ON THE CONSTRUCTION OF RENORMALIZED QUANTUM FIELD THEORY USING RENORMALIZATION GROUP TECHNIQUES*

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ABSTRACT

The aim of these lectures is to describe a construction, as self-contained as possible, of renormalized quantum field theory. Following a suggestion of Polchinski we base our analysis on the Wilson renormalization group method.

After a discussion of the infinite cut-off limit we study the short distance properties of the Green functions verifying the validity of Wilson short distance expansion. We also consider the problem of the extension to the quantum level of the classical symmetries of the theory. With this purpose we analyze in details the breakings induced by the cut-off in a $SU(2)$ gauge symmetry and we prove the possibility of compensating these breakings by a suitable choice of non-gauge invariant counter terms.

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1. Introduction

In a period of approximately ten years, about twenty years ago, the perturbative construction of renormalized quantum field theory has achieved a remarkable level of rigor and efficiency. based on many technical achievements that are widely explained in the current literature. We mention among others the extension to all orders of perturbation theory of subtraction schemes suitable to avoid infinities in the Feynman amplitudes. This has put on a rigorous basis the method of counter-terms, although in the framework of a formal perturbation theory which is not absolutely summable.

A second significant technical progress consists in the discovery of very clever regularization schemes, in particular the dimensional one which has made possible calculations of renormalization effects of remarkably high order, as e. g. the fourth of QCD. This has greatly improved the efficiency of renormalized quantum field theory .

The great majority of the above mentioned achievements have been based on a deep and complicated analysis of Feynman diagrams a typical ingredient of which is the concept of "forest". The difficulties with Feynman diagrams are amplified in the gauge models of fundamental interactions in which the number of contributions to a given amplitude increases rapidly with the perturbative order and hence it is often prohibitive to push the calculations beyond one loop. In these models dimensional regularization too can become a source of problems due to the difficulty of extending the concept of chirality to complex space-time dimensions.

Waiting for new ideas and tools to compute higher orders in gauge theories, there remains, in our opinion, the need of an essential simplification of the proofs of the relevant general properties of renormalized quantum field theory.

A few years ago Polchinski [1] has shown how the existence of the ultra-violet limit of a scalar theory, regularized by means of a momentum cut-off, can be proved using Wilson renormalization group techniques [2] . The method of Polchinski is remarkably simple and can be trivially extended to spinor and vector fields; however one still needs to recover in the same framework the whole set of general results that have made possible the above mentioned progresses in renormalized quantum field theory.

In these lectures we present an attempt, following the ideas of Polchinski, to give a self-contained proof of the existence of a perturbatively renormalized quantum field theory and of two "general properties" of it. First the validity of Wilson short distance expansion [3] which is a fundamental tool in the analysis of Green functions. Secondly we discuss the "quantum action principle", from which rather general results on the renormalized structure of theories with continuous symmetries can be obtained [4],[5] .

The lecture notes are so organized: in section 2 we discuss the construction of the Feynman functional generator. In section 3 we present the renormalization group method, whose perturbative solution is discussed in section 4. In section 5 we make some comments on the construction of composite operators and we discuss Wilson short distance expansions. In section 6 we derive the "quantum action principle" that we apply to the study of the $SU(2)$ Yang-Mills model.

2. The Feynman formula

The existence of a theory of scattering is one of the fundamental results of field theory. It is based on the classical axiomatic results on the asymptotic evolution of states and on the well known Lehmann-Symanzik-Zimmermann reduction formulae relating S matrix elements to time-ordered Green functions. If ϕ_{in} is a set of asymptotic ingoing fields whose wave operator is W we can write the S operator in the Fock space of ϕ_{in} as:

$$S =: exp\left(\int d^4x\phi_{in}(x)W_x z^{-1}\frac{\delta}{\delta j(x)}\right) : Z[j]|_{j=0} \ , \tag{2.1}$$

where z is the residue of the Fourier transformed two-point function on the mass shell pole and the functional $Z[j]$ is the generator of the time-ordered Green functions:

$$Z[j] = (\Omega, Te^{i\int d^4x j(x)\phi(x)}\Omega) \ , \tag{2.2}$$

and Ω is the vacuum state.

The determination of the functional generator Z is therefore the main dynamical problem in the construction of a field theory. The Feynman formula is universally considered as the solution of this problem.

To put into evidence the role of locality and covariance and the difficulties that follow the assumption of these axioms, it is particularly convenient to characterize the Feynman formula by means of the field evolution equation. On a completely formal level one assumes for the field the local and covariant equation:

$$(\partial^2 + m^2)\phi(x) \equiv W_x\phi(x) = I'\left[\phi(x)\right] \ . \tag{2.3}$$

We have chosen the simplest possible framework referring to a scalar field theory in spite of its possible triviality. The field equation is translated into a functional differential equation for the functional generator Z by the substitution of the field with $-i\frac{\delta}{\delta j(x)}$:

$$\left[-iW_x\frac{\delta}{\delta j(x)} + I'\left[-i\frac{\delta}{\delta j(x)}\right]\right] Z[j] = j(x)Z[j] \ . \tag{2.4}$$

If one identifies the classical version of (2.3) with the Euler-Lagrange equation corresponding to the action S_{cl}, the functional generator Z can be identified with

$$Z[j] = \int \prod_x d\phi(x) e^{i\left[S_{cl} - \int d^4x j(x)\phi(x)\right]} \ , \tag{2.5}$$

provided that this functional integral to make sense and the measure $\prod_x d\phi(x)$ to be translation invariant. (2.5) is the Feynman formula.

In the first part of this course we shall discuss the possibility giving a meaning to this formula. For this we have to overcome a sequence of difficulties that can be traced back to the functional measure and to the integrand.

The problem with the measure is technically related to the lack of local compactness, more naively, to the presence of an infinite number of variables. Concerning the integrand, if the measure is translation invariant as assumed, it has to introduce a uniform convergence factor for large field amplitudes. This is not verified in the present case since the integrand has absolute value equal to one.

The standard solution to this convergence problem is based on the transformation of the minkowskian theory into an euclidean one. This is achieved "Wick rotating" the time variables from the positive real axis to the negative imaginary one. The time-ordered Green functions are then replaced by the euclidean Schwinger functions [6] whose functional generator is defined whose functional generator is defined in complete analogy with (2.5), absorbing the imaginary unit into the euclidean space measure. That is:

$$Z[j] = \int \prod_x d\phi(x) e^{-\left[S_e - \int d^4x j(x)\phi(x)\right]} . \tag{2.6}$$

If the euclidean classical action S_e is a positive functional increasing with the field amplitude the wanted convergence factor is guaranteed. Of course, we have overcome the first difficulty at the price of computing something that is different from our goal. However, Osterwalder and Schrader have identified the conditions ensuring the possibility of recovering the wanted physical information from an euclidean theory. [7]

Coming back to the infinity of the number of integration variables, we notice that one can distinguish two sources of this difficulty. First, euclidean invariance requires the space volume to be infinite. This is the infra-red (IR) difficulty. Giving up the euclidean invariance one could quantize the theory in a hypercube choosing as integration variables the Fourier amplitudes of the field. However these are still infinite in number and locality requires the interaction to involve all the field Fourier components. This is the ultra-violet (UV) difficulty, it can also be seen from a different point of view. If the interacting Green (Schwinger) functions are distributions, as in the free case, the same is true for the functional derivatives of Z and hence the strictly local functional $I'\left[-i\frac{\delta}{\delta j}\right]$ is ill-defined.

To avoid these IR and UV difficulties, giving up for the moment locality and covariance, we introduce a system of regularizations. First, as said above, we restrict our theory into a four-dimensional hypercube. Periodic boundary conditions are chosen for the fields in order to preserve translation invariance. As a matter of fact in this way we are considering a quantum mechanical system in a three-dimensional cube of side L in thermal equilibrium at the temperature $\beta^{-1} = \frac{1}{L}$. An infinite volume and zero temperature limit will eventually reproduce the original relativistic euclidean field theory.

The second regularization concerns directly the UV difficulties. We modify the interaction decoupling the short wavelength field components, which , however, continue to appear into the dynamical framework as free degrees of freedom. Notice that in standard approaches these degrees of freedom are simply not taken into account.

We regularize the euclidean field equation by replacing into the interaction the field $\phi(x)$ with

$$K_{\Lambda_0}\phi(x) = \frac{1}{L^2}\sum_{\vec{p}} k(\frac{p^2}{\Lambda_0^2})\phi_{\vec{p}}e^{ipx} \tag{2.7}$$

where

$$\phi_{\vec{p}} = \frac{1}{L^2}\int d^4x\phi(x)e^{ipx} \ , \tag{2.8}$$

and, of course the integral is limited within the above mentioned four-dimensional hypercube of side L. The ultraviolet (UV) cut-off factor k is a C^∞ function assuming the value 1 below 1 and vanishing above 2.

It should be clear that the introduction of the UV cut-off should interfere as weakly as possible with the observables of our theory. This requires, in particular, that the chosen value of the cut-off be much higher than the greatest wave number appearing in the Fourier decomposition of the sources j and of the other sources that one could introduce to define composite operators. More precisely let Λ_R be the greatest observable wave number, we select the sources so that

$$K_{\Lambda_R}j = j \ . \tag{2.9}$$

and we ask

$$\Lambda_0 \gg \Lambda_R \ . \tag{2.10}$$

Eventually Λ_0 will be sent to infinity.

Now we come to the choice of the action. This is done having in mind the short-distance properties of the free theory that we want only weakly perturbed by the interaction. With this purpose we assign to every field ϕ a mass-dimension d_ϕ and to the corresponding source $d_j = D - d_\phi$, with D the dimension of the euclidean space (in our case 4). d_ϕ and hence d_j are computed from the mass-dimension of the wave operator W requiring that the free field equations be dimensionally homogeneous. Thus, in the scalar field case, the mass-dimension of the wave operator is two, that of the laplacian and hence the mass-dimension of the field is computed from:

$$2 + d_\phi = d_j = D - d_\phi \ . \tag{2.11}$$

That is one in the 4-dimensional case.

The action in (2.6) is chosen as the sum of the free part

$$\int d^4x \frac{(\partial\phi)^2 + m^2\phi^2}{2} \ , \tag{2.12}$$

and of the interaction L_0, an integrated local polynomial in the regularized field $K_\Lambda\phi$ and its derivatives. The short-distance "power counting" condition limits to 4 the dimension of the operators appearing in L_0. Therefore in the scalar case we have:

$$L_0 = \int d^4x \left[\frac{\lambda_0}{4!}\phi^4 + \frac{z_0 - 1}{2}(\partial\phi)^2 + \frac{m_0^2 - m^2}{2}\phi^2\right] \ . \tag{2.13}$$

We introduce for example the operator ϕ^2 to which we assign the source ω whose dimension is the complement to 4 of the dimension of the operator, that is 2. We add to the interaction L_0 the ω-dependent terms:

$$\int d^4x \left[\zeta_0\omega\phi^2 + \eta_0\omega^2(x) + \xi_0\omega(x)\right] \ . \tag{2.14}$$

This functional identifies the general solution of an extended power counting condition taking into account also the dimension of the sources.

One should wonder about the meaning of these field independent terms. It is clear from the definition of the functional generator Z that the new term induces a contribution proportional to a Dirac δ in the correlation function of two composite operators. Back to the minkowskian world this corresponds to a redefinition of the time-ordering of the two operators.

Now, taking into account both UV and IR regularizations, we write the interaction of our model according:

$$\begin{aligned} L_0 = \int_\Omega d^4x &\left[\frac{\lambda_0}{4!}(K_{\Lambda_0}\phi)^4 + \frac{z_0-1}{2}(\partial K_{\Lambda_0}\phi)^2 + \frac{m_0^2-m^2}{2}(K_{\Lambda_0}\phi)^2 + \right. \\ &\left.\zeta_0\omega(K_{\Lambda_0}\phi)^2 + \eta_0\omega^2 + \xi_0\omega\right] = \frac{\lambda_0}{4!L^4}\sum_{\vec p_1,..\vec p_4}\delta_{\vec 0,\vec p_1+..+\vec p_4}\, K_{\Lambda_0}\phi_{\vec p_1}...K_{\Lambda_0}\phi_{\vec p_4} + \\ &\sum_{\vec p}\frac{(z_o-1)p^2+m_0^2-m^2}{2}K_{\Lambda_0}\phi_{\vec p}K_{\Lambda_0}\phi_{-\vec p} + \\ &\frac{\zeta_0}{L^2}\sum_{\vec p_1,..\vec p_3}\delta_{\vec 0,\vec p_1+..+\vec p_3}\,\omega_{\vec p_1}K_{\Lambda_0}\phi_{\vec p_2}K_{\Lambda_0}\phi_{-\vec p_3} + \\ &\eta_0\sum_{\vec p}\omega_{\vec p}\omega_{-\vec p} + L^2\xi_0\omega_{\vec 0} \ . \end{aligned} \tag{2.15}$$

Notice that by the IR regularization we have automatically broken the euclidean invariance of the theory, since a cube is not rotation invariant. Therefore there is no reason to preserve the euclidean invariance of every single term of the interaction. Following the fine-tuning strategy that will be discussed in section 6 it is possible to prove the compensability of the possible breakdown of euclidean invariance induced by the IR regularization by the introduction into the interaction of non-invariant counter-terms. This compensability holds true for all continuous symmetries if the symmetry group is semisimple. This is a typical consequence of the quantum action principle [4],[5].

It is apparent that in our example L_0 depends on the 6 parameters λ_0, z_0, m_0, ζ_0, η_0 and ξ_0 that can be extracted from the interaction by suitable normalization operators. That is:

$$\rho_{0,1} = L^4\frac{\partial^4}{\partial\phi_{\vec 0}^4}L_0|_{\phi=\omega=0} \equiv N_1L_0 = \lambda_0 \ , \tag{2.16}$$

and, setting

$$\partial_{\vec p}\partial_{-\vec p}L_0|_{\phi=\omega=0} = \Pi_{\vec p} \ , \tag{2.17}$$

$$\rho_{0,2} = \frac{\Pi_{\vec p}-\Pi_{\vec 0}}{p^2} \equiv N_2L_0 = z_0 - 1 \ , \tag{2.18}$$

for some suitably chosen $\vec{p}$ and

$$\rho_{0,3} = \Pi_{\vec{0}} \equiv N_3 L_0 = m_0^2 - m^2 \ , \tag{2.19}$$

$$\rho_{0,4} = \frac{L^2}{2} \frac{\partial^3}{\partial \phi_{\vec{0}}^2 \partial \omega_{\vec{0}}} L_0|_{\phi=\omega=0} \equiv N_4 L_0 = \zeta_0 \ , \tag{2.20}$$

$$\rho_{0,5} = \frac{1}{2} \frac{\partial^2}{\partial \omega_{\vec{0}}^2} L_0|_{\phi=\omega=0} \equiv N_5 L_0 = \eta_0 \ , \tag{2.21}$$

$$\rho_{0,6} = L^{-2} \frac{\partial}{\partial \omega_{\vec{0}}} L_0|_{\phi=\omega=0} \equiv N_6 L_0 = \xi_0 \ . \tag{2.22}$$

The functional generator corresponding to the interaction (2.15) is given by

$$Z[j,\omega] = N \int \prod_{\vec{p}} d\phi_{\vec{p}} e^{-\left[\sum_{\vec{p}} \phi_{-\vec{p}} \frac{C(p)}{2} \phi_{\vec{p}} + L_0 - \sum_{\vec{p}} j_{-\vec{p}} \phi_{\vec{p}}\right]} \equiv N \int \prod_{\vec{p}} d\phi_{\vec{p}} \ e^{-S} \ , \tag{2.23}$$

where the normalization factor N ensures the integration condition

$$Z[0,0] = 1 \ . \tag{2.24}$$

Notice that, taking into account the introduced regularizations, the integral in (2.23) factorizes in an infinite dimensional, purely gaussian, contribution corresponding to the Fourier components $\phi_{\vec{p}}$ with $\vec{p} > \sqrt{2}\Lambda_0$, that is reabsorbed into the normalization factor N and in a finite dimensional part, that is absolutely convergent provided that λ_0 is positive.

