasing of the damping time for these modes means that they are more effectively trapped by the curvature of the star.

The existence of the **s**-modes was proved by using homogeneous stars as a model, and we have seen that they appear only if $\frac{R}{M}$ is sufficiently close to the limiting value 2.25. It would be interesting to understand whether this constraint on $\frac{R}{M}$ derives from the particular choice of the model we have used, or whether it could be relaxed by the use of a different equation of state. And further, is the existence of the **s**-modes related to some characteristic property of the equation of state, as, for example, on how stiff this equation is?

To answer these questions, in collaboration with Maria Alessandra Papa [25] we have studied the quasi-normal modes of polytropic stars having at the center a very small core with the equation of state of stiff matter $\epsilon = p$.

We chose this model because, as firstly suggested by Zeldovich [26], the equation $\epsilon = p$ represents the most extreme equation of state for high density matter compatible with the requirements of special relativity. For example, the Tsuruta-Cameron [27] equation of state has this asymptotic behaviour near the center of the star. Furthermore, we wanted to understand whether the presence of a stiff core would give any particular signature to the spectrum of the gravitational waves the star emits.

We determined the equilibrium configurations of such stars, and the range in which the radius of the stiff core can vary in order the star to be stable. The main characteristics of the models we have studied are summarized in tables 3 and 4. It should be noted that since the core is extremely small, neither the mass nor the radius change significantly as a function of R_{core} (they change at most by a few percents), when it varies in the stability range. Typical values of R and M for these stars are given in table 4. From table 3 we see that as the polytropic index of the envelope decreases, the core is allowed to occupy a larger fraction of the star. Moreover the ratio $\frac{R}{M}$ decreases and the star becomes smaller and more compact. We did not consider values of n lower than 0.5 because the star would become too small and the stiff core too big, and we did not want to deal with extreme situations.

