One-loop three-point functions of BMN operators at weak and strong coupling

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Three-point functions

- Two-point functions/anomalous dimensions have been studied thoroughly (to all loops)
- Knowledge of three-point functions would have vested us with full knowledge of the $\mathcal{N} = 4$ SYM.
- More or less everything is known on correlators of “light” states; less is known about heavy-heavy-light operators; very little is known about heavy-heavy-heavy correlators.

Motivation

- Three-point functions are non-protected objects, thus they can have all kinds of corrections
- There is no guarantee that the standard strings-to-fields equivalence methods will work
- The operator-to-string equivalence definition for purposes of three-point function construction is not unambiguous
- There are significant discrepancies at one-loop level *[Bissi, Harmark, Orselli 2011]*
- Thus it is important to classify all the sectors in which there are discrepancies so that their nature could be established.

Objectives

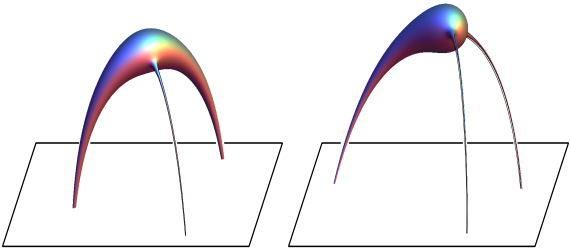
A long-range goal is to come closer to understanding the correlator of three giant magnons, which is a semiclassical calculation still on the to-do list. In some sense it would be the "heaviest possible" correlator. One of the motivations of the present work is to provide a counterpart for testing it in the future, which must be feasible in the Frolov-Tseytlin limit. That is why we are interesting in a correlator with all three states BMN (and not BMN-BMN-BPS, which have been extensively studied previously).

How to obtain three-loop functions

- From field theory *[Beisert, Kristjansen, Plefka, Semenoff, Staudacher 2002]*: by direct perturbative calculation
- From Bethe Ansatz*[Gromov, Vieira 2012]*
- From string field theory: as matrix elements of the Dobashi-Yoneya vertex *[Dobashi,Yoneya 2004]*
- From string theory in the semiclassic regime: as action value on a worldsheet solution with three delta-sources *[Zarembo 2010]*

Not this work: semiclassics

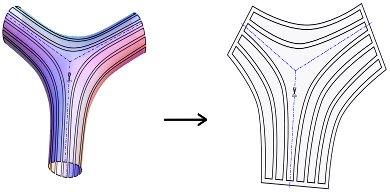
An illustration to what this work is NOT: to get a three-point function semiclassically one should evaluate string theory action on a surface like these ones



Here on the left is a typical heavy-heavy-light worldsheet, and on the right a heavy-light-light. Figure courtesy of Kolya Gromov.

Not this work: integrability

This work is also NOT an integrability work. Here the Gromov-Vieira tailoring procedure is illustrated, for details see *[Gromov, Vieira 1205.5288]*:



The procedure expresses perturbative magnon dynamics in terms of concise formulae, which is advantageous for large number of magnons reducing greatly the computer- and man-power consumption of the calculation, yet still needs a direct perturbative verification, due to the complexity of the “integrability” result derivation. Figure courtesy of Kolya Gromov.

Asymptotics vs. Theories

Three-point functions can be obtained from field theory, field theory assisted by Bethe Ansatz, string field theory, string theory semiclassics in the following regimes:

Based on	λ	R-charge J	$\lambda' = \frac{\lambda}{J^2}$	Nr. of magnons
FT	small	any	any	small
Bethe Ansatz	small	any	any	any
SFT	large	any	any	small
ST semiclassics	large	large	any	large

Thus the unique possibility to compare these objects is the λ' expansion in Frolov-Tseytlin limit, where asymptotically large-coupling and small-coupling limits can be unified.

Definitions

We consider two-magnon BMN operators

$$\mathcal{O}^J_{ij,n} = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} \mathrm{tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right) \psi_{n,l},$$

which fall into the three irreducible representations of $SO(4)$; we choose the symmetric one for which

$$\psi^S_{n,l} = \cos\frac{(2l+1)\pi n}{J+1}$$

We consider three operators: $\mathcal{O}_1 = \mathcal{O}^{J_1,12}_{n_1}$, $\mathcal{O}_2 = \mathcal{O}^{J_2,23}_{n_2}$, $\mathcal{O} = \mathcal{O}^{J,31}_{n}$, where n_1, n_2, n_3 are the magnon momenta, J_1, J_2, J_3 are their R-charges R_3, $J = J_1 + J_2$, $J_1 = Jy$,$J_2 = J(1-y)$.