The crucial problem in quantum field theory is to prove the existence of an IR ($L \to \infty$) and an UV ($\Lambda_0 \to \infty$) limit of (2.23). This goal has been achieved in the framework of the perturbative method for the whole family of theories satisfying the power counting criterion.

The major aim of these lectures is to describe the main lines of this result. For this is convenient to remember that perturbation theory is constructed introducing into the Feynman functional integral the ordering parameter $\hbar$ according

$$Z[j,\omega] \equiv e^{\frac{Z_c[j,\omega]}{\hbar}} = N \int \prod_{\vec{p}} d\phi_{\vec{p}} \ e^{\frac{-S}{\hbar}} \ , \tag{2.25}$$

and developing the connected functional Z_c as a formal power series in $\hbar$. This power series is obtained applying to (2.25) the method of the steepest descent. Analyzing the terms of the series as the sum of Feynman diagrams one sees that Z_c receives contributions only from the connected diagrams and that $\hbar$ can be interpreted as a loop counting parameter.

3. The Wilson renormalization group method.

A fundamental tool in the analysis of the UV limit is the Wilson renormalization group method. [2] This method, in its original version, is based on a sharp UV cut-off (Λ_0), or equivalently on a lattice regularization, limiting the number of degrees of freedom, i. e. that of the integration variables in the Feynman formula. It consists in the iterative reduction of the number of degrees of freedom through the integration over the field Fourier components $\phi_{\vec{p}}$ with $\Lambda_0 \geq p \geq \frac{\Lambda_0}{2}$. The resulting, partially integrated, Feynman formula is brought back to the original form substituting the interaction lagrangian with an effective one which now depends on the fields cut-off at $\frac{\Lambda_0}{2}$. The existence of the wanted UV limit is related to that of a "fixed point" in the space of effective lagrangians which is approached afteranalyzing a great number of iterations of the partial integration procedure.

Following Polchinski, [1] we shall use a modified version of this method based on a continuous lowering of the cut-off, which, in our case, corresponds to a continuous switching-off of the interaction of the field components with higher wave number.

This is accompanied by a continuous evolution of the effective lagrangian L_{eff} replacing the bare interaction L_0 in the Feynman formula as it is exhibited in :

$$Z[j,\omega] = N \int \prod_{\vec{p}} d\phi_{\vec{p}} e^{-\left[\sum_{\vec{p}} \phi_{-\vec{p}} \frac{C(p)}{2} \phi_{\vec{p}} + L_{eff}(K_\Lambda \phi, \omega, \Lambda, \Lambda_0, \rho_0) - \sum_{\vec{p}} j_{-\vec{p}} \phi_{\vec{p}}\right]} \equiv$$
$$N \int \prod_{\vec{p}} d\phi_{\vec{p}}\, e^{-S} \ , \tag{3.1}$$

which holds true for $\Lambda_R < \Lambda < \Lambda_0$. The identity of (3.1) and (2.23) is guaranteed if L_{eff} satisfies the following evolution equation:

$$\Lambda \partial_\Lambda L_{eff} = \frac{1}{2} \sum_{\vec{p}} \Lambda \partial_\Lambda k^2 \left(\frac{p}{\Lambda}\right) C^{-1}(p) \left[\partial_{-\vec{p}} L_{eff} \partial_{\vec{p}} L_{eff} - \partial_{\vec{p}} \partial_{-\vec{p}} L_{eff}\right] \ , \tag{3.2}$$

with the initial condition

$$L_{eff}(K_{\Lambda_0}\phi, \omega, \Lambda_0, \Lambda_0, \rho_0) = L_0 \ . \tag{3.3}$$

In (3.2) we have introduced the simplified notation:

$$\partial_{\vec{p}} L_{eff} = \frac{\partial L_{eff}}{\partial K_\Lambda \phi_{\vec{p}}} \ . \tag{3.4}$$

Using (3.2) it is easy to verify (3.1). Indeed

$$\Lambda \partial_\Lambda Z[j,\omega] = -\int \prod_{\vec{p}} d\phi_{\vec{p}} \left[\sum_{\vec{p}} \Lambda \partial_\Lambda K_\Lambda \phi_{\vec{p}} \partial_{\vec{p}} L_{eff} + \Lambda \partial_\Lambda L_{eff}\right] e^{-S} \ , \tag{3.5}$$

and

$$e^S\, C^{-1}(p) \partial_{\phi_{-\vec{p}}} e^{-S} = -\phi_{\vec{p}} - C^{-1}(p) \left[k\left(\frac{p}{\Lambda}\right) \partial_{-\vec{p}} L_{eff} - j_{\vec{p}}\right] \ . \tag{3.6}$$

Thus we have:

$$\Lambda\partial_\Lambda Z[j,\omega] = -\int \prod_{\vec{p}} d\phi_{\vec{p}}\,[\Lambda\partial_\Lambda L_{eff} -$$

$$\sum_{\vec{p}} \Lambda\partial_\Lambda k\left(\frac{p}{\Lambda}\right)\partial_{\vec{p}} L_{eff} C^{-1}(p)\left[\partial_{\phi_{-\vec{p}}} + k\left(\frac{p}{\Lambda}\right)\partial_{-\vec{p}} L_{eff}\right]\Bigg] e^{-S}\ , \tag{3.7}$$

where we have taken into account that from (2.9) and (2.10) one has

$$\Lambda\partial_\Lambda k\left(\frac{p}{\Lambda}\right) j_{\vec{p}} = 0\ . \tag{3.8}$$

Now, integrating by parts the second term in the right-hand side of (3,7) we get:

$$\Lambda\partial_\Lambda Z[j,\omega] = -\int \prod_{\vec{p}} d\phi_{\vec{p}}\,[\Lambda\partial_\Lambda L_{eff} -$$

$$\sum_{\vec{p}} \Lambda\partial_\Lambda k\left(\frac{p}{\Lambda}\right) C^{-1}(p) k\left(\frac{p}{\Lambda}\right)\left[\partial_{-\vec{p}} L_{eff}\partial_{\vec{p}} L_{eff} - \partial_{\vec{p}}\partial_{-\vec{p}} L_{eff}\right]\Bigg] e^{-S}\ . \tag{3.9}$$

This vanishes owing to (3.2).

The evolution equation (3.2) defines a family of lines, identified by the parameters $\rho_{0,a}$ of the bare lagrangian for a certain choice of the cut-off Λ_0, in the space of the field functionals L_{eff}. The running parameter along the lines is the cut-off Λ. However these lines do not describe the evolution toward the limit we are interested in. In order to prove the existence of an UV limit of our Green functional Z, we should rather study the evolution of our theory when Λ_0 increases and some "low energy" properties of the effective interaction are kept fixed. We can define these low energy properties by means of the six numbers:

$$\rho_a(\Lambda,\Lambda_0,\rho_0) = N_a L_{eff}\ , \tag{3.10}$$

where N_a $(a=1,..,6)$ are the normalization operators defined in (2.16)-(2.22). These effective parameters are in one-to-one correspondence with the parameters of the bare interaction and we assume that the functional relation between $\rho_{0,a}$ and ρ_a be invertible. This will certainly be true at least for a limited range of Λ. We denote by

$$\rho_{0,a}(\Lambda,\Lambda_0,\rho) \tag{3.11}$$

the inverse function of (3.10). Let us now consider the new field functional:

$$\begin{aligned}
&V(\Lambda) \equiv \Lambda_0\partial_{\Lambda_0} L_{eff}\left(K_\Lambda\phi,\omega,\Lambda,\Lambda_0,\rho_0(\Lambda,\Lambda_0,\rho)\right) = \\
&\Lambda_0\partial_{\Lambda_0} L_{eff} - \Lambda_0\partial_{\Lambda_0}\rho_a\partial_{\rho_a} L_{eff} = \\
&\Lambda_0\partial_{\Lambda_0} L_{eff}(K_\Lambda\phi,\omega,\Lambda,\Lambda_0,\rho_0) - \\
&\Lambda_0\partial_{\Lambda_0}\rho_a\left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \partial_{\rho_{0,b}} L_{eff}(K_\Lambda\phi,\omega,\Lambda,\Lambda_0,\rho_0)\ .
\end{aligned} \tag{3.12}$$

This functional gives an indication of the dependence of L_{eff} on the UV cut-off when the parameters ρ are kept fixed. We would like to show that V vanishes stronger than a positive power of $\frac{\Lambda}{\Lambda_0}$. This would clearly indicate that, once the low energy parameters are fixed, the whole theory tends toward a fixed point depending only on ρ. This is what is usually meant by renormalizability of the theory.

Let us then study the evolution equation of V. Given any field functional X, we define:

$$M[X] \equiv \sum_{\vec{p}} \Lambda\partial_\Lambda k\left(\frac{p}{\Lambda}\right) C^{-1}(p) k\left(\frac{p}{\Lambda}\right) \left[2\partial_{-\vec{p}} X \partial_{\vec{p}} L_{eff} - \partial_{\vec{p}}\partial_{-\vec{p}} X\right] , \tag{3.13}$$

where the action of $\partial_{\vec{p}}$ on X is defined in the same way as on L_{eff}.

From (3.2) we have:

$$\Lambda\partial_\Lambda \Lambda_0\partial_{\Lambda_0} L_{eff} = M\left[\Lambda_0\partial_{\Lambda_0} L_{eff}\right] . \tag{3.14}$$

This gives the evolution equation of the first term of V in (3.12). Considering the second term, on gets three contributions which can be computed taking into account that, by definition:

$$\Lambda\partial_\Lambda \rho_a = N_a\left[\Lambda\partial_\Lambda L_{eff}\right] . \tag{3.15}$$

The first contribution is

$$\begin{aligned}
&\Lambda\partial_\Lambda \Lambda_0\partial_{\Lambda_0}\rho_a \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \partial_{\rho_{0,b}} L_{eff} = \\
&N_a\left[M\left[\Lambda_0\partial_{\Lambda_0} L_{eff}\right]\right] \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \partial_{\rho_{0,b}} L_{eff} = \\
&N_a\left[M\left[\Lambda_0\partial_{\Lambda_0} L_{eff}\right]\right] \partial_{\rho_a} L_{eff} .
\end{aligned} \tag{3.16}$$

The second is:

$$\begin{aligned}
&\Lambda_0\partial_{\Lambda_0}\rho_a \Lambda\partial_\Lambda \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \partial_{\rho_{0,b}} L_{eff} = \\
&-\Lambda_0\partial_{\Lambda_0}\rho_a \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_c \Lambda\partial_\Lambda \frac{\partial\rho_d}{\partial\rho_{0,c}} \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^d_b \partial_{\rho_{0,b}} L_{eff} = \\
&-\Lambda_0\partial_{\Lambda_0}\rho_a \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_c N_d\left[M\left[\frac{\partial L_{eff}}{\partial\rho_{0,c}}\right]\right] \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^d_b \partial_{\rho_{0,b}} L_{eff} .
\end{aligned} \tag{3.17}$$

The third is:

$$\begin{aligned}
&\Lambda_0\partial_{\Lambda_0}\rho_a \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \Lambda\partial_\Lambda \partial_{\rho_{0,b}} L_{eff} = \\
&\Lambda_0\partial_{\Lambda_0}\rho_a \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b M\left[\frac{\partial L_{eff}}{\partial\rho_{0,b}}\right] .
\end{aligned} \tag{3.18}$$

Combining (3.14) and (3.18) and recalling that M is a linear operator we get $M[V]$. In much the same way (3.14) and (3.15) give $N_a\left[M[V]\right]\partial_{\rho_a} L_{eff}$. Altogether we get:

$$\Lambda\partial_\Lambda V = M[V] - N_a\left[M[V]\right]\partial_{\rho_a} L_{eff} . \tag{3.19}$$

In order to compute the initial condition to this evolution equation we notice that, when Λ tends to Λ_0, L_{eff} tends to L_0 and that

$$\rho(\Lambda_0, \Lambda_0, \rho_0) = \rho_0 \ . \tag{3.20}$$

Thus:

$$\begin{aligned} &L_{eff}\left(K_{\Lambda_0}\phi, \omega, \Lambda, \Lambda_0, \rho_0\right) = L_0\left(K_{\Lambda_0}\phi, \omega, \rho_0\right) + \\ &(\Lambda - \Lambda_0)\partial_\Lambda L_{eff}\left(K_{\Lambda_0}\phi, \omega, \Lambda, \Lambda_0, \rho_0\right)|_{\Lambda=\Lambda_0} + O\left((\Lambda-\Lambda_0)^2\right) \ . \end{aligned} \tag{3.21}$$

Then, taking the derivative of both members of (3.21) with respect to Λ_0 and setting $\Lambda = \Lambda_0$ we have:

$$\begin{aligned} &V[\Lambda_0] = \Lambda_0\partial_{\Lambda_0} L_{eff}\left(K_\Lambda\phi, \omega, \Lambda, \Lambda_0, \rho_0\right)|_{\Lambda=\Lambda_0} - \\ &\Lambda_0\partial_{\Lambda_0}\rho_a\partial_{\rho_a} L_{eff}\left(K_\Lambda\phi, \omega, \Lambda, \Lambda_0, \rho_0(\Lambda, \Lambda_0, \rho)\right|_{\Lambda=\Lambda_0} = \\ &-\Lambda\partial_\Lambda L_{eff}\left(K_{\Lambda_0}\phi, \omega, \Lambda, \Lambda_0, \rho_0\right)|_{\Lambda=\Lambda_0} + \\ &\Lambda\partial_\Lambda N_a\left[L_{eff}\left(K_{\Lambda_0}\phi, \omega, \Lambda, \Lambda_0, \rho_0\right)\right]|_{\Lambda=\Lambda_0}\partial_{\rho_a} L_0 = \\ &-\frac{1}{2}\sum_{\vec{p}} \Lambda_0\partial_{\Lambda_0} k^2\left(\left(\frac{p}{\Lambda_0}\right)^2\right) C^{-1}(p)\partial_{-\vec{p}} L_0 \partial_{\vec{p}} L_0 + \\ &\frac{1}{2} N_a\left[\sum_{\vec{p}} \Lambda_0\partial_{\Lambda_0} k^2\left(\left(\frac{p}{\Lambda_0}\right)^2\right) C^{-1}(p)\partial_{-\vec{p}} L_0 \partial_{\vec{p}} L_0\right]\partial_{\rho_{0,a}} L_0 + c \ , \end{aligned} \tag{3.22}$$

where c is field independent.

In deducing (3.22) we have taken into account that, for any integrated local functional X of dimension up to 4, one has:

$$X = N_a\left[X\right]\partial_{\rho_{0,a}} L_0 + c \ , \tag{3.23}$$

where c is again field independent. We still need the evolution equation of

$$\left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \partial_{\rho_{0,b}} L_{eff} \equiv \partial_{\rho_a} L_{eff} \ . \tag{3.24}$$

We have

$$\begin{aligned} &\Lambda\partial_\Lambda\partial_{\rho_a} L_{eff} = \left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_b \Lambda\partial_\Lambda\partial_{\rho_{0,b}} L_{eff} - \\ &\left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^a_c \Lambda\partial_\Lambda\frac{\partial\rho_d}{\partial\rho_{0,c}}\left[\left(\frac{\partial\rho}{\partial\rho_0}\right)^{-1}\right]^d_b \partial_{\rho_{0,b}} L_{eff} = \\ &M\left[\partial_{\rho_a} L_{eff}\right] - N_b\left[M\left[\partial_{\rho_a} L_{eff}\right]\right]\partial_{\rho_b} L_{eff} \ , \end{aligned} \tag{3.25}$$

and the initial condition is:

$$\partial_{\rho_a} L_{eff}|_{\Lambda=\Lambda_0} = \partial_{\rho_{0,a}} L_0 \ . \tag{3.26}$$

It remains to discuss the solutions of the system of differential equations (3.2), (3.25) and (3.19) whose corresponding initial conditions are L_0, (3.26) and (3.22).

A first step in this discussion consists in the infinite volume limit. This is harmless at the level of the effective lagrangian since the presence of a derivative of the cut-off function limits the range of momenta in (3.2) and (3.13) between Λ and 2Λ. This and the euclidean character of our theory exclude any IR singularity.

