Amplitudes, Wilson loops, Symbols and Coproducts in N=4 Super Yang-Mills

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Motivation

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  over 30 years since first non trivial computations K. Ellis Ross Terrano 81

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- can we envisage a similar leap forward ?

N=4 Super Yang-Mills

- maximal supersymmetric theory (without gravity)
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  - spin 1 gluon
    4 spin 1/2 gluinos
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- AdS/CFT duality Maldacena 97
  - large-λ limit of 4dim CFT ↔ weakly-coupled string theory
    (aka weak-strong duality)

AdS/CFT duality, amplitudes & Wilson loops

- planar scattering amplitude at strong coupling — Alday Maldacena 07

$$\mathcal{M} \sim \exp\left[i \frac{\sqrt{\lambda}}{2\pi} (Area)_{cl} \right]$$

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- computation ``formally the same as ... the expectation value of a Wilson loop given by a sequence of light-like segments"

MHV amplitudes in planar N=4 SYM

- at any order in the coupling, colour-ordered MHV amplitude in N=4 SYM can be written as tree-level amplitude times helicity-free loop coefficient $M_n^{(L)} = M_n^{(0)} m_n^{(L)}$

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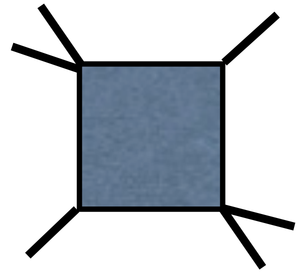
$$m_n^{(1)} = \sum_{pq} F^{\rm 2me}(p,q,P,Q) \qquad n \geq 6$$



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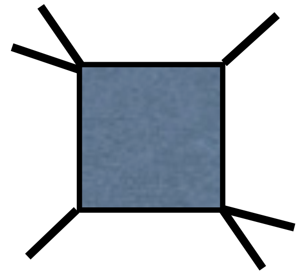

- at 2 loops, iteration formula for the *n*-pt amplitude

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

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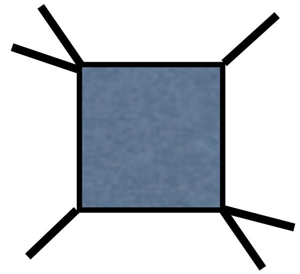

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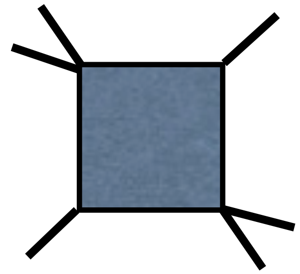

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remainder function

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ansatz for MHV amplitudes in planar N=4 SYM

Bern Dixon Smirnov 05

$$M_n = M_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon) \right]$$

$$= M_n^{(0)} \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^\epsilon$ $\lambda = g^2 N$ ‘t Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \, \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 \, f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

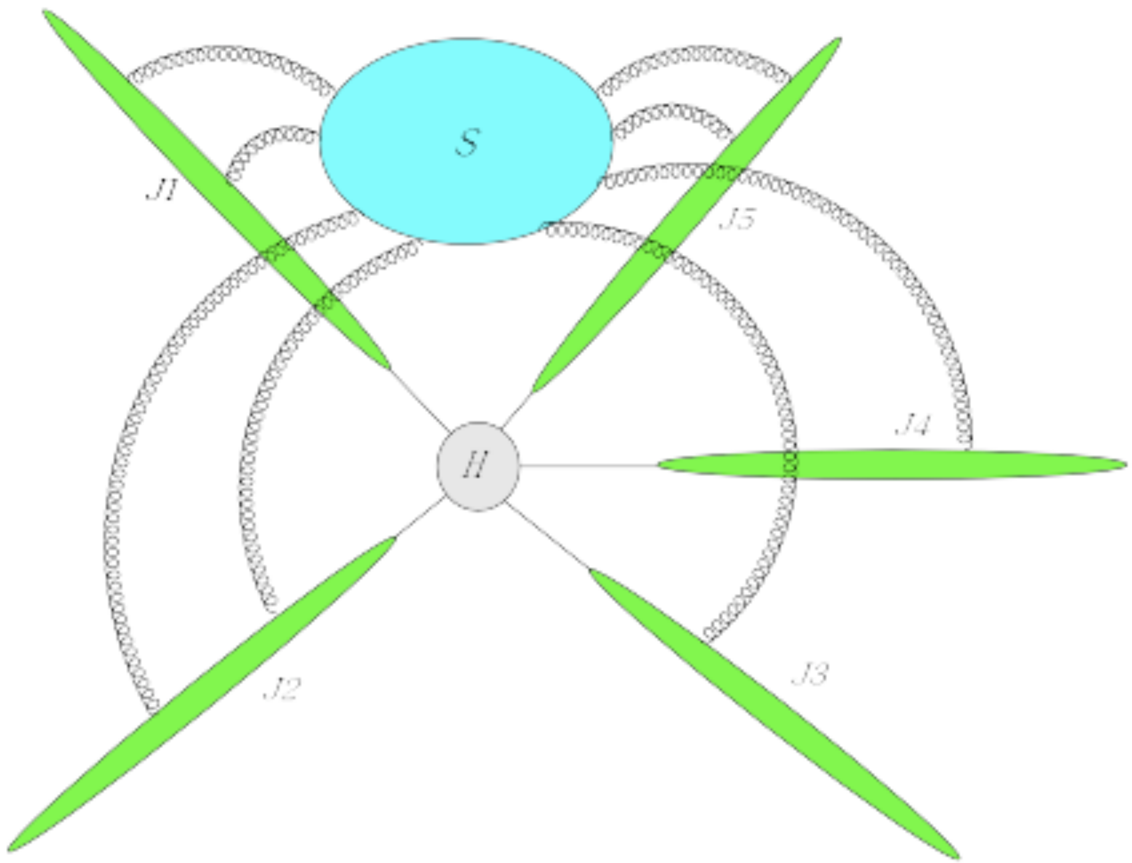
$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a*

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06

$\hat{G}^{(l)}$ collinear anomalous dimension, known through O(a^4)

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07

Factorisation of a multi-leg amplitude in QCD



Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
Aybat Dixon Sterman 2006
Becher Neubert 2009
Gardi Magnea 2009

$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}(\beta_i\cdot\beta_j,\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)\prod_i \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

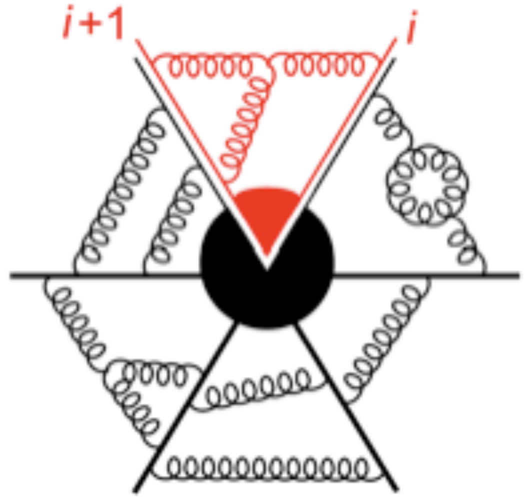
$p_i = \beta_i Q_0/\sqrt{2}$ value of Q_0 is immaterial in *S, J*

to avoid double counting of soft-collinear region (IR double poles),
$\mathcal{J}_i$ removes eikonal part from J_i, which is already in *S*
$J_i/\mathcal{J}_i$ contains only single collinear poles

$\mathcal{N} = 4$ SYM in the planar limit

- colour-wise, the planar limit is trivial: can absorb S into J_i

- each slice is square root of Sudakov form factor



$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon\right)\right]^{1/2} h_n(\{p_i\}, \mu^2, \alpha_s, \epsilon)$$

- β fn = 0 ⇒ coupling runs only through dimension $\bar{\alpha}_s(\mu^2)\mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon}$

Sudakov form factor has simple solution

$$\ln\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \left(\frac{-Q^2}{\mu^2}\right)^{-n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right]$$

Magnea Sterman 90
Bern Dixon Smirnov 05

⇒ IR structure of $\mathcal{N} = 4$ SYM amplitudes

- the ansatz checked for the 3-loop 4-pt amplitude — Bern Dixon Smirnov 05
  2-loop 5-pt amplitude — Cachazo Spradlin Volovich 06; Bern Czakon Kosower Roiban Smirnov 06

- the ansatz fails on 2-loop 6-pt amplitude — Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08; Alday Maldacena 07; Bartels Lipatov Sabio-Vera 08

- at 2 loops, the remainder function characterises the deviation from the ansatz

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

  - for $n = 4, 5$, R is a constant
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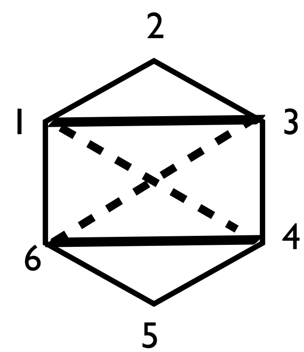
  - for $n = 6$, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$



x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

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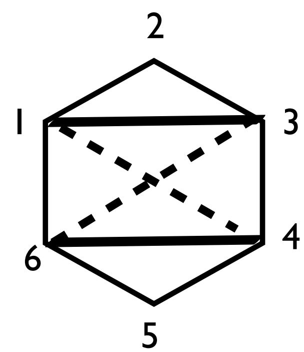

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$R_6^{(2)}$ known

numerically — Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08; Drummond Henn Korchemsky Sokatchev 08; Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

analytically — Duhr Smirnov VDD 09

Wilson loops

- $$W[\mathcal{C}_n] = \mathrm{Tr}\ \mathcal{P}\ \exp\left[ig \oint \mathrm{d}\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))\right]$$

closed contour $\mathcal{C}_n$ made by light-like external momenta

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Alday Maldacena 07

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- non-Abelian exponentiation theorem: vev of Wilson loop as an exponential, allows us to compute the log of W

  Gatheral 83
  Frenkel Taylor 84

  $$\langle W[\mathcal{C}_n]\rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)}$$

  through 2 loops $\quad w_n^{(1)} = W_n^{(1)} \qquad w_n^{(2)} = W_n^{(2)} - \frac{1}{2}\left(W_n^{(1)}\right)^2$

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- relation between 1 loop amplitudes & Wilson loops

  $$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - n\frac{\zeta_2}{2} + \mathcal{O}(\epsilon)$$

  Brandhuber Heslop Travaglini 07

Wilson loops & Ward identities

Drummond Henn Korchemsky Sokatchev 07

- N=4 SYM is invariant under $SO(2,4)$ conformal transformations
- the Wilson loops fulfill conformal Ward identities
- the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + R
  - at 2 loops

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

with $\quad f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\epsilon - 5\zeta_4\epsilon^2$

(to be compared with $\quad f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \quad$ for the amplitudes)

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- $R_{n,WL}^{(2)}$ arbitrary function of conformally invariant cross ratios

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- duality Wilson loop ⇔ MHV amplitude is expressed by

$$R_{n,WL}^{(2)} = R_n^{(2)}$$

MHV amplitudes ⇔ Wilson loops

- agreement between *n*-edged Wilson loop and *n*-point MHV amplitude at weak coupling (aka weak-weak duality)
  - verified for *n*-edged 1-loop Wilson loop up to 6-edged 2-loop Wilson loop

Brandhuber Heslop Travaglini 07

Drummond Henn Korchemsky Sokatchev 07

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

- *n*-edged 2-loop Wilson loops computed (numerically)

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

- no amplitudes are known beyond the 6-point 2-loop amplitude!