We shall be looking for the quantity

$$C_{123} = \langle \bar{\mathcal{O}}_3 \mathcal{O}_1 \mathcal{O}_2 \rangle$$

as a function of y, J, n_1, n_2, n_3, and compare it at one loop in FT with ST in Penrose limit.

Problems on our way

Let us point out some of the obstacles that may be encountered on the way to three-point functions:

1. Double-trace admixture
2. Fermionic operators admixture
3. Magnon number nonconserving admixture

Happily enough, problems (1) and (2) are resolved by choosing the symmetric sector operators in $SO(6)$, and (3) is resolved by invoking the large-J limit.

String field theory calculation

In terms of the BMN basis $\{\alpha_m\}$ our operators look like

$$\mathcal{O}_m = \alpha^\dagger_m \alpha^\dagger_{-m}|0\rangle$$

The three-point function is related to the matrix element of the string field Hamiltonian as follows

$$\langle \bar{\mathcal{O}}_3 \mathcal{O}_1 \mathcal{O}_2 \rangle = \frac{4\pi}{-\Delta_3 + \Delta_1 + \Delta_2} \sqrt{\frac{J_1 J_2}{J}} H_{123}$$

where

$$\Delta_i = J_i + 2\sqrt{1 + \lambda' n_i^2},$$

and the matrix element is defined as

$$H_{123} = \langle 123 | V \rangle.$$

Dobashi–Yoneya prefactor

We use the findings of *[Grignani et al. 2006]* to start with the Dobashi–Yoneya prefactor *[Dobashi, Yoneya 2004]* in the natural string basis $\{a_m^r\}$.

$$V = P\, e^{\frac{1}{2}\sum_{m,n} N^{rs}_{mn}\delta^{IJ} a_m^{rI\dagger} a_n^{sJ\dagger}}\,.$$

Here I, J are $SU(4)$ flavour indices, r, s run within $1, 2, 3$ and refer to the first, second and third operator. The natural string basis is related to the BMN basis for $m > 0$ as follows

$$\alpha_m = \frac{a_m + ia_{-m}}{\sqrt{2}}\,,\ \alpha_{-m} = \frac{a_m - ia_{-m}}{\sqrt{2}}$$

The Neumann matrices are given as

$$N^{rs}_{m,n} = \frac{1}{2\pi}\,\frac{(-1)^{r(m+1)+s(n+1)}}{x_s\omega_{rm} + x_r\omega_{sn}}\sqrt{\frac{x_r x_s(\omega_{rm} + \mu x_r)(\omega_{sn} + \mu x_s)s_{rm}s_{qn}}{\omega_{rm}\omega_{sn}}}\,,$$

$$N^{rs}_{-m,-n} = -\frac{1}{2\pi}\,\frac{(-1)^{r(m+1)+s(n+1)}}{x_s\omega_{rm} + x_r\omega_{sn}}\sqrt{\frac{x_r x_s(\omega_{rm} - \mu x_r)(\omega_{sn} - \mu x_s)s_{rm}s_{qn}}{\omega_{rm}\omega_{sn}}}\,,$$

where m, n are always meant positive, $s_{1m} = 1$, $s_{2m} = 1$, $s_{3m} = -2\sin(\pi m y)$, $x_1 = y$, $x_2 = 1 - y$, $x_3 = -1$, the frequencies are $\omega_{r,m} = \sqrt{m^2 + \mu^2 x_r^2}$,, and the expansion parameter is $\mu = \frac{1}{\sqrt{\lambda'}}$. The Dobashi-Yoneya prefactor we are using is the prefactor supported with positive modes only:

$$P = \sum_{m>0}\sum_{r,I}\frac{\omega_r}{\mu\alpha_r} a_m^{Ir\dagger} a_m^{Ir}.$$

String result

Due to the flavour structure of C_{123} the only combinations of terms from the exponent that could contribute are $N^{12}_{n_1 n_2} N^{23}_{n_2 n_3} N^{31}_{n_3 n_1}$. The leading order contribution is

$$C^0_{123} = \frac{1}{\pi^2} \frac{\sqrt{J}}{N} \frac{n_3^2 y^{3/2} (1-y)^{3/2} \sin^2(\pi n_3 y)}{(n_3^2 y^2 - n_1^2)(n_3^2 (1-y)^2 - n_2^2)}$$

The next-order coefficient in the expansion

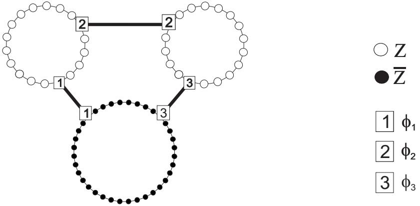
$$C_{123} = C^0_{123} \left(1 + \lambda' c^1_{123}\right),$$

where $c^1_{123} \equiv \frac{C^1_{123}}{C^0_{123}}$ is

$$c^1_{123} = -\frac{1}{4} \left(\frac{n_1^2}{y^2} + \frac{n_2^2}{(1-y)^2} + n_3^2 \right).$$

Let us compare this calculation to the field theory calculation.