We define the reduced functional $\hat{L}_{eff}$ according:

$$L_{eff}(K_\Lambda \phi, \omega, \Lambda, \Lambda_0, \rho_0) = (L\Lambda)^4 \hat{L}_{eff}\left(\frac{K_\Lambda \phi}{\Lambda L^2}, \frac{\omega}{\Lambda^2 L^2}, \Lambda, \Lambda_0, \rho_0\right) \ . \tag{3.27}$$

Writing L_{eff} and $\hat{L}_{eff}$ as (formal) power series in $K_\Lambda \phi$ and ω whose coefficients, the effective vertices, are $\Gamma^{(n,m)}$ and $\hat{\Gamma}^{(n,m)}$. That is, for L_{eff}:

$$\begin{aligned} L_{eff} = &\sum_{n,m=0}^{\infty} \frac{1}{n!m!} \sum_{\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m} \delta_{\vec{0},\sum \vec{p}_i + \sum \vec{q}_j} \\ &\Gamma^{(n,m)}(\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m) K_\Lambda \phi_{\vec{p}_1}..K_\Lambda \phi_{\vec{p}_n} \omega_{\vec{q}_1}..\omega_{\vec{q}_m} \ , \end{aligned} \tag{3.28}$$

and a completely analogous equation for $\hat{L}_{eff}$. Translating the evolution equation of L_{eff}, (3.2), in terms of the effective vertices $\hat{\Gamma}$ of $\hat{L}_{eff}$, we get:

$$\begin{aligned} &(\Lambda\partial_\Lambda + 4 - n - 2m)\, \hat{\Gamma}^{(n,m)}(\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m) = \\ &\Lambda^2 \sum_{\vec{p}} \frac{1}{2} \Lambda\partial_\Lambda k^2\Big(\left(\frac{p}{\Lambda}\right)^2\Big) C^{-1}(p) \sum_{k=1}^{n}\sum_{l=1}^{m} \sum_{[i_1,..,i_k][i_{k+1},..,i_n]} \sum_{[j_1,..,j_l][j_{l+1},..,j_m]} \delta_{\vec{p},\sum \vec{p}_i + \sum \vec{q}_j} \\ &\hat{\Gamma}^{(k+1,l)}\left(\vec{p},\vec{p}_{i_1},..,\vec{p}_{i_k},\vec{q}_{j_1},..,\vec{q}_{j_l}\right) \hat{\Gamma}^{(n-k+1,m-l)}\left(-\vec{p},\vec{p}_{i_{l+1}},..,\vec{p}_{i_n},\vec{q}_{j_{l+1}},..,\vec{q}_{j_m}\right) - \\ &\sum_{\vec{p}} \frac{1}{2} \Lambda\partial_\Lambda k^2\Big(\left(\frac{p}{\Lambda}\right)^2\Big) C^{-1}(p) \frac{1}{\Lambda^2 L^4} \hat{\Gamma}^{(n+2,m)}(\vec{p},-\vec{p},\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m) \ . \end{aligned} \tag{3.29}$$

For $\Lambda >> \frac{1}{L}$ the sums in the right-hand side of (3.29) can be safely replaced with integrals whose measure is: $d^4p\left(\frac{L}{2\pi}\right)^4$. In the same limit the Kronecker $\delta_{\vec{p},\sum \vec{p}_i + \sum \vec{q}_j}$ has to be replaced by the Dirac measure: $\left(\frac{2\pi}{L}\right)^4 \delta(\vec{p} - \sum \vec{p}_i - \sum \vec{q}_j)$. This yields:

$$\begin{aligned} &(\Lambda\partial_\Lambda + 4 - n - 2m)\, \hat{\Gamma}^{(n,m)}(\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m) = \\ &\frac{1}{2}\int \frac{d^4p}{(2\pi\Lambda)^4}\, \vec{p}\cdot\vec{\partial}_p k^2\left(\left(\frac{p}{\Lambda}\right)^2\right) \frac{\Lambda^2}{p^2+m^2} \hat{\Gamma}^{(n+2,m)}(\vec{p},-\vec{p},\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m) - \\ &\frac{1}{2}\int d^4p\ \vec{p}\cdot\vec{\partial}_p k^2\left(\left(\frac{p}{\Lambda}\right)^2\right) \frac{\Lambda^2}{p^2+m^2} \sum_{k=1}^{n}\sum_{l=1}^{m} \sum_{[i_1,..,i_k][i_{k+1},..,i_n]} \\ &\sum_{[j_1,..,j_l][j_{l+1},..,j_m]} \delta(\vec{p} - \sum \vec{p}_i - \sum \vec{q}_j) \hat{\Gamma}^{(k+1,l)}\left(\vec{p},\vec{p}_{i_1},..,\vec{p}_{i_k},\vec{q}_{j_1},..,\vec{q}_{j_l}\right) \\ &\hat{\Gamma}^{(n-k+1,m-l)}\left(-\vec{p},\vec{p}_{i_{l+1}},..,\vec{p}_{i_n},\vec{q}_{j_{l+1}},..,\vec{q}_{j_m}\right) \ . \end{aligned} \tag{3.30}$$

Now, rescaling the integration variable, by setting:

$$\hat{\Gamma}^{(n,m)}(\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m) = \gamma^{(n,m)}\left(\frac{\vec{p}_1}{\Lambda},..,\frac{\vec{p}_n}{\Lambda},\frac{\vec{q}_1}{\Lambda},..,\frac{\vec{q}_m}{\Lambda}\right) \ , \tag{3.31}$$

we can make the right-hand side of (3.30) Λ-independent at high momenta ($p^2 >> m^2$).

With the purpose of analysing perturbatively our approach, we reintroduce into the evolution equation for $\gamma^{(n,m)}$ the loop counting parameter $\hbar$ appearing in (2.25). This is done multiplying by a factor $\frac{1}{\hbar}$ both the covariance C and the interaction L_{eff}: we obtain:

$$\begin{aligned}
&(\Lambda\partial_\Lambda + 4 - n - 2m - D)\,\gamma^{(n,m)} = -\frac{1}{2}\int d^4p \frac{1}{p^2+\frac{m^2}{\Lambda^2}}\ \vec{p}\cdot\vec{\partial}_p k^2(p^2) \\
&\sum_{k=1}^{n}\sum_{l=1}^{m}\sum_{[i_1,..,i_k][i_{k+1},..,i_n]}\ \sum_{[j_1,..,j_l][j_{l+1},..,j_m]} \delta(\vec{p}-\sum\vec{p}_i-\sum\vec{q}_j) \\
&\gamma^{(k+1,l)}\left(\vec{p},\vec{p}_{i_1},..,\vec{p}_{i_k},\vec{q}_{j_1},..,\vec{q}_{j_l}\right)\gamma^{(n-k+1,m-l)}\left(-\vec{p},\vec{p}_{i_{l+1}},..,\vec{p}_{i_n},\vec{q}_{j_{l+1}},..,\vec{q}_{j_m}\right) + \\
&\frac{\hbar}{2}\int\frac{d^4p}{(2\pi)^4}\ \vec{p}\cdot\vec{\partial}_p k^2(p^2)\frac{1}{p^2+\frac{m^2}{\Lambda^2}}\gamma^{(n+2,m)}(\vec{p},-\vec{p},\vec{p}_1,..,\vec{p}_n,\vec{q}_1,..,\vec{q}_m)\ ,
\end{aligned} \tag{3.32}$$

where D is the dilation operator:

$$D = \sum_i \vec{p}_i\cdot\vec{\partial}_{p_i} + \sum_j \vec{q}_j\cdot\vec{\partial}_{q_j}\ . \tag{3.33}$$

To proceed further, it is convenient to introduce the functional generator l_{eff} of the vertices $\gamma^{(n,m)}$, that is:

$$\begin{aligned}
&l_{eff}[\phi,\omega] = \sum_{n=0,m=0}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n}\left(d^4p_i\phi(\vec{p}_i)\right) \\
&\prod_{j=1}^{m}\left(d^4q_j\omega(\vec{q}_j)\right)(2\pi)^4\delta(\sum\vec{p}_i+\sum\vec{q}_j) \\
&\gamma^{(n,m)}\left(\vec{p}_{i_1},..,\vec{p}_{i_n},\vec{q}_{j_1},..,\vec{q}_{j_m}\right)\ .
\end{aligned} \tag{3.34}$$

Notice that the whole set of transformations which has led us from L_{eff} to l_{eff} consists in the introduction of dimensionless variables combined with the infinite volume limit. Indeed setting:

$$\phi(x) \equiv \Lambda\hat{\phi}(\Lambda x)\ ,\ \omega(x) \equiv \Lambda^2\hat{\omega}(\Lambda x)\ , \tag{3.35}$$

we have set:

$$L_{eff}(K_\Lambda\phi,\omega,\Lambda,\Lambda_0,\rho_0) = l_{eff}\left(K_\Lambda\hat{\phi},\hat{\omega},\Lambda,\Lambda_0,\rho_0\right)\ . \tag{3.36}$$

Introducing;

$$\Delta'(p) = \vec{p}\cdot\vec{\partial}_p k^2(p^2)\frac{1}{p^2+\frac{m^2}{\Lambda^2}}\ , \tag{3.37}$$

and the functional differential operators:

$$\mathcal{N} = \int d^4p\ \phi(\vec{p})\frac{\delta}{\delta\phi(\vec{p})}\ , \tag{3.38}$$

$$\mathcal{M} = \int d^4p\ \omega(\vec{p})\frac{\delta}{\delta\omega(\vec{p})}\ , \tag{3.39}$$

$$\mathcal{D} = \int d^4p\left[\phi(\vec{p})\vec{p}\cdot\vec{\partial}_p\frac{\delta}{\delta\phi(\vec{p})} + \omega(\vec{p})\vec{p}\cdot\vec{\partial}_p\frac{\delta}{\delta\omega(\vec{p})}\right]\ , \tag{3.40}$$

we can write (3.32) in the form:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D}\right] e^{\frac{-l_{eff}}{\hbar}} = \frac{\hbar}{2}\int d^4p \frac{\Delta'(p)}{(2\pi)^4}\frac{\delta}{\delta\phi(\vec{p})}\frac{\delta}{\delta\phi(-\vec{p})} e^{\frac{-l_{eff}}{\hbar}} \ . \tag{3.41}$$

Representing graphically by blobs (vertices) with n legs the n-th functional derivative of l_{eff} and with a solid line $\Delta'(p)$, we see that the first term in the right-hand side of (3.32) can be represented by the fusion of two vertices into a single one joining two lines, while, in the second term, two lines of the same vertex joining together form a loop.

$$\left[\Lambda\tfrac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D}\right] \qquad = -\tfrac{1}{2}\int \tfrac{d^4p}{(2\pi)^4}\left[\quad \big| \quad -\hbar \qquad\qquad \right] \ . \tag{3.42}$$

The infinite volume limit and momentum scale transformations performed above can be also applied to $\partial_{\rho_a} L_{eff}$ and to $V(\Lambda)$ giving $\partial_{\rho_a} l_{eff}$ and:

$$v(\Lambda) = \Lambda_0 \partial_{\Lambda_0} l_{eff}|_{\rho,\ \Lambda} \ . \tag{3.43}$$

The evolution equations of these functionals are easily deduced from (3.19) and (3.25), by replacing $\Lambda\partial_\Lambda$ with $\Lambda\frac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D}$ and the operator M with m defined by:

$$m[X] \equiv \int d^4p \frac{\Delta'(p)}{(2\pi)^4}\left[\frac{\delta}{\delta\phi(\vec{p})}X\frac{\delta}{\delta\phi(-\vec{p})}l_{eff} - \frac{\hbar}{2}\frac{\delta}{\delta\phi(\vec{p})}\frac{\delta}{\delta\phi(-\vec{p})}X\right] \ . \tag{3.44}$$

The normalization functionals N_a are also changed, owing to the fact that, after the infinite volume limit, the momentum $\vec{p}$ is no longer an index but a continuous variable. One has also to take into account that both the fields and the effective lagrangian have been rescaled. Therefore, after this limit, one has in particular:

$$N_1 = 2^8 \int d^4p \frac{1}{(2\pi)^4}\frac{\delta^4}{\delta\phi^4(\vec{p})} \ , \tag{3.45}$$

$$N_2 = 2\vec{\partial}_q \cdot \vec{\partial}_q \int d^4p \frac{1}{(2\pi)^4}\frac{\delta}{\delta\phi(\vec{p}+\vec{q})}\frac{\delta}{\delta\phi(\vec{p}-\vec{q})}|_{\vec{q}=0} \ , \tag{3.46}$$

while one has:

$$N_3 = 2^4\Lambda^2 \int d^4p \frac{1}{(2\pi)^4}\frac{\delta^2}{\delta\phi^2(\vec{p})} \equiv \Lambda^2 n_3 \ , \tag{3.47}$$

and in general a factor Λ^{d_a} appears in the normalization operator of the parameter ρ_a with mass dimension d_a.

Therefore the normalization conditions become

$$\gamma^{(4,0)}(0) = \rho_1 \quad , \quad \partial_{p^2}\gamma^{(2,0)}(0) = \rho_2 \quad , \quad \gamma^{(2,0)}(0) = \frac{\rho_1}{\Lambda^2} \ , \tag{3.48}$$

and

$$\gamma^{(2,1)}(0) = 2\rho_4 \quad , \quad \gamma^{(0,2)}(0) = 2\rho_5 \quad , \quad \gamma^{(0,1)}(0) = \frac{\rho_6}{\Lambda^2} \ . \tag{3.49}$$

Unluckily, until now, nobody has been able to solve our equations in any interesting case in 4 dimensions. The analytical Wilson renormalization group method has been successfully applied only "near" two dimensions [8] . What we are going to do in the following is to apply to our equations a purely dimensional analysis and, on this basis, to discuss their pertubative solution.

4. Analysis of the perturbative solution.

We shall now briefly discuss the perturbative solution to (3.41).

In the perturbative framework the effective lagrangian and its initial value L_0 are considered as a formal power series in $\hbar$. Therefore in particular the bare constants ρ_0 and the vertices Γ, $\hat{\Gamma}$ and γ are formal power series. In this framework the solution to (3.41) can be directly constructed as follows.

We notice, first of all, that the initial value $l_0(\Lambda_0)$ of l_{eff} is directly obtainable by applying to (2.15) the transformations (3.27) and (3.31) which yield:

$$l_0(\Lambda) = \int d^4x \left[\frac{\lambda_0}{4!}\hat{\phi}^4 + \frac{z_0-1}{2}(\partial\hat{\phi})^2 + \frac{m_0^2-m^2}{2\Lambda^2}\hat{\phi}^2 + \zeta_0\hat{\omega}\hat{\phi}^2 + \eta_0\hat{\omega}^2 + \frac{\xi_0}{\Lambda^2}\hat{\omega}\right] , \qquad (4.1)$$

where we have used the Fourier transformed variables:

$$\hat{\phi}(\vec{x}) = \int d^4p e^{-i(\vec{p}\cdot\vec{x})}\phi(\vec{p}) \; . \qquad (4.2)$$

We also introduce the cut-off propagator:

$$\Delta(p) = \frac{k^2(\frac{p\Lambda}{\Lambda_0}) - k^2(p)}{p^2 + \frac{m^2}{\Lambda^2}} \; . \qquad (4.3)$$

Now it is sufficient to notice that:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D}\right] l_0(\Lambda) = 0 \; , \qquad (4.4)$$

and the commutation relation:

$$\left[\left[\Lambda\frac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D}\right], \int d^4p\frac{\Delta(p)}{(2\pi)^4}\frac{\delta}{\delta\phi(\vec{p})}\frac{\delta}{\delta\phi(-\vec{p})}\right] =$$
$$\int d^4p\frac{\Delta'(p)}{(2\pi)^4}\frac{\delta}{\delta\phi(\vec{p})}\frac{\delta}{\delta\phi(-\vec{p})} \; , \qquad (4.5)$$

to prove that;

$$l_{eff} = -\hbar \; ln\left[e^{\frac{\hbar}{2}\int d^4p\frac{\Delta(p)}{(2\pi)^4}\frac{\delta}{\delta\phi(\vec{p})}\frac{\delta}{\delta\phi(-\vec{p})}}e^{-\frac{l_0(\Lambda)}{\hbar}}\right] \; , \qquad (4.6)$$

is the solution of (3.41) with initial value $l_0(\Lambda_0)$. It is also easy to verify that l_{eff} is the functional generator of the connected and amputated Feynman amplitudes corresponding to the propagator $\Delta(p)$ and to the vertices described by l_0.

