THE QUANTUM HALL EFFECT*

R. Ferrari
Dipartimento di Fisica, Univ. di Trento
I.N.F.N., Sezione di Trento

ABSTRACT

The Two-Dimensional electrons in heterostructures is a unique many-body system with extreme features (high mobility, small electron effective mass, small Landé factor (g-factor), etc.). On this system Quantum Hall Effect (both Integer and Fractional) can be observed with high accuracy . We think that a lot of theoretical work has still to be done for a satisfactory understanding of this phenomenon. Therefore much emphasis will be put on the experimental work done in the last ten years. The most popular theories are briefly sketched.

* *Lectures given at the*
3^{th} *National Seminar of Theoretical Physics*
Parma, 1-12 September 1993.

1 Basic language

We assume that the system consists of N electrons confined in the $x-y$ plane with mass $m \simeq 0.067 m_e$, polarization constant $\varepsilon \simeq 13$, g-factor $\simeq -0.44$, density $\rho_0 = 3\ 10^9, \ldots, 10^{12}$ cm^{-2} (low density in Ref. [1]) and high mobility $\mu \simeq 10^6$cm^2/(Volt sec) (for the heterostructure $Al_xGa_{1-x}As - GaAs$; in other heterojunctions the parameters can be quite different: $Ga_xIn_{1-x}As - InP$ [2, 3, 4] has $m = 0.047 m_e$, $|g| = 4.1$, $\rho_0 = 0.5\ 10^{11}$ cm^{-2} and $\mu \simeq 10^5$cm^2/(Volt sec). With different doping the charge carriers are holes. In Si-MOSFET $m = 0.19 m_e$, $\mu = 3.63\ 10^4$cm^2/(Volt sec) $\varepsilon = 7.7$ $\rho_0 = 5\ 10^{10} \ldots 10^{11}$ cm^{-2} [5]). Fig. 1 gives the geometry of the experimental setup (Hall bar and Corbino). Fig. 2 shows the energy band structure. The results of a typical magnetotransport experiment are shown in Figs. 3 and 4. The plateaus in the Hall resistance are described by

$$R_H = \frac{h}{\nu_f e^2}, \tag{1.1}$$

where ν_f is an integer (IQHE) [8] or a rational number q/p with p odd (FQHE) [9] (apart few exceptions: $\nu_f = 5/2$).

Mutual interaction via Coulomb and interaction with the constant magnetic (tilted by θ from the z-axis) are then described by the Hamiltonian (second quantization). For Bloch electrons, minimal coupling for Bloch electrons, Wannier functions and all that see Ref. [10] chap. 9.

$$\begin{aligned} H &= \int_A d^2\mathbf{r}\psi^\dagger_{s'} \left\{ \frac{1}{2m}\Big(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\Big)^2 \delta_{s's} - \frac{g}{2}\mu_B \sigma_{s's}\cdot \mathbf{B} \right\} \psi_s + H_C, \\ H_C &= \frac{e^2}{2\varepsilon}\int_A d^2\mathbf{r} d^2\mathbf{r}' V(\mathbf{r}-\mathbf{r}') \Big[\psi^\dagger_s(\mathbf{r})\psi^\dagger_{s'}(\mathbf{r}')\psi_{s'}(\mathbf{r}')\psi_s(\mathbf{r}) \\ &\quad -2\frac{N}{|A|}\psi^\dagger_s(\mathbf{r})\psi_s(\mathbf{r}) + \Big(\frac{N}{|A|}\Big)^2\Big], \end{aligned} \tag{1.2}$$

(interaction with the uniform positive background distributed over the domain A is included) where the electromagnetic potential describes only the perpendicular part of the magnetic field:

$$\begin{aligned} \mathbf{A} &= (-B_\perp y, 0) \qquad \text{Landau gauge} \\ \mathbf{A} &= \frac{B_\perp}{2}(-y, x) \qquad \text{symmetric gauge.} \end{aligned} \tag{1.3}$$

Spin indices will be neglected from now on. There are natural units of energy and length

$$\begin{aligned} &\hbar\omega_c \qquad \omega_c \equiv \frac{eB_\perp}{mc} \\ &\lambda \equiv \left(\frac{\hbar c}{eB_\perp}\right)^{\frac{1}{2}}; \end{aligned} \tag{1.4}$$

thus if we measure energies in units of $\hbar\omega_c$, lengths in units of λ and fields in units of $1/\lambda$, the Hamiltonian becomes

$$H = \int_A d^2\mathbf{r}\psi^\dagger \left\{ \frac{1}{2}\Big(-i\nabla + \mathbf{A}\Big)^2 - \kappa_p \sigma \cdot \hat{\mathbf{B}} \right\} \psi + H_C,$$

$$H_C = \frac{\kappa_c}{2}\int_A d^2\mathbf{r}d^2\mathbf{r}'V(\mathbf{r}-\mathbf{r}')\Big[\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \\ -2\frac{N}{|A|}\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) + \Big(\frac{N}{|A|}\Big)^2\Big] \tag{1.5}$$

where

$$\begin{aligned} \kappa_c &= \frac{e^2}{\varepsilon\lambda\hbar\omega_c} \\ \kappa_p &= \frac{g\mu_B B}{2\hbar\omega_c}. \end{aligned} \tag{1.6}$$

(in most experiments $\kappa_c \geq 1$ and $\kappa_p << 1$) and

$$\begin{aligned} \mathbf{A} &= (-y, 0) \qquad \text{Landau gauge} \\ \mathbf{A} &= \frac{1}{2}(-y, x) \qquad \text{symmetric gauge.} \end{aligned} \tag{1.7}$$

Canonical commutation relations are imposed

$$\{\psi(x), \psi^\dagger(x')\}_{ET} = \delta_2(\mathbf{x}-\mathbf{x}'). \tag{1.8}$$

The construction of the ground state for $\nu_f < 1$ is very difficult since all state with $n_L = 0$ are degenerate (see next subsection). It is believed that single particle approximation (Slater determinant state) does not work. Nevertheless let us play the game: suppose that the state is given by

$$|\Omega\rangle = \prod_{i=1}^{N}\left(\int d\mathbf{r}\psi^\dagger(\mathbf{r})f_i(\mathbf{r})\right)|0\rangle \tag{1.9}$$

with $\{f_i\}$ orthonormal, then one gets easily

$$\langle\Omega|H_C|\Omega\rangle = \frac{\kappa_c}{2}\sum_{i,j}\int d\mathbf{r}d\mathbf{r}'V(\mathbf{r}-\mathbf{r}')\left[|f_i|^2(\mathbf{r})\left(|f_j|^2(\mathbf{r}') - \frac{1}{A}\right) - f_i^*(\mathbf{r})f_j(\mathbf{r})f_j^*(\mathbf{r}')f_i(\mathbf{r}')\right]. \tag{1.10}$$

The first term is very small if the electrons are distributed uniformly in space. The second term (exchange energy) is negative and usually large ($\simeq \kappa_c$). Of course this depends on a judicious choice of $\{f_i\}$.

By using Lagrange multipliers we get the self-consistent equation

$$\begin{aligned} \epsilon_i f_{is}(\mathbf{r}) &= \left\{\frac{1}{2}\Big(-i\nabla + \mathbf{A}\Big)^2\delta_{ss'} - \kappa_p\sigma_{ss'}\cdot\hat{\mathbf{B}}\right\}f_{is'}(\mathbf{r}) \\ &+ \kappa_c\sum_j\int d\mathbf{r}'V(\mathbf{r}-\mathbf{r}')\left[f_{is}\left(\mathbf{r})(|f_j|^2(\mathbf{r}') - \frac{1}{A}\right) - f_{js}(\mathbf{r})f_{js'}^*(\mathbf{r}')f_{is'}(\mathbf{r}')\right] \end{aligned} \tag{1.11}$$

From the above equation one sees that the exchange part does not mix spins: a spin up electron *sees* only the other up spins (the direct part is usually very small).

Important issues are the boundary conditions and the introduction of an external electric field and of a current. These points will be discussed later.

1.1 Free electrons model

In the **Landau gauge** (eq. (1.3)) the free part of the Hamiltonian is easily diagonalized. The eigenfunctions

$$\chi_{n_L}(y-k)\exp(ikx), \tag{1.12}$$

where χ_n is the Hermite function for the one dimensional oscillator, has energy

$$E_{n_L} = n_L + \frac{1}{2}. \tag{1.13}$$

n_L is called the Landau number and there is degeneracy respect to k (position of the center of the oscillator). To find the degeneracy we impose some boundary conditions. In a box with dimensions (L_x, L_y) we require (to be compatible with the algebra of the field: eqs. of motion, etc.)

$$\begin{aligned} \psi(x+L_x, y) &= \psi(x,y) \\ \psi(x, y+L_y) &= \exp(iL_y x)\psi(x,y). \end{aligned} \tag{1.14}$$

Thus periodicity in x requires

$$k = 2\pi\frac{n}{L_x}, \tag{1.15}$$

while the condition in y necessitates the introduction of a set of eigenfunctions

$$\psi_{n_L k}(x,y) \equiv \exp(ikx)\sum_{n=-\infty}^{\infty}\exp(-inL_y x)\chi_{n_L}(y-k+nL_y) \tag{1.16}$$

which satisfy the boundary conditions. Then, if

$$g_L \equiv \frac{1}{2\pi}L_x L_y \qquad (\text{in units } \lambda^2) \tag{1.17}$$

is an integer, k runs over the values

$$k = 0, \ldots, L_y(1-g_L^{-1}). \tag{1.18}$$

Thus g_L is the degeneracy of the Landau level (for all levels). It says how many oscillators one can put on the y-axis.

