Energy of closed semi-classical short spinning strings from algebraic curve

CarloAlberto Ratti

Based on

- M. Beccaria, G. Macorini, C. Ratti, S. Valatka

  Semi-classical folded string in $AdS_4 \times CP^3$

  JHEP 1205 (2012) 030 [arXiv:1203.3852 [hep-th]]

- M. Beccaria, C. Ratti, A.A. Tseytlin

  Leading quantum correction
  to the energy of "short" spiky strings

  J.Phys.A A45 (2012) 155401 [arxiv:1201.5033 [hep-th]]

Introduction and motivations

Target: Computation of anomalous dimensions of operators within AdS/CFT correspondence

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Tools from Integrability: Y-systems and TBA

- ◇ Valid in principle both for weak and strong coupling regimes
- ◇ They include wrapping for *"short"* operators

Main analytic results found for

- ♣ Long operators at weak and strong coupling
- ♣ Short operators at weak coupling

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Problem:

Strong coupling regime for short strings (deeply quantum states)

- Difficult with standard perturbative string techniques
  (as expected...)
- Difficult with the Y-system
  (Y-functions with a more and more complicated analytic structure as λ increases)

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$$E \;=\; \underbrace{\sqrt{\lambda}E_0(\mathcal{S},\mathcal{J})}_{\Downarrow \text{ classical}} \;+\; \underbrace{E_1(\mathcal{S},\mathcal{J})}_{\Downarrow \text{ semi-clas from algebric curve}} \;+\; \frac{1}{\sqrt{\lambda}}E_2(\mathcal{S},\mathcal{J}) \;+\; \cdots$$

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In the short string regime $S, J \sim 1 \Longrightarrow$ re-expansion for $\mathcal{S}, \mathcal{J} \ll 1$:

$$E \;=\; \underset{\substack{\Uparrow \\ E_0}}{\lambda^{1/4} a_0} \;+\; \underset{\substack{\Uparrow \\ (E_0,\; E_1)}}{\frac{1}{\lambda^{1/4}}\, a_1} \;+\; \cdots$$

⇒ Short strings available through semi-classical algebraic curve!

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Konishi operator ⟷ folded string spinning in $AdS_3 \times S^1$

↙ spin S

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  Algebraic curve studied in $\longrightarrow$ [Gromov, Serban, Shenderovich, Volin 1102.1040 Gromov, Valatka 1109.6305]

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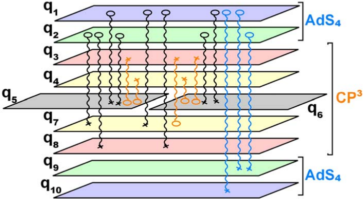
Algebraic curve for strings in $AdS_4 \times CP^3$: overview

[Gromov, Vieira 0807.0437]

- Algebraic curve in $AdS_4 \times CP^3$: Rieman surface with 10 sheets



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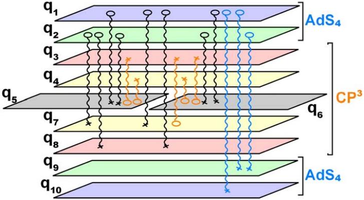

- Each sheet is described by a quasi-momentum $q_i(x)$ $(i = 1, \ldots, 10)$
  - $\longrightarrow$ Logs of the eigenvalues of the monodromy matrix
  - $\longrightarrow$ Only 2 independent ($q_1(x)$ and $q_3(x)$): $q_i(x) = -q_{11-i}(x)$, etc ...

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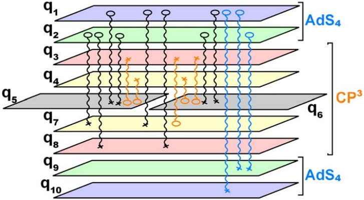

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- Physical polarizations associated with couples of sheets

	AdS4	Fermions	$\mathbb{CP}^3$
heavy	$(1,10)(2,9)(1,9)$	$(1,7)(1,8)(2,7)(2,8)$	$(3,7)$
light		$(1,5)(1,6)(2,5)(2,6)$	$(3,5)(3,6)(4,5)(4,6)$

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- Admit fluctuations in the classical quasi-momenta: $q_i(x) \to q_i(x) + \delta q_i(x)$
- $\delta q_i(x) \Longleftrightarrow$ fluctuation frequencies $\Omega_{(i,j)}(x)$ on each polarization (i,j)