Using this result one could try to study the existence of an UV limit of the perturbative theory. However we shall see in the following that, in order to have a regular UV limit, we have to consider as fixed parameters the renormalized ρ's, given in (3.10) computed at $\Lambda = \Lambda_R$.

From (4.6) we find in particular:

$$\gamma^{(2,0)}(p^2) = \frac{\rho_{0,2}p^2 + \frac{\rho_{0,3}}{\Lambda^2}}{1 + \left(\rho_{0,2}p^2 + \frac{\rho_{0,3}}{\Lambda^2}\right)\Delta(p^2)} + O(\hbar) ,$$
$$\gamma^{(4,0)} = \rho_{0,1} + O(\hbar) , \tag{4.7}$$

and

$$\gamma^{(6,0)}(\vec{p}_1, .., \vec{p}_6) \sum_{i_1 \geq i_2 \geq i_3 = 1}^{6} \rho_{0,1}^2 \Delta\left((\vec{p}_{i_1} + \vec{p}_{i_2} + \vec{p}_{i_3})^2\right) . \tag{4.8}$$

The analysis of the UV limit will develop along the following lines. We shall introduce a suitable class of norms for l_{eff} and its Λ_0-derivative $v(\Lambda)$ in the new parametrization. We then show that, for Λ_0 large with respect to Λ_R and Λ and at any finite perturbative order, this functional vanishes in the UV limit faster than $\left(\frac{\Lambda}{\Lambda_0}\right)^{2-\epsilon}$ times l_{eff} for any positive ϵ. This will immediately imply the UV convergence of the theory. Now (4.6) is explicitly parametrized in terms of the bare constants which, written as functions of ρ_R, coincide with the counter-terms associated with the corresponding vertices. These are $\hbar$-ordered series. Thus the reparametrization of l_{eff} in terms of the renormalized constants is equivalent to the introduction of a subtraction procedure. This is however extremely cumbersome. In particular checking the mechanism of subtraction of the UV divergent contributions of subdiagrams with the corresponding counter-terms, requires a detailed analysis of the so-called overlapping divergences. The advantage of the method presented here, which is inspired by the work of Polchinski, is that, using explicitly the evolution equations (3.2) and (3.19), one can reach the same results without any diagrammatic analysis.

Concerning the just mentioned change of parametrization, let us notice that, as shown above in the tree approximation, the two-field vertex $\gamma^{(2,0)}$ involves tree parameters, m, ρ_2 and ρ_3. However the quantity that is physically relevant is the correlation length that in our approach turns out to be a function of the above mentioned parameters. This seems to leave a wide freedom in particular in the choice of ρ_2 and ρ_3 that can be compensated by that of the covariance $m^2 + p^2$, leaving the correlation length unchanged.

However we shall see in a moment that in a loop ordered perturbation theory, to simplify the recursive solution of the evolution equations, it is highly convenient to choose the two-field vertex $\Gamma^{(2,0)}$ vanishing at the tree level. This means that ρ_2 and ρ_3 are not "really" free parameters. They must be of order $\hbar$. This assumption is perfectly compatible with our evolution equations from which we can compute, in the case of a sharp cut-off

$$\rho_2 = \rho_{0,2} - \frac{\hbar}{2(4\pi)^2}\rho_{0,1} , \tag{4.9}$$

$$\rho_3 = \rho_{0,3} + O(\hbar^2) \ , \tag{4.10}$$

$$\rho_1 = \rho_{0,1} + O(\hbar)\rho_{0,1}^2 \ . \tag{4.11}$$

It is also clear from (4.7) and this result that the jacobian matrix $\left(\frac{\partial \rho}{\partial \rho_0}\right)$ and the functional $\partial_\rho l_{eff}$ remain perfectly regular after this choice.

Owing to the nature of the normalization conditions, we have to study the Λ dependence of the vertices generated by l_{eff}, $\partial_{\rho_a} l_{eff}$ and v together with their momentum derivatives.

The evolution equations for the momentum derivatives of the vertices are immediately deducible from (3.32). One can verify in particular from this equation how the derivatives distribute in its the right-hand side. For purely formal reasons we briefly discuss here how the insertion of momentum derivatives can be performed at the functional level. This point, that is not crucial to the study of the perturbative UV limit, is motivated by the formal need to translate the whole normalization group action into a finite number of differential equations for the functional generators.

To define these momentum derivatives within the functional framework we introduce the family of differential operators:

$$D_{\nu,n,m}\left(\vec{p}_1, \ldots, \vec{p}_{n+m}\right) = P_\nu\left(\vec{\partial}_{p_k} - \vec{\partial}_{p_l}\right) \prod_{i=1}^{n} \frac{\delta}{\delta\phi(\vec{p}_i)} \prod_{j=n+1}^{n+m} \frac{\delta}{\delta\omega(\vec{p}_j)} \ , \tag{4.12}$$

where P_ν is a polynomial of degree $d(\nu)$. Considering in particular the evolution equations for $D_\nu l_{eff}$ one finds that, due to the non-linearity of (3.32), these equations contain a great number of terms. Let us consider for example:

$$\vec{D}_{1,2,0}\left(\vec{p}_1, \vec{p}_2\right) = \left(\vec{\partial}_{p_1} - \vec{\partial}_{p_2}\right) \frac{\delta}{\delta\phi(\vec{p}_1)} \frac{\delta}{\delta\phi(\vec{p}_2)} \ . \tag{4.13}$$

and notice that for any pair of functionals $\mathcal{A}$ and $\mathcal{B}$ one has:

$$\begin{aligned}
&\vec{D}_{1,2,0}\left(\vec{p}_1, \vec{p}_2\right) \int d^4p \frac{\Delta'(p)}{(2\pi)^4} \frac{\delta}{\delta\phi(\vec{p})} \mathcal{A} \frac{\delta}{\delta\phi(-\vec{p})} \mathcal{B} = \\
&\int d^4p \frac{\Delta'(p)}{(2\pi)^4} \frac{\delta}{\delta\phi(\vec{p})} \vec{D}_{1,2,0}\left(\vec{p}_1, \vec{p}_2\right) \mathcal{A} \frac{\delta}{\delta\phi(-\vec{p})} \mathcal{B} + \\
&\int d^4p \frac{\Delta'(p)}{(2\pi)^4} \vec{D}_{1,2,0}\left(\vec{p}_1, \vec{p}\right) \mathcal{A} \frac{\delta}{\delta\phi(\vec{p}_2)} \frac{\delta}{\delta\phi(-\vec{p})} \mathcal{B} + \\
&\int d^4p \frac{\Delta'(p)}{(2\pi)^4} \frac{\delta}{\delta\phi(\vec{p}_1)} \frac{\delta}{\delta\phi(\vec{p})} \mathcal{A} \vec{D}_{1,2,0}\left(-\vec{p}, \vec{p}_2\right) \mathcal{B} - \\
&\int d^4p \vec{\partial}_p \frac{\Delta'(p)}{(2\pi)^4} \frac{\delta}{\delta\phi(\vec{p}_1)} \frac{\delta}{\delta\phi(\vec{p})} \mathcal{A} \frac{\delta}{\delta\phi(\vec{p}_2)} \frac{\delta}{\delta\phi(-\vec{p})} \mathcal{B} + (\mathcal{B} \leftrightarrow \mathcal{A}) \ .
\end{aligned} \tag{4.14}$$

Iterating the same equation, we see that for a generic operator D_ν:

$$\begin{aligned}
&D_{\nu,n,m} \int d^4p \frac{\Delta'(p)}{(2\pi)^4} \frac{\delta}{\delta\phi(\vec{p})} \mathcal{A} \frac{\delta}{\delta\phi(-\vec{p})} \mathcal{B} = \\
&\sum_{\rho,\sigma,\tau,k,l} \int d^4p P_\rho(\vec{\partial}_p) \frac{\Delta'(p)}{(2\pi)^4} D_{\sigma,k+1,l} \mathcal{A} D_{\tau,n+1-k,m-l} \mathcal{B} \ ,
\end{aligned} \tag{4.15}$$

where the sum is restricted to the differential operators and polynomials for which:

$$d(\rho) + d(\sigma) + d(\tau) = d(\nu) \ . \tag{4.16}$$

We have thus the evolution equation:

$$\begin{aligned}&\left[\Lambda\frac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D} - n - 2m - d(\nu)\right] D_{\nu,n,m} l_{eff} = \\ &\frac{\hbar}{2}\int d^4p \frac{\Delta'(p)}{(2\pi)^4} D_{\nu,n,m} \frac{\delta}{\delta\phi(\vec{p})}\frac{\delta}{\delta\phi(-\vec{p})} l_{eff} - \\ &\frac{1}{2}\sum_{\rho,\sigma,\tau,k,l}\int d^4p P_\rho(\vec{\partial}_p)\frac{\Delta'(p)}{(2\pi)^4} D_{\sigma,k+1,l} l_{eff} D_{\tau,n+1-k,m-l} l_{eff} \ ,\end{aligned} \tag{4.17}$$

where again the sum in the right-hand side is restricted by (4.16). Analogous equations hold true for $\partial_{\rho_a} l_{eff}$ and v.

This completes the construction of the evolution equations. To study their solutions we have to translate these evolution equations for the generating functional into those for the corresponding N field and M operator vertices. For this we have to clarify a technical point. Since the functional generators are translation invariant, the corresponding vertices are proportional to a Dirac delta function in the sum of the external leg momenta. To get rid of this delta function we notice that, given any translation invariant functional $\mathcal{A}$, the (ν, n, m) momentum derivative of the corresponding (N, M) vertex is given by

$$\begin{aligned}&D_{\nu,n,m}\mathcal{A}^{(N,M)}(\vec{p}_1, ..., \vec{p}_{n+m}) = (N+M)^4 \int \frac{d^4k}{(2\pi)^4} \prod_{i=n+m}^{n+m+N} \frac{\delta}{\delta\phi(\vec{p}_i + \vec{k})} \\ &\prod_{j=n+m+N+1}^{n+m+N+M} \frac{\delta}{\delta\omega(\vec{p}_j + \vec{k})} D_{\nu,n,m}\left(\vec{p}_1 + \vec{k}, ..., \vec{p}_{n+m} + \vec{k}\right) \mathcal{A}|_{\phi=\omega=0} \ .\end{aligned} \tag{4.18}$$

For these vertices we define the norms:

$$\|D_{\nu,n,m}\mathcal{A}\|^{(N,M)} \equiv sup_{p_i^2<2, \sum \vec{p}_i = 0} |D_{\nu,n,m}\mathcal{A}^{(N,M)}| \ . \tag{4.19}$$

The value of these norms in the tree approximation can be easily computed from (4.6), (4.7) and (4.8) from which it turns out that the norms of l_{eff} can be bounded above and below by constants for a generic choice of the parameters ρ and for:

$$\Lambda_R \leq \Lambda << \Lambda_0 \ . \tag{4.20}$$

Furthermore from (4.6) and (3,42) one easily finds that:

$$\|v(\Lambda)\|^{(N,M)} < C_{N,M}\left(\frac{\Lambda}{\Lambda_0}\right)^2 \|l_{eff}\|^{(N,M)} \ . \tag{4.21}$$

Notice that the condition $\Lambda_0 >> \Lambda$ is needed to guarantee the assumed lower bound on the vertices; indeed when Λ tends to Λ_0 almost all of them vanish together with the propagator (4.3). In the following we shall forget (4.21) whenever we discuss upper bounds.

To push our analysis to all orders, the first step is to evaluate the norms of l_{eff} and $\partial_\rho l_{eff}$. We shall then discuss those of v.

We study the evolution equations recursively for increasing order (k) in $\hbar$ and degrees (N and M) in the fields.

From now on we shall label by the index k the k^{th} order term of a vertex.

For l_{eff} we assume the induction hypothesis that:

$$\|D_\nu \gamma_k\|^{(N,M)} < P_k\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right) \ , \tag{4.22}$$

where P_k is a polynomial of degree k with positive cofficients. (4.22) is, of course, verified for $k = 0$.

Let us notice now that the essential simplification introduced by perturbation theory consists in the possibility of performing a recursive analysis of (3.32) for increasing loop order and for any given order in $\hbar$ increasing N and M starting from $(2,0)$. This is possible since the right-hand side of the evolution equation for the (N,M) vertex at the perturbative order k depends on the $(N-2,M)$ and $(N,M-1)$ vertices at the same order, while the other vertices appearing in it have lower orders. Indeed, as already noticed, the first term in the right-hand side of (3.32) corresponds to the product of two vertices joined by one line, this has a tree graph structure, hence the sum of the orders of the two vertices is k, the total number of ϕ-legs is $N+2$, that of ω-legs is M. Owing to the particular choice of the parametrization discussed above, which implies the vanishing of the $(2,0)$ vertex in the tree approximation, this first term does not contain vertices of order k with more than $N-1$ ϕ-legs and M ω-legs. In the second term in the right-hand side of (3,32) there appears only the $(N+2,M)$ vertex at the order $k-1$.

A further remark which is essential to obtain an upper bound for the absolute value of the right-hand side of (3.32) at each recursive step, is that $\Delta'(p)$ is absolutely bounded together with all its derivatives and that it vanishes identically outside the hypersphere of radius 2 and inside that of radius 1.

Taking into account these remarks and (4.22) we conclude that at the order k the absolute value of the right-hand side of (3.32) is bounded by a polynomial of degree k in $ln\left(\frac{\Lambda}{\Lambda_R}\right)$.

Coming to the study of the single vertices, let us consider, first of all:

$$\begin{aligned}
&D_4^{\mu\nu\rho\sigma}\gamma_{k+1}^{(2,0)} \equiv \partial_p^\mu\partial_p^\nu\partial_p^\rho\partial_p^\sigma\gamma_{k+1}^{(2,0)}(\vec{p}) = \\
&4\left(\delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\rho}\delta^{\sigma\nu} + \delta^{\mu\sigma}\delta^{\rho\nu}\right)\gamma_{k+1}^{(2,0)}(p^2)^{(ii)} + \\
&8\left(\delta^{\mu\nu}p^\rho p^\sigma + \delta^{\mu\rho}p^\sigma p^\nu + \delta^{\mu\sigma}p^\rho p^\nu + \delta^{\rho\sigma}p^\mu p^\nu + \delta^{\sigma\nu}p^\mu p^\rho + \right. \\
&\left.\delta^{\rho\nu}p^\mu p^\sigma\right)\gamma_{k+1}^{(2,0)}(p^2)^{(iii)} + 16 p^\mu p^\nu p^\rho p^\sigma \gamma_{k+1}^{(2,0)}(p^2)^{(iv)} \ ,
\end{aligned} \tag{4.23}$$

where $\gamma_{k+1}^{(2,0)}(p^2)^{(x)}$ stays for the x^{th} derivative of $\gamma_{k+1}^{(2,0)}(p^2)$ with respect to p^2. We have:

$$\|\left(\Lambda\frac{\partial}{\partial\Lambda} - D - 2\right) D_4\gamma_{k+1}\|^{(2,0)} < P_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right) \ , \tag{4.24}$$

which has to be integrated with the initial condition:

$$D_4\gamma_{k+1}^{(2,0)}|_{\Lambda=\Lambda_0} = 0 \ . \tag{4.25}$$

Now we can integrate (4.25) as follows. Setting:

$$D_4\gamma_{k+1}^{(2,0)}(\vec{p}) = \left(\frac{\Lambda}{\Lambda_0}\right)^2 F(\Lambda\vec{p},\Lambda)\ , \tag{4.26}$$

one has:

$$|\Lambda\frac{\partial}{\partial\Lambda}F| < \left(\frac{\Lambda_0}{\Lambda}\right)^2 P_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ , \tag{4.27}$$

then, integrating, one gets:

$$|F| < \left(\frac{\Lambda_0}{\Lambda}\right)^2 P'_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ , \tag{4.28}$$

and hence one recovers (4.22) for $D_4\gamma_{k+1}^{(2,0)}$.