If we include the spin, the energy becomes

$$E_{n_L s} = n_L + \frac{1}{2} \pm \kappa_p \tag{1.19}$$

The **symmetric gauge** allows a more complete treatment of the single particle case (although the results are at the end gauge independent). Complex notation is useful (more about this algebra in Ref. [11])

$$w = x + iy \qquad \partial_w = \frac{1}{2}(\partial_x - i\partial_y) \tag{1.20}$$

(our notation is the opposite of the commonly used one: $w = x - iy$). The first quantization operators:

$$
\begin{aligned}
a^\dagger &= \frac{1}{\sqrt{2}}\Big(\frac{w}{2} - 2\partial_{w^*}\Big) \\
b &= \frac{1}{\sqrt{2}}\Big(\frac{w}{2} + 2\partial_{w^*}\Big)
\end{aligned}
\tag{1.21}
$$

satisfy the commutation relations

$$[a, a^\dagger] = [b, b^\dagger] = 1 \tag{1.22}$$

$$[a, b] = [a, b^\dagger] = 0 \tag{1.23}$$

and the free Hamiltonian reads

$$H_0 = a^\dagger a + \frac{1}{2}. \tag{1.24}$$

Thus $a, a^\dagger$ are the step-down and step-up operators for the Landau number, while $b, b^\dagger$ commute with the unperturbed Hamiltonian. The single-particle states can be labelled by two numbers: n_L and $n_b \equiv b^\dagger b$. One gets easily the $n_b = 0$ states

$$\varphi_{n_L,0}(\mathbf{r}) = (2\pi n_L!)^{-\frac{1}{2}}\Big(\frac{x+iy}{\sqrt{2}}\Big)^{n_L}\exp\Big(-\frac{r^2}{4}\Big) \tag{1.25}$$

and similarly for $n_L = 0$

$$\varphi_{0,n_b}(\mathbf{r}) = (2\pi n_b!)^{-\frac{1}{2}}\Big(\frac{x-iy}{\sqrt{2}}\Big)^{n_b}\exp\Big(-\frac{r^2}{4}\Big). \tag{1.26}$$

It is useful to introduce the coherent-state operators

$$
\begin{aligned}
S(c) &\equiv \exp\Big[\frac{1}{\sqrt{2}}(c^* b - c b^\dagger)\Big] \\
T(c) &\equiv \exp\Big[\frac{1}{\sqrt{2}}(c a - c^* a^\dagger)\Big]
\end{aligned}
\tag{1.27}
$$

which can be written in term of translations

$$
\begin{aligned}
S(c) &= \exp\Big[+\frac{i}{2}\mathbf{c}\times\mathbf{r}\cdot\hat{\mathbf{z}}\Big] exp\Big(c_1\partial_x + c_2\partial_y\Big) \\
T(c) &= \exp\Big[-\frac{i}{2}\mathbf{c}\times\mathbf{r}\cdot\hat{\mathbf{z}}\Big] exp\Big(c_1\partial_x + c_2\partial_y\Big)
\end{aligned}
\tag{1.28}
$$

and satisfy the algebra

$$S(c)S(d) = S(c+d)exp\Big[-\frac{i}{2}\mathbf{c}\times\mathbf{r}\cdot\hat{\mathbf{z}}\Big]. \tag{1.29}$$

$$T(c)T(d) = T(c+d)exp\Big[\frac{i}{2}\mathbf{c}\times\mathbf{r}\cdot\hat{\mathbf{z}}\Big]. \tag{1.30}$$

Notice the useful relation

$$\exp(i\mathbf{c}\cdot\mathbf{r}) = T(-\tilde{c})S(\tilde{c}) \tag{1.31}$$

where

$$\tilde{c} = (-c_2, c_1). \tag{1.32}$$

Since S operators commute with H (the potential V has to be made periodic over the domain A), they are good for imposing the boundary conditions

$$\begin{aligned} S(L_1)\psi &= e^{i\theta_1}\psi \\ S(L_2)\psi &= e^{i\theta_2}\psi \end{aligned} \tag{1.33}$$

which are compatible if

$$S(L_1)S(L_2) = S(L_2)S(L_1) \tag{1.34}$$

i.e. condition of eq. (1.17):

$$\mathbf{L}_1 \times \mathbf{L}_2 \cdot \hat{\mathbf{z}} = 2\pi g_L \tag{1.35}$$

(integer number of quantum flux through the parallelogram with side vectors L_1, L_2).

The operators S form the Magnetic Translation Group. The elements $S(w)$ are fixed by the boundary conditions:

$$[S(w), S(L_j)] = 0 \qquad j = 1, 2 \Longrightarrow w = \frac{1}{g_L}(n_2 L_1 - n_1 L_2). \tag{1.36}$$

There is an interesting representation of the MTG, obtained by solving the equation

$$\begin{aligned} S(f)\phi^{\alpha\beta}_{n_L} &= e^{i\alpha}\phi^{\alpha\beta}_{n_L} \\ S(g)\phi^{\alpha\beta}_{n_L} &= e^{i\beta}\phi^{\alpha\beta}_{n_L} \end{aligned} \tag{1.37}$$

with (to guarantee the consistency of the above equations)

$$\mathbf{f} \times \mathbf{g} \cdot \hat{\mathbf{z}} = 2\pi \Longrightarrow [S(f), S(g)] = 0. \tag{1.38}$$

Notice that for any allowed element w of the MTG

$$\begin{aligned} S(f)S(w)\phi^{\alpha\beta}_{n_L} &= e^{i\alpha'}S(w)\phi^{\alpha\beta}_{n_L} \\ S(g)S(w)\phi^{\alpha\beta}_{n_L} &= e^{i\beta'}S(w)\phi^{\alpha\beta}_{n_L} \end{aligned} \tag{1.39}$$

with

$$\begin{aligned} \alpha' &= \alpha - \mathbf{f} \times \mathbf{w} \cdot \hat{\mathbf{z}} \quad \mathrm{mod}(2\pi) \\ \beta' &= \beta - \mathbf{g} \times \mathbf{w} \cdot \hat{\mathbf{z}} \quad \mathrm{mod}(2\pi). \end{aligned} \tag{1.40}$$

Thus one can start from a cyclic element and then obtain all the others by using eqs. (1.39) and (1.40)

$$\phi^{\alpha_0\beta_0}_{n_L} = (g_L)^{-\frac{1}{2}} \sum_{m,n=-\infty}^{+\infty} \left[e^{-i\alpha_0}S(f)\right]^m \left[e^{-i\beta_0}S(g)\right]^n \varphi_{n_L} \tag{1.41}$$

(see eq. (1.25)). The angles can be obtained by using the commensurability of the lattice (f, g) with the domain:

$$\begin{aligned} L_1 &= l_p\, f + l'_q\, g \\ L_2 &= l'_p\, f + l_q\, g \end{aligned} \tag{1.42}$$

where all coefficients are integers and

$$l_p l_q - l'_p l'_q = g_L. \tag{1.43}$$

The boundary conditions in eq. (1.33) give a finite set of angles. One of them is

$$\begin{aligned} \alpha_0 &= \frac{1}{g_L}\left(l_q\theta_1 - l'_q\theta_2 + \pi l_q l'_q(l'_p - l_p)\right) \\ \beta_0 &= \frac{1}{g_L}\left(-l'_p\theta_1 + l_p\theta_2 + \pi l_p l'_p(l'_q - l_q)\right). \end{aligned} \tag{1.44}$$

Finally from eq. (1.40) we get all values of α, β:

$$\begin{aligned} \alpha &= \alpha_0 + \frac{2\pi}{g_L}\left(l_q n_1 - l'_q n_2\right) \\ \beta &= \beta_0 + \frac{2\pi}{g_L}\left(-l'_p n_1 + l_p n_2\right). \end{aligned} \tag{1.45}$$

A canonical set of integers can be fixed by the requirements

$$\begin{aligned} 0 &\le \left(l_q n_1 - l'_q n_2\right) < g_L \\ 0 &\le \left(-l'_p n_1 + l_p n_2\right) < g_L. \end{aligned} \tag{1.46}$$

The wave function in eq. (1.41) is not normalized

$$\int_A d^2r|\phi^{\alpha_0\beta_0}_{n_L}|^2 = \sum_{m,n=-\infty}^{+\infty}(-1)^{mn}\mathrm{e}^{i(m\alpha_0+n\beta_0)}\exp\left(-\frac{1}{4}|mf+ng|^2\right). \tag{1.47}$$

There is a little mystery:

$$\|\phi^{\pi\pi}\|^2 = 0. \tag{1.48}$$

2 Some theorems

There are some important theorems for electrons in presence of a static magnetic field. They involve features of the system over large distances (low momentum), in contrast with other features (e.g. interaction with impurities) where the short distances involved are pertinent for the high momentum.