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- Position of the poles $x^n_{(i,j)}$: $q_i(x^n_{(i,j)}) - q_j(x^n_{(i,j)}) = 2\pi n \Leftrightarrow$ mode number n
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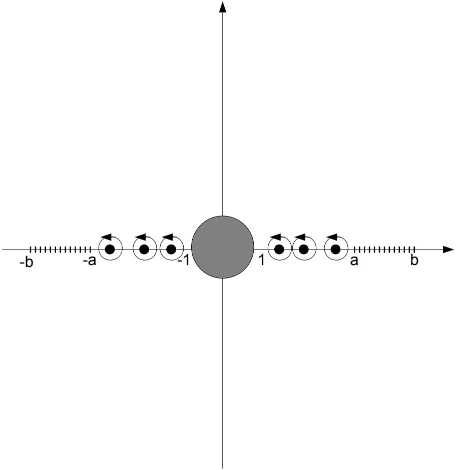
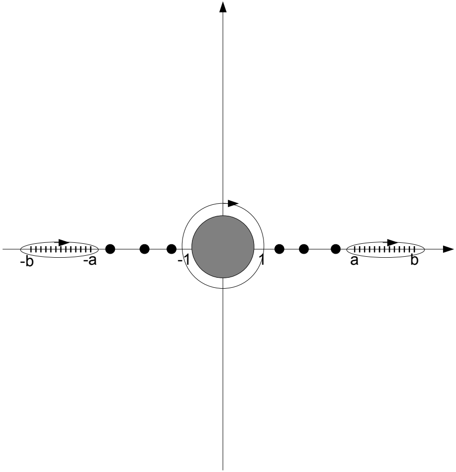
$$\begin{aligned} E_1 &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\sum_{(i,j)}(-1)^{F_{i,j}}\Omega_{(i,j)}(x^n_{(i,j)}) \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}}\sum_{(i,j)}(-)^{(i,j)}\Omega_{(i,j)}(x)\,\partial_x \log\sin\left(\frac{q_i(x)-q_j(x)}{2}\right) \end{aligned}$$

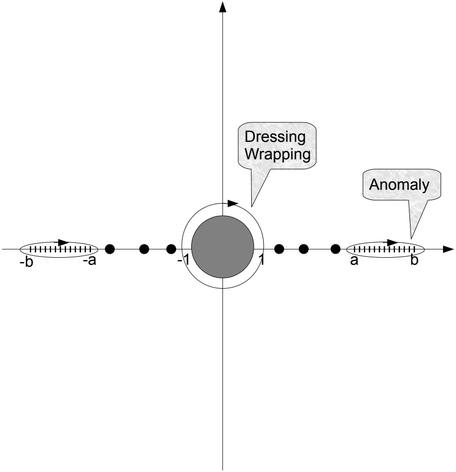
- Integral representation for E_1:



- Integral representation for E_1:
  by deforming the contour



- Integral representation for E_1:
  by deforming the contour: $E_1 = E_1^{\text{dressing}} + E_1^{\text{wrapping}} + E_1^{\text{anomaly}}$



$$E_1^{\text{dressing}} = \sum_{(i,j)} (-1)^{F_{ij}} \int_{-1}^{1} \frac{dx}{2\,\pi\, i}\, \Omega^{(i,j)}(x)\, \partial_x \frac{i\,[q_i(x) - q_j(x)]}{2}$$

$$E_1^{\text{wrapping}} = \sum_{(i,j)} (-1)^{F_{ij}} \int_{-1}^{1} \frac{dx}{2\,\pi\, i}\, \Omega^{(i,j)}(x)\, \partial_x \log(1 - e^{-i\,(q_i(x) - q_j(x))})$$

$$E_1^{\text{anomaly}} = \sum_{\text{"some"}\ (i,j)} \int_{\text{cuts}} \frac{dx}{2\,\pi\, i} \Omega^{(i,j)}(x)\, \partial_x \log\sin \frac{q_i(x) - q_j(x)}{2}$$