It is important to notice, at this point, that the initial condition (4.25) has not been relevant for the final result (4.22). This is due to the fact that the initial condition contributes to the solution of (4.17) proportionally to that of the associated homogeneous equation (the one with vanishing right-hand side). Considering a momentum derivative of order d of the vertex (N,M) this homogeneous solution is proportional to $\left(\frac{\Lambda}{\Lambda_0}\right)^{N+2M+d-4}$: it is therefore irrelevant if the exponent is positive. Now we compute:

$$\begin{aligned}&\frac{1}{3!}\int dt(1-t)^3 p_\mu p_\nu p_\rho p_\sigma D_4^{\mu\nu\rho\sigma}\gamma_{k+1}^{(2,0)}(t\vec{p}) = \\ &\gamma_{k+1}^{(2,0)}(p^2) - \gamma_{k+1}^{(2,0)}(0) - p^2\gamma_{k+1}^{(2,0)(i)}(0)\ .\end{aligned} \tag{4.29}$$

Notice that, independently of the presence of a mass, our vertices are not affected with infra-red singularities since the propagator (4.3) is cut-off both at high and low momenta.

Then we get:

$$\|\gamma_{k+1}\|^{(2,0)} < P'_k\left(ln\left(\frac{\Lambda_0}{\Lambda}\right)\right) + |\gamma_{k+1}^{(2,0)}(0)| + 4|\gamma_{k+1}^{(2,0)(i)}(0)|\ . \tag{4.30}$$

The second and third term in the right-hand side can be computed using again the evolution equations. However now these equations have to be integrated between Λ_R and Λ since the value of both terms is fixed by the renormalization conditions (3.10) at Λ_R.

In much the same way as above we consider:

$$D_2^{\mu\nu}\gamma_{k+1}^{(2,0)}(0) \equiv \partial_p^\mu\partial_p^\nu\gamma_{k+1}^{(2,0)}(0) = 2\delta^{\mu\nu}\gamma_{k+1}^{(2,0)(i)}(0)\ , \tag{4.31}$$

for which the evolution equation gives:

$$|\Lambda\frac{\partial}{\partial\Lambda}D_2\gamma_{k+1}^{(2,0)}(0)| < P_k\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ , \tag{4.32}$$

T he right-hand side of this equation receives contributions only from the term proportional to $\hbar$ in (3.32) since $\Delta'(p)$ vanishes identically for $p<1$.

Integrating (4.32) between Λ_R and Λ and taking into account (4.31) yields:

$$|\gamma_{k+1}^{(2,0)(i)}(0)| < P_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ . \tag{4.33}$$

Notice that the degree of the polynomial P has increased by one. This will happen every time we shall integrate a vertex evolution equation whose left-hand side is of the form of that in (4.32).

In an analogous way we have:

$$|\left(\Lambda\frac{\partial}{\partial\Lambda}+2\right)\gamma_{k+1}^{(2,0)}(0)|<P_k\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ , \tag{4.34}$$

leading, after integration, to:

$$|\gamma_{k+1}^{(2,0)}(0)|<P"_k\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ . \tag{4.35}$$

Notice that here the normalization at Λ_R has an irrelevant influence on the value of the right-hand side.

Now, combining (4.30), (4.33) and (4.35), we get back the induction hypothesis at order $k+1$ for $\|l_{eff}\|^{(2,0)}$.

We have discussed in some detail this step of the recursive proof to give an example of the general strategy, which is essentially the following. For every (N,M) we consider a momentum-derivative of the vertex of high enough order to make it "convergent" (N+2M+the order of derivation larger than 4) and we compute this momentum derivative integrating the evolution equation between Λ and Λ_0 where it has to vanish.

The original vertex, or its derivatives of low order, are then computed from the derivative of higher order integrating it with respect to a momentum scale variable, as in (4.29). The resulting formula contains integration constants, corresponding to the values at zero momentum of the vertex and its derivatives of low order (its "divergent part"). These are computed integrating their evolution equations between Λ and Λ_R where they are assigned by the normalization conditions.

The procedure turns out to be much simpler in the case of "convergent" vertices ($N+2M>4$) which are determined directly by integrating the evolution equation between Λ and Λ_0 and in the case ($N=0\ ,\ M=1$) where it is sufficient to integrate between Λ and Λ_R.

The results that we have so far reached concerning l_{eff} could easily be resumed formulating a theorem. We avoid this assuming that the crucial steps of our study have been sufficiently clarified.

The above method can be employed to evaluate the norms of $\partial_\rho l_{eff}$ for which we assume the induction hypothesis that:

$$\|\partial_{\rho_a}D_\nu\gamma_k\|^{(N,M)}<\Lambda^{-d_a}P_k\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ , \tag{4.36}$$

where, as mentioned above, d_a is the mass dimension of the parameter ρ_a.

The validity of (4.36) in the tree approximation can be read directly from (4,1). To extend (4.36) to all orders we repeat the analysis described above for l_{eff} noticing that the initial conditions at Λ_R and Λ_0 for $\partial_{\rho_a}l_{eff}$ can be easily deduced from those for l_{eff}.

After infinite volume limit and momentum rescaling the evolution equation for $\partial_\rho l_{eff}$ becomes:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D}\right]\partial_{\rho_a} l_{eff} =$$
$$m[\partial_{\rho_a} l_{eff}] - \Lambda^{-d_b} n_b\left[m[\partial_{\rho_a} l_{eff}]\right]\partial_{\rho_b} l_{eff}\ . \tag{4.37}$$

Let us remark that in the second term of the right-hand side of this equation the factor Λ^{-d_b} coming from the normalization operator compensates Λ^{d_b} appearing in (4.36). Therefore the whole right-hand side turns out to be proportional to Λ^{-d_a}.

As above we start from the study of $\partial_{\rho_a} D_4 l_{eff}$ for which we have:

$$\|\left(\Lambda\frac{\partial}{\partial\Lambda} - D - 2\right)\partial_{\rho_a} D_4 l_{eff}\|^{(2,0)} < \Lambda^{-d_a} P_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ . \tag{4.38}$$

The integration between Λ and Λ_0 (4.38) leads back to (4.36) for $\partial_{\rho_a} D_4 l_{eff}$. It is now evident that if ρ_a is a dimensionless parameter ($d_a = 0$) the recursive proof of (4.36) is identical to that of (4.22). However, if $d_a > 0$, we have to change the strategy of our proof. Indeed, for example, the evolution equation of $\partial_{\rho_a} D_2 \gamma_{k+1}^{(2,0)}(0)$ yields:

$$|\Lambda\frac{\partial}{\partial\Lambda}\partial_{\rho_a} D_2 \gamma_{k+1}^{(2,0)}(0)| < \Lambda^{-d_a} P_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ . \tag{4.39}$$

Comparing this with (4.26) and (4.32) we see that integrating down from Λ_0 we can profit of the irrelevancy of the high energy value in the present situation. Thus we get:

$$|\partial_{\rho_a} D_2 \gamma_{k+1}^{(2,0)}(0)| < \Lambda^{-d_a} P'_{k+1}\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ . \tag{4.40}$$

In much the same way we can show that:

$$|\partial_{\rho_a} \gamma_{k+1}^{(2,0)}(0)| < \Lambda^{-d_a} P_{k+1}"\left(ln\left(\frac{\Lambda}{\Lambda_R}\right)\right)\ . \tag{4.41}$$

Finally, using all these results to compute $\partial_{\rho_a}\gamma_{k+1}^{(2,0)}(p^2)$, by means of an integration procedure completely analogous to (4,25), we prove the validity of the induction hypothesis for this derivative. Following the same strategy, one can study $\partial_{\rho_a}\gamma_{k+1}^{(4,0)}$ starting from a suitable second derivative.

Then, increasing step by step N and M, one can prove (4.36) for all the coefficients of $\partial_{\rho_a} l_{eff}$ at the order $k+1$ completing the proof of this hypothesis.

We now come to the study of v defined in (3.43) and notice, first of all, that in its evolution equation every factor Λ^{d_a} coming from a normalization operator compensates the one appearing in the norms of $\partial_{\rho_a} l_{eff}$.

In much the same way as in the already discussed cases, the analysis begins in the tree approximation where v can be computed directly from (3.43) and (4.6)-(4.11). Indeed, in this approximation, l_{eff} depends on Λ_0 only through the propagator (4.3) since the bare and renormalized parameters coincide. Now we have:

$$\Lambda_0\frac{\partial}{\partial\Lambda_0}\Delta(p^2) = -\frac{1}{p^2+\frac{m^2}{\Lambda^2}}\vec{p}\cdot\vec{\partial}_p k^2\left(\left(\frac{p\Lambda}{\Lambda_0}\right)^2\right) = -\left(\frac{\Lambda}{\Lambda_0}\right)^2\Delta'\left(\frac{p\Lambda}{\Lambda_0}\right)\ , \tag{4.42}$$

thus:

$$sup|\Lambda_0 \frac{\partial}{\partial \Lambda_0} \Delta| = \left(\frac{\Lambda}{\Lambda_0}\right)^2 sup|\Delta'| \ . \tag{4.43}$$

One has also to notice that a cut at low momenta appears in $\Lambda_0 \frac{\partial}{\partial \Lambda_0} \Delta$ at $p = \frac{\Lambda_0}{\Lambda}$ (while in Δ it appears at $p = 1$), therefore the Λ_0-derivative of a vertex vanishes unless the momenta of a great number of external momenta add together to overcome this cut. Thus this derivative does not vanish only for vertices whose external legs increase proportionally to $\frac{\Lambda_0}{\Lambda}$.

Anyway, recalling the structure of the tree diagrams and (4.38), we can conclude that:

$$\|D_\nu v_0\|^{(N,M)} < C_{N,M,\nu} \left(\frac{\Lambda}{\Lambda_0}\right)^2 \ . \tag{4.44}$$

In order to study the higher perturbative orders using, as above, the evolution equation, we have to discuss, first of all, the initial condition at $\Lambda = \Lambda_0$. Before the infinite volume limit this is given in (3.22). After this limit and the rescaling of momenta shown in (3.31) we have:

$$\begin{aligned} v(\Lambda_0) = \frac{1}{2} \int d^4p \frac{\Delta'(p)}{(2\pi)^4} \Bigg[& \frac{\delta}{\delta\phi(\vec{p})} l_0(\rho_0) \frac{\delta}{\delta\phi(-\vec{p})} l_0(\rho_0) - \\ N_a \left[\frac{\delta}{\delta\phi(\vec{p})} l_0(\rho_0) \frac{\delta}{\delta\phi(-\vec{p})} l_0(\rho_0) \right] & \partial_{\rho_a} l_0(\rho_0) \Bigg] + \hbar C \ , \end{aligned} \tag{4.45}$$

where C is field independent. Recalling (4.22) we find for this initial condition the upper bound:

$$\|D_\nu v_k(\Lambda_0)\|^{(N,M)} < P_k \left(ln \left(\frac{\Lambda_0}{\Lambda_R} \right) \right) \ . \tag{4.46}$$

It is now possible to apply the evolution equations for v repeating the analysis already developed for l_{eff} and $\partial_{\rho_a} l_{eff}$. In the present case the recursive procedure will be based on the induction hypothesis:

$$\|D_\nu v_k\|^{(N,M)} < \left(\frac{\Lambda}{\Lambda_0}\right)^2 P_k \left(ln \left(\frac{\Lambda_0}{\Lambda_R} \right) \right) \ , \tag{4.47}$$

which is suggested by (4.44) and by the fact that the initial condition at Λ_0 gives irrelevant contributions to v (in this case irrelevant means of order $\left(\frac{\Lambda}{\Lambda_0}\right)^2$).

Considering the right-hand side of the evolution equation of v, that has essentially the same structure of that of l_{eff} but is linear in v, we find that:

$$\begin{aligned} \| \left[\Lambda \frac{\partial}{\partial \Lambda} - \mathcal{N} - 2\mathcal{M} - \mathcal{D} - n - 2m - d(\nu) \right] D_{\nu,n,m} v_{k+1} \|^{(N,M)} < \\ \left(\frac{\Lambda}{\Lambda_0}\right)^2 P_{k+1} \left(ln \left(\frac{\Lambda_0}{\Lambda_R} \right) \right) \ . \end{aligned} \tag{4.48}$$

We then begin, order by order and in complete analogy with l_{eff}, from $D_4 v_k$, with D_4 defined in (4.23), and we find:

$$\|D_4 v_{k+1}\|^{(N,M)} < \left(\frac{\Lambda}{\Lambda_0}\right)^2 P'_{k+1} \left(ln \left(\frac{\Lambda_0}{\Lambda_R} \right) \right) \ . \tag{4.49}$$

Notice that here the initial value at Λ_0 gives a relevant contribution to (4.49).

Applying on $D_4 v^{(2,0)}_{k+1}$ the integration procedure exhibited in (4.29) we find directly $\|v_{k+1}\|^{(2,0)}$ since $v^{(2,0)}_{k+1}(0)$ and $v^{(2,0)(i)}_{k+1}(0)$ vanish. Indeed from (3.12) one finds

$$N_a V(\Lambda) = 0 \ . \tag{4.50}$$

Then, step by step, we consider the coefficients of v with increasing N and M completing the proof of (4.47).

From this inequality we can prove the existence of an UV limit for the theory. Indeed from the definition of v we have:

$$\begin{aligned} &l_{eff}\left(\Lambda_R, \Lambda_0, \rho_0\left(\Lambda_R, \Lambda_0, \rho_R\right)\right) - l_{eff}\left(\Lambda_R, \Lambda_0\prime, \rho_0\left(\Lambda_R, \Lambda_0\prime, \rho_R\right)\right) = \\ &\int_{\Lambda_0}^{\Lambda_0\prime} \frac{dx}{x} v\left(\Lambda_R, x, \rho_0\left(\Lambda_R, x, \rho_R\right)\right) \ . \end{aligned} \tag{4.51}$$

This equation can be translated in terms of the coefficients and, using (4,43), it leads to:

$$\begin{aligned} &\| l_{eff}\left[\Lambda_R, \Lambda_0, \rho_0(\Lambda_R, \Lambda_0, \rho_R)\right] - l_{eff}\left[\Lambda_R, \Lambda_0\prime, \rho_0(\Lambda_R, \Lambda_0\prime, \rho_R)\right] \|^{(N,M)} < \\ &|\Lambda_0^2 - \Lambda_0\prime^2| \frac{\Lambda_R^2}{\Lambda_0^4} P_r\left(ln\left(\frac{\Lambda_0}{\Lambda_R}\right)\right) \ , \end{aligned} \tag{4.52}$$

which proves, to any perturbative order r, the wanted convergence property. Indeed (4.52) shows that, if the effective parameters $\rho_{R,a}$ are kept fixed, the vertices of the effective theory satisfy the Cauchy convergence criterion for $\Lambda_0 \to \infty$.

A further comment about the nature of the UV limit is here possible if we translate (4.22) in terms of the effective vertices Γ in (3.28). Indeed we have that, for $p < \Lambda$ and $q < \Lambda$ the coefficients in the expansion of L_{eff} in series of $K_\Lambda \phi$ and ω satisfy, at the order r:

$$|\Gamma_r^{(n,m)}| \leq P_r\left(log\left(\frac{\Lambda}{\Lambda_R}\right)\right)(\Lambda)^{4-n-2m} \ . \tag{4.53}$$

For the momentum derivatives of order ν of these vertices the power behavior in the right-hand side of (4.53) is increased by ν.

From this inequality we can see in particular the effect of the introduction into L_0 of terms with dimension d higher that 4 (e. g. 6). Indeed these terms can be introduced first as composite operators together with their sources of negative dimension (e. g. -2) and then they can be inserted into the bare interaction by replacing their source with a coupling coefficient proportional to Λ_0^{4-d}. Now (4.53) shows that at the first order in the coupling the effect on $\Gamma^{(n,m)}$ of the insertion into the bare lagrangian of an operator of dimension $d > 4$ is proportional to Λ^{d-n-2m}, thus, the ratio of this versus the unperturbed effective coefficient vanishes at low energy as $\left(\frac{\Lambda}{\Lambda_0}\right)^{d-4}$. Therefore the effect of a "non-renormalizable" term in L_0 is "irrelevant" at low energies.

This result agrees with the previous remark that in the evolution of "finite" vertices or derivatives of vertices a change in the initial conditions at Λ_0 gives irrelevant contributions to l_{eff} for $\Lambda << \Lambda_0$.