2.1 Kohn's theorem

This theorem [12] is very powerful: cyclotron resonance frequency (Landau splitting $\hbar\omega_c$) is not modified by perturbation due to a translation invariant potential:

$$U = \int d^2x d^2y u(\mathbf{x}-\mathbf{y})\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x). \tag{2.49}$$

We give some arguments about this theorem. If one excites the system by a radiofrequency (weak field), the semiclassical approximation can be used, i.e. an oscillatory term is added to the external electromagnetic potential (complex notation):

$$\begin{aligned} A &\rightarrow A + A^{(r)} \\ A^{(r)} &\equiv \frac{e\lambda}{\hbar\omega}(-iE_x + E_y)\exp -i(\omega t - kz) \end{aligned} \tag{2.50}$$

This amounts to consider a perturbation of the form

$$\int d^2r\mathbf{j}\mathbf{A}^{(r)} \tag{2.51}$$

(integration in dz implies some *Ansatz* on the z-part of the electron wave function) where (in dimensionless quantities $j_i/[(e\hbar)/(m\lambda^3)]$)

$$\begin{aligned} j_x &= \frac{1}{2}\{[-\frac{i}{\sqrt{2}}(a - a^\dagger)\psi]^\dagger\psi + \psi^\dagger[-\frac{i}{\sqrt{2}}(a - a^\dagger)]\psi\} \\ j_y &= \frac{1}{2}\{[\frac{1}{\sqrt{2}}(a + a^\dagger)\psi]^\dagger\psi + \psi^\dagger[\frac{1}{\sqrt{2}}(a + a^\dagger)]\psi\}. \end{aligned} \tag{2.52}$$

(for small electromagnetic field). Thus the typical operator is

$$\hat{a} \equiv \int d^2r\psi^\dagger a\psi. \tag{2.53}$$

Now we consider the operator

$$T(w) = \exp[\frac{1}{\sqrt{2}}(wa - w^*a^\dagger)] = \exp[-\frac{i}{2}\mathbf{w}\times\mathbf{r}\cdot\hat{\mathbf{z}}]\exp(w_1\partial_1 + w_2\partial_2) \tag{2.54}$$

and its second quantized equivalent

$$\hat{T}(w) = \exp[\frac{1}{\sqrt{2}}(w\hat{a} - w^*\hat{a}^\dagger)], \tag{2.55}$$

which is gives

$$\hat{T}(w)\psi\hat{T}^{-1}(w) = T(w)\psi. \tag{2.56}$$

By using translation invariance

$$\hat{T}(w)U\hat{T}^{-1}(w) = U \tag{2.57}$$

and

$$\hat{T}(w)H_0\hat{T}^{-1}(w) = \int d^2r\psi^\dagger(r)\left[(a^\dagger + \frac{w}{\sqrt{2}})(a + \frac{w^*}{\sqrt{2}}) + \frac{1}{2}\right]\psi. \tag{2.58}$$

By differentiating respect to w^* and putting $w = 0$ one gets

$$[\hat{a}^\dagger, U] = 0 \tag{2.59}$$

$$[\hat{a}^\dagger, H_0] = -\hat{a}^\dagger. \tag{2.60}$$

Let $|\Omega\rangle$ be the ground state of the Hamiltonian (with no radiofrequency), then we get

$$H\hat{a}^\dagger|\Omega\rangle = ([H, \hat{a}^\dagger] + \hat{a}^\dagger H)|\Omega\rangle = (E+1)\hat{a}^\dagger|\Omega\rangle. \tag{2.61}$$

Then the excited energy (cyclotron frequency) is not modified by the interaction U.

2.2 Larmor's theorem

A similar theorem is valid also for the Zeeman splitting: the mutual interaction among the electrons (if no spin-orbit potential is present) does not change the electron spin resonance.

The argument to prove this theorem is much similar to the previous discussion: one consider a radiofrequency coupled to magnetic moment of the electrons:

$$\frac{g\mu_B}{2\hbar\omega_c}\int d^2r\psi^\dagger\sigma_x\psi B^{(r)}(t). \tag{2.62}$$

The radiation magnetic field is small. Then the relevant matrix element at first order in perturbation (time dependent) expansion is

$$\langle\Omega'|\int d^2r\psi^\dagger\sigma_x\psi|\Omega\rangle \tag{2.63}$$

By using the invariance under spin rotations of H_0 and U one gets

$$(E'-E)\langle\Omega'|\int d^2r\psi^\dagger\sigma_x\psi|\Omega\rangle = -2i\kappa_p\langle\Omega'|\int d^2r\psi^\dagger\sigma_y\psi|\Omega\rangle. \tag{2.64}$$

Now, **if** we can relate the two matrix elements, e.g. by using rotation invariance under σ_z rotations (this is not always easy: mixed spin states and/or boundary conditions might pose some problems):

$$[\sigma_z,\sigma_x] = 2i\sigma_y \tag{2.65}$$

Finally we get

$$E'-E = \pm 2\kappa_p. \tag{2.66}$$

This tells that at low momentum the Zeeman splitting is not modified by the interaction among the electrons. In particular the Coulomb exchange energy cannot be detected by Electron Spin Resonance experiment [13] (in forward scattering experiments! If the momentum transfer is large, as in large angle scattering, then the exchange energy shows up. See Ref. [14])

3 Boundary conditions

The boundary conditions is a nasty problem: the connection with experiments is somewhat obscure. Periodic conditions, as on a torus, have troubles from interactions with copies; the system on a sphere might have difficulties to crystallize in the triangular lattice (Wigner crystal); realistic conditions (as boundary external potential) are too difficult from the theoretical point of view.

If the electric field is present the situation gets worse, as we would like to illustrate here. Let us start from far away. Current and electric field are related by

$$J_i = \sigma_{ij}E_j \tag{3.67}$$

and the inverse

$$E_i = \rho_{ij}J_j. \tag{3.68}$$

If there is $x-y$ symmetry then

$$\sigma_{xx} = \sigma_{yy}, \qquad \sigma_{xy} = -\sigma_{yx}. \tag{3.69}$$

Please notice that

$$\sigma_{xx} = \frac{\rho_{yy}}{\rho_{xx}^2 + \rho_{xy}^2} \tag{3.70}$$

thus $\rho_{xx} = 0$ implies $\sigma_{xx} = 0$. Moreover

$$\sigma_{xy} = -\frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} \tag{3.71}$$

On the Hall plateaus one has $\rho_{xx} \simeq 0$; therefore

$$\sigma_{xy} \simeq -\rho_{xy}^{-1}. \tag{3.72}$$

In the Corbino geometry (disk) the longitudinal current and potential difference are along a radius. The transverse quantities are not measured. **The external potential is given**. Moreover from rotational symmetry

$$E_\theta = \rho_{\theta r} J_r + \rho_{\theta\theta} J_\theta = 0. \tag{3.73}$$

If we are sitting on a Hall plateau then

$$\rho_{\theta\theta} = 0 \Longrightarrow J_r = 0, \tag{3.74}$$

i.e. there is no longitudinal current. The argument should be valid also for a cylinder (axis along y): at the Hall plateau then

$$E_\theta = 0 \ \text{ and } \ \rho_{\theta\theta} = 0 \Longrightarrow J_y = 0. \tag{3.75}$$

Thus on a torus the transport of an electron by $\mathbf{L}_2$ increases the system energy by

$$e\mathbf{L_2} \cdot \mathbf{E}. \tag{3.76}$$

To implement this by the boundary conditions we require ($\mathbf{E} \cdot \mathbf{L}_1$=0)

$$\begin{aligned} S(L_1)\psi'' &= \mathrm{e}^{i\theta_1}\psi'' \\ S(L_2)\psi'' &= \mathrm{e}^{i(\theta_2 - t\mathbf{L}_2\cdot\mathcal{E})}\psi'' \end{aligned} \tag{3.77}$$

where time is measured in units of ω_c^{-1} and

$$\mathcal{E} = \frac{e\lambda E}{\hbar\omega_c} \tag{3.78}$$

($''$ is used for the field ψ in presence of the electric field). θ_1 and θ_2 are time independent. We introduce the electric field by the scalar potential

$$A_0 = -\mathbf{E} \cdot \mathbf{r} \tag{3.79}$$

i.e.

$$H_0 \to H_0 + \int d^2r \psi''^\dagger \mathbf{E} \cdot \mathbf{r} \psi''. \tag{3.80}$$

A Galilei transformation allows to go back to the static problem. Define

$$\psi'' = S(i\mathcal{E}t)T(\mathcal{E})exp(i/2|\mathcal{E}|^2 t)\psi. \tag{3.81}$$

With a little algebra one gets the original Schrödinger equation with $E = 0$ and boundary conditions

$$\begin{aligned} S(L_j)\psi &= S(L_j)S(-i\mathcal{E}t)T(-\mathcal{E})exp(-i/2|\mathcal{E}|^2 t)\psi'' \\ &= exp(i\theta_j)\psi. \end{aligned} \tag{3.82}$$

In the Hall bar geometry, **the current along x-axis is given** and the potential difference is measured across y-axis. Thus

$$J_y = \sigma_{yx}E_x + \sigma_{yy}E_y = 0. \tag{3.83}$$

If we are on a Hall plateau

$$\rho_{xx} = 0 \Longrightarrow \sigma_{xx} = \sigma_{yy} = 0 \tag{3.84}$$

therefore

$$E_x = 0. \tag{3.85}$$

The transport along L_1 and L_2 does not require any mechanical work. Thus we can impose the time independent boundary conditions

$$\begin{aligned} SL_1\psi' &= \mathrm{e}^{i\theta_1}\psi' \\ SL_2\psi' &= \mathrm{e}^{i\theta_2}\psi' \end{aligned} \tag{3.86}$$

The current J_x is induced by some **current** generator along the x-axis (same direction as L_1). Thus the stationary state $|\Omega'\rangle$ satisfies

$$\langle\Omega'|\mathbf{j}|\Omega'\rangle = (J_x, 0) \tag{3.87}$$

(at least as mean value over the device), where

$$\mathbf{j} = \frac{e}{m}\{\frac{1}{2}[(\mathbf{p}\psi)^\dagger\psi + \psi^\dagger\mathbf{p}\psi] + \frac{e}{c}\psi^\dagger\mathbf{A}\psi\}. \tag{3.88}$$

A is the potential for the magnetic field and for the small electric field introduced to drive the current. Now we can go back to zero current case, by using the transformation

$$\psi \equiv T(-(1+it)\mathcal{E})exp(-\frac{i}{2}|\mathcal{E}|^2 t)\psi'. \tag{3.89}$$

with $\mathcal{E}_x \simeq 0$ and

$$\mathcal{E}_y \equiv \frac{|A|}{N}J_x. \tag{3.90}$$

In fact we get the same boundary conditions as before, the current is zero, but the Schroödinger equation contains a new term in the vector potential

$$\mathbf{A} \to \mathbf{A} + ct\mathbf{E}. \tag{3.91}$$

Finally the problem to be solved is that of a static system of many electrons in an external electric on the plane and a magnetic field as usual.

4 Some phenomenology

For Quantum Hall Effect, two devices have been used: $Si-MOSFET$ and heterostructures (e.g. junctions as $Al_xGa_{1-x}As - GaAs$). In the first material Bloch-electrons have higher mass as in the second ($0.2\ m_e$ versus $0.067\ m_e$), thus Landau splitting is smaller. Moreover the mobility in the first material is at least a factor ten lower as in the second. Although QHE started with MOSFET, most of recent experiments are done in heterojunctions. The Hall resistance has a simple classical explanation: if we match Lorentz and electrostatic force for electrons moving with drift velocity v_x:

$$v_x\ B_z = cE_y. \tag{4.92}$$

The current is given by

$$j_x = -eL_y\rho v_x \tag{4.93}$$

and therefore

$$R_H \equiv \frac{V_y}{J_x} = -\frac{B}{ec\rho} \tag{4.94}$$

The Integer Quantum Hall Effect comes along if ν_f quantum levels are filled with degeneracy

$$g_L = \frac{L_xL_yeB}{hc}. \tag{4.95}$$

Homogeneity of the system gives then

$$\rho = \nu_f\frac{eB}{hc}, \tag{4.96}$$

then from eq. (4.94)

$$R_H = \frac{h}{\nu_f e^2} \tag{4.97}$$

with

$$\frac{h}{e^2} = 25,812.8063\ \Omega \tag{4.98}$$

The Hall plateaus are along the classical line (we forget the sign)

$$R_H\frac{e^2}{h} = \frac{Be}{hc\rho}. \tag{4.99}$$

The scale of the magnetic field is

$$B_0 = \rho_0\frac{ch}{e}, \tag{4.100}$$

thus the density at $B = 0$ gives the natural unit for the magnetic field ($1Tesla = 10^4Gauss = 0.24178\ 10^{11}\ \mathrm{cm}^{-2}$). By varying the magnetic field (usual precision $\Delta B/B \simeq 10^{-3}$), the Landau levels move on the energy axis and their degeneracy changes. If the number of electron is fixed, then the transition between filled levels states is continuous (de Haas-van Alpen); however, if the number of electrons is free, then the filled levels are those with energy lower then the chemical potential (this after a bunch of approximations: single particle state, zero temperature, no impurities, infinite reservoir, etc..) and the transition is discontinuous. Please notice that the majority of experts prefer to keep the number of electrons fixed and to account for the sudden variation in the number of charge carriers, by localizing (or de-localizing) the electrons around the impurities of the material. For our purposes the two descriptions are equivalent: in both cases a parameter has to be introduced (the energy necessary to bring-in or to de-localize one electron).

4.1 Plateaus shape: the spin problem

This simple model (free electron model) encounters here some difficulties. The filling sequence, for decreasing magnetic field, is supposed to be $0\downarrow$, $0\downarrow +0\uparrow$, $0\downarrow +0\uparrow +1\downarrow$, $0\downarrow +0\uparrow +1\downarrow +1\uparrow$, ... The transition from odd filled plateaus to even filled plateaus corresponds to complete filling of a Landau level. However the spin splitting is much smaller than the Landau splitting:

$$\frac{g\mu_B B}{2\hbar\omega_c} \simeq -7.3 \times 10^{-3} \tag{4.101}$$

therefore the odd filled plateaus are expected to be much smaller than the even filled ones. This fact is contradicted by experiments [6] (see Fig. 5). The Coulomb interaction does not cure the problem either, since the interaction (at the first non zero perturbation level) leaves the gas of the up-spins independent from the gas of the down-spins. Only a *ad hoc* energy contribution coming from the bulk can account [15] (together with the Coulomb interaction) for the anomaly in the plateaus size. See Fig. 6. This bulk term describes the electron reservoir as a buffer of finite capacity.

4.2 ESR

In this experiments the spin flip is induced in odd-filled plateaus by some radiofrequency (up to $70GHz$ and several hundred mW in Ref. [13]). Absorption of microwave is measured. Since $g \simeq -0.44$ then Larmor theorem gives (see section 2)

$$\Delta E = g\mu_B B \simeq 20GHz \tag{4.102}$$

at $B = 4.5\ 10^4 Gauss$. In fact an effective $g(B, n_L)$ is found

$$g(B, n_L) = g_0 + c(n_L + \frac{1}{2})B \tag{4.103}$$

with typical values: $g_0 = -0.42$ and $c = 0.0111\ 10^{-4} Gauss^{-1}$.

The light diffusion experiment [14] reveals that the spin splitting is dependent from the momentum transfer. See Fig. 7. In the experiment (at $\nu_f \simeq 1$ and $B = 8.02\ 10^4 Gauss$) the peak at $E \simeq 14meV$ is identified with $\Delta E = \hbar\omega_c$, i.e. an inter Landau level transition ($q = 0$), another interesting peak at $E = 17.5meV$ is assigned to the transition $0\downarrow\rightarrow 1\uparrow$ and numerically corresponds to a large exchange energy, as predicted by Kallin and Halperin [16] at large momentum transfer $q \simeq \frac{1}{\lambda}$.

4.3 Activation energies

There is some indirect way to get informations on the excitation energies of the system, this is by measuring the dependence on the temperature T of the longitudinal part of the conductivity tensor

$$\sigma_{xx} = \frac{\rho_{yy}}{\rho_{xx}^2 + \rho_{xy}^2}. \tag{4.104}$$

A typical experiment is described in Ref. [17], where the dependence is measured at the filling factors $\nu_f = 1, 2, 3$ (Integer case). There $\rho_{xy} = h/(\nu_f e^2)$ and ρ_{yy} is in comparison small, thus conductivity and resistivity behave the same. If $\Delta E >> k_B T$ then the expected behavior is

$$\sigma_{xx} = \sigma_0 \exp\left[-\frac{\Delta E}{2k_B T}\right]. \tag{4.105}$$

Plotting $\log \sigma_{xx}$ versus $1/T$ one expects a straight line yielding the two parameters σ_0 and ΔE. See Fig. 8. In the odd-filled plateaus the spin flip is supposed to be the mechanism of excitation, while for even-filled one expects that inter Landau transitions are involved. According to Kallin and Halperin [16], the exchange Coulomb energy gives a sizable contribution (large momentum transfers are expected in the interaction electron-impurities). The experiment shows that the law in eq. (4.105) is valid over a wide range of temperatures, but ΔE is smaller than the theoretical prediction (50% of the theoretical figures for even-filled plateaus and only 20% for the odd-filled, where spin-flip enters). Typical values are: $\Delta E \simeq= 100\ ^0K$ and $\Delta E \simeq= 20\ ^0K$ respectively for even- and odd-filled plateaus. Fig. 9.

4.4 Tilted field

From eqs. (1.2) and (1.3) it is clear that the Landau term depends from $B_\perp$ and the Pauli term from B. This allows to perform two type of experiments which are crucial for the TDE.

A) **Coincidence experiments**

Let us consider IQHE and parametrize the energy of the levels by (not so correct, see later)

$$E_{n_L s} = \hbar\omega_c(n_L + \frac{1}{2}) - \frac{g^*}{2} s \mu_B B \tag{4.106}$$

where ω_c contains $B_\perp$. If we tilt the field by an angle θ from the normal to the plane, then

$$B_\perp = B\cos\theta \tag{4.107}$$

and therefore two levels can coincide in energy for some particular value of θ, e.g. $n_L, s = 1$ and $n_L + 1, s = -1$ (Fig. 10). In this situation the two plateaus are expected to disappear since the vanishing of the energy gap will increase the longitudinal resistivity ρ_{xx}. This will provide an experiment measurement of g^* to be confronted with theory. The experiment in Ref. [18] gives evidence of this effect ($\theta \simeq 87^0$) (Fig. 11). Unfortunately the theoretical analysis is difficult both because the Coulomb energy has not been taken into account in eq. (4.106) and even more difficult because the bulk effect is necessary in order to evaluate the magnetic field at which the transition between two plateaus occurs.

B) **New spin phase**

In $GaInAs - InP$ the effective g is ten times larger than in $AlGaAs - GaAs$. Thus it is easy to increase the Pauli term in tilted field, perhaps to a point where the filling sequence is $0\downarrow, 0\downarrow +1\downarrow, 0\downarrow +1\downarrow +2\downarrow, \ldots$ (ferromagnetism) at variance with the alternate spin filling (paramagnetism). Coulomb energy tends to favor this transition, since at the lowest order the exchange energy for an ensemble of electrons is lower if all spins are in the same direction than in the alternate spin filling. In this phase the coincidence of levels does not occur. In Ref. [19] this phase transition is demonstrated experimentally.

C) **Mixed spin states**

At $\nu_f < 1$ and high density the magnetic field is very large (see eq. (4.100)) and therefore the states are supposed to be formed with electrons with $s = -1$ (Pauli term goes with $\sim B$, while Coulomb with $\sim B_\perp^{\frac{1}{2}}$). Thus in devices where the density is low we expect that the Pauli term will become small enough to allow mixed spin states. In this situation a tilted field experiment could reveal the presence of spin up electrons. By varying the angle the state changes and the energy gap with the excited states is expected to vary also, thus a variation of ρ_{xx} respect to tilting angle is to be found. In Ref. [20] this effect is shown experimentally for $\nu_f = \frac{2}{3}, \frac{2}{5}$ ($\rho_0 = 2.4\ 10^{10}\ cm^{-2}$). From theoretical point of view this problem is very difficult since nobody is able to construct mixed spin states, practically.

4.5 $\nu_f = \frac{1}{2}$

The odd denominator rule (FQHE) is said to be understood. This self-assuring statement [21] is disturbed by a recurrent nightmare: the absent $\nu_f = \frac{1}{2}$ plateau is investigated regularly by some experimentalist. We mention some of them.

A) **Direct search**

The direct search of the $\nu_f = \frac{1}{2}$ plateau has provided various false alarms. The more recent investigation is a throughout study of the plateaus accumulating around that value [22]. They are parametrized by

$$\nu_f = \frac{p}{2p \pm 1}. \tag{4.108}$$

Very close levels are detected ($\nu_f = 9/19$ and $9/17$) and moreover the activation energies are measured by looking at the temperature dependence of ρ_{xx} (see eq. (4.105)). No plateau at $\nu_f = \frac{1}{2}$ is found, but a very intriguing dependence of the activation energy from B is found (Fig. 12). There are various theoretical speculations about these results [23, 24].

B) **Double-Layer Electron System**

Up to now we did not care about the z-direction: some potential well should be at work with energy levels so far apart that only the lowest (with no nodes in the wave function carrying z-dependence) is occupied. However also on this side some beautiful experiments have been done. In Ref. [25] the profile of the potential well is modified in such a way that a double-layer (in the z-direction) is formed. In Ref. [26] two wells are separated and tunneling is used to couple the two layers. In both experiments a plateau with the filling factor $\nu_f = \frac{1}{2}$ is detected (Fig. 13). This requires particular values in the geometry of the wells. The appearance of this plateau goes together with a rather anomalous behavior of the IQHE. The theoretical explanations of this effect are somewhat involved. See Ref. [27] and the papers quoted therein.

C) **Surface acoustic wave**

In the d-c experiments the static properties of the TDE system are investigated. The use of surface acoustic waves (up to 1 GHz), transmitted very close to the TDE ($< 5000\mathring{A}$), allows the analysis of the dynamical properties. The amplitude and the velocity of the outgoing wave are measured. The attenuation Γ (amplitude $\sim \exp(-\Gamma x)$) and velocity can be related by a simple model to the static conductivity σ_{xx}. The experimental results [28] agree with the prediction of the model, in particular amplitude and velocity increase at the Hall plateau, both in the IQHE and FQHE. There is a striking anomaly at $\nu_f = \frac{1}{2}$ (and perhaps at $\frac{3}{2}$), where a sharp minimum both in amplitude and velocity is detected (in contrast with the prediction of the model, that predicts an opposite behavior). See Fig. 14. The anomaly decreases by increasing the temperature. This phenomenon has started an intense theoretical analysis of the TDE state at this particular filling factor [24].

4.6 Edge currents

Various theoretical speculations [29, 30] suggest that the longitudinal current flows mostly along the edges. Numerous experiments have been performed, but no definite conclusion can be drawn (e.g. surface acoustic wave experiment favor a bulk current picture). As representative of this intense research we quote Ref. [31, 32], where the authors claim a strong evidence for current-carrying edge states. See Fig. 15.

4.7 Wigner crystal

At low temperature and small filling factor, a two dimensional electron system without disorder is supposed to form a Wigner crystal (due to Coulomb interaction). The transition from liquid to crystal is expected for $\nu_f \leq 1/7$. The distinctive properties of a crystal are many, e.g. resistance to sheer, Bragg scattering, temperature behavior of specific heat, electric insulation, etc. Unfortunately many of these properties are out of range for experiments. It has been suggested to measure the threshold voltage in the non linear I-V plot, where the differential resistance drops off. Moreover by increasing the temperature an Insulating Phase shows a decreasing in the electrical resistance and the opposite is expected for a fluid (which is supposed to be the phase at the Hall plateaus). We quote two experimental works where some evidence is given for Wigner crystallization. In Ref. [33] the insulating phase is found around the plateau at $\nu_f = 1/5$, where the resistance has very high value ($\sim 5\ 10^5 \Omega$). The temperature dependence is as described above (Fig. 16). In Ref. [34] the same evidence is found around the plateau at $\nu_f = 1/3$ for a device where the *holes* are the electric carriers (mass is 5 times larger and mobility is one order of magnitude lower).

4.8 Photoluminescence

In a series of beautiful experiments the Stuttgart and Chernogolovka clubs have analysed the optical properties of the TDE system. The radiative recombination of 2D electrons with photoexcited holes in a monolayer of acceptors has been studied. The trick of creating a spatially separated layer (δ layer) of acceptors is used in order to avoid the masking effect of strong electron-hole correlation (excitons). The experiment (see for instance Ref. [35], Fig. 17) shows clearly the optical transitions to IQHE levels (Landau fan), where, according to Kohn's theorem, the energies are in quanta of $\hbar\omega_c$, and steps (in the excitation energies) at various fractional fillings (FQHE). Fig. 18 (left).

In a recent paper [36] the group showed that time resolved luminescence measurements allow to investigate the properties of liquid and solid phase independently and to derive a phase diagram for the Wigner crystallization (the recombination time for electrons in the pinned Wigner-solid phase is up to several orders of magnitude longer than for electrons in the liquid phase). Fig.18 (right).

5 Theorist's side

In order to follow the common notation, we change the definition

$$z = x + iy \rightarrow z = x - iy \tag{5.109}$$

and correspondingly

$$\begin{aligned} a^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} - 2\partial_z\right) \\ b &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} + 2\partial_z\right) \end{aligned} \tag{5.110}$$

(see eqs. (1.20), (1.21)).

The IQHE seems to be understood, although (contrary to common belief) not in terms of free electrons model. The basic approach is the Hartee-Fock approximation, together with lowest order radiative corrections (in particular excitons energies) [16]. Some problems remain, particularly the low value of the measured activation energies (Sect. 4, subsection C) and the profile of the Hall resistance: odd filled plateaus have strong anomalies [15] (Sect. 4, subsection A).

The FQHE is a very difficult problem, because any perturbative approach has to face the high degeneracy (if all electrons, with $\nu_f < 1$, are in $n_L = 0$ state, they have the same unperturbed energy). Various approaches have been attempted. Exact solutions for small systems ($N \simeq 10$) have been done. They are somewhat unreliable, because of the very strong influence of boundary conditions (we think that it should be $N \geq 200$). Hartree-Fock calculations give rather high value for the total energy (for $\nu_f = 1/3$ one gets $E_C/N = -0.387e^2/(\varepsilon\lambda)$ to be compared with Laughlin state value of -0.41) and moreover they cannot explain the odd denominator rule.

This section gives a brief account of the Laughlin *Ansatz* and of further developments.

5.1 Laughlin state

The Laughlin *Ansatz* concerns the wave function of a (hopefully) ground state $|L\rangle$

$$\psi_m \equiv \frac{1}{\sqrt{N!}}\langle 0|\psi(z_1)\cdots\psi(z_N)|L\rangle. \tag{5.111}$$

Let us assume that all spins are down and moreover that all electrons are in the lowest Landau level. Then the ground state is a superposition of single particle states

$$|j_1 \cdots j_N\rangle \equiv \prod_{i=1}^{N} c^\dagger_{j_i}|0\rangle. \tag{5.112}$$

Where the operator c_j destroys an electron described by a wave function given in eq. (1.26) with $n_b = j$. Then the general form of ψ_m is

$$\psi_m = f(z_1, \ldots, z_N) \exp\left(-\frac{1}{4}\sum_{i=1}^{N}|z_i|^2\right), \tag{5.113}$$

where f is an analytic function of z_i (remember the change in notations given in eq. (5.109)) and moreover it should be antisymmetric (Fermi statistics). Laughlin [37] proposed

$$\psi_m = \prod_{i<j}(z_i - z_j)^m \exp\left(-\frac{1}{4}\sum_{i=1}^{N}|z_i|^2\right), \tag{5.114}$$

with m odd (not normalized). For $m = 1$ f is just the Vandermonde determinant and corresponds to fill all the states $j_1 = 0, \ldots, j_N = N - 1$ in eq. (5.112). This is the state with $\nu_f = 1$. For $m > 1$ the calculation of the expectation value of a dynamical variable A proceeds as follows

$$\langle L|A|L\rangle = \|\psi_m\|^{-2} \int d^2z_1 \cdots d^2z_N \psi_m^* A \psi_m. \tag{5.115}$$

Notice that

$$|\psi_m|^2 = \exp(-\beta H_{cl}), \tag{5.116}$$

with

$$\beta = \frac{2}{m} \quad \text{and} \quad H_{cl} = \frac{m}{4}\sum_i |z_i|^2 - m^2 \sum_{i<j} \ln|z_i - z_j|. \tag{5.117}$$

H_{cl} describes a One Component Plasma (or *jellium*) in two dimensions of classical charges m in a background with charge density

$$\sigma = -\frac{1}{2\pi}\Delta\frac{r^2}{4} = -\frac{1}{2\pi}. \tag{5.118}$$

The evaluation of the mean value in eq. (5.115) is then equivalent to the problem of the statistical mechanics of the *classical* OCP at temperature β^{-1}. In plasma physics the important parameter is $\Gamma \equiv e^2\beta$; here $\Gamma = 2m$ (try with a scale transformation ...). At this point any method of classical statistical mechanics is good: classical Monte Carlo, Langevin equation, hypernetted chain approximation, etc.. An amusing game can bring you to surprisingly good results. The fictitious particles will form a droplet (for high Γ a solid), of radius R, which neutralizes the charge of the (infinite) disk. Thus

$$R = (2\ m\ N)^{\frac{1}{2}}. \tag{5.119}$$

Then the density of the electrons is

$$\rho = \frac{N}{\pi R^2} = \frac{1}{2\pi m} \tag{5.120}$$

Then the filling factor is

$$\nu_f = 2\pi\rho = \frac{1}{m} \tag{5.121}$$

(here $\lambda = 1$). Thus the Laughlin state describes a TDE system as a liquid (incompressible, if the excited states are separated by a energy gap). There is some problems, however. For finite N, the distribution in space of the droplet is not uniform: at the border the density has some oscillations [38] (in the variable r) and these oscillations are more pronounced for high m. This problem (edge states) is far from been solved and the connection with the edge excitations might be important. See Figs. 19 and 20

The energy per particle has been evaluated with many methods and turns out to be very low (in comparison with other proposed states). For instance at $m = 3$ and for large systems the energy is

$$E_C/N = -0.41\ \frac{e^2}{\varepsilon\lambda}. \tag{5.122}$$

The Laughlin function describes states with filling factor $\nu_f = 1/m$. Particle-hole symmetry allows to use the function for $\nu_f = \frac{m-1}{m}$.

5.2 Quasi-holes and quasi-electrons

Let us consider the wave function

$$\psi_h = \left(\prod_{i=1}^{N} z_i\right) \prod_{i<j} (z_i - z_j)^m \exp\left(-\frac{1}{4}\sum_{i=1}^{N} |z_i|^2\right) \tag{5.123}$$

and use the trick of the OCP. This corresponds to a new classical hamiltonian

$$H^h_{cl} = -m \sum_i \ln|z_i| + \frac{m}{4}\sum_i |z_i|^2 - m^2 \sum_{i<j} \ln|z_i - z_j|. \tag{5.124}$$

The new term describes a particle in the centre with (classical) charge 1. This external charge depletes an equal amount of charge, i.e. a region of area

$$\delta = \frac{1}{\sigma} = 2\pi. \tag{5.125}$$

Since the density of the electrons is $\rho = 1/(2\pi m)$ (see eq. (5.120)), the depleted true charge (not the charge of the OCP) is $-e\rho\delta = -e/m$. We conclude that the new function describes a hole in the centre with charge

$$q_h = \frac{e}{m}. \tag{5.126}$$

The process could be iterated by multiplying ψ_h by

$$h(\zeta) \equiv \prod_i (z_i - \zeta). \tag{5.127}$$

Notice that

$$[h(\zeta), h(\zeta')] = 0, \tag{5.128}$$

thus the quasi-holes are Bosons. They carry charge e/m if they are far apart.

The construction of quasi-electrons is not so straightforward. Laughlin proposed

$$\psi_e(\zeta^*) = \exp\left(-\frac{1}{4}\sum_{i=1}^{N}|z_i|^2\right)\left(\prod_{i=1}^{N}(2\frac{\partial}{\partial z_i}-\zeta^*)\right)\prod_{i<j}(z_i-z_j)^m. \tag{5.129}$$

The interpretation of ψ_e as a system of N electrons with a quasi-electron carrying charge $-e/m$ is not as simple as for the quasi-hole. See Ref. [39]. There the energy to create a pair of quasi-hole and quasi-electron is estimated to be $\simeq 0.05e^2/(\varepsilon\lambda)$ (Morf and Halperin [40] found $0.1e^2/(\varepsilon\lambda) \simeq 5.8^0K$ at $B = 150\ kGauss$, a good value for the activation energy). Also for quasi-electrons Bose statistics applies.

Other types of excited states have been considered (but always in the subspace $n_L = 0$) by Girvin *et al.* [41]:

$$\int d^2r\psi^\dagger(\mathbf{r})S(ik)\psi(\mathbf{r})|L\rangle. \tag{5.130}$$

These states are the projection on the $n_L = 0$ subspace of

$$\int d^2r\psi^\dagger(\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r})\psi(\mathbf{r})|L\rangle. \tag{5.131}$$

The last equation is reminiscent (in the usual Translations group) of the Single Mode Excitations

$$\sum_q \hat{c}^\dagger_{q+k}\hat{c}_q|L\rangle \tag{5.132}$$

where $\hat{c}^\dagger_q$ creates an electron with momentum q. The authors quote values of excitation energy ranging from 0.1 to 0.15 in units $e^2/(\varepsilon\lambda)$.

5.3 Hierarchy and related

From the previous subsection we learned that $\nu_f = 1/m$ filling can be described by a Laughlin state and that excited states can be constructed with charge $\pm e/m$. The idea of hierarchy, introduced to explain the $\nu_f = p/q$ fillings, consists in the use of quasi-holes (or quasi-electrons) as elementary objects. A similar procedure was used when we forgot the lattice potential and considered the Bloch electrons as elementary. The price is generalized statistics (anyons of Wilczek [42]). Hierarchy has been introduced by various authors [43, 44, 45]; we follow the approach of Ref. [45].

Jain [23] has developed a scheme for the FQHE, which seems to describe better, than the hierarchy model, the observed filling factor sequence.

6 Kubo formula or linear response theory

The fundamental tool for the theoretical discussion of the QHE is provided by the Kubo formula [46], which gives the response function ϕ_{BA}, describing the variation of B in presence of a perturbation (small) due to a force F coupled to a variable A

$$H_F = H - A\cdot F(t) \tag{6.133}$$

Thus

$$\Delta B(t) = \int_{-\infty}^{t} \phi_{BA}(t-t')F(t')dt'. \tag{6.134}$$

This is based on linearity and causality.