2-loop 6-edged remainder function $R_6^{(2)}$

Duhr Smirnov VDD 09

- the remainder function $R_6^{(2)}$ is explicitly dependent on the cross ratios u_1, u_2, u_3
- it is symmetric in all its arguments
  (for $n > 6$, it is symmetric under cyclic permutations and reflections)
- it is of uniform transcendental weight 4
  transcendental weights: $w(\ln x) = w(\pi) = 1 \qquad w(\mathrm{Li}_2(x)) = w(\pi^2) = 2$
- it vanishes under collinear and multi-Regge limits (in Euclidean space)
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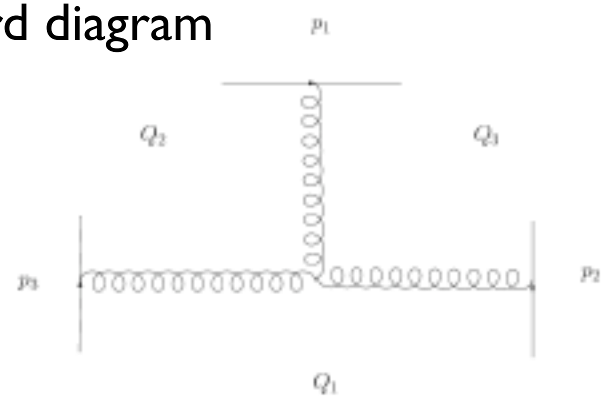
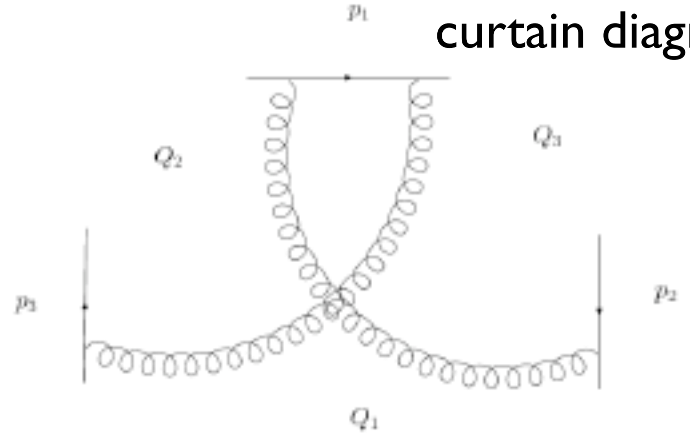
↓ finite answer, but in intermediate steps many divergences
output is punishingly long

Diagrams of 2-loop Wilson loops

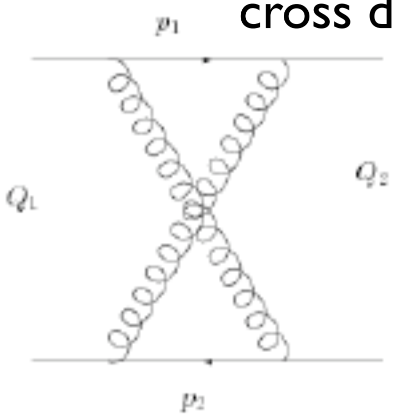
hard diagram



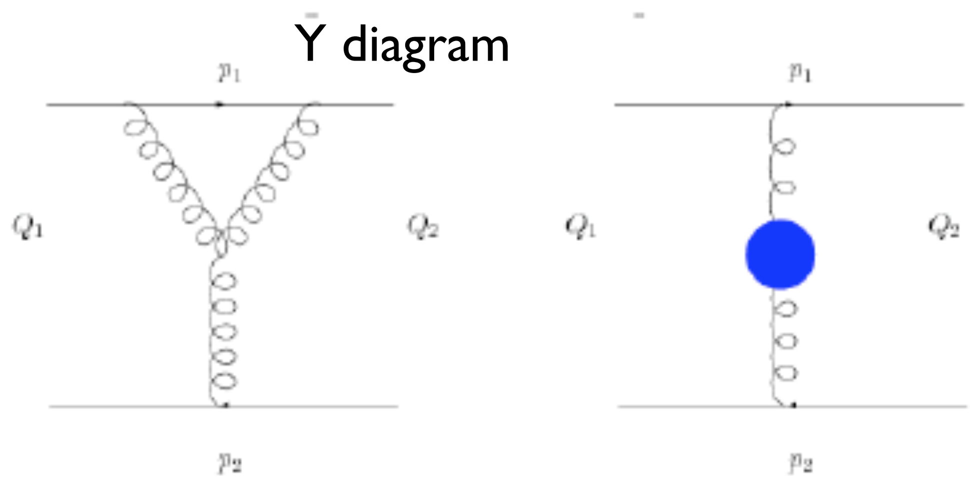
curtain diagram

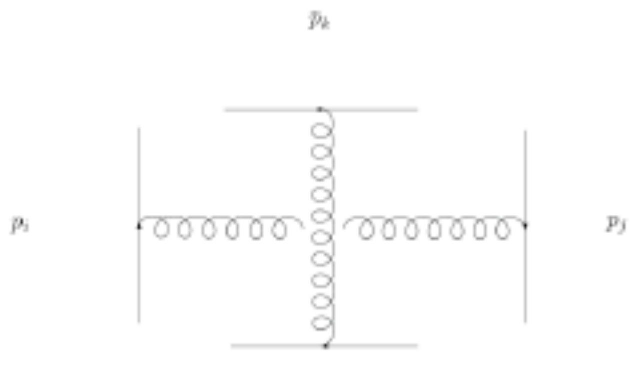


cross diagram



Y diagram





factorised cross diagram

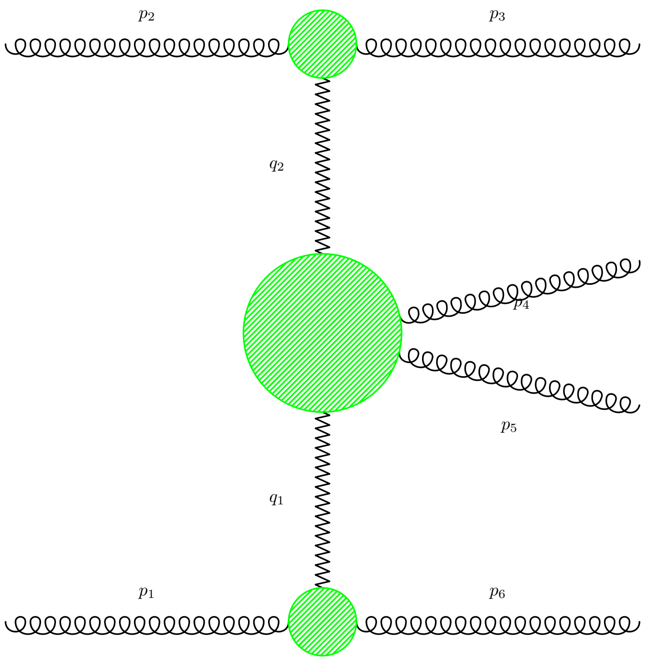
Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

each diagram yields an integral,
similar to a Feynman-parameter integral

Quasi-multi-Regge limit of hexagon Wilson loop

- 6-pt amplitude in the qmR limit of a pair along the ladder

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|$$



the conformally invariant cross ratios are

$$u_{36} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$u_{14} = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

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the cross ratios are all $O(1)$
→ R_6 does not change its functional dependence on the u's

- R_6 is invariant under the qmR limit of a pair along the ladder

Duhr Glover Smirnov VDD 08

Quasi-multi-Regge limit of Wilson loops

- *L*-loop Wilson loops are Regge exact

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09

$$w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon)\, w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + \mathcal{O}(\epsilon)$$

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$$\ln(s_{ij}) \quad + \quad \mathrm{Li}_2(1-u_{ij})$$

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- we may compute the Wilson loop in qmRk
  the result will be correct in general kinematics !!!

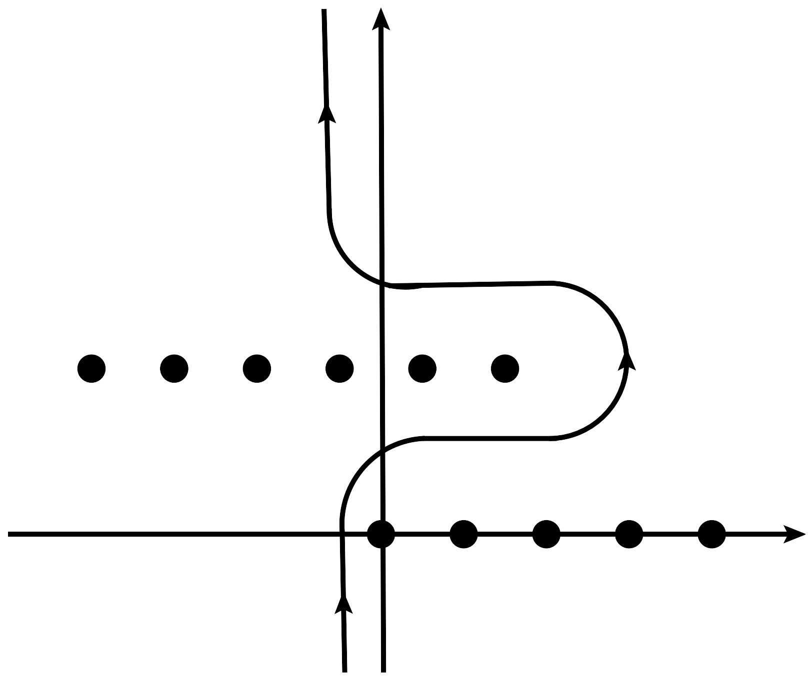
Wilson loops: analytic calc

1. Use Mellin-Barnes (MB) representation of the Feynman-parameter integrals: replace each denominator by a contour integral