As a further example of irrelevant operators that will be important in the study of quantum breaking of symmetries. Let us consider a bare operator that in the limit $\Lambda_0 \to \infty$ is local and has finite dimension d, while for finite Λ_0 it is non-local and contains contributions of any dimension. Suppose also that the operator satisfies vanishing normalization conditions at Λ_R at all perturbative orders. It is possible to show that the operator vanishes in the UV limit.

Indeed let us consider the evolution equation of the corresponding effective operator, the derivative of the effective lagrangian with respect to the corresponding source in the origin. The evolution equation is directly obtained from (3.41) and it is strictly analogous to it. However it is linear in the effective operator. This equation can thus be recursively solved increasing the dimension and the perturbative order of the vertices. In the first step, increasing the dimension of the vertices and keeping their order fixed, one begins integrating the evolution equation from Λ_R. Due to the vanishing normalization conditions and to the linearity of the equation one finds vanishing vertices. However, if the dimension of the vertex overcomes that of the operator, one has to integrate down from Λ_0 and hence one finds non-vanishing results due to the non-locality of the of the bare operator. The vertices of high dimension are however proportional to a positive integer power (n) of $\frac{\Lambda}{\Lambda_0}$ as are the irrelevant contributions to the solution of (3.41). Thus one finds a global upper bound proportional to $\left(\frac{\Lambda}{\Lambda_0}\right)^n$ for the vertices of the effective operator. It is clear that this upper bound holds true to all orders. This proves our claim.

5. Composite operators and Wilson expansion.

Having so concluded the general study of the UV limit, we have now to deepen the analysis of composite operators. Until now the general method to introduce these operators in the framework of the functional technique has been illustrated referring to the operator ϕ^2 . The method is based on the introduction of a source (ω) of given dimension (2). We have also seen that, in order to identify completely an operator one has to assign a suitable set of normalization conditions. Among these conditions those corresponding to lagrangian terms non-linear in the source concern possible multi-local contributions to the Green functions of many composite operators.

We shall now make some comments on this procedure, limiting however our attention to the Green functions of a single composite operator and an arbitrary number of fields.

The first point to be considered is the choice of the dimension of the source. In our example we have requested the bare lagrangian to have dimension 4 and therefore we have assigned dimension 2 to the source of an operator of dimension 2. Following our study of the effective lagrangian and of the corresponding evolution equation, it is possible to have a better understanding of this choice. Indeed the dimension of the source has influenced its rescaling in the infinite momentum limit (3.31) and hence the evolution equation of l_{eff}. On the other hand, in the tree approximation, the bare structure of the composite operator determines directly the scaling behavior of the vertices containing its source; this behavior has to match with the evolution equation in order to be reproduced to all orders of perturbation theory. In this way we have an upper bound for the source dimension.

As we have seen in the analysis of the evolution equation of l_{eff}, once the dimension δ of the source has been fixed, in order to define the operator completely, we have to assign a normalization condition for every "divergent" vertex involving the source, that is, for every monomial in the quantized field and its derivatives with dimension up to $4-\delta$, that of the operator we are willing to define.

Concerning the bare structure of an operator, that fixes the initial conditions at Λ_0, it has been repeatedly remarked that, after infinite volume limit and momentum rescaling, the addition of terms of higher dimension with Λ_0-dependent coefficients gives irrelevant contributions to the operator.

Thus, in conclusion, in the UV limit a composite operator is specified by assigning its dimension $4-\delta$, or, equivalently, the dimension of its source δ, and a polynomial operator P with dimension up to $4-\delta$ which defines its normalization conditions at Λ_R. An analogous characterization of composite operators has been introduced by Zimmermann [3] who has represented them with the symbol $N_{4-\delta}\,[P]$. We can use here the same symbol provided we take into account the normalization point Λ_R which in Zimmermann's definition is equal to zero.

It is interesting to notice here that, given an operator of dimension $4-\delta$, if one considers only the vertices linear in tis source and if multiplies all these vertices by $\left(\frac{\Lambda}{\Lambda_R}\right)^\eta$ for $\eta > 0$, one gets the vertices of a new operator with dimension $4-\delta+\eta$. Indeed the new vertices satisfy the evolution equations of the vertices of an operator of this dimension since these evolution equations are linear in the operator vertices. The normalization conditions for the vertices of this new operator can be directly computed from the vertices of the original one at Λ_R. The number of the conditions for the new operator is larger than that of the original one since the dimension is higher. Comparing the vertices of the two operators one gets a linear relation analogous to that connecting in Zimmermann's scheme operators with different dimension.

A further important remark concerning our approach to composite operators is that nowhere in our analysis the locality of the bare operator has played an essential role. Indeed, to identify the relevant normalization parameters of a certain operator, we have used the evolution equations of its vertices. These equations are obtained, independently of the locality of the operator, by taking the source derivative of both members of (3.2) and hence they have the same structure of (3.14). Once the evolution equations have been determined one has to identify a recursive hypothesis for the upper bounds of the vertices. As we have said above these bounds are set together with the dimension of the source in the tree approximation. Then the analysis follows strictly that of the previous section identifying the relevant normalization parameters with the zero momentum value of the vertices whose dimension does not exceed 4 taking into account also the dimension of the source.

We can therefore apply our results to multilocal products of operators to study their Wilson short-distance expansion. We can therefore apply our results to To put into evidence the origin and the nature of this expansion we simplify as much as possible our discussion referring to the product of two field operators, that is: to the operator whose bare structure is the product of two bare fields at different points $(\phi(x)\phi(0))$. The crucial new aspect of our analysis consists in considering this product, for what concerns the effective lagrangian, as a single composite operator. In particular we associate a source ω_X to it and we study the dependence of l_{eff} on this source.

Notice that considering the product of two fields in different points as a single composite operator implies a modification of the concept of connectedness of a diagram.

As discussed above, the first step in the analysis of our composite operator consists in assigning a dimension to ω_X. Owing to the fact that the contributions of Feynman diagrams to the tree vertices involving our operator depend on x through a phase factor e^{ipx}, where p is a partial sum of the external momenta, we have that the largest possible value of the dimension of ω_X is 2, the same the local operator ϕ^2.

Having specified the dimension of the source of our bilocal operator, we consider now its normalization conditions. According to the above discussion in order to define $(\phi(x)\phi(0))$ as a single composite operator, we have to assign its vacuum vertex, that is: the coefficient of l_{eff} linear in the source ω_X and of degree zero in the field, and the vertex with two legs with vanishing momentum. These vertices can be easily deduced from the vertices of l_{eff} involving only ϕ-legs. It is sufficient to consider respectively the vertices $\gamma^{(2,0)}(\vec{p})$ and $\gamma^{(4,0)}(\vec{p},-\vec{p},0,0)$, to multiply then by $e^{i\vec{p}\cdot\vec{x}}\Delta(p)^2$ and to integrate with respect to $\vec{p}$.

It is therefore possible, in our theory, to define for every x a renormalized operator of dimension 2 corresponding to the product of two bare fields at different points. This is identified with an operator of dimension two satisfying the following normalization conditions:

i) For the vacuum vertex, :

$$\int \frac{d^4k}{(2\pi)^4}\frac{\delta}{\delta\omega_X(\vec{k})} l_{eff}|_{\phi=\omega_X=0} = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\Lambda_R^2\Delta(p)\left[1+\gamma^{(2,0)}(p^2)\Delta(p)\right] \ . \tag{5.1}$$

ii) For the coefficient with two field legs at zero momentum:

$$\int \frac{d^4k}{(2\pi)^4}\frac{\delta}{\delta\omega_X(\vec{k})}\frac{\delta^2}{\delta\phi^2(\vec{k})} l_{eff}|_{\phi=\omega_X=0} = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\Delta^2(p)\gamma^{(4,0)}(\vec{p},-\vec{p},\vec{0},\vec{0}) \ . \tag{5.2}$$

As it is evident already at the tree level, these normalization conditions are by no means uniform in x, in particular the vacuum vertex is proportional to $\frac{1}{x^2}$ at short distances. This lack of uniformity in x can however be taken into account in the following way. We consider the linear combination of composite operators:

$$N_2^{(\Lambda_R)}[\phi(x)\phi(0)] \equiv \phi(x)\phi(0) - A(x) - B(x)\ N_2^{(\Lambda_R)}[\phi^2(0)] \ . \tag{5.3}$$

Here $A(x)$ and $B(x)$ are c-numbers chosen in such a way that the new composite operator $N_2^{(\Lambda_R)}[\phi(x)\phi(0)]$, satisfies normalization conditions uniform in x. More precisely:

i) Its vacuum vertex has to vanish.

ii) The two field vertex at zero momentum has to be equal to two.

Now, remembering the analysis of the evolution equations, we notice that the new operator has vertices uniformly bounded in x owing to the uniformity of the normalization conditions at Λ_R. Therefore the introduction of $N_2^{(\Lambda_R)}[\phi(x)\phi(0)]$ can be seen as a decomposition of a singular operator product into the sum of a singular and a regular part:

$$\phi(x)\phi(0) = A(x) + B(x)\ N_2^{(\Lambda_R)}[\phi^2(0)] + N_2^{(\Lambda_R)}[\phi(x)\phi(0)]\ . \tag{5.4}$$

The first two terms in the right-hand side give the singular part. Indeed taking into account (5.1) and (5.2) it is possible to verify that:

$$A(x) \sim \frac{1}{x^2} P\left(ln(x\Lambda_R)\right)\ , \tag{5.5}$$

and that:

$$B(x) \sim Q\left(ln(x\Lambda_R)\right)\ , \tag{5.6}$$

where P and Q are polynomials of increasing degree with the perturbative order. This requires a little work on the evolution equation (4.22) that, after UV limit, can be transformed into the Callan-Symanzik equation [7] and then used to study the presence of log's.

The decomposition (5.4) is the essential part of Wilson operator product expansion [3] , it can be obviously extended to multilocal products of composite operators. Starting from this expansion it is possible to analyze the Green functions of the theory [4] as distributions recovering the results of Bogoliubov, Parashiuk, Hepp, Zimmermann, Epstein, Glaser and Stora and hence proving that our theory leads to the same Green functions as the subtraction methods mentioned in the introduction.

6. The quantum action principle.

Now we discuss how one can characterize the invariance of a theory under a system of, in general non-linear, continuous field transformations.

In the Feynman functional framework, the invariance of the theory corresponds to that of the functional measure and its consequences are simply exhibited by introducing the field transformations as a change of variables in (2.23).

A simple but significant example of a model with non-linear symmetry is provided by the two-dimensional non-linear σ-model. With a simple choice of coordinates the classical action of this model is written:

$$S = \int d^2x \left[z\left[(\partial\vec{\pi})^2 + \left(\partial\sqrt{F^2-\pi^2}\right)^2 \right] + m^2\left(\sqrt{F^2-\pi^2} - F\right) \right]\ , \tag{6.1}$$

where $\vec{\pi}$ is an isovector field with vanishing dimension. This action is invariant under the linear field transformations corresponding to rotations in isotopic space. However to characterize completely (6.1) we have to exploit the transformation property of the action under the non-linear transformation:

$$\vec{\pi} \to \vec{\pi} + \vec{\epsilon}\sqrt{F^2 - \pi^2} \ , \tag{6.2}$$

where $\vec{\epsilon}$ is an infinitesimal, space independent isovector. It is easy to verify that the corresponding variation of the classical action is:

$$-\int d^2x \; m^2 \; \vec{\epsilon}\cdot\vec{\pi} \ . \tag{6.3}$$

Furthermore, if one asks, together with the linear isotopic invariance, that the variation of the action under (6.2) be (6.3), one identifies the action (6.1) up to the choice of z. This example is presented to put into evidence the role of symmetries in the identification of the action and hence in the construction of quantum field theories.

In the general case we shall consider the infinitesimal, non-linear field transformations:

$$\phi_{\vec{p}} \to \phi_{\vec{p}} + \sum_{\vec{q}} k\left(\frac{p}{\Lambda_0}\right) \epsilon_{\vec{q}} P_{0,\vec{p}-\vec{q}}(K_{\Lambda_0}\phi, \omega, 0) \ . \tag{6.4}$$

To define the composite operator P within the functional framework, we introduce its source γ by the substitution:

$$L_0(K_{\Lambda_0}\phi,\omega) \to L_0(K_{\Lambda_0}\phi,\omega) - \sum_{\vec{p}} \gamma_{\vec{p}} P_{0,-\vec{p}}(K_{\Lambda_0}\phi,\omega) + O(\gamma^2) \equiv L_0(K_{\Lambda_0}\phi,\omega,\gamma) \ . \tag{6.5}$$

Notice that the introduction of a composite operator of source γ is in general accompanied by terms of degree higher than one in γ which appear in the term $O(\gamma^2)$ in (6.5). The presence of these terms induces a change of the infinitesimal transformation law (6.4) that we want to write in the form:

$$\begin{aligned}\phi_{\vec{p}} \to \phi_{\vec{p}} - \sum_{\vec{q}} \epsilon_{\vec{q}} k\left(\frac{p}{\Lambda_0}\right) \partial_{\gamma_{\vec{q}-\vec{p}}} L_0(K_{\Lambda_0}\phi,\omega,\gamma) \equiv \\ \phi_{\vec{p}} + \sum_{\vec{q}} \epsilon_{\vec{q}} k\left(\frac{p}{\Lambda_0}\right) P_{0,\vec{p}-\vec{q}}(K_{\Lambda_0}\phi,\omega,\gamma) \ .\end{aligned} \tag{6.6}$$

Introducing (6.6) as a change of variables into the Feynman formula (2,30) and requiring the invariance of the functional integral, we get, up to first order in ϵ:

$$\begin{aligned}\int \prod_{\vec{p}} d\phi_{\vec{p}} \sum_{\vec{p},\vec{q}} k\left(\frac{p}{\Lambda_0}\right) \epsilon_{\vec{q}} \Big[\Big(j_{\vec{p}} - \phi_{\vec{p}} C(p) - \partial_{\phi_{-\vec{p}}} L_0\Big) \partial_{\gamma_{\vec{q}+\vec{p}}} L_0 + \\ \partial_{\phi_{-\vec{p}}} \partial_{\gamma_{\vec{q}+\vec{p}}} L_0\Big] e^{-S} = 0 \ .\end{aligned} \tag{6.7}$$

Let us now introduce the bare action:

$$S_0 \equiv \sum_{\vec{p}} \phi_{-\vec{p}} \frac{C(p)}{2} \phi_{\vec{p}} + L_0 \ , \tag{6.8}$$

and the local composite operator of dimension $3 + d_P$:

$$\Delta_{-\vec{q}} \equiv \sum_{\vec{p}} k\left(\frac{p}{\Lambda_0}\right) \Big[\partial_{\phi_{-\vec{p}}} \partial_{\gamma_{\vec{p}-\vec{q}}} S_0 - \partial_{\phi_{-\vec{p}}} S_0 \partial_{\gamma_{\vec{p}-\vec{q}}} S_0\Big] \ , \tag{6.9}$$

and define

$$\Delta_{-\vec{q}} \cdot Z \tag{6.10}$$

the generator of the Schwinger functions with the insertion of the operator $\Delta_{\vec{q}}$. This is computed in the functional framework adding to the bare action $\sum_{\vec{q}} \epsilon_{\vec{q}} \Delta_{-\vec{q}}$ and taking the derivative of Z with respect to $\epsilon_{\vec{q}}$ at $\epsilon = 0$. remembering that the support of ϵ and j is contained in a sphere of radius Λ_R, which is smaller than Λ_0, we can disregard the cut-off factor $k\left(\frac{p}{\Lambda_0}\right)$ in the first term in the left-hand side of (6.7) which can be written:

$$\sum_{\vec{p},\vec{q}} \epsilon_{\vec{q}} j_{\vec{p}} \partial_{\gamma_{\vec{p}+\vec{q}}} Z = \sum_{\vec{q}} \epsilon_{\vec{q}} \Delta_{-\vec{q}} \cdot Z \ . \tag{6.11}$$

Now, if the couplings of the source γ are suitably normalized and therefore the theory has UV limit in the presence of γ (the operator P is finite), from (6.11) we see that the operator Δ is finite.

Equation (6.11) is a broken Ward identity, it is sometimes referred to as "the Quantum Action Principle". For an introduction to its consequences in a regularization independent framework we refer to [4],[5].

Notice that in many instances of physical interest the field transformations are linear at the bare level. However this property is not maintained at the effective level due to the integration over the short wavelength field amplitudes. The choice to discuss directly non-linear transformations is justified by our interest in the analysis of the effective theory.