Contrary to our expectations, we found that the depth of the potential

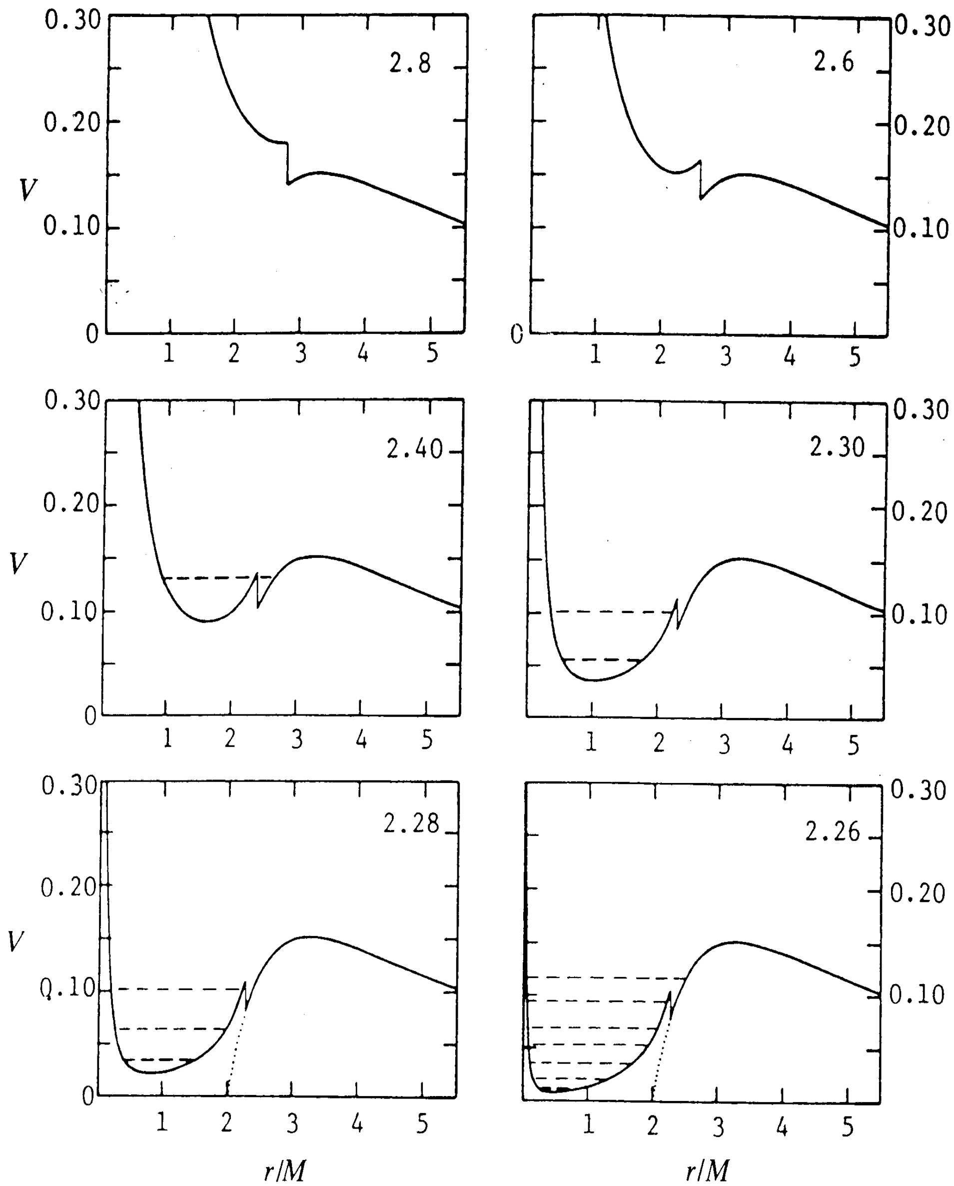

Figure 1: *The potential barrier of the axial perturbations of homogeneous stars is plotted for different values of the ratio* $\frac{R}{M}$ *ranging from 2.8 to 2.26.*

n	$\frac{R_{core}}{R}$	$\frac{R}{M}$
1.5	0.25% - 0.50%	4.21167 -4.21170
1.0	1.29% - 2.78%	3.2649 - 3.2645
0.5	5.68% - 11.22%	2.843 - 2.836

Table 3: *Parameters of the structure of a polytropic star with an* $\epsilon = p$ *core and different values of the polytropic index in the envelope. In order the star to be stable, the radius of the core must range in the interval given in column 2. In column 3 the corresponding range of variation of* $\frac{R}{M}$ *is given.*

	$\rho_c = 5 \cdot 10^{15} g/cm^3$		$\rho_c = 10^{16} g/cm^3$	
n	$\frac{M}{M_\odot}$	R in km	$\frac{M}{M_\odot}$	R in km
1.5	2.5	15.8	1.8	11.2
1.0	2.0	9.9	1.5	7.0
0.5	1.6	6.9	1.2	4.9

Table 4: *Typical values of mass and radius for two values of the central density. As the core radius varies in the allowed range given in Table 3,* M *and* R *change by at most a few percents.*

well inside the star does not significantly increase as the ratio $\frac{R}{M}$ decreases, and that these stars do not possess axial slowly damped modes. This result suggests that the appearence of the **s**-modes in the spectrum of the axial perturbations is more likely to be due to the incompressibility of the equation of state rather than to its stiffness. However, this point remains to be clarified, as well as whether it is the core or the envelope which play a fundamental role in this respect.

The influence of a small stiff core on the spectrum of the polar modes is much more significant. In order to locate the frequencies of the quasi-normal modes, one usually plots a "resonance curve" $[\alpha^2(\omega) + \beta^2(\omega)]$, that represents the amplitude of the standing wave at radial infinity obtained by numerically integrating the perturbed equations for real frequency. It can be shown that the values of frequency at which this curve exhibits a sharp minimum correspond to the real part of a quasi-normal modes, provided the imaginary part of the corresponding eigenfrequency is small enough ($\omega_i <<$ ω_0.) The damping time associated to a mode is related to the curvature of the parabola that fits the curve near a minimum: smoother minima correspond to shorter damping times [28]. It should be noted that this algorithms is designed to determine essentially the slowly damped modes. In figure 2 the resonance curve is shown for an $n = 1.5$ polytropic star with $\frac{R}{M} = 4.2$, as a function of the frequency. By analyzing the behaviour of the thermodynamical variables in correspondence of the frequencies of the quasi-normal modes, one can easily identify the **g**-, **f**-and **p**-modes that one defines in newtonian theory according to the Cowling classification [29], [30]. In figure 3 we plot the resonance curve for an $n = 1.5$ polytropic star having in its center a very tiny stiff core extending only up to the 0.30% of the total radius, and with the same ratio $\frac{R}{M} = 4.2$.

Compared to the case illustrated in figure 2, the structure of the spectrum becomes incredibly rich, and in particular a large number of **g**-modes appear that were not present in the fully polytropic star. In addition, smoother mimima are present, indicating that the composite star possesses both slowly-damped and highly-damped modes. This example powerfully illustrate how the spectrum of the quasi-normal modes of a star carry relevant information on its internal structure and on the manner in which the fluid and the gravitational field couple at supernuclear regimes.