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\,\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\Gamma(-z)\,\Gamma(\lambda+z)\,\frac{A^z}{B^{\lambda+z}}$$

integral turns into a sum of residues

$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$$



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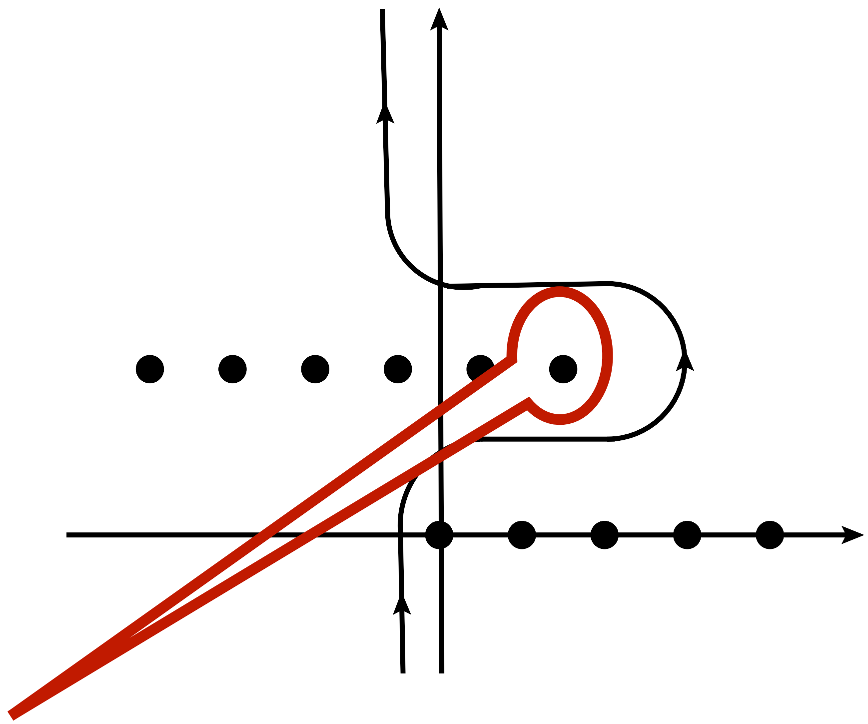
2. Use Regge exactness in the qmR limit: retain only leading behaviour (*i.e.* leading residues) of the integral



leading residue

Wilson loops: analytic calc

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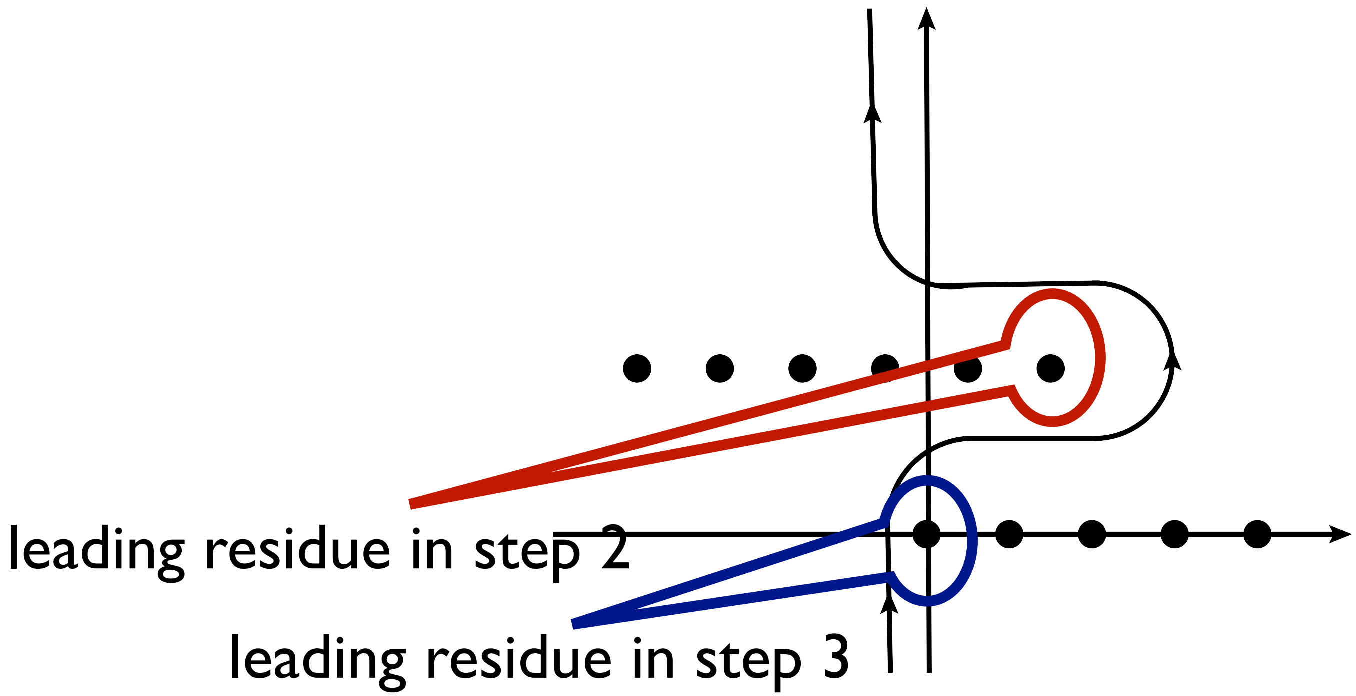

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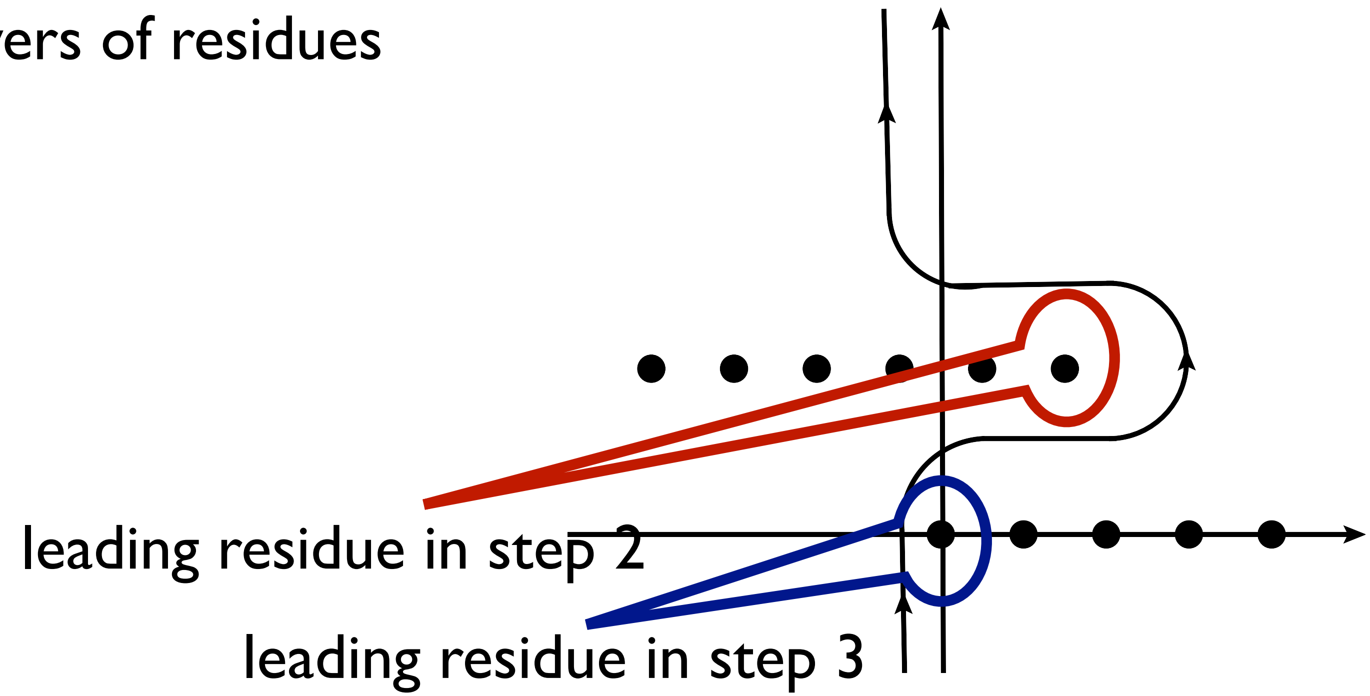
4. Sum remaining towers of residues

$$\sum_{n=1}^{\infty} \frac{u^n}{n} = -\ln(1-u)$$

$$\sum_{n=1}^{\infty} \frac{u^n}{n^k} = \mathrm{Li}_k(u)$$



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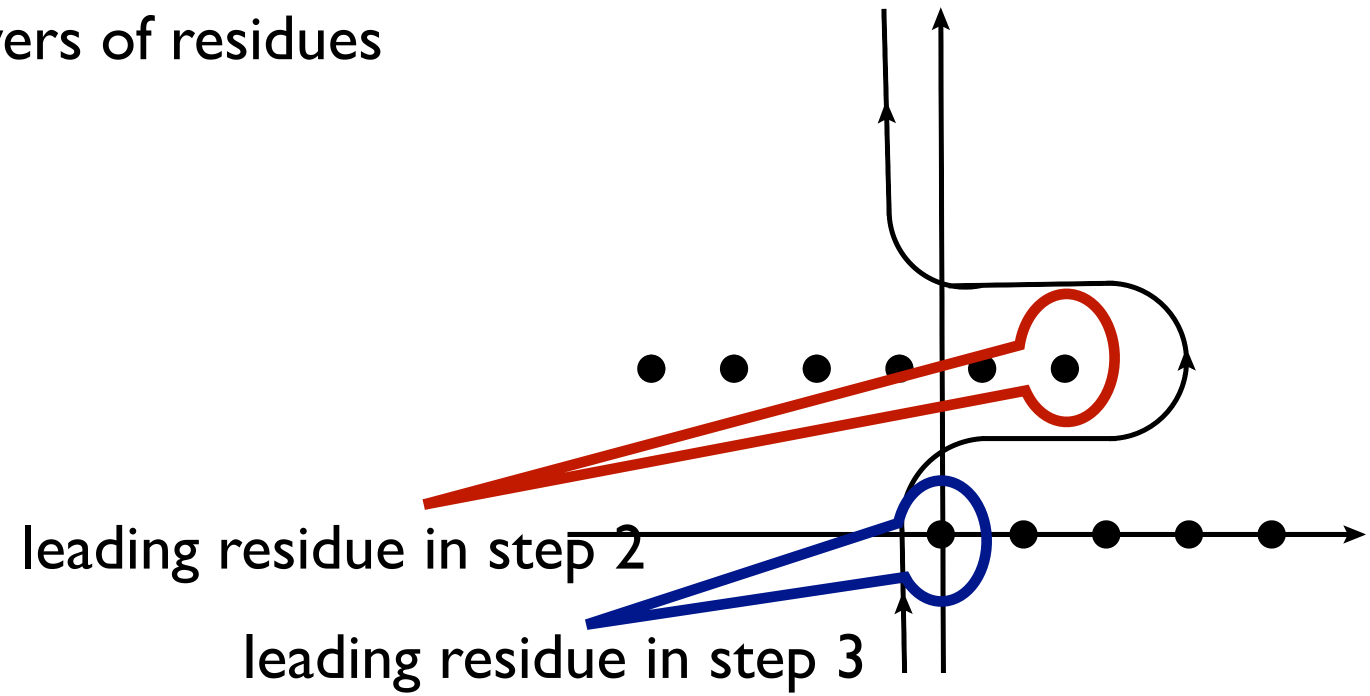

in general, get nested harmonic sums → multiple polylogarithms

$$\sum_{n_1=1}^{\infty} \frac{u_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \cdots \sum_{n_k=1}^{n_{k-1}-1} \frac{u_k^{n_k}}{n_k^{m_k}} = (-1)^k\, G\Bigg(\underbrace{0,\ldots,0}_{m_1-1}, \frac{1}{u_1}, \ldots, \underbrace{0,\ldots,0}_{m_k-1}, \frac{1}{u_1 \ldots u_k}; 1\Bigg)$$