Of course, the finiteness of Δ is interesting since it ensures the meaningfulness of the broken Ward identity (6.11). However this is not the final goal of our study which is devoted to the construction of quantum theories with symmetry properties. Therefore the equation that we have to discuss and possibly to solve is:

$$\Delta = 0 \ . \tag{6.12}$$

This is a system of equations for the bare, or better, the normalization constants of the theory that should allow to identify the invariant theory. The general idea is that the symmetry breaking effect induced by the regularization of the Feynman integral should be compensated fine-tuning non-invariant terms in the bare action. This is why we shall call (6.12) the fine-tuning equation.

Until now we have only shown the possibility of defining the left-hand side of (6.12) both before and after UV limit. The first point to be clarified in our analysis of this equation is to decide if we are going to study it before or after UV limit. From a general point of view this can be a crucial choice. Indeed, if the existence of the UV limit were guaranteed by simple power counting arguments, , as it is in the framework of perturbation theory, it would be natural to refer to the situation after this limit. Then one would discuss, instead of (6.11), the corresponding equation for the effective action since the bare action is singular in the UV limit.

Let us briefly discuss the fine-tuning equation after the UV limit. To extend our results to the effective theory we repeat the above analysis in this framework and we introduce the effective breaking:

$$\Delta_{eff} \equiv \int d^4p d^4q k\left(\frac{p}{\Lambda}\right) \epsilon(q) \left[\frac{\delta}{\delta\phi(-p)} \frac{\delta}{\delta\gamma(p-q)} S_{eff} - \right.$$
$$\left. \frac{\delta}{\delta\phi(-p)} S_{eff} \frac{\delta}{\delta\gamma(p-q)} S_{eff}\right] , \tag{6.13}$$

where Λ is the cut-off and:

$$S_{eff} \equiv \int d^4x \frac{(\partial\phi(x))^2 + m^2\phi^2(x)}{2} + L_{eff} \ . \tag{6.14}$$

Then (6.12) is naturally replaced with:

$$\Delta_{eff} = 0 \ . \tag{6.15}$$

Notice that we can write (6.13) (and hence (6.9)) in the form:

$$\Delta_{eff} = -\hbar^2 \int d^4p d^4q k\left(\frac{p}{\Lambda}\right) \epsilon(q) e^{\frac{S_{eff}}{\hbar}} \frac{\delta}{\delta\phi(-p)} \frac{\delta}{\delta\gamma(p-q)} e^{\frac{-S_{eff}}{\hbar}} \ , \tag{6.16}$$

where $\hbar$ has been reintroduced in view of the study of the perturbative case.

The fine-tuning condition (6.15) is in fact equivalent to an infinite number of equations for Δ_{eff}. However we have shown above ((6.9)) that Δ has limited dimension provided the field transformations (6.6) change the field dimension by a limited amount. More precisely the bare operator becomes local in the limit $\Lambda_0 \to \infty$. Its dimension is equal to 4 plus the increase of field dimension in the transformations (6.4). Furthermore, computing explicitly the Λ-derivative of the right-hand side of (6.13) and taking into account (3.2)it can be verified directly that Δ_{eff} satisfies, before the UV limit, the evolution equation:

$$\Lambda\partial_\Lambda \Delta_{eff} = M\left[\Delta_{eff}\right] \ . \tag{6.17}$$

Therefore, after the general discussion concluding section 4, we can say that (6.15) is guaranteed in the UV limit if Δ_{eff} satisfies vanishing normalization conditions.

We can express the vanishing of these normalization conditions in a simple form introducing the local approximant of dimension d of a translation invariant functional F, that we shall label by $\mathcal{T}_d F$. This is defined as the integrated local functional of the fields and sources, and of ϵ, in the case of Δ_{eff}, whose vertices have the same Taylor expansion around the origin of momentum space, as those of F up to a degree equal to d minus the total dimension of the fields and sources corresponding to the legs of the vertex. Here we assign the source of a local operator the complement to four of the dimension of the operator. For example if F depends only on ϕ_i we have:

$$\begin{aligned} \mathcal{T}_3 F = F[0] + \int d^4x \phi_i(x) \Bigg[& \frac{\delta}{\delta\phi_i(0)} F + \frac{1}{2}\phi_j(x) \frac{\delta}{\delta\tilde{\phi}_j(0)} \frac{\delta}{\delta\phi_i(0)} F \\ & + \frac{i}{2}\partial_\mu\phi_j(x)\partial_{p_\mu} \frac{\delta}{\delta\tilde{\phi}_j(0)} \frac{\delta}{\delta\phi_i(0)} F + \\ & \frac{1}{6}\phi_j(x)\phi_k(x) \frac{\delta^2}{\delta\tilde{\phi}_j(0)\delta\tilde{\phi}_k(0)} \frac{\delta}{\delta\phi_i(0)} F \Bigg] |_{\phi=0} \ , \end{aligned} \tag{6.18}$$

where we have set

$$\phi(x) = \int d^4p e^{-ipx} \tilde{\phi}(p) \ . \tag{6.19}$$

In the case of $\mathcal{T}_{d_{\Delta_{eff}}} \Delta_{eff}$ and of $\mathcal{T}_4 S_{eff}$ we shall, from now on, understand the dimension of the local approximant.

Since the couplings appearing in $\mathcal{T}\Delta_{eff}$ are in one-to-one correspondence with the normalization parameters of Δ_{eff} it is now clear that if the theory is finite in the UV limit (6.15) is equivalent to

$$\mathcal{T}\Delta_{eff} = 0 \ . \tag{6.20}$$

Beyond the perturbative level the existence of an UV limit is far from being guaranteed. However there is some evidence of a rather general connection between the existence of this limit and asymptotic freedom [10]. In this case the study of (6.12) or (6.15) must precede the UV limit since the theory with broken symmetry in general is not expected to be asymptotically free.

Notice however that in the presence of an UV cut-off Λ_0 the wanted symmetry can be unreachable at the bare level. This happens typically in a gauge theory in which (6.12) is unsolvable for finite UV cut-off.

Therefore at first sight the construction of a fully quantized gauge theory seems impossible. However in the framework of the effective theory it might be possible to fine-tune the low energy parameters under the condition that Δ_{eff} given in (6.13) satisfy normalization conditions vanishing with $\frac{\Lambda}{\Lambda_0}$.

If this condition is met, considering the solution of the evolution equations discussed in section 4, we see that the breaking is irrelevant and hence, if the theory has a regular UV limit, Δ vanishes in this limit at least as $\frac{\Lambda}{\Lambda_0}$. To satisfy this last condition one should ask the fine-tuned theory to be asymptotically free. Postponing the dream of the non-perturbative construction of a gauge theory to better times, we shall limit our present discussion to the perturbative situation.

7. Analysis of the $SU(2)$ Yang-Mills model

To proceed further with the analysis of (6.15) it is convenient to refer to the highly non-trivial example of a pure $SU(2)$ gauge theory, whose field content consists of the gauge field A^a_μ, of the Lagrange multiplier b^a and of the anticommuting Faddeev-Popov ghosts c^a and $\bar{c}^a$. All the fields are isotopic vectors. In terms of the gauge field strength:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu \ , \tag{7.1}$$

where ϵ^{abc} is the antisymmetric Ricci tensor, and of the covariant derivative:

$$D_\mu c^a = \partial_\mu c^a - g\epsilon^{abc} A^b_\mu c^c \ , \tag{7.2}$$

the "classical action of the theory is given:

$$S_{cl} = \int d^4x \ \left[\frac{z}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{b^a b^a}{2} + i b^a \partial_\mu A^a_\mu + \partial_\mu \bar{c}^a D_\mu c^a\right] \ . \tag{7.3}$$

From this equation one can easily deduce the covariance matrix of the fields. It is also possible to single out the symmetries characterizing the theory at the quantum level. It turns out that, given the b^a-dependent part of (7.3), S_{cl} is identified by the invariance condition under the non-linear, nilpotent field transformations:

$$A^a_\mu \to A^a_\mu + \eta D_\mu c^a \ , \tag{7.4}$$

$$c^a \to c^a + \eta \frac{g}{2} \epsilon^{abc} c^b c^c \ , \tag{7.5}$$

$$\bar{c}^a \to \bar{c}^a + i\eta b^a \ , \tag{7.6}$$

$$b^a \to b^a \ . \tag{7.7}$$

Notice that here η is an anticommuting constant that replaces ϵ in the definition of Δ.

A further necessary comment concerning (7.3), is that the field b^a plays the role of an auxiliary field since the action is at most quadratic in it and hence it does not appear into the interaction lagrangian. It follows that the Wilson renormalization group starting from a bare action with this property will generate a b-field independent effective lagrangian.

Owing to the anticommuting character of the transformations (7.4- 7.7), quantizing the theory, we have to introduce an anticommuting vector-isovector source γ^a_μ of dimension two, coupled to $D_\mu c^a$ and a commuting isovector ζ^a of dimension two, coupled to the variation of c^a. Adding to the action these source terms together with the sources of the fields we have:

$$S_{cl} = \int d^4x \ \Big[\frac{z}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{b^a b^a}{2} + i b^a \partial_\mu A^a_\mu + \left(\partial_\mu \bar{c}^a - \gamma^a_\mu \right) D_\mu c^a - \\ \zeta^a \frac{g \epsilon^{abc}}{2} c^b c^c - j^a_\mu A^a_\mu - J^a b^A - \xi^a \bar{c}^a - \bar{\xi}^a c^a \Big] \ . \tag{7.8}$$

Notice that ξ^a and $\bar{\xi}^a$ are anticommuting sources and that there is a conserved ghost charge that vanishes for the vector field, is equal to $+1$ for the c field, -1 for the $\bar{c}$ and γ source and -2 for the ζ source.

One should also remark that the classical action (7.8) is left invariant by the transformations:

$$\bar{c}^a \to \bar{c}^a + \bar{\xi}^a \ , \ \gamma^a_\mu \to \gamma^a_\mu + \partial_\mu \bar{\xi}^a \ . \tag{7.9}$$

In order to quantize this theory we introduce a cut-off in much the same way as for the scalar field, choosing the same function k for all the field components. In this way we maintain at the quantum level the invariance of the theory under rigid isotopic symmetry. however this automatically introduces an unavoidable breaking into the Ward identity corresponding to (7.4-7.7). Indeed considering the cut-off transformations:

$$A^a_{\mu\vec{p}} \to A^a_{\mu\vec{p}} + \eta k \left(\frac{p}{\Lambda_0} \right) \partial_{\gamma^a_{\mu\vec{p}}} L_0 \ , \tag{7.10}$$

$$c^a_{\vec{p}} \to c^a_{\vec{p}} + \eta k \left(\frac{p}{\Lambda_0} \right) \partial_{\zeta^a_{\vec{p}}} L_0 \ , \tag{7.11}$$

$$\bar{c}^a_{\vec{p}} \to \bar{c}^a_{\vec{p}} + i\eta k \left(\frac{p}{\Lambda_0} \right)^2 b^a_{\vec{p}} \ , \tag{7.12}$$

and following the analysis described above, we get the broken Ward identity:

$$\sum_{\vec{p}} \left(j^a_{\mu\vec{p}} \partial_{\gamma^a_{\mu\vec{p}}} - \bar{\xi}^a_{\vec{p}} \partial_{\zeta^a_{\vec{p}}} - i\xi^a_{\vec{p}} \partial_{J^a_{\vec{p}}} \right) Z \equiv \mathcal{S}Z = \Delta \cdot Z \ , \tag{7.13}$$

where:

$$\begin{aligned}
&\Delta = \\
&\sum_{\vec{p}} \Big[\left((\delta_{\mu\nu} p^2 - p_\mu p_\nu) A^a_{\nu\vec{p}} + p_\mu b^a_{\vec{p}} + \partial_{A^a_{\mu-\vec{p}}} L_0 \right) k\left(\frac{p}{\Lambda_0}\right) \left(\partial_{\gamma^a_{\mu\vec{p}}} L_0 + i p_\mu c^a_{-\vec{p}} \right) - \\
&\left(p^2 \bar{c}^a_{\vec{p}} - \partial_{c^a_{-\vec{p}}} L_0 \right) k\left(\frac{p}{\Lambda_0}\right) \partial_{\zeta^a_{\vec{p}}} L_0 - i \left(p^2 c^a_{\vec{p}} + \partial_{\bar{c}^a_{-\vec{p}}} L_0 \right) b^a_{-\vec{p}} - \\
&k\left(\frac{p}{\Lambda_0}\right) \left(\partial_{A^a_{\mu-\vec{p}}} \partial_{\gamma^a_{\mu\vec{p}}} + \partial_{c^a_{-\vec{p}}} \partial_{\zeta^a_{\vec{p}}} \right) L_0 \Big] \ ,
\end{aligned} \tag{7.14}$$

For a generic choice of the bare action it is apparent that this breaking is b-dependent. Now this is a true disaster since, even if the breaking were compensable by a suitable modification of the action, this would introduce b-dependent interaction terms. However this difficulty can be avoided if one chooses the bare action so that:

$$\begin{aligned}
&-ip_\mu \partial_{\gamma^a_{\mu\vec{p}}} L_0 = k\left(\frac{p}{\Lambda_0}\right) \left(\partial_{\bar{c}^a_{\vec{p}}} L_0 + p^2 c^a_{\vec{p}} \right) = \\
&p^2 K_{\Lambda_0} c^a_{\vec{p}} + \partial_{K_{\Lambda_0} \bar{c}^a_{\vec{p}}} L_0 \ .
\end{aligned} \tag{7.15}$$

Indeed substituting (7.15) into (7.14) it is apparent that the b-dependent terms in Δ annul each other.

Now, assuming that the theory remain finite in the UV limit, we translate the above results in terms of the effective theory in the infinite volume limit.

We have in particular that the constraint (7.15) remains true if we replace in them L_0 with L_{eff}. That is

$$\begin{aligned}
&-ip_\mu \partial_{\gamma^a_{\mu\vec{p}}} L_{eff} = k\left(\frac{p}{\Lambda}\right) \left(\partial_{\bar{c}^a_{\vec{p}}} L_{eff} + p^2 c^a_{\vec{p}} \right) = \\
&p^2 K_\Lambda c^a_{\vec{p}} + \partial_{K_\Lambda \bar{c}^a_{\vec{p}}} L_{eff} \ .
\end{aligned} \tag{7.16}$$

Indeed we can verify that both sides of (7.16) satisfy the same linear evolution equation. In particular it is evident that the left-hand side satisfies:

$$\Lambda \partial_\Lambda \partial_{\gamma^a_{\mu\vec{p}}} L_{eff} = M\left[\partial_{\gamma^a_{\mu\vec{p}}} L_{eff} \right] \ , \tag{7.17}$$

while the right-hand side satisfies:

$$\begin{aligned}
&\Lambda \partial_\Lambda \left[p^2 K_\Lambda c^a_{\vec{p}} + \partial_{K_\Lambda \bar{c}^a_{\vec{p}}} L_{eff} \right] = \\
&\Lambda \partial_\Lambda k\left(\frac{p}{\Lambda}\right)^2 \partial_{K_\Lambda \bar{c}^a_{\vec{p}}} L_{eff} + k\left(\frac{p}{\Lambda}\right)^2 M\left[\partial_{K_\Lambda \bar{c}^a_{\vec{p}}} L_{eff} \right] \ .
\end{aligned} \tag{7.18}$$

Remember that the evolution equation does not take into account the Λ-dependence of the cut-off field variables.

Now it is sufficient to notice that:

$$M\left[p^2 K_\Lambda c^a_{\vec{p}}\right] = \Lambda\partial_\Lambda k\left(\frac{p}{\Lambda}\right)^2 \partial_{K_\Lambda \bar{c}^a_{\vec{p}}} L_{eff} \ , \tag{7.19}$$

to verify our claim.