There are further information that one can derive from the knowledge of the frequencies and damping times of the quasi-normal. In newtonian theory the frequency of the **f**-mode scales with the mean density of the star. In geometric units

$$\omega_f = \sqrt{\frac{2\ell(\ell+1)}{2\ell+1}\left(\frac{M}{R^3}\right)}. \qquad (36)$$

This relation has been generalized by N. Andersson and K. Kokkotas [31] who have determined both the frequency and the damping time of the **f**-mode for several equations of state proposed in the literature for neutron stars. They find the following relations

$$\omega_f = 0.39 + 44.45\sqrt{\left(\frac{M}{R^3}\right)} \qquad (37)$$

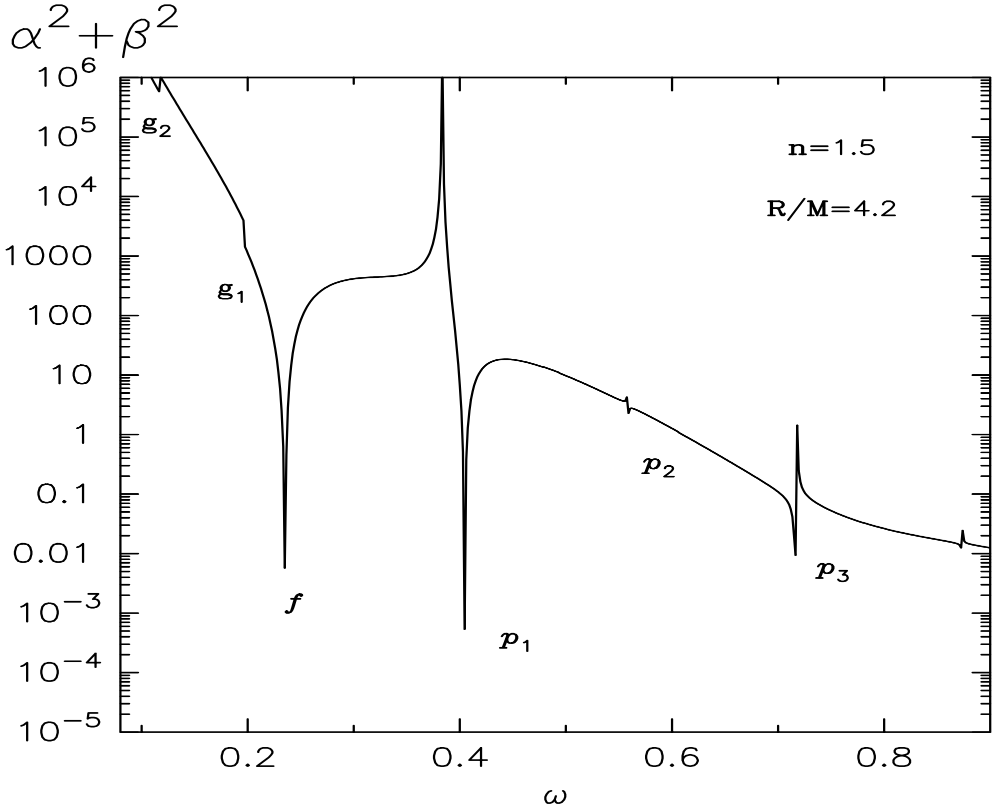

Figure 2: The resonance curve $\alpha^2 + \beta^2$ of a fully polytropic star, is plotted versus the real frequency ω, for $\ell = 2$. ω is measured in unities of $\epsilon_0^{-1/2}$.