The tree-level diagram is shown below:



and evaluates in the leading order to

$$N\sqrt{J_1 J_2 J}\sum_{l_1,l_2}\cos\frac{\pi(2l_1+1)}{J_1+1}\cos\frac{\pi(2l_2+1)}{J_2+1}\cos\frac{\pi(2(l_1+l_2)+1)}{J+1},$$

which after the $1/J$ expansion and the due normalization of the operator to unity yields

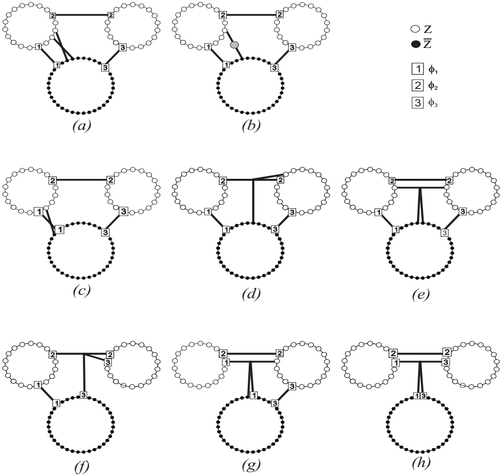
$$C^0_{123}=\frac{1}{\pi^2}\frac{\sqrt{J}}{N}\frac{n_3^2 y^{3/2}(1-y)^{3/2}\sin^2(\pi n_3 y)}{(n_3^2y^2-n_1^2)(n_3^2(1-y)^2-n_2^2)},$$

corresponding exactly to the ST result above.

One Loop

At the one loop level we estimate all possible insertions of the interaction terms of the Hamiltonian

$H_2 = \frac{\lambda}{8\pi^2}(I - P)$ depicted below:



We systematize these contributions in the table below. J order shown without the overall norm prefactor

Diagram	Vertex type	Order in $1/J$
a	Z^4	J^3
b	Self-energy	J^3
c	$Z^2\phi^2$	J^2
d	$Z^2\phi^2$	J^2
e	Z^4	J^2
f	ϕ^4	J
g	$Z^2\phi^2$	J
h	ϕ^4	1

The final result has the order J^0, thus cancellation of higher orders is a test for correct perturbative calculations.

One loop result from FT

After summation (details not shown here) we get

$$c_{123}^{1} = -\frac{1}{4}\left(\frac{n_1^2}{y^2} + \frac{n_2^2}{(1-y)^2} + n_3^2\right).$$

exactly as in the string theory above.

Result

The three-point correlation function for all dynamical BMN operators matches precisely the perturbative weakly coupled planar field theory and the Penrose limit of the strongly coupled string field theory at one loop level in the Frolov–Tseytlin limit.

Discussion

- The result is quite unexpected, since, on one hand, a correlator of two heavy and one light operators has been previously demonstrated by Agnese to disagree the semiclassical string calculation in the Frolov–Tseytlin limit.
- On the other hand, a heavy-heavy-light correlator calculated via integrability by Gromov has been shown to beautifully agree with the string theory in the Frolov–Tseytlin limit, yet **only in the thermodynamical regime**, when the number of excitations tends to infinity.

Conclusions

- Our result is thus the only one-loop analytic calculation of a three-point function so far, where complete agreement between fields and strings is observed.
- As noted by Agnese, such a matching is not necessarily present even at tree level, since C_{123} is unprotected.
- Thus our case should be considered as another “wonder” of AdS/CFT and must be explained somehow. The well known state/operator identification for BMN states, which in other cases is not so well established, certainly helps in providing this matching. However, we do not yet possess a generic argument why this must work in a more general setting; neither can we guess which corner of the parameter space may be covered by the conjecture on exact matching between the three-point functions on gauge and gravity sides.

Acknowledgments

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for interesting and stimulating discussions.

Extra

In the spin chain picture, each of the three single trace operators corresponds to a state on a closed chain. The closed chains are cut into right and left open chains where the external states are represented. The three states are sewed together into the three-point function by overlapping the wave functions on each right chain with the wave function on the left subchain of the next operator.

- Cut
- Flip
- Overlap