The folded string in $AdS_4 \times CP^3$

Classical string solution: folded string moving in $AdS_3 \times S^1$

- Classical charges: $\mathcal{S}$, $\mathcal{J}$

$$\begin{aligned}
\mathcal{S} &= \frac{m}{2}\frac{ab+1}{ab}\left[b\,\mathbb{E}\left(1-\frac{a^2}{b^2}\right)-a\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\right]\\
\mathcal{J} &= \frac{m}{\pi b}\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\sqrt{(a^2-1)\,(b^2-1)}\\
\mathcal{E}_0 &= \frac{m}{2\pi}\frac{ab-1}{ab}\left[b\,\mathbb{E}\left(1-\frac{a^2}{b^2}\right)+a\mathbb{K}\left(1-\frac{a^2}{b^2}\right)\right]
\end{aligned}$$

- $m \longrightarrow$ number of folds
  In what follows $m=1$ (see [Beccaria, CR, Tseytlin] for $m \neq 1$)

- Branching points a, b $\longleftrightarrow$ oscillatory frequencies + string extension
- Quasimomenta

$$\begin{aligned} q_1(x) &= \pi\, f(x) \left\{ -\mathcal{J} \left(\frac{1}{f(1)\,(1-x)} - \frac{1}{f(-1)(1+x)} \right) + \right. \\ &\left. - \frac{4}{\pi\,(a+b)(a-x)(a+x)} \left[(x-a)\, \mathbb{K} \left(\frac{(b-a)^2}{(b+a)^2} \right) + \;\; +2\, a\, \Pi \left(\frac{(b-a)(a+x)}{(a+b)(x-a)} \,\middle|\, \frac{(b-a)^2}{(b+a)^2} \right) \right] \right\} - \pi \\ f(x) &= \sqrt{x-a}\,\sqrt{x+a}\,\sqrt{x-b}\,\sqrt{x+b} \end{aligned}$$

$$q_3(x) = 2\pi \mathcal{J}\, \frac{x}{x^2-1} \qquad q_5(x) = 0$$

$$q_2(x) = -q_1(1/x) \qquad q_4(x) = q_3(x) \qquad q_{11-i}(x) = -q_i(x)$$

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f(x) \quad &= \quad \sqrt{x-a}\,\sqrt{x+a}\,\sqrt{x-b}\,\sqrt{x+b}
\end{aligned}
$$

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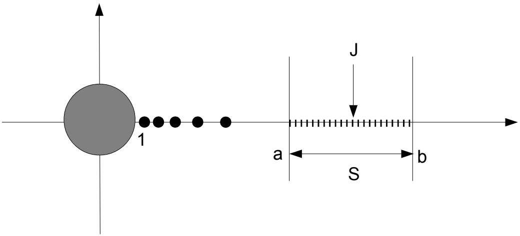

- Independent physical frequencies

$$\begin{aligned}\Omega_{1\ 10}(x) &= \frac{2}{ab-1}\left(1-\frac{f(x)}{x^2-1}\right)\\ \Omega_{37}(x) &= 2\frac{\sqrt{a^2-1}\sqrt{b^2-1}}{ab-1}\,\frac{1}{x^2-1}\end{aligned}$$

- Dependent frequencies

$$\begin{aligned}\Omega_{15}(x) &= 1/2\ \Omega_{1\ 10}(x)\\ \Omega_{25}(x) &= -\Omega_{15}(1/x)+\Omega_{15}(0)\\ \Omega_{35}(x)=\Omega_{45}(x) &= 1/2\ \Omega_{37}(x)\\ \Omega_{29}(x) &= 2\ \Omega_{25}(x)\\ \Omega_{19}(x) &= \Omega_{15}(x)+\Omega_{25}(x)\\ \Omega_{17}(x)=\Omega_{18}(x) &= \Omega_{15}(x)+\Omega_{45}(x)\\ \Omega_{27}(x)=\Omega_{28}(x) &= \Omega_{25}(x)+\Omega_{45}(x)\\ \Omega_{i6}(x) &= \Omega_{i5}(x) \qquad\qquad i=1,2,3,4\end{aligned}$$

Short string (I)

Short string: $\mathcal{S} \to 0$.
In relation with $\mathcal{J}$: Two possibilities

1) $\mathcal{S} \to 0$ at fixed $\mathcal{J} \quad \oplus \quad \mathcal{J} \to 0$

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- Even and odd powers of $\mathcal{J}$ vs only odd terms in $\mathcal{N} = 4$
- Terms proportional to log(2) vs no such terms in $\mathcal{N} = 4$

Origin of the differences:

- Dressing

$$
\begin{aligned}
\delta E^{\text{dressing}}_{ABJM} &= \frac{1}{2} E^{\text{dressing}}_{\mathcal{N}=4} \\
&= \tfrac{1}{2}\mathcal{S}^2 \left(\frac{\left(\mathcal{J}^2+2\right)\coth^{-1}\left(\sqrt{\mathcal{J}^2+1}+\mathcal{J}\right)}{\mathcal{J}^3\left(\mathcal{J}^2+1\right)^{3/2}} - \frac{1}{2\mathcal{J}^3\left(\mathcal{J}^2+1\right)} \right) + \mathcal{O}(\mathcal{S}^3)
\end{aligned}
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- Anomaly: $\delta E^{\text{anomaly}}_{ABJM} = \delta E^{(1,10)}_{anomaly} + \delta E^{(1,5)}_{anomaly}$

$$\delta E^{(1,10)}_{anomaly} = \delta E^{\text{anomaly}}_{\mathcal{N}=4}$$

$$= -\frac{1}{2\mathcal{J}(\mathcal{J}^2+1)}\mathcal{S} + \left[\frac{2\mathcal{J}^4+15\mathcal{J}^2+4}{16\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} - \frac{\pi^2}{12\mathcal{J}^3\sqrt{\mathcal{J}^2+1}} \right]\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3)$$

$$\delta E^{(1,5)}_{anomaly} = -\frac{\pi^2}{32\mathcal{J}^3\sqrt{J^2+1}}\mathcal{S}^2 + O(\mathcal{S}^3)$$

- Wrapping: analytic expression only up to $\mathcal{O}(\mathcal{S})$

$$\begin{aligned}\delta E_{ABJM}^{\text{wrapping}} &= \mathcal{S}\sum_{n=-\infty}^{\infty}\sigma_{wrapping}^{n}=\mathcal{S}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2\mathcal{J}\sqrt{\left(\mathcal{J}^2+1\right)\left(\mathcal{J}^2+n^2\right)}}+\mathcal{O}(\mathcal{S}^2)\\ &\sim -\frac{\sqrt{2}}{\mathcal{J}^{5/2}}\,e^{-\pi\,\mathcal{J}}+\cdots \qquad \text{as } \mathcal{J}\to\infty\end{aligned}$$

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Even powers of $\mathcal{J}$

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Even powers of $\mathcal{J}$

$$\begin{aligned}\sum_{n\neq 0}\sigma^{n\neq 0}_{wrapping} &= \frac{\log(2)}{\mathcal{J}} + \mathcal{J}\left(-\frac{3\zeta(3)}{8} - \frac{\log(2)}{2}\right) + \mathcal{J}^3\left(\frac{3\zeta(3)}{16} + \frac{45\zeta(5)}{128} + \frac{3\log(2)}{8}\right) +\\ &+ \mathcal{J}^5\left(-\frac{9\zeta(3)}{64} - \frac{45\zeta(5)}{256} - \frac{315\zeta(7)}{1024} - \frac{5\log(2)}{16}\right) + O\left(\mathcal{J}^6\right)\end{aligned}$$

All the irrational terms (in particular the log(2) ones)

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$\delta E^{\text{wrapping}}_{\mathcal{N}=4} = \mathcal{O}(\mathcal{S}^2)$ $\Longrightarrow$ Crucial difference for the slope

Slope function: for $S \ll 1$ in the $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM

[Basso 1109.3154]

$$\begin{aligned} E &= J + \alpha_J(\lambda) S + \mathcal{O}(S^2) \\ \alpha_J(\lambda) &= 1 + \frac{\sqrt{\lambda}}{J} \frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} \end{aligned}$$

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- Valid at any coupling λ
- Derived from ABA equations $\Longleftrightarrow$ no contribution from wrapping!