Analytic 2-loop 6-edged Wilson loop

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  in fact, only one 3-fold integral, which comes from $f_H(p_1,p_3,p_5;p_4,p_6,p_2)$

$$\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}z_1}{2\pi i}\frac{\mathrm{d}z_2}{2\pi i}\frac{\mathrm{d}z_3}{2\pi i}\left(z_1\,z_2+z_2\,z_3+z_3\,z_1\right)u_1^{z_1}\,u_2^{z_2}\,u_3^{z_3}$$
$$\times\,\Gamma\left(-z_1\right)^2\,\Gamma\left(-z_2\right)^2\,\Gamma\left(-z_3\right)^2\,\Gamma\left(z_1+z_2\right)\,\Gamma\left(z_2+z_3\right)\,\Gamma\left(z_3+z_1\right)$$

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the result is in terms of multiple polylogarithms

$$G(a,\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a}\,G(\vec{w};t)\,, \qquad G(a;z) = \ln\left(1-\frac{z}{a}\right)$$

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$$\times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1 + z_2) \Gamma(z_2 + z_3) \Gamma(z_3 + z_1)$$

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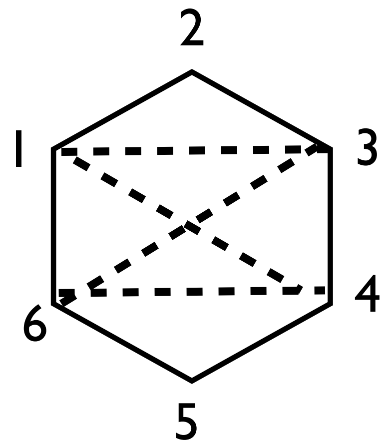
- the remainder function $R_6^{(2)}$ is given in terms of $O(10^3)$ multiple polylogarithms $G(u_1, u_2, u_3)$

Duhr Smirnov VDD 09

Z_n symmetric regular hexagons

regular hexagons are characterised by

$$x_{13}^2 = x_{24}^2 = x_{35}^2 = x_{46}^2 = x_{51}^2 = x_{62}^2; \qquad x_{14}^2 = x_{25}^2 = x_{36}^2$$



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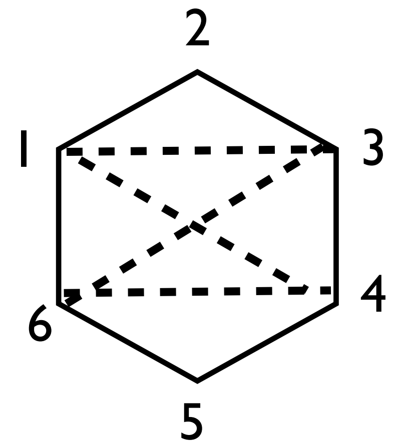

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At strong coupling, remainder function is obtained from ``minimal area surfaces in $AdS_5$ which end on a null polygonal contour at the boundary". One gets ``integral equations which determine the area as a function of the shape of the polygon. The equations are identical to those of the Thermodynamics Bethe Ansatz. The area is given by the free energy of the TBA system. The high temperature limit of the TBA system can be exactly solved"

$$R_6^{strong}(u,u,u) = \underbrace{\frac{\pi}{6} - \frac{1}{3\pi}\phi^2}_{\text{free energy}} - \frac{3}{8}\left(\underbrace{\ln^2(u) + 2\,\mathrm{Li}^2(1-u)}_{\text{BDS - BDSlike}}\right) \qquad u = \frac{1}{4\cos^2(\phi/3)}$$

Alday Gaiotto Maldacena 09

Remainder function at weak and strong coupling

compare remainder functions at weak and strong coupling introducing coefficients in the strong coupling result and try to curve fit the 2 results

$$R_6^{strong}(u,u,u) = c_1\left(\frac{\pi}{6} - \frac{1}{3\pi}\phi^2\right) + c_2\left(\frac{3}{8}\left(\ln^2(u) + 2\,\mathrm{Li}^2(1-u)\right)\right) + c_3$$

$$c_1 = 0.263\pi^3 \qquad c_2 = 0.860\pi^2 \qquad c_3 = -\frac{\pi^2}{12}c_2$$

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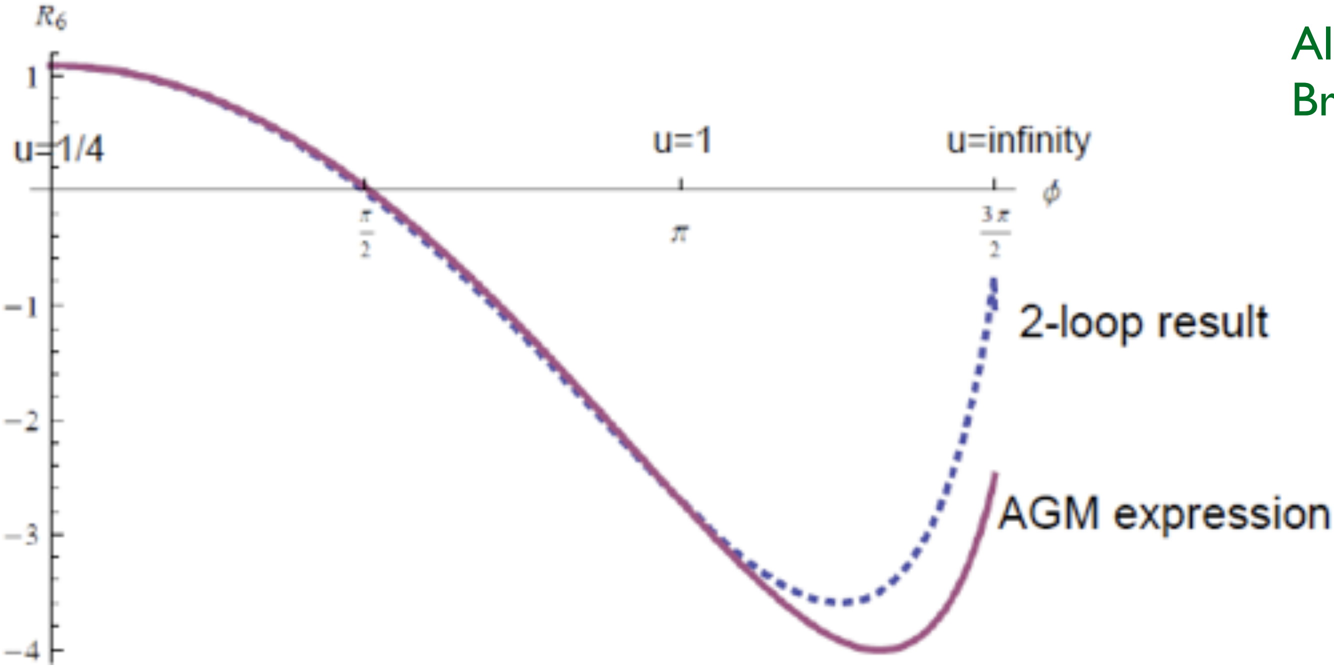

- the 2 curves are strikingly similar

the remainder $R_6^{(2)}$ has been simplified and given in terms of polylogarithms

Goncharov Spradlin Vergu Volovich 10

$$R_{6,WL}^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

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$$L_4(x^+,x^-) = \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-)) + \frac{1}{8!!}\log(x^+x^-)^4$$

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↑ answer is short and simple
introduces *symbols* in TH physics

Symbols

- take a fn. defined as an iterated integral of logs of rational functions R_i

$$T^{(k)} = \int_a^b \mathrm{d}\, lnR_1 \circ \cdots \circ \mathrm{d}\, lnR_k = \int_a^b \left(\int_a^t \mathrm{d}\, lnR_1 \circ \cdots \circ \mathrm{d}\, lnR_{k-1} \right) \mathrm{d}\, lnR_k(t)$$

then the total differential can be written as

$$d\,T^{(k)} = \sum_i T_i^{(k-1)} d \ln R_i$$

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- the symbol is defined recursively as $\mathrm{Sym}[T^{(k)}] = \sum_i \mathrm{Sym}[T_i^{(k-1)}] \otimes R_i$

Goncharov

as such, the symbol is defined on the tensor product of the group of rational functions, modulo constants

$$\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots$$

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- if *T* is a multiple polylogarithm *G*, then

$$dG(a_{n-1}, \ldots, a_1; a_n) = \sum_{i=1}^{n-1} G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n) d \ln \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right)$$

the symbol is

$$\mathrm{Sym}\left(G(a_{n-1}, \ldots, a_1; a_n)\right) = \sum_{i=1}^{n-1} \mathrm{Sym}\left(G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n)\right) \otimes \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right)$$

- Euler and Nielsen polylogarithms are multiple polylogarithms with special arguments

$$G(\vec{0}_n;x) = \frac{1}{n!}\ln^n x \qquad G(\vec{a}_n;x) = \frac{1}{n!}\ln^n\left(1-\frac{x}{a}\right)$$

$$G(\vec{0}_{n-1},a;x) = -\,\mathrm{Li}_n\left(\frac{x}{a}\right) \qquad G(\vec{0}_n,\vec{a}_m;x) = (-1)^m\,S_{n,m}\left(\frac{x}{a}\right) \qquad S_{n-1,1}(x) = \mathrm{Li}_n(x)$$