By taking the Fourier transform of eq. (6.134) we get the complex *admittance* (or generalized susceptibility)

$$\Delta B(\omega) = \chi(\omega)F(\omega) \tag{6.135}$$

with

$$\chi_{BA}(\omega) = \lim_{\epsilon\to 0+} \int_{0}^{\infty} \phi_{BA}(t)\exp\left(-i\omega t - \epsilon t\right). \tag{6.136}$$

We derive the response function. At $t=-\infty$ the perturbation is off and the system is at equilibrium, i.e. $\rho(-\infty)=\rho_0$, with

$$[H,\rho_0]=0 \tag{6.137}$$

at any t

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H_F,\rho]. \tag{6.138}$$

Let $\Delta\rho = \rho-\rho_0$, then

$$\frac{d\Delta\rho}{dt} = \frac{1}{i\hbar}\left([H,\Delta\rho] - F(t)[A,\Delta\rho] - F(t)[A,\rho_0]\right). \tag{6.139}$$

In the limit of very small perturbation the second term is neglected and the equation can be formally integrated

$$\Delta\rho(t) = \frac{1}{i\hbar}\int_{-\infty}^{t} \exp\left(-i(t-t')\frac{H}{\hbar}\right)[A,\rho_0]\exp\left(i(t-t')\frac{H}{\hbar}\right)F(t')dt' \tag{6.140}$$

and finally

$$\Delta B(t) = tr(\Delta\rho(t)B) = -\frac{1}{i\hbar}tr\int_{-\infty}^{t}[A,\rho_0]B(t-t')F(t')dt' \tag{6.141}$$

The response function is then

$$\phi_{BA}(t) = -\frac{1}{i\hbar}tr[A,\rho_0]B(t) = \frac{1}{i\hbar}tr[A,B(t)]\rho_0. \tag{6.142}$$

Notice that the evaluation of the response function is performed on the **unperturbed** system ($F(t)=0$)!

Sometimes the Kubo formula is given in the following notation (which should not be scaring): in the canonical ensemble ($\rho_0 = \exp(-\beta H)$)

$$\phi_{BA}(t) = -\frac{1}{i\hbar}tr\left(e^{-\beta H}\left(e^{\beta H}Ae^{-\beta H}B(t) - AB(t)\right)\right) = tr\rho_0\int_0^{\beta}d\lambda\dot{A}(-i\hbar\lambda)B(t). \tag{6.143}$$

In the zero temperature limit $\beta\to\infty$ and for non degenerate ground state

$$\lim_{\beta=\infty}\phi_{BA}(t) = \frac{1}{i\hbar}\langle\Omega|[A,B(t)]|\Omega\rangle. \tag{6.144}$$

The conductivity tensor at $\omega = 0$ (static field) and $\beta \to \infty$ is then

$$\sigma_{BA} = \lim_{\epsilon=0} \int_0^\infty dt e^{-\epsilon t} \frac{1}{i\hbar} \langle \Omega | [A, B(t)] | \Omega \rangle. \tag{6.145}$$

If the system consists of N electrons in presence of an electric field

$$H_F = H - (-e) \sum_i \mathbf{r}_i \cdot \mathbf{E} \tag{6.146}$$

then

$$A = -e \int d\mathbf{r} \psi^\dagger r_i \psi \tag{6.147}$$

for some i. The measured quantity is the current. Since

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{1}{m}(p_x + \frac{e}{c}A_x) = v_x \tag{6.148}$$

one gets

$$\sigma_{ij} = \frac{1}{A}\frac{e^2}{i\hbar} \sum_l \int_0^\infty dt e^{-\epsilon t} \left(\langle \Omega | r_j | l \rangle \langle l | v_i | \Omega \rangle e^{-\frac{it}{\hbar}(E_0 - E_l)} - \langle \Omega | v_i | l \rangle \langle l | r_j | \Omega \rangle e^{\frac{it}{\hbar}(E_0 - E_l)} \right). \tag{6.149}$$

We use now

$$\langle \Omega | r_j | l \rangle (E_0 - E_l) \frac{1}{i\hbar} = \langle \Omega | v_j | l \rangle. \tag{6.150}$$

Finally the conductivity tensor is

$$\sigma_{ij} = \frac{1}{A}(ie^2\hbar) \sum_{l \neq \Omega} \frac{1}{(E_0 - E_l)^2} \left(\langle \Omega | v_j | l \rangle \langle l | v_i | \Omega \rangle - \langle \Omega | v_i | l \rangle \langle l | v_j | \Omega \rangle \right). \tag{6.151}$$

6.1 Thouless + 3

It is tempting to try to solve the FQHE by introducing an interaction which removes the degeneracy of the Landau levels. Thus if the chemical potential is in a energy gap, there is fractional filling. The use of eq. (4.94) would yield plateaus at non-integer filling. Thouless *et al.* [47] proved that this hope is not justified for a model of independent electrons in an external periodic potential. The result has been generalized to the case of interacting electrons [48, 49], with the main assumption that the ground state is non-degenerate.

In the proof of Thouless *et al.* it is assumed that the flux through the unit cell (a, b) is a rational number in units of quantum flux: $\Phi = abB\frac{e}{ch} = p/q$. Then one gets q energy bands. The solution of the dynamical problem is given in terms of the functions (generalized Bloch conditions for periodic potential, with $0 \leq k_1 < 2\pi/a$ and $0 \leq k_2 < 2\pi/(qb)$)

$$u_{k_1 k_2} = \psi_{k_1 k_2} \exp(-ik_1 x - ik_2 y) \tag{6.152}$$

where $u_{k_1 k_2}$ satisfy the usual boundary conditions given in eq. (1.14) and are eigenstates of the Hamiltonian

$$\hat{H}(k_1, k_2) = \frac{1}{m} \left[\left(p_x - \frac{e}{c}By + \hbar k_1 \right)^2 + (p_y + \hbar k_2)^2 \right] + u(x, y). \tag{6.153}$$

The velocity is

$$v_i = \frac{1}{\hbar}\frac{\partial}{\partial k_i}\hat{H}(k_1, k_2). \tag{6.154}$$

The Hall conductivity (eq. (6.151)) becomes

$$\sigma_H = \frac{ie^2}{A\hbar}\sum_{\epsilon_\alpha<\epsilon_F}\sum_{\epsilon_\beta>\epsilon_F}(\epsilon_\alpha-\epsilon_\beta)^{-2}\left\{\left(\frac{\partial}{\partial k_1}\hat{H}\right)_{\alpha\beta}\left(\frac{\partial}{\partial k_2}\hat{H}\right)_{\beta\alpha} - \left(\frac{\partial}{\partial k_2}\hat{H}\right)_{\alpha\beta}\left(\frac{\partial}{\partial k_1}\hat{H}\right)_{\beta\alpha}\right\}. \tag{6.155}$$

Now consider

$$\left(\hat{H}(k_1, k_2) - \epsilon_\alpha\right) u_\alpha = 0, \tag{6.156}$$

take the derivative and finally the scalar product with $u_\beta \neq u_\alpha$:

$$\left(\frac{\partial}{\partial k_1}\hat{H}\right)_{\beta\alpha} = (\epsilon_\alpha - \epsilon_\beta)\langle\beta, \frac{\partial}{\partial k_1}\alpha\rangle = (\epsilon_\alpha - \epsilon_\beta)\int d^2 r u_\beta^* \frac{\partial}{\partial k_1}u_\alpha. \tag{6.157}$$

With this the conductivity in (6.155) becomes

$$\sigma_H = -\frac{ie^2}{A\hbar}\sum_{\epsilon_\alpha<\epsilon_F}\sum_{\epsilon_\beta>\epsilon_F}\left\{\langle\alpha, \partial_1\beta\rangle\langle\beta, \partial_2\alpha\rangle - \langle\alpha, \partial_2\beta\rangle\langle\beta, \partial_1\alpha\rangle\right\}. \tag{6.158}$$

But $\langle\alpha, \partial_1\beta\rangle = -\langle\partial_1\alpha, \beta\rangle$ gives

$$\sigma_H = \frac{ie^2}{A\hbar}\sum_{\epsilon_\alpha<\epsilon_F}\sum_{\epsilon_\beta>\epsilon_F}\left\{\langle\partial_1\alpha, \beta\rangle\langle\beta, \partial_2\alpha\rangle - \langle\partial_2\alpha, \beta\rangle\langle\beta, \partial_1\alpha\rangle\right\}. \tag{6.159}$$

A little algebra shows that the sum can be extended to all states

$$\sigma_H = \frac{ie^2}{A\hbar}\sum_{\alpha,\epsilon_\alpha<\epsilon_F}\left\{\langle\partial_1\alpha, \partial_2\alpha\rangle - \langle\partial_2\alpha, \partial_1\alpha\rangle\right\}. \tag{6.160}$$

The sum over the states can be written as a sum over the filled bands and the integral over the wave vectors:

$$\sigma_H = \frac{ie^2}{2\pi h}\sum_{\text{filledbands}}\int d^2 k\left\{\langle\partial_1\alpha, \partial_2\alpha\rangle - \langle\partial_2\alpha, \partial_1\alpha\rangle\right\}. \tag{6.161}$$

The term in brackets is half the curl of

$$\zeta_i = \langle\alpha, \partial_i\alpha\rangle - \langle\partial_i\alpha, \alpha\rangle = \int d^2 r\left\{u_\alpha^*\partial_i u_\alpha - \partial_i u_\alpha^* u_\alpha\right\}. \tag{6.162}$$

Then the Stokes theorem gives

$$\sigma_H = \frac{ie^2}{4\pi h}\sum\int dk_i\int d^2 r\left\{u^*\partial_i u - \partial_i u^* u\right\}. \tag{6.163}$$

This quantity is in some sense *gauge invariant*: by changing

$$u \rightarrow \exp(i\theta(k_1, k_2))u \tag{6.164}$$

σ_H does not change. Moreover according to the authors one has

$$\int dk_i \int d^2r \, \{u^* \partial_i u - \partial_i u^* u\} = 4\pi i \times \text{integer}. \tag{6.165}$$

The argument runs as follows. u is an analytic function of k_1, k_2 (no band crossing). Then the following periodicity conditions must be valid (on the Brillouin zone)

$$u(k_1, k_2 = \frac{2\pi}{qb}) = \exp(i\phi(k_1))u(k_1, k_2 = 0) \tag{6.166}$$

and

$$u(k_1 = \frac{2\pi}{a}, k_2) = \exp(i\psi(k_2))u(k_1 = 0, k_2). \tag{6.167}$$

Then the integral is

$$\int dk_i \int d^2r \, \{u^* \partial_i u - \partial_i u^* u\} = 2i \left(\psi(\frac{2\pi}{qb}) - \psi(0) + \phi(0) - \phi(\frac{2\pi}{a}). \right) \tag{6.168}$$

The matching of the phases ϕ and ψ on the border of the unit cell (k-space) imposes that the term in brackets is an *integer* multiple of 2π. Thus the Quantum Hall Effect is always integer. This theorem can be evaded if the ground state is degenerate.

References

[1] T. Sajoto, Y.W. Suen, L.W. Engel, M.B. Santos, and M. Shayegan, Phys. Rev. B **41**, 8449 (1990).

[2] S. Koch, R.J. Haug, K. v. Klitzing, and M. Razeghi, *Suppression of the Landau Level Coincidence: a Novel Phase Transition in Tilted Magnetic Fields*, preprint.

[3] S. Yarlagadda, Phys. Rev. B **44**, 13101 (1991).

[4] C. Weisbuch and C. Hermann, Phys. Rev. B **15**, 816 (1977).

[5] V.M. Pudalov, M. D'Iorio, S.V. Kravchenko, J.W. Campbell, Phys. Rev. Lett., **70**, 1866 (1993).

[6] M.A. Paalanen, D.C. Tsui, and A.C. Gossard, Phys. Rev. B **25**, 5566 (1982).

[7] R. Willet, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard, J.H. English, Phys. Rev. Lett. **59**, 1776 (1987).

[8] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[9] , D.C. Tsui, H.L. Störmer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[10] C. Kittel, *Quantum Theory of Solids*, second edition, John Wiley & Sons (New York 1987).

[11] R. Ferrari, Phys. Rev. B **42**, 4598 (1990).

[12] W. Kohn, Phys. Rev. **123**, 1242 (1961).

[13] M. Dobers, K. v. Klitzing, and G. Weimann, Phys. Rev. B**38**, 5453 (1988).

[14] A. Pinczuk, B.S. Dennis, D. Heiman, C. Kallin, L. Brey, C. Tejedor, S. Schmitt-Rink, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. **24**, 3623 (1992).

[15] R. Ferrari and M.I. Trioni, Phys. Rev. B **47**, 9577 (1993).

[16] C. Kallin and B.I. Halperin, Phys. Rev. B **30**, 5655 (1984).

[17] A. Usher, R.J. Nicholas, J.J. Harris, and C.T. Foxon, Phys. Rev. B **41**, 1129 (1990).

[18] R.J. Nicholas, R.J. Haug, and K. v. Klitzing, Phys. Rev. B **37**, 1294 (1988).

[19] S. Koch, R.J. Haug, M. Razeghi, and K. v. Klitzing, Phys. Rev. B**47**, 4048 (1993).

[20] L.W. Engel, S.W. Hwang, T. Sajoto, D.C. Tsui, and M. Shayegan, Phys. Rev. B**45**, 3418 (1992).

[21] Physics Today **47**, 17 (1993).

[22] R.R Du, H.L. Störmer, D.C. Tsui, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. **70**, 2944 (1993).

[23] J.K. Jain, Phys. Rev. B **41**, 7653 (1990).

[24] B.I. Halperin, P. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).

[25] Y.W. Suen, L.W. Engel, M.B. Santos, M. Shayegan, and D.C. Tsui, Phys. Rev. Lett. **68**, 1379 (1992).

[26] J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer, K.W. West, and Song He, Phys. Rev. Lett. **68**, 1383 (1992).

[27] A.H. MacDonald, D. Yoshioka, and S.M. Girvin, Phys. Rev. B **39**, 8044 (1989).

[28] R.L. Willet, M.A. Paalanen, R.R. Ruel, K.W. West, L.N. Pfeiffer, and D.J. Bishop, Phys. Rev. Lett. **65**, 112 (1990).

[29] B.I. Halperin, Phys. Rev. B **25**, 2185 (1982).

[30] A.H. MacDonald, Phys. Rev. Lett. **64**, 220 (1990).

[31] G. Müller, D. Weiss, S. Koch, K. v. Klitzing, H. Nickel, W. Schlapp, and R. Lösch, Phys.

[32] K. v. Klitzing, J. Nieder, R.J. Haug, G. Müller, S. Koch, D. Weiss, and K. Ploog, Semicond. Sci. Technol. **7**, 82 (1992).

[33] L.W. Engel, T. Sajoto, Y.P.Li, D.C. Tsui, and M. Shayegan, Surf. Sci. **263**, 44 (1992).

[34] M.B. Santos, Y.W. Suen, M. Shayegan, Y.P.Li, L.W. Engel, and D.C. Tsui, Phys. Rev. Lett. **68**, 1188 (1992).

[35] I.V. Kukushkin, K. v. Klitzing, A.S. Plaut, K. Ploog, H. Baumann, W. Joss, G. Martinez, and V.B. Timofeev, JETP Lett. **51**, 655 (1990).

[36] I.V. Kukushkin, N.J. Pulsford, K. v. Klitzing, R. Haug, K. Ploog, and V.B. Timofeev, Physica B **184**, 38 (1993).

[37] R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[38] N. Datta and R. Ferrari, *On the Mean Energy and Particle Density of the Laughlin State (m=3)*, University of Trento preprint (MPI-Ph/ 92-16, U.T.F.-254), February 1992.

[39] R.B. Laughlin, in *The Quantum Hall Effect*, R.E. Prange and S.M. Girvin eds., second edition. Springer Verlag (New York 1990).

[40] R. Morf and B.I. Halperin, Phys. Rev. B **33**, 2221 (1986).

[41] S.M. Girvin, A.H. MacDonald, and P.M. Platzman, Phys. Rev. B **33**, 2481 (1986).

[42] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).

[43] F.D.M. Haldane, Phys. Rev. Lett. **51**, 957 (1983).

[44] R.B. Laughlin, Surf. Sci. **141**, 11 (1984).

[45] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); **52**, 2390 (1984) (E).

[46] R. Kubo, J. Phys. Soc. Japan, **12**, 570 (1957).

[47] , D.J. Thouless, M. Kohmoto, M.P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[48] Q. Niu and D.J. Thouless, J. Phys. A**17**, 2453 (1984)

[49] J.E. Avron and R. Seiler, Phys. Rev. Lett. **54**, 259(1985).

7 Figure Captions

Figure 1: Top: the experimental setup of the Hall bar geometry. Bottom: Corbino geometry .

Figure 2: Band structure in heterostructure .

Figure 3: Hall plateaus for IQHE [6] .

Figure 4: Hall plateaus showing FQHE (notice the $\nu_f = 5/2$) [7] .

Figure 5: The prediction of the free electron model (continuous) compared with the data of Ref. [6] (dashed) .

Figure 6: Hall plateaus for IQHE in a model with Coulomb and finite buffer capacity of the bulk [15] (continuous) compared with the data of Ref. [6] (dashed). .

Figure 7: The light diffusion experiment at large angle shows the contribution of exchange energy in the transition SF (at $17.5 meV$) [14] .

Figure 8: Arrenius plots for activation energies [17] .

Figure 9: Activation energies [17] .

Figure 10: Coincidence of levels in tilted field [18] .

Figure 11: Disappearance of plateaus in a coincidence experiment [18] .

Figure 12: Plateaus around $\nu_f = 1/2$. Ref. [22] .

Figure 13: Hall plateaus in double layer (presence of $\nu_h = 1/2$ structure). Ref. [26] (top) and Ref. [25] (bottom) .

Figure 14: Anomaly in surface acoustic wave experiment [28] .

Figure 15: Edge current experiment: influence of ohmic contacts on longitudinal resistivity [32] .

Figure 16: Insulating Phase near $\nu_f = 1/5$. Ref. [33] .

Figure 17: δ doping and IQHE levels (left). Energy position of the peak due to recombination (right). Ref. [35] .

Figure 18: Energy position of the peak due to recombination (left) [35]. Phase diagram for Wigner crystallization (right) [36] .

Figure 19: Profile of the single particle density versus radius in the Laughlin state $m = 3$ for $N = 7, 10, 20, 43, 91, 200$. Ref. [38] .

Figure 20: Energy per particle for subsystems (of N_s particles, x-axis) embedded in a larger system (with $N = 43, 91, 200$ particles). Ref. [38]