[Gromov 1205.0018, Basso 1205.0054]

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In ABJM:

- Inclusion of the interpolating h function: $\sqrt{\lambda} \longrightarrow 2\pi h(N_c/k)$
- At strong coupling (but also at weak): wrapping contributes to the slope!

$$\sigma(\mathcal{J}) = \lim_{\mathcal{S} \to 0} \frac{E_1(\mathcal{S}, \mathcal{J})}{\mathcal{S}} = \delta E^{(1,10)}_{anomaly} + \delta E_{wrapping}$$

No general formula for the ABJM slope

Short string (II)

2) $\mathcal{S} \to 0$ with $\rho = \mathcal{J}/\sqrt{\mathcal{S}}$ fixed

$$E_1 = -\frac{1}{2}\,\mathcal{C}(\rho,\mathcal{S}) + a_{01}(\rho)\,\sqrt{\mathcal{S}} + a_{1,1}(\rho)\,\mathcal{S}^{3/2} + \mathcal{O}(\mathcal{S}^{5/2})$$

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a_{1,0}(\rho) &= \frac{2\ \log(2)-1}{2\,\sqrt{\rho^2+2}} \\
a_{1,1}(\rho) &= -\frac{\log(2)\left(2\rho^4+6\rho^2+3\right)}{4\left(\rho^2+2\right)^{3/2}} + \frac{8\rho^4+25\rho^2+16}{16\left(\rho^2+2\right)^{3/2}} - \frac{3\left(\rho^2+3\right)\zeta(3)}{8\sqrt{\rho^2+2}} \\
\mathcal{C}(\rho,\mathcal{S}) &= \frac{\sqrt{(a^2-1)\,(b^2-1)}}{1-a\,b}+1 \\
&= 1-\frac{\rho}{\sqrt{\rho^2+2}} - \frac{2\,\rho^3+5\,\rho}{4\,(\rho^2+2)^{3/2}}\,\mathcal{S} + \frac{\rho\,(12\,\rho^6+68\,\rho^4+126\,\rho^2+73)}{32\,(\rho^2+2)^{5/2}}\,\mathcal{S}^2+\cdots
\end{aligned}
$$

Prediction for short states:

- trading $(\mathcal{S}, \mathcal{J})$ with (S, J)
- expanding $E = E_0 + E_1$ at large λ

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$\Longrightarrow$ S-free term due to the presence of $\mathcal{C}$

- Relation with short limit (I): by expanding at large $\rho^2 = \mathcal{J}^2/\mathcal{S}$

$$E_1 + \mathcal{O}(\mathcal{S}) = -\frac{1}{2}\left(1 - \frac{\rho}{\sqrt{\rho^2+2}}\right) = -\frac{1}{2\rho^2} + \frac{3}{4\rho^4} - \frac{5}{4\rho^6} + \frac{35}{16\rho^8} - \frac{63}{16\rho^{10}} + \cdots$$

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$$E_1 = \sum_{n=1}^{\infty} K_n \qquad K_n = \begin{cases} \omega_n^{\text{heavy}} + \omega_{n/2}^{\text{light}} & n \in 2\,\mathbb{Z} \\ \omega_n^{\text{heavy}} & n \notin 2\,\mathbb{Z} \end{cases}$$

$$\begin{tikzcd} K_p = & (-1)^p\,\mathcal{C} \arrow[dl] & +\widehat{K}_p \arrow[dr] & \\ -\mathcal{C}+\mathcal{C}-\mathcal{C}+\mathcal{C}+\cdots & & & \text{convergent } \mathcal{O}(\mathcal{S}) \text{ sum} \end{tikzcd}$$

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Why $-\frac{1}{2}\mathcal{C}$? Arbitrary but:

- Coherent with the results from the integral representation
- Necessary to match with ABA (long string $\mathcal{S} \gg 1$ limit)

Long string

Large $\mathcal{S}$ limit with $\mathcal{J} = \frac{\ell}{\pi} \log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right)$

In $\mathcal{N} = 4$ SYM

[Gromov, Serban, Shenderovich, Volin 1102.1040]

$$E_1^{AdS_5} = f_{10}^{AdS_5}(\ell) \log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right) + f_{11}^{AdS_5}(\ell) + \frac{c^{AdS_5}}{\log\left(\frac{8\,\pi\,\mathcal{S}}{\sqrt{\ell^2+1}}\right)} + \cdots$$

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$$f_{10}^{AdS_5}(\ell) = \frac{\sqrt{\ell^2+1}+2\left(\ell^2+1\right)\log\left(\frac{1}{\ell^2}+1\right)-\left(\ell^2+2\right)\log\left(\frac{\sqrt{\ell^2+2}}{\sqrt{\ell^2+1}-1}\right)-1}{\pi\sqrt{\ell^2+1}}$$

$$f_{11}^{AdS_5}(\ell) = \frac{