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- when the root equals +1,-1,0 multiple polylogarithms become harmonic polylogarithms (HPLs)

$$H(a, \vec{w}; z) = \int_0^z \mathrm{d}t\, f(a;t)\, H(\vec{w};t) \qquad f(-1;t) = \frac{1}{1+t}, \quad f(0;t) = \frac{1}{t}, \quad f(1;t) = \frac{1}{1-t}$$

with $\{a, \vec{w}\} \in \{-1, 0, 1\}$

Remiddi Vermaseren

when the root equals +1,0 HPLs reduce to Euler and Nielsen polylogarithms

$$\mathrm{Li}_n(x) = H(\vec{0}_{n-1}, 1; x) \qquad S_{n,m}(x) = H(\vec{0}_n, \vec{1}_m; x)$$

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- ... on to symbols

$$\mathrm{Sym}[\ln x] = x \qquad \mathrm{Sym}\left[\frac{1}{n!}\ln^n x\right] = \underbrace{x\otimes\cdots\otimes x} \equiv x^{\otimes n}$$

$$\mathrm{Sym}[\mathrm{Li}_n(x)] = -(1-x)\otimes x^{\otimes(n-1)}$$

$$\mathrm{Sym}[S_{n,m}(x)] = (-1)^m(1-x)^{\otimes m}\otimes x^{\otimes n}$$

$$\mathrm{Sym}[H(a_1,\ldots,a_n;x)] = (-1)^k(a_n-x)\otimes\cdots\otimes(a_1-x) \qquad \{a_i\}\in\{0,1\}$$

k is the number of a's equal to 1

- using symbols, one can reduce the HPLs to a minimal set — Buehler Duhr 11

weight 1: $B_1^{(1)}(x) = \ln x\,, \qquad B_1^{(2)}(x) = \ln(1-x)\,, \qquad B_1^{(3)}(x) = \ln(1+x)$

weight 2: $B_2^{(1)}(x) = \mathrm{Li}_2(x)\,, \qquad B_2^{(2)}(x) = \mathrm{Li}_2(-x)\,, \qquad B_2^{(3)}(x) = \mathrm{Li}_2\left(\frac{1-x}{2}\right)$

weight 3: polylogarithms of type Li_3 of various arguments

weight 4: polylogarithms of type Li_4 of various arguments,
plus a few polylogarithms of type $\mathrm{Li}_{2,2}$, like $\mathrm{Li}_{2,2}(-1, x)$ etc.
Alternatively, the polylogarithms of type $\mathrm{Li}_{2,2}$ can be replaced by the HPLs: $H(0,1,0,-1;x)$ and $H(0,1,1,-1;x)$

if needed numerically, any combination of HPLs up to weight 4 can be evaluated in terms of a minimal set of numerical routines

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if needed numerically, any combination of HPLs up to weight 4 can be evaluated in terms of a minimal set of numerical routines

- multiple polylogarithms are also defined through nested harmonic sums

$$\mathrm{Li}_{m_1,\dots,m_k}(u_1,\dots,u_k) = \sum_{n_k=1}^{\infty} \frac{u_k^{n_k}}{n_k^{m_k}} \sum_{n_{k-1}=1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{u_1^{n_1}}{n_1^{m_1}} = (-1)^k G_{m_k,\dots,m_1}\left(\frac{1}{u_k},\dots,\frac{1}{u_1\cdots u_k}\right)$$

$$G_{m_1,\dots,m_k}(u_1,\dots,u_k) = G\Bigg(\underbrace{0,\dots,0}_{m_1-1},u_1,\dots,\underbrace{0,\dots,0}_{m_k-1},u_k;1\Bigg)$$

- also multiple polylogarithms can be reduced to a minimal set — Duhr Gangl Rhodes 11

weight 1: one needs functions of type $\ln x$

weight 2: $\mathrm{Li}_2(x)$

weight 3: $\mathrm{Li}_3(x)$

weight 4: $\mathrm{Li}_4(x)$, $\mathrm{Li}_{2,2}(x,y)$

weight 5: $\mathrm{Li}_5(x)$, $\mathrm{Li}_{2,3}(x,y)$

weight 6: $\mathrm{Li}_6(x)$, $\mathrm{Li}_{2,4}(x,y)$, $\mathrm{Li}_{3,3}(x,y)$, $\mathrm{Li}_{2,2,2}(x,y,z)$

- the symbol knows about the discontinuities of *T*; if

$$\mathrm{Sym}[T^{(k)}] = R_1 \otimes \cdots \otimes R_k$$

then *T* has a branch cut at $R_l = 0$, and the symbol of the discontinuity is

$$\mathrm{Sym}[Disc_{R_1}(T^{(k)})] = R_2 \otimes \cdots \otimes R_k$$

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- $Disc(\ln x\ ln\ y) = \begin{cases} 2\pi i\ ln\ x & \text{along the } y \text{ cut } [-\infty, 0] \\ 2\pi i\ ln\ y & \text{along the } x \text{ cut } [-\infty, 0] \end{cases}$

$$\mathrm{Sym}[\ln x\ \ln y] = x \otimes y + y \otimes x$$

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- in general, if *Disc(f g) = Disc(f) g + f Disc(g)*

and $\mathrm{Sym}[f] = \otimes_{i=1}^{n} R_i \qquad \mathrm{Sym}[g] = \otimes_{i=n+1}^{m} R_i$

then $\mathrm{Sym}[f g] = \sum_\sigma \otimes_{i=1}^{n} R_{\sigma(i)}$

where σ denotes the set of all shuffles of *n*+(*m*-*n*) elements

e.g. $\mathrm{Sym}[f] = R_1 \otimes R_2 \qquad \mathrm{Sym}[g] = R_3 \otimes R_4$

$$\begin{aligned} \mathrm{Sym}[f g] = \; & R_1 \otimes R_2 \otimes R_3 \otimes R_4 + R_1 \otimes R_3 \otimes R_2 \otimes R_4 + R_1 \otimes R_3 \otimes R_4 \otimes R_2 \\ + \; & R_3 \otimes R_1 \otimes R_2 \otimes R_4 + R_3 \otimes R_1 \otimes R_4 \otimes R_2 + R_3 \otimes R_4 \otimes R_1 \otimes R_2 \end{aligned}$$

- the symbol knows about the discontinuities of T; if

$$\mathrm{Sym}[T^{(k)}] = R_1 \otimes \cdots \otimes R_k$$

then T has a branch cut at $R_l = 0$, and the symbol of the discontinuity is

$$\mathrm{Sym}[Disc_{R_1}(T^{(k)})] = R_2 \otimes \cdots \otimes R_k$$

- *Disc*(ln x *ln* y) = $\begin{cases} 2\pi i \ln x & \text{along the } y \text{ cut } [-\infty, 0] \\ 2\pi i \ln y & \text{along the } x \text{ cut } [-\infty, 0] \end{cases}$

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- symbols form a shuffle algebra, *i.e.* a vector space with a shuffle product
(also iterated integrals and multiple polylogarithms form shuffle algebras)

polylogarithm identities satisfied by the function f
become algebraic identities satisfied by its symbol

let us prove the identity $\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \ln x \ln(1-x) + \frac{\pi^2}{6}$

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proof $\mathrm{Sym}[\mathrm{Li}_2(x)] = -(1-x) \otimes x$ $\quad$ $\mathrm{Sym}[\mathrm{Li}_2(1-x)] = -x \otimes (1-x)$

$$\mathrm{Sym}[\ln x \ln(1-x)] = x \otimes (1-x) + (1-x) \otimes x$$

thus $\mathrm{Sym}[\mathrm{Li}_2(1-x)] = \mathrm{Sym}[-\mathrm{Li}_2(x) - \ln x \ln(1-x)]$

which determines the function up to functions of lesser degree

$$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \ln x \ln(1-x) + c\,\pi^2 + i\pi\,(c' \ln x + c'' \ln(1-x))$$

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but the equation is real for $0 < x < 1$, so $c' = c'' = 0$

at $x = 1$ $\qquad 0 = -\dfrac{\pi^2}{6} - 0 + c\,\pi^2 \quad \Longrightarrow \quad c = \dfrac{1}{6}$

- take f, g with $w(f) = w(g) = n$ and $\text{Sym}[f] = \text{Sym}[g]$
  then $f\text{-}g = h$ with $w(h) = n - 1$
  the symbol does not know about h
  info on the degree $n\text{-}1$ is lost by taking the symbol

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- in N=4 SYM, polynomials exhibit a uniform weight
  $w(\ln x) = 1,\ \ w(\mathrm{Li}_k(x)) = k,\ \ w(\pi) = 1$
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- take f, g with $w(f) = w(g) = n$ and $\mathrm{Sym}[f] = \mathrm{Sym}[g]$
  then f-$g = h$ with $w(h) = n - 1$
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  ➙ symbols fix polynomials up to factors of π times functions of lesser weight

Thus, we have a procedure to simplify a generic function of polylogarithms:

- find suitable variables (through momentum twistors or else) such that the arguments of the multiple polylogarithms become rational functions

- determine the symbol of the function

- through some symbol-processing procedure, find a simpler form of the integral in terms of multiple polylogarithms

Duhr Gangl Rhodes 11

Recent results on symbols

- symbol of *n*-point 2-loop MHV amplitudes/Wilson loops (in principle one can get the *n*-point 2-loop Wilson loop, but the symbol is complicated) Caron-Huot 11

- symbol of 6-point 3-loop MHV amplitude, up to 2 constants (and function in the multi-Regge limit) Dixon Drummond Henn 11

- symbol of 6-point 2-loop NMHV amplitude (and function up to a 1-dim integral) Dixon Drummond Henn 11

- symbol of non-planar massive double box (to be used in *qq*, *gg* → *ttbar*)

  von Manteuffel *presented at ACAT2011*

- symbol of 3-gluon 2-loop form factor Brandhuber Travaglini Yang 12

6-dim one-loop 6-point integrals

- $2n$-dim one-loop $2n$-pt integrals ($n > 2$) are finite and conformal invariant
- For n=3, its symbol contributes to the symbol of two-loop Wilson loop

Caron-Huot 11

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  Caron-Huot 11

- explicit expression of massless one-loop 6-pt integral
  is reminiscent of 2-loop 6-edged Wilson loop, but it has weight 3