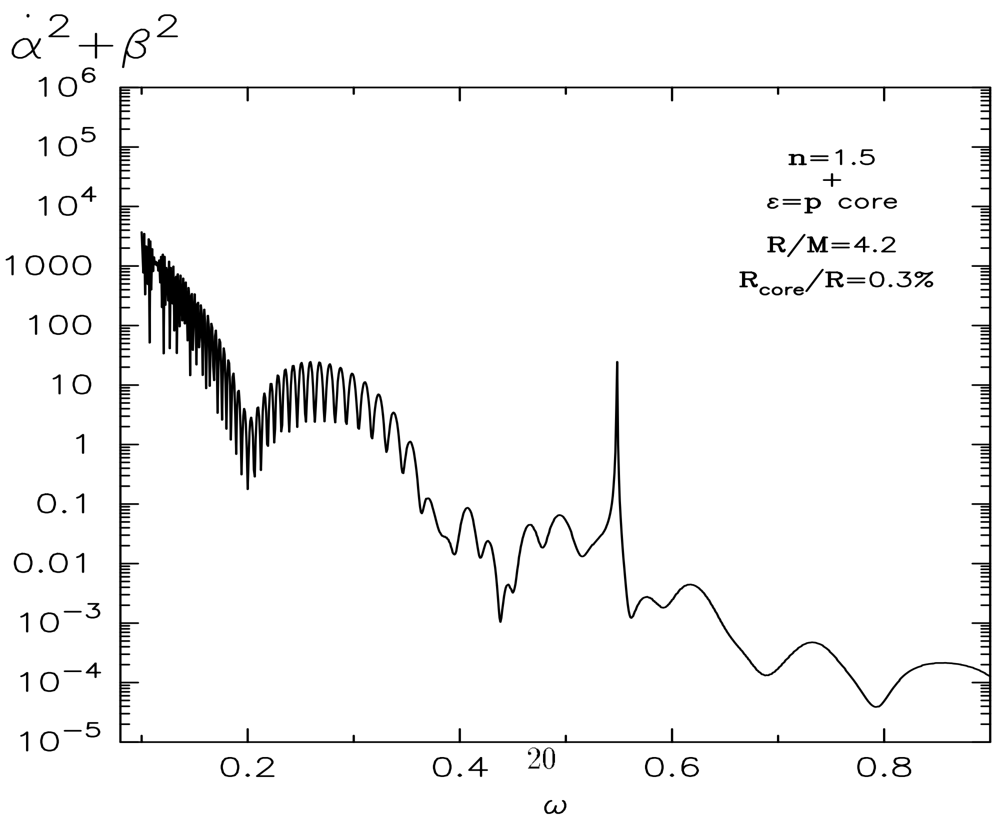

Figure 3: The resonance curve of the same polytropic star with a stiff core in its center.

$$\tau_f = 0.1 - \left(\frac{M}{R}\right) + 2.69\left(\frac{M}{R}\right)^2,$$

where M and R are expressed in km, ω_f in kHz and τ_f in ms. These two relations provide an estimate both for M and R, good within 5% if compared with the true values. It should be noted that the frequency of the **f**-mode ranges in the interval $\approx 1 - 2kHz$, and the damping time is $\approx 0.1 - 0.5s$. A further relation is provided by the damping time of the lowest **w**-mode computed for the same models

$$\frac{1}{\tau_{w_0}} = 0.104 - 0.063\left(\frac{M}{R}\right). \tag{38}$$

Andersson and Kokkotas have also studied how the axial quasi-normal modes are excited when an initial Gaussian pulse is scattered by the potential barrier of the axial perturbations of homogeneous stars. Their simulation shows that, in principle, the various modes can be excited. However, it would be interesting to know how the different modes are excited in some realistic situations. For example during the last stages of the gravitational collapse, when the newborn star wildly oscillate releasing gravitational waves, or when a mass, smaller than the star mass, is scattered or captured by the big one [32], [33].

6. Slowly rotating stars

The theory of stellar perturbations developed for static stars can be generalized to the case when the star is rotating so slowly that the distortion from spherical symmetry is quadratic in the angular velocity Ω, and may be ignored [34]. The unperturbed configuration is described by the following metric [35],[36]

$$ds^2 = e^{2\nu}(dt)^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\mu_2}(dr)^2 - e^{2\mu_3}(d\theta)^2, \tag{39}$$

where ν, ψ, μ_2, μ_3 differ from those of a static star by quantities of order Ω^2, while ω (that is zero in the non-rotating case) is a first order quantity in Ω. The equations governing ν, ψ, μ_2, μ_3 are given in sections 3. The equation for ω is

$$\varpi_{,r,r} + \frac{4}{r}\varpi_{,r} - (\mu_2 + \nu)_{,r}\left(\varpi_{,r} + \frac{4}{r}\varpi\right) = 0, \tag{40}$$

where

$$\varpi = \Omega - \omega. \tag{41}$$

In the vacuum outside the star, $\mu_2 + \nu = 0$ and the solution of eq. (40) reduces to $\varpi = \Omega - 2Jr^{-3}$, where J is the angular momentum of the star. In ref. [34] we showed that the axial perturbations of a slowly rotating star couple to the polar perturbations, and viceversa.