Now, introducing the effective action:

$$S_{eff} = \int d^4p\left[\tilde{A}^a_\mu(p)\frac{\delta_{\mu\nu}p^2 - p_\mu p_\nu}{2}\tilde{A}^a_\nu(p) + \tilde{\bar{c}}^a(p)p^2\tilde{c}^a(p)\right] + L_{eff} \ , \tag{7.20}$$

we find the fine-tuning condition for this effective action:

$$\begin{aligned}&\Delta_{eff} = \\ &-\hbar^2 e^{\frac{S_{eff}}{\hbar}}\int d^4pk\left(\frac{p}{\Lambda}\right)\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)e^{\frac{-S_{eff}}{\hbar}} = \\ &\int d^4pk\left(\frac{p}{\Lambda}\right)\left[\frac{\delta}{\delta\tilde{A}^a_\mu(p)}S_{eff}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}S_{eff} + \frac{\delta}{\delta\tilde{c}(p)^a}S_{eff}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}S_{eff} - \right. \\ &\left.\hbar\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)S_{eff}\right] = 0 \ ,\end{aligned} \tag{7.21}$$

whose local version is:

$$\begin{aligned}&\mathcal{T}\Delta_{eff} = \\ &-\hbar^2\mathcal{T}_5\left[e^{\frac{S_{eff}}{\hbar}}\int d^4pk\left(\frac{p}{\Lambda}\right)\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)e^{\frac{-S_{eff}}{\hbar}}\right] = \\ &\int d^4p\left[\mathcal{T}_5\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}S_{eff}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}S_{eff} + \frac{\delta}{\delta\tilde{c}(p)^a}S_{eff}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}S_{eff}\right) - \right. \\ &\left.\hbar\mathcal{T}_5 k\left(\frac{p}{\Lambda}\right)\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)S_{eff}\right] = 0 \ .\end{aligned} \tag{7.22}$$

Notice that the cut-off factor in the first two terms under integral has disappeared since, by momentum conservation, the $\mathcal{T}$ operator restricts the p variable corresponding to these terms to zero. In the present case the differential operator

$$\begin{aligned}&\int d^4pk\left(\frac{p}{\Lambda}\right)\frac{\delta}{\delta\phi(-p)}\frac{\delta}{\delta\gamma(p)} \equiv \\ &\int d^4pk\left(\frac{p}{\Lambda}\right)\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right) \ ,\end{aligned} \tag{7.23}$$

is nilpotent owing to the anticommuting nature of the ghost field c^a and of the source γ^a_μ.

This nilpotency of the differential operator (7.23) leads to the following equation for Δ_{eff} :

$$-\hbar\int d^4pk\left(\frac{p}{\Lambda}\right)\frac{\delta}{\delta\phi(-p)}\frac{\delta}{\delta\gamma(p)}\Delta_{eff}e^{\frac{-S_{eff}}{\hbar}} =$$

$$\int d^4pk\left(\frac{p}{\Lambda}\right)\left[\frac{\delta}{\delta\phi(-p)}S_{eff}\frac{\delta}{\delta\gamma(p)}\Delta_{eff}+\frac{\delta}{\delta\phi(-p)}\Delta_{eff}\frac{\delta}{\delta\gamma(p)}S_{eff}-\right.$$
$$\left.\hbar\frac{\delta}{\delta\phi(-p)}\frac{\delta}{\delta\gamma(p)}\Delta_{eff}\right]e^{\frac{-S_{eff}}{\hbar}}=0\ , \tag{7.24}$$

where we have taken into account (6.16) and the anticommuting character of Δ.

The last identity, that in the gauge case is written:

$$\int d^4pk\left(\frac{p}{\Lambda}\right)\left[\frac{\delta}{\delta\tilde{A}^a_\mu(p)}S_{eff}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}\Delta_{eff}+\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}S_{eff}\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\Delta_{eff}+\right.$$
$$\frac{\delta}{\delta\tilde{c}(p)^a}S_{eff}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\Delta_{eff}+\frac{\delta}{\delta\tilde{\zeta}(-p)^a}S_{eff}\frac{\delta}{\delta\tilde{c}(p)^a}\Delta_{eff}-$$
$$\left.\hbar\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}+\frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)\Delta_{eff}\right]=0\ , \tag{7.25}$$

plays the role of a consistency condition for the breaking analogous to the Wess-Zumino [11] consistency condition for the chiral anomaly.

Now we show how in the perturbative case this consistency condition turns out to be sufficient to prove the solvability of the fine-tuning equation (6.15).

As repeated many times in the previous sections, perturbation theory is built by a recursive procedure in $\hbar$. Hence we have to apply this procedure to the fine-tuning equation. Therefore we develop the effective action S_{eff} and the effective breaking Δ_{eff} in power series of $\hbar$ and we label the k^{th}-order terms of these series by the index k.

Now, if we assume the tree approximation normalization condition

$$\mathcal{T}S_{eff_0}=\int d^4x\ \left[\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}+\left(\partial_\mu\bar{c}^a-\gamma^a_\mu\right)D_\mu c^a-\zeta^a\frac{g\epsilon^{abc}}{2}c^bc^c\right]\ , \tag{7.26}$$

we have

$$\mathcal{T}\Delta_{eff_0}=\int d^4p\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\mathcal{T}S_{eff_0}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}\mathcal{T}S_{eff_0}+\right.$$
$$\left.\frac{\delta}{\delta\tilde{c}(p)^a}\mathcal{T}S_{eff_0}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\mathcal{T}S_{eff_0}\right)=0\ , \tag{7.27}$$

as one can directly verify using the fact that S_{eff_0} is invariant under the transformations (7.4-7.5).

Assuming that

$$\mathcal{T}\Delta_{eff_{n-1}}=0\ , \tag{7.28}$$

it follows that $\Delta_{eff_{n-1}}=0$ and hence the consistency condition

$$\int d^4pk\left(\frac{p}{\Lambda}\right)\left[\frac{\delta}{\delta\tilde{A}^a_\mu(p)}S_{eff_0}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}\Delta_{eff_n}+\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}S_{eff_0}\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\Delta_{eff_n}+\right.$$
$$\left.\frac{\delta}{\delta\tilde{c}(p)^a}S_{eff_0}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\Delta_{eff_n}+\frac{\delta}{\delta\tilde{\zeta}(-p)^a}S_{eff_0}\frac{\delta}{\delta\tilde{c}(p)^a}\Delta_{eff_n}\right]=0\ . \tag{7.29}$$

At this point of our analysis it is convenient to write explicitly the local approximant of dimension 5 to Δ_{eff_n}. From now on we shall put $\bar{c}=0$ without any loss of generality since the dependence on this field is strictly connected to that on the source Γ^a_μ. Owing to the euclidean, isotopic ($SU(2)$) and translation invariance and recalling that Δ_{eff} has ghost charge $+1$, we have:

$$
\begin{aligned}
\mathcal{T}\Delta_{eff_n} = \int d^4x \Big[& a_n \epsilon^{abc}\gamma^a_\mu c^b \partial_\mu c^c + b_n \gamma^a_\mu A^b_\mu c^a c^b + \\
& c_n A^a_\mu \partial^2 \partial_\mu c^a + d_n A^a_\mu \partial_\mu c^a + \\
& e_n \epsilon^{abc} A^a_\mu \partial_\nu A^b_\mu \partial_\nu c^c + f_n \epsilon^{abc} A^a_\mu \partial_\mu A^b_\nu \partial_\nu c^c + \\
& g_n A^a_\mu A^a_\mu \partial_\nu A^b_\nu c^b + h_n A^a_\mu A^b_\mu \partial_\nu A^a_\nu c^b + \\
& k_n A^a_\mu A^a_\nu \partial_\mu A^b_\nu c^b + j_n A^a_\mu A^b_\nu \partial_\mu A^a_\nu c^b + \\
& l_n A^a_\mu A^b_\nu \partial_\mu A^b_\nu c^a \Big] \; .
\end{aligned} \tag{7.30}
$$

Thereby we see that this functional does not contain monomials of dimension lower than 3.

Selecting the local approximant of dimension 6 to (7.29) and taking into account (7.26) and (7.30) we get the perturbative consistency condition:

$$
\begin{aligned}
& \int d^4p \left[\frac{\delta}{\delta \tilde{A}^a_\mu(p)} \mathcal{T} S_{eff\,0} \frac{\delta}{\delta \tilde{\gamma}^a_\mu(-p)} \mathcal{T}\Delta_{eff_n} + \frac{\delta}{\delta \tilde{\gamma}^a_\mu(-p)} \mathcal{T} S_{eff\,0} \frac{\delta}{\delta \tilde{A}^a_\mu(p)} \mathcal{T}\Delta_{eff_n} + \right. \\
& \left. \frac{\delta}{\delta \tilde{c}(p)^a} \mathcal{T} S_{eff\,0} \frac{\delta}{\delta \tilde{\zeta}(-p)^a} \mathcal{T}\Delta_{eff_n} + \frac{\delta}{\delta \tilde{\zeta}(-p)^a} \mathcal{T} S_{eff\,0} \frac{\delta}{\delta \tilde{c}(p)^a} \mathcal{T}\Delta_{eff_n} \right] \equiv \\
& \mathcal{D}\mathcal{T}\Delta_{eff_n} = 0 \; .
\end{aligned} \tag{7.31}
$$

Here we have used the fact that, as it is evident from (4.7) and (4.8), in the tree approximation any local approximant of dimension higher than 4 of S_{eff} is equal to $\mathcal{T}S_{eff}$. It is also important to notice that the differential operator $\mathcal{D}$ is nilpotent. Therefore any element of the image of $\mathcal{D}$ solves (7.31).

Inserting into (7.31) (7.26) and (7.30) one finds, after some algebraic manipulations, the following constraints:

$$
\begin{aligned}
& b_n = -g a_n \quad , \quad f_n + e_n + g c_n = 0 \\
& h_n = k_n = j_n \quad , \quad l_n = 2 g_n \; ,
\end{aligned} \tag{7.32}
$$

therefore the general solution of the perturbative consistency condition contains six free parameters.

Now, given the most general local approximant to S_{eff} satisfying (7.15):

$$
\begin{aligned}
\mathcal{T}S_{eff} = \int d^4x \; \Big[& \frac{z_1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - z_2 \gamma^a_\mu D_\mu c^a - z_3 \zeta^a \frac{g\epsilon^{abc}}{2} c^b c^c \\
& + x_1 \epsilon^{abc} \gamma^a_\mu A^b_\mu c^c + x_2 \partial_\mu A^a_\mu \partial_\nu A^a_\nu + x_3 A^a_\mu A^a_\mu + \\
& x_4 \epsilon^{abc} A^a_\mu A^b_\nu \partial_\mu A^c_\nu + x_5 A^a_\mu A^b_\mu A^a_\nu A^b_\nu + x_6 \left(A^a_\mu A^a_\mu \right)^2 \Big] \; ,
\end{aligned} \tag{7.33}
$$

it is a little lengthy but straightforward to verify that $\mathcal{T}\Delta_{eff_n}$ in (7.31) with the constraints (7.32) satisfies:

$$\mathcal{T}\Delta_{eff_n} = \mathcal{D}\mathcal{T}\bar{S}_{eff_n} \ , \tag{7.34}$$

if the coefficients of $\mathcal{T}\bar{S}_{eff_n}$ are chosen according

$$\begin{aligned} &a_n = -\bar{x}_1 \quad , \quad c_n = -2\bar{x}_2 \\ &d_n = 2\bar{x}_3 \quad , \quad e_n = \bar{x}_4 \\ &g_n = -\bar{x}_4 - 4\bar{x}_6 \quad , \quad h_n = \bar{x}_4 - 4\bar{x}_5 \ . \end{aligned} \tag{7.35}$$

Let us now consider the structure of $\mathcal{T}\Delta_{eff}$ at the n^{th} order in $\hbar$ as it can be computed from (7.22). This is written:

$$\begin{aligned} &\mathcal{T}\Delta_{eff_n} = \\ &\int d^4p \left[\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\mathcal{T}S_{eff\,0}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}\mathcal{T}S_{eff_n} + \frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}\mathcal{T}S_{eff\,0}\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\mathcal{T}S_{eff_n} + \right. \\ &\frac{\delta}{\delta\tilde{c}(p)^a}\mathcal{T}S_{eff\,0}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\mathcal{T}S_{eff_n} + \frac{\delta}{\delta\tilde{\zeta}(-p)^a}\mathcal{T}S_{eff\,0}\frac{\delta}{\delta\tilde{c}(p)^a}\mathcal{T}S_{eff_n} + \\ &\sum_{k=1}^{n-1}\mathcal{T}_5\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}S_{eff_k}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}S_{eff_{n-k}} + \frac{\delta}{\delta\tilde{c}(p)^a}S_{eff_k}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}S_{eff_{n-k}}\right) - \\ &\left.\hbar\mathcal{T}_5 k\left(\frac{p}{\Lambda}\right)\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)S_{eff_{n-1}}\right] \ . \end{aligned} \tag{7.36}$$

On account of (7.34), (7.36) can be written:

$$\begin{aligned} &\mathcal{D}\mathcal{T}S_{eff_n} + \\ &\int d^4p \left[\sum_{k=1}^{n-1}\mathcal{T}_5\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}S_{eff_k}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)}S_{eff_{n-k}} + \right.\right. \\ &\left.\frac{\delta}{\delta\tilde{c}(p)^a}S_{eff_k}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}S_{eff_{n-k}}\right) - \\ &\left.\hbar\mathcal{T}_5 k\left(\frac{p}{\Lambda}\right)\left(\frac{\delta}{\delta\tilde{A}^a_\mu(p)}\frac{\delta}{\delta\tilde{\gamma}^a_\mu(-p)} + \frac{\delta}{\delta\tilde{c}(p)^a}\frac{\delta}{\delta\tilde{\zeta}(-p)^a}\right)S_{eff_{n-1}}\right] \equiv \\ &\mathcal{D}\mathcal{T}S_{eff_n} + \Omega_n = \mathcal{D}\mathcal{T}\bar{S}_{eff_n} \ , \end{aligned} \tag{7.37}$$

from which we see that there exists a 4-dimensional integrated local functional $\Sigma_n = \mathcal{T}\bar{S}_{eff_n} - \mathcal{T}S_{eff_n}$ such that

$$\Omega_n = \mathcal{D}\Sigma_n \ . \tag{7.38}$$

Taking into account this result and (7.36) the n^{th} order fine-tuning equation is written:

$$\mathcal{D}\left(\mathcal{T}S_{eff_n} + \Sigma_n\right) = 0 \ , \tag{7.39}$$

and it is solved by

$$\mathcal{T}S_{eff_n} = -\Sigma_n \ . \tag{7.40}$$

This proves the iterative solvability of the fine-tuning equation.

8. Further comments.

As it is apparent from the above analysis the crucial step of this proof has been the fact that the consistency condition can be written in the form

$$\mathcal{D}\mathcal{T}\Delta_{eff_n} = 0 \ , \tag{8.1}$$

and the result that the general solution of this equation is

$$\mathcal{T}\Delta_{eff_n} = \mathcal{D}\mathcal{T}\bar{S}_{eff_n} \ . \tag{8.2}$$

In the framework of a more general gauge theory one can always reduce the perturbative problem to the study of an equation completely analogous to (8.1) and in particular to the comparison of the kernel of the nilpotent operator with its image. In this way the solvability of the fine-tuning equation is in general reduced to the triviality of a certain class of the so-called BRS cohomology. This is exactly what we have verified in the present example.

Of course, our proof is strongly dependent on the chosen perturbative framework. However the idea of making the low-energy vertices of the breaking to vanish by a fine tuning of the parameters of the theory could well work beyond perturbation theory.

The first question to answer in this direction is how this possibility is related to the linear consistency condition (7.24). In particular it can be interesting to notice that, if we write (7.24) in the form:

$$\hat{\mathcal{D}}\Delta_{eff} = 0 \ , \tag{8.3}$$

the differential operator $\hat{\mathcal{D}}$ is not nilpotent. Indeed in the case of anticommuting γ and commuting ϕ one has ($\hbar = 1$)

$$\hat{\mathcal{D}}^2 = \int d^4p \left[\frac{\delta}{\delta\gamma(p)} \Delta_{eff} \frac{\delta}{\delta\phi(-p)} - \frac{\delta}{\delta\phi(-p)} \Delta_{eff} \frac{\delta}{\delta\gamma(p)} \right] \ . \tag{8.4}$$

Here we have assumed ϕ commuting and γ anticommuting.

Thus the nilpotency of the perturbative coboundary operator $\mathcal{D}$ is violated beyond the pertubative level by terms of order Δ_{eff}.

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