Duhr Smirnov VDD 11
Dixon Drummond Henn 11

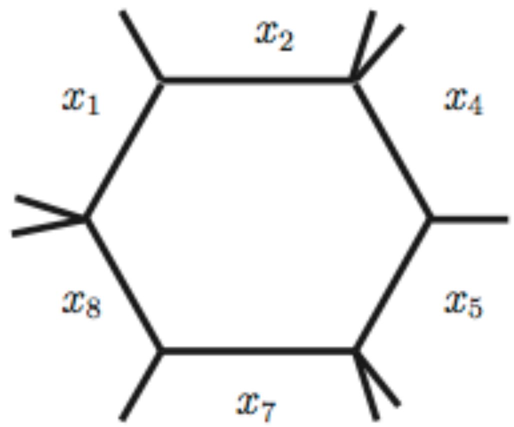
$$I_6(u_1,u_2,u_3) = \frac{1}{\sqrt{\Delta}}\Bigg[-2\sum_{i=1}^{3} L_3(x_i^+,x_i^-) + \frac{1}{3}\left(\sum_{i=1}^{3}\ell_1(x_i^+)-\ell_1(x_i^-)\right)^3 + \frac{\pi^2}{3}\chi\sum_{i=1}^{3}(\ell_1(x_i^+)-\ell_1(x_i^-))\Bigg]$$

$$L_3(x^+,x^-) = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!!}\ln^k(x^+x^-)\left(\ell_{3-k}(x^+)-\ell_{3-k}(x^-)\right)$$

6-dim *3-mass easy* one-loop 6-pt integral

- hexagon with 3 massive sides, x_{24}, x_{57}, x_{81}

  the cross ratios are



$$u_1 = \frac{x_{25}^2 x_{17}^2}{x_{15}^2 x_{27}^2}\,, \quad u_2 = \frac{x_{58}^2 x_{41}^2}{x_{48}^2 x_{15}^2}\,, \quad u_3 = \frac{x_{82}^2 x_{74}^2}{x_{27}^2 x_{48}^2}\,,$$

$$u_4 = \frac{x_{24}^2 x_{15}^2}{x_{14}^2 x_{25}^2}\,, \quad u_5 = \frac{x_{57}^2 x_{48}^2}{x_{47}^2 x_{58}^2}\,, \quad u_6 = \frac{x_{81}^2 x_{72}^2}{x_{82}^2 x_{17}^2}$$

  - in the massless limit, $u_4, u_5, u_6 \to 0$
  - $D_3 \cong S_3$ symmetry made of cyclic rotations *c* and reflections *r*

$$u_1 \xrightarrow{c} u_2 \xrightarrow{c} u_3 \xrightarrow{c} u_1\,, u_4 \xrightarrow{c} u_5 \xrightarrow{c} u_6 \xrightarrow{c} u_4\,,$$

$$u_1 \xleftrightarrow{r} u_3\,, u_4 \xleftrightarrow{r} u_5\,,$$

$$u_2 \xleftrightarrow{r} u_2\,, u_6 \xleftrightarrow{r} u_6\,.$$

Dixon Drummond Duhr Henn Smirnov VDD 11

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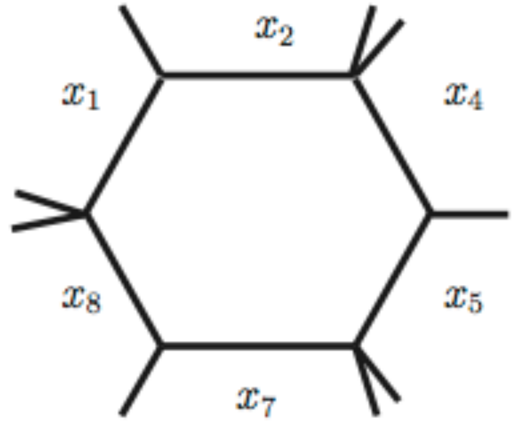

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$$u_1 \xleftrightarrow{r} u_3\,, u_4 \xleftrightarrow{r} u_5\,,$$

$$u_2 \xleftrightarrow{r} u_2\,, u_6 \xleftrightarrow{r} u_6\,.$$

Dixon Drummond Duhr Henn Smirnov VDD 11

- after using diff. eqs, the symbol map and momentum twistors, the integral is

$$\Phi_9(u_1,\ldots,u_6) = \frac{1}{\sqrt{\Delta_9}} \sum_{i=1}^{4} \sum_{g\in S_3} \sigma(g)\, \mathcal{L}_3(x^+_{i,g}, x^-_{i,g})$$

$$\sigma(g) = \begin{cases} +1 \text{ for } \{1, c, c^2\} \\ -1 \text{ for } \{r, rc, rc^2\} \end{cases}$$

$$x^\pm_{i,g} = g(x^\pm_i) \qquad x^\pm_i = x^\pm_i(u_1,u_2,u_3,u_4,u_5,u_6)$$

$$\mathcal{L}_3(x^+,x^-) = \frac{1}{18}\left(\ell_1(x^+) - \ell_1(x^-)\right)^3 + L_3(x^+,x^-)$$

$$\Delta_9 = (1-u_1-u_2-u_3+u_4u_1u_2+u_5u_2u_3+u_6u_3u_1-u_1u_2u_3u_4u_5u_6)^2 - 4u_1u_2u_3(1-u_4)(1-u_5)(1-u_6)$$

reduces to Δ in the massless limit

8-edged Wilson loop in AdS_3

- at strong coupling, Alday & Maldacena have considered $2n$-sided polygons embedded into the boundary of AdS_3
- $2n$-sided remainder function depends on $2(n-3)$ variables

8-edged Wilson loop in AdS3

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- 2*n*-sided remainder function depends on 2(*n*-3) variables
- for the octagon, the remainder function is

$$R_{8,WL}^{strong} = -\frac{1}{2}\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^+}\right)+\frac{7\pi}{6} + \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{|m|\,\sinh t}{\tanh(2t+2i\phi)}\ln\left(1+e^{-2\pi|m|\cosh t}\right)$$

Alday Maldacena 09

where $\chi^+ = e^{2\pi \mathrm{Im}\, m}$ $\chi^- = e^{-2\pi \mathrm{Re}\, m}$ $m = |m|e^{i\phi}$

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- at weak coupling, the 2-loop octagon remainder function is

$$R^{(2)}_{8,WL}(\chi^+,\chi^-) = -\frac{\pi^4}{18} - \frac{1}{2}\ln\left(1+\chi^+\right)\ln\left(1+\frac{1}{\chi^+}\right)\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^-}\right)$$

Duhr Smirnov VDD 10

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Duhr Smirnov VDD 10

- 2-loop 2*n*-sided polygon *R* conjectured through collinear limits (Heslop Khoze 10); proven through OPE (Gaiotto Maldacena Sever Vieira 10)

Coproducts

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- look for *something* with more structure
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- algebra is a vector space with a multiplication $\mu\colon A \otimes A \to A$ $\mu(a\otimes b) = a \cdot b$
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iterate: sum over ways to split it into three

$$w\,x \otimes y\,z \to (w \otimes x) \otimes y\,z$$

$$w\,x \otimes y\,z \to w\,x \otimes (y \otimes z)$$

if sum over all possibilities, get to the same result

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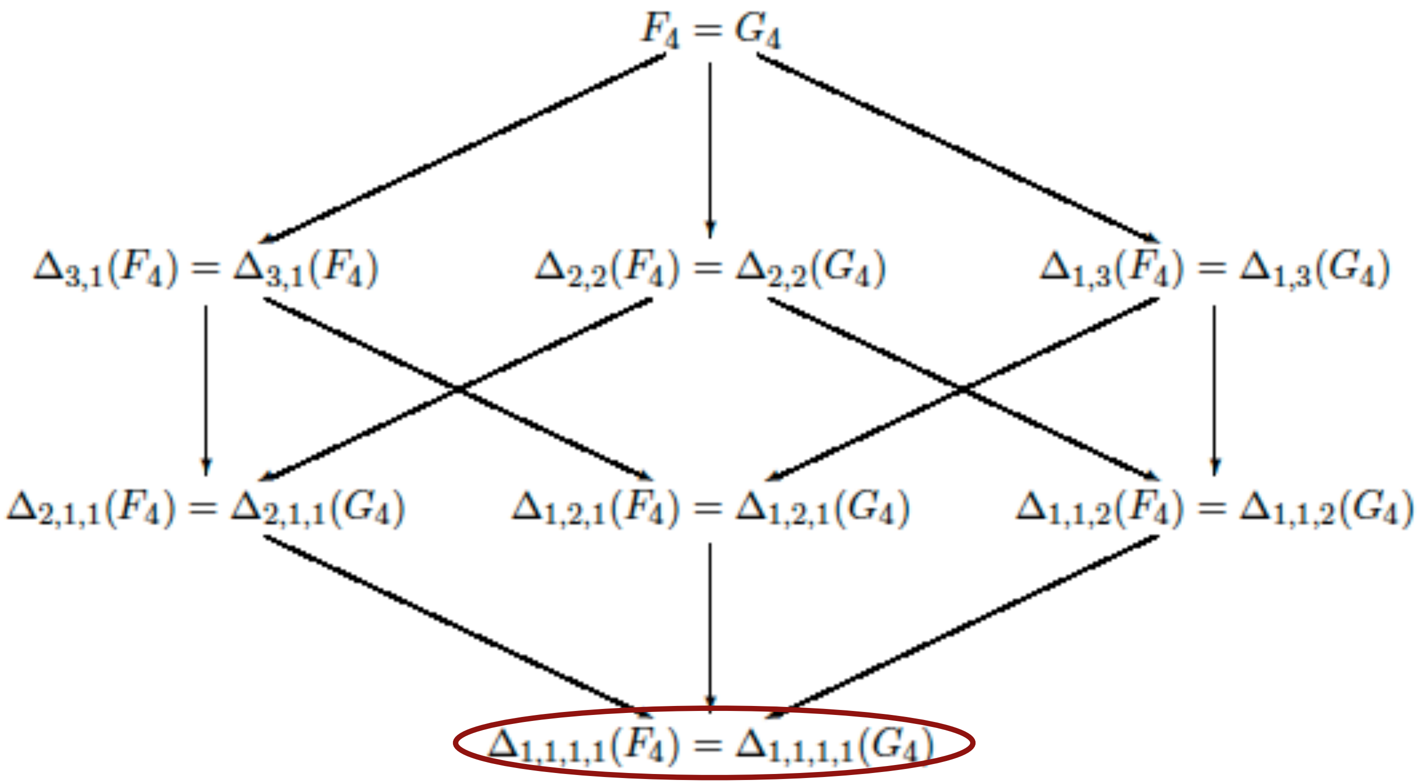
- example on a function of weight 4

Duhr 12



symbols represent the maximal iteration of a coproduct

- ... but there is a problem

put $z = 1$ in $\Delta\big(\mathrm{Li}_n(z)\big) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$

get $\Delta(\zeta_n) = 1 \otimes \zeta_n + \zeta_n \otimes 1$

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- define $\Delta(\zeta_{2n}) = \zeta_{2n}\otimes 1$ Francis Brown 11