The way this coupling works for the axial perturbations is illustrated by the following equation [4]

$$\sum_{l=2}^{\infty}\left\{\frac{d^2 Z_l^1}{dr_*^2} + \omega^2 Z_l^1 - \frac{e^{2\nu}}{r^3}[l(l+1)r + r^3(\epsilon - p) - 6m(r)]Z_l^1\right\} C_{l+2}^{-\frac{3}{2}}(\mu) = re^{2\nu-2\mu_2}(1-\mu^2)^2 \sum_{l=2}^{\infty} S_l^0(r,\mu), \tag{42}$$

where

$$S_l^0 = \varpi_{,r}[(2W_l^0 + N_l^0 + 5L_l^0 + 2nV_l^0 P_{l,\mu} + 2\mu V_l^0 P_{l,\mu,\mu}] + 2\varpi W_l^0 (Q-1)\nu_{,r} P_{l,\mu}, \tag{43}$$

and Q has been defined in eq. (29). $\mu = \cos\theta$, and $C_{l+2}^{-\frac{3}{2}}(\mu)$ and $P_l(\mu)$ are respectively the Gegenbauer and the Legendre polynomials.

Eq. (42) holds from the center of the star up to radial infinity, provided outside the star ϵ, p and W are set to zero. As described in previous sections, if the star does not rotate the axial and the polar perturbations are described by two distinct sets of equations: eqs. (31) for the polar variables $N_l^0, L_l^0 V_l^0, W_l^0$ etc., and eqs. (34) for the axial function Z_ℓ^0. If the rotation is switched on ($\varpi \neq 0$), the axial function of first order in Ω, Z_ℓ^1, couple as indicated in eq. (42) with the polar functions $(W_l^0, N_l^0, L_l^0 V_l^0)$ of zero order in Ω, i.e. evaluated in the case of no rotation.
It should be noted that the coupling function is the quantity ϖ which is responsible for the dragging of inertial frames in the Lense-Thirring effect. Thus, rotating stars extert not only a dragging of the bodies, but also of the waves, and consequently an incoming polar gravitational wave can convert, through the fluid oscillations it excites, some of its energy into outgoing axial waves.

[4] The equations describing the coupling of the polar with the axial perturbations were subsequently determined by Y.Kojima [37].

I would like to stress that this phenomenon is a purely relativistic effect with no counterpart in newtonian theory.

Equation (42) is not yet separated. When the angular dependence is removed, one finds that the axial and the polar perturbations couple according to the following rules:

- *The Laporte rule* - the polar modes belonging to *even* ℓ can couple only with the axial modes belonging to *odd* ℓ, and conversely.
- *The selection rule* - $l = m + 1$, or $l = m - 1$.
- *The propensity rule [38]* - the transition $l \rightarrow l + 1$ is strongly favoured over the transition $l \rightarrow l - 1$. This derives from the manner in which the behaviour of the axial function is affected by the polar source near the origin.

As a consequence of this coupling, new families of modes are likely to emerge. For example, in ref. [34] we studied the axial perturbations of a slowly rotating polytropic star with polytropic index $n = 1.5$, and we showed that if one scatters an $\ell = 2$ polar gravitational waves on the potential barrier of eq. (42), for some value of the frequency of the incident wave the $m = 3$ axial perturbation induced by the coupling behaves as a pure outgoing wave at radial infinity. These "induced" axial resonances are characterized by damping times considerably longer than those of the polar modes of order zero in Ω (up to hundred times).

7. Concluding remarks

The existence of an energy conservation law governing the non-radial oscillations of a spherical star, which was derived in analogy with the conservation law governing the scattering of gravitational waves by a Schwarzschild black hole, provides an additional constraint to the theory and allows to recast the problem of stellar perturbations as a problem in scattering theory. The scattering approach proves extremely powerfull in enlighthening some aspects of the theory that where obscured in previous formulations. The existence of the slowly-damped axial modes in ultra-compact stars, the coupling between the polar and axial perturbations in slowly-rotating stars and the resonances induced by this coupling naturally emerge in this framework, though they could have also been discovered by other approaches.

The scattering approach is applicable also when the star is newtonian, i.e. when its equilibrium configuration is built in the Newtonian framework and the curvature it generates is very shallow. Indeed we showed that the frequencies of oscillation of a newtonian star can be determined by integrating the polar equations in the limit of small curvature, under the condition that no radiation emerges, as in the case of the dipole oscillations [39].

At the end of this lecture I would like to add to the scientific illustration of my work with Chandra some personal recollection on our collaboration. It developed over twelve years, and it was certainly based on reciprocal respect, esteem, trust and common scientific interests. But the real engine was Chandra's genuine enthusiasm for science which he was able to communicate to me by making me feel that, no matter how difficult a problem was, together we could make it. I am grateful to Chandra for his precious gift of sharing with me his patrimony of knowledge, experience, craftsmanship, fruits of a life entirely dedicated to science.

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