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- this allows us to account consistently for $\zeta, i\pi$ terms (which the symbol misses) so the coproduct fixes all but the primitive elements

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iπ more than the symbol

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one can do better

$$\Delta_{2,1}\left(\mathrm{Li}_3\left(\frac{1}{z}\right) - \left(\mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z\right)\right) = -\frac{\pi^2}{3}\otimes\ln z$$
$$= \Delta_{2,1}\left(-\frac{\pi^2}{3}\ln z\right)$$

so $$\mathrm{Li}_3\left(\frac{1}{z}\right) = \mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z - \frac{\pi^2}{3}\ln z + c_1\zeta_3 + c_2 i\pi^3$$ $z=1 \to c_1 = c_2 = 0$

Higgs + 3 gluons

- the 2-loop amplitudes for Higgs + 3 gluons have been computed in terms of 2-dim HPLs

Koukoutsakis 03
Gehrmann Jacquier Glover Koukoutsakis 11

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- using coproducts, the whole 2-loop amplitude for Higgs + 3 gluons can be expressed in terms of classical polylogarithms up to weight 4

  Duhr 12

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  - in a particular (1+1)-dim kinematic set-up, the problem of computing the 2-loop MHV amplitudes/Wilson loops is now completely solved
- a major progress has come from the introduction of symbols, which capture most of the analytic properties of a function, and help us in simplifying what the final result should be like. Symbols are being introduced in the analytic results of 2-loop quantities in QCD, and will certainly be used there more and more

Conclusions

- Planar N=4 SYM is an ideal lab where to learn how an integrable field theory works
- one can make comparisons between quantities at weak and strong couplings: the 2-loop 6-edged Wilson loop
- one can learn about 2-loop *n*-point (N)MHV amplitudes, and think of recycling that knowledge in realistic gauge field theories
  - in a particular (1+1)-dim kinematic set-up, the problem of computing the 2-loop MHV amplitudes/Wilson loops is now completely solved
- a major progress has come from the introduction of symbols, which capture most of the analytic properties of a function, and help us in simplifying what the final result should be like. Symbols are being introduced in the analytic results of 2-loop quantities in QCD, and will certainly be used there more and more
- ... but symbols loose much info about the target function. Most of that info can be recovered using coproducts, which include the symbols, and much more ...

Back-up slides

Resummation: Sudakov form factor

- Sudakov (quark) form factor as matrix element of EM current

$$\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) \equiv < 0|J_\mu(0)|p_1,p_2 > = \bar{v}(p_2)\gamma_\mu u(p_1)\,\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

obeys evolution equation

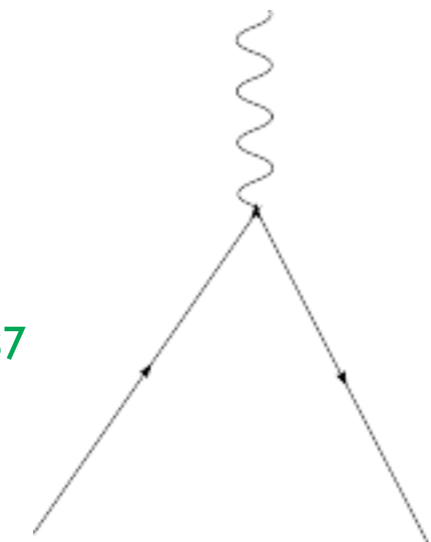
$$Q^2\frac{\partial}{\partial Q^2}\ln\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] = \frac{1}{2}\left[K\left(\alpha_s(\mu^2),\epsilon\right) + G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right]$$

K is a counterterm; G is finite as $\varepsilon \to 0$

RG invariance requires

$$\mu\frac{dG}{d\mu} = -\mu\frac{dK}{d\mu} = \gamma_K(\alpha_s(\mu^2))$$

Korchemsky Radyushkin 1987



γ_K is the cusp anomalous dimension

solution is

$$\Gamma\left(Q^2,\epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[G\left(-1,\bar{\alpha}_s(\xi^2,\epsilon),\epsilon\right) - \frac{1}{2}\gamma_K\left(\bar{\alpha}_s(\xi^2,\epsilon)\right)\ln\left(\frac{-Q^2}{\xi^2}\right)\right]\right\}$$

Collinear limits of Wilson loops

collinear limit $a||b$

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

$R_6 \to 0$ $\quad R_7 \to R_6$ $\quad R_n \to R_{n-1}$

triple collinear limit $a||b||c$

$R_6 \to R_6$ $\quad R_7 \to R_6$ $\quad R_8 \to R_6 + R_6$ $\quad R_n \to R_{n-2} + R_6$

quadruple collinear limit $a||b||c||d$

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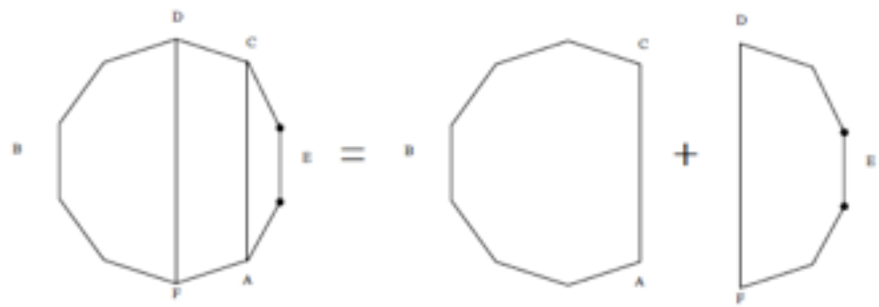
(*k*+1)-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

$R_n \to R_{n-k} + R_{k+4}$



(*n*-4)-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$R_{n-1} \to R_{n-1}$ $R_n \to R_{n-1}$

(*n*-3)-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

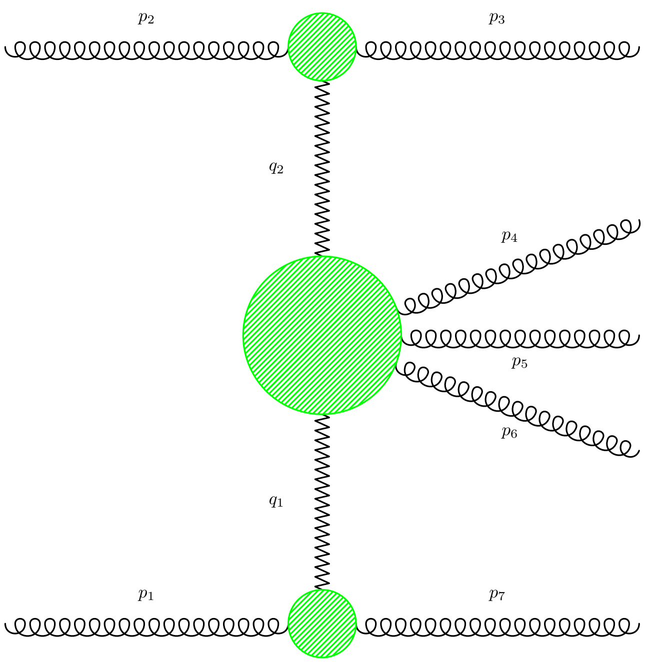
$R_n \to R_n$

- thus R_n is fixed by the (*n*-3)-ple collinear limit

Quasi-multi-Regge limit of *n*-sided Wilson loop

- 7-pt amplitude in the qmR limit of a triple along the ladder

$$y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}| \simeq |p_{7\perp}|$$



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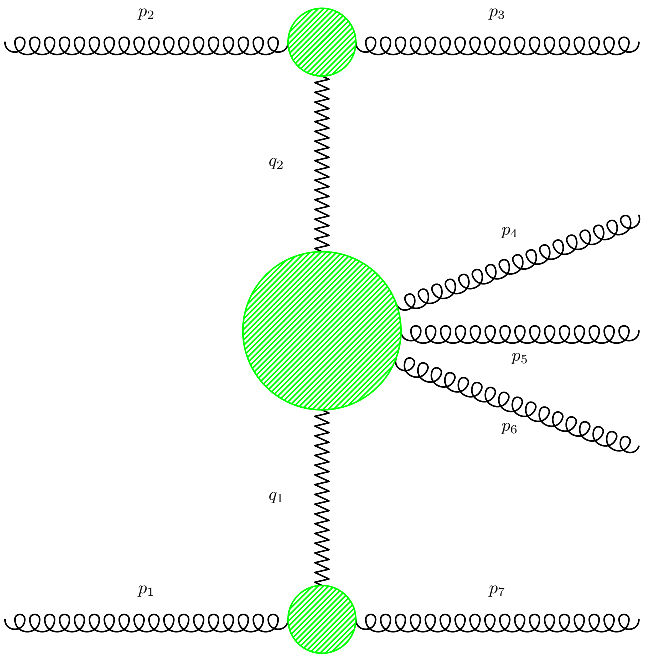

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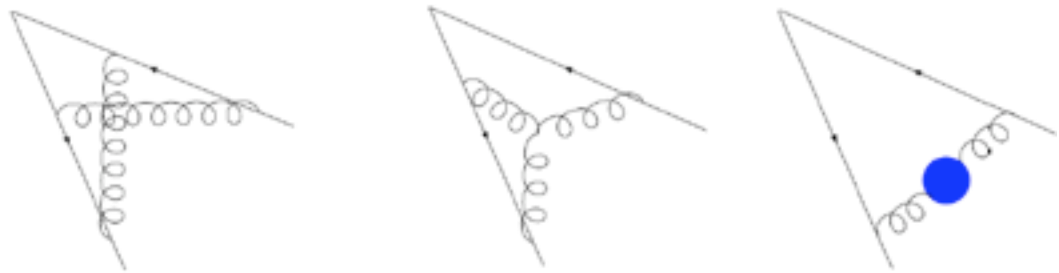
- can be generalised to the *n*-pt amplitude in the qmR limit of a (*n*-4)-ple along the ladder

$$y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n; \qquad |p_{3\perp}| \simeq \ldots \simeq |p_{n\perp}|$$

Duhr Smirnov VDD 09

Computing 2-loop Wilson loops

cusp diagrams are given by cross and Y diagrams with gluons attaching to consecutive sides



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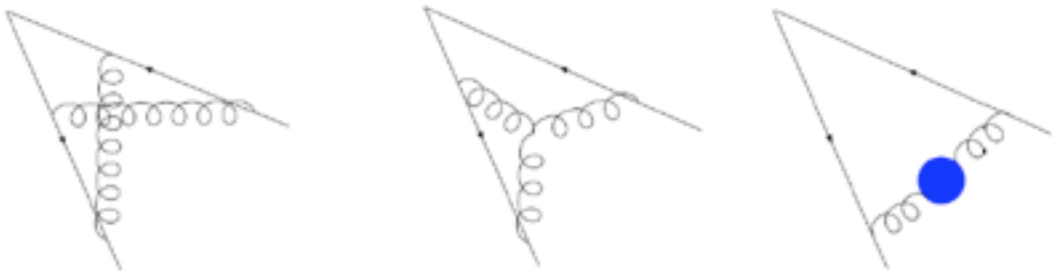
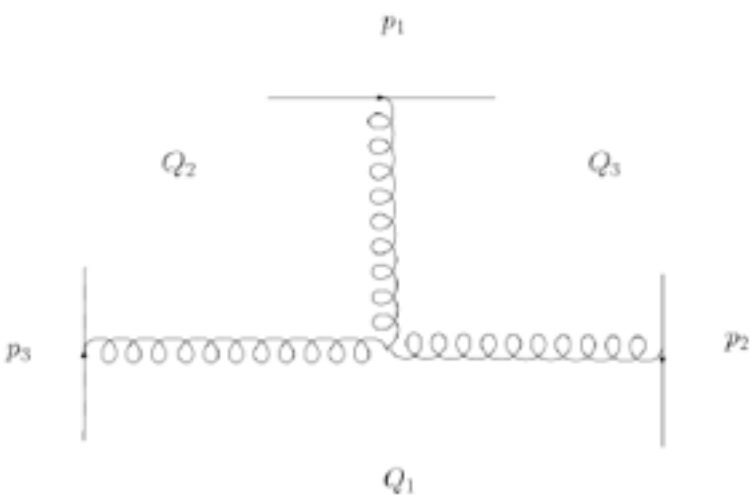


- most difficult diagrams to compute are hard diagrams



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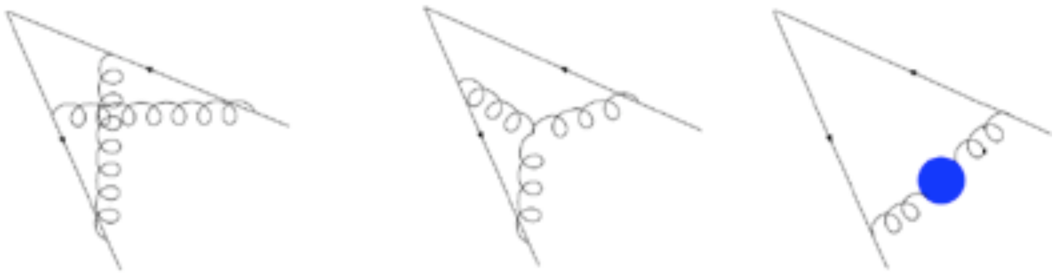

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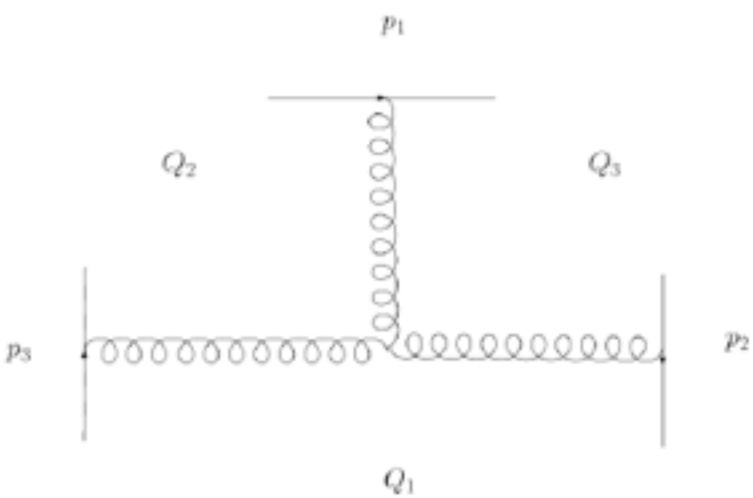

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- most general hard diagram has $Q_1^2, Q_2^2, Q_3^2 \neq 0$; it occurs for $n \geq 9$

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- For $10 \leq n \leq 12$, the only new contributions come from the *factorized cross diagram* topology, which is the simplest

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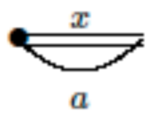
at $x = 1$ $\quad 0 = -0 - 0 + c\,\pi^2 \quad \Longrightarrow \quad c = 0$

Symbols in the DGR construction

Duhr Gangl Rhodes 11

- DGR associate *decorated (n+1)-gons* to multiple polylogarithms of weight *n*

  - $G(a;x) \leftrightarrow$ 

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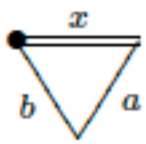
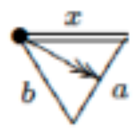
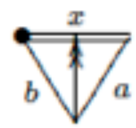
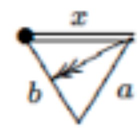
  - $G(a,b;x) \leftrightarrow$ 

$S(G(a,b;x)) \leftrightarrow$ 





$+ax|ba$ $\quad +bx|ax$ $\quad -bx|ab$

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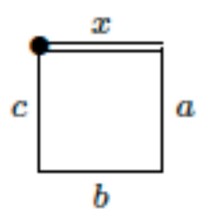
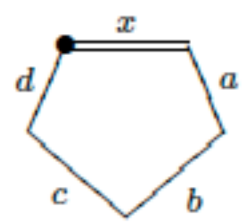
  - $G(a,b,c;x) \leftrightarrow$ 

    $G(a,b,c,d;x) \leftrightarrow$ 

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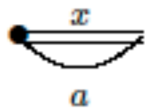
  - $G(a;x) \leftrightarrow$ 

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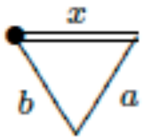
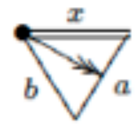
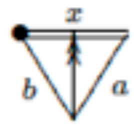
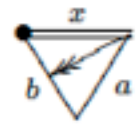
  - $G(a,b;x) \leftrightarrow$ 

$S(G(a,b;x)) \leftrightarrow$ 





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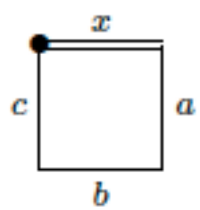
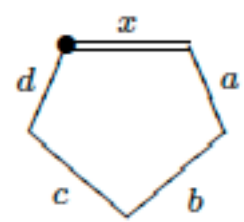
  - $G(a,b,c;x) \leftrightarrow$ 

 $G(a,b,c,d;x) \leftrightarrow$ 

- the symbol in the DGR construction is basically equivalent to GSVV's, *except* that one needs not treat *d log c* as zero

$$C \otimes 2^m\, 3^n\, x^{-5} \otimes D = m\,(C \otimes 2 \otimes D) + n\,(C \otimes 3 \otimes D) - 5(C \otimes x \otimes D)$$

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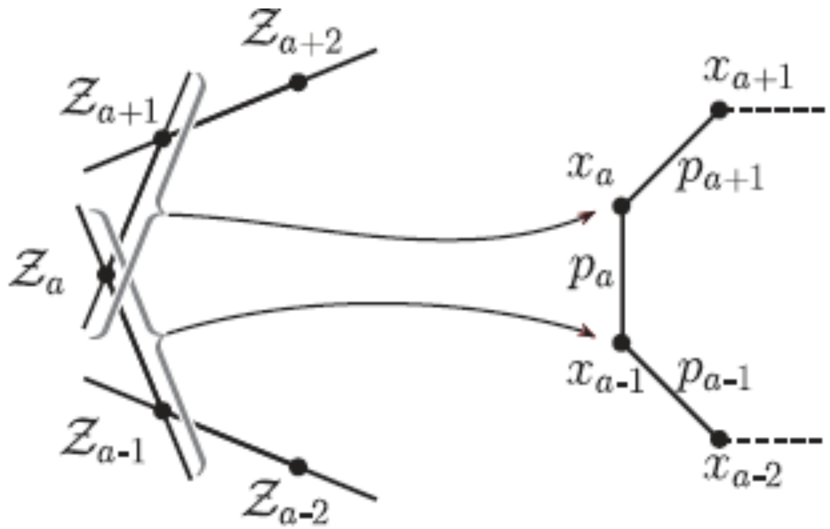

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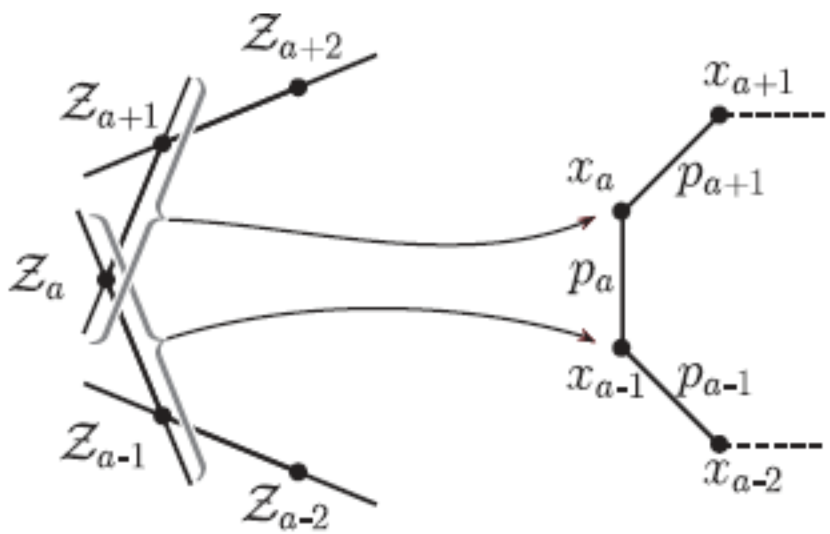

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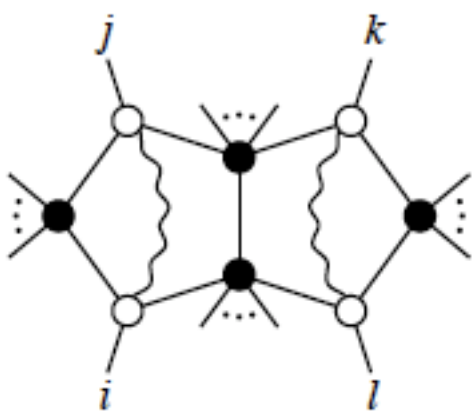
2-loop *n*-pt MHV amplitudes can be written as sum of pentaboxes in twistor space

$$m_n^{(2)} = \frac{1}{2} \sum_{i<j<k<l<i}$$



Arkani-Hamed Bourjaily Cachazo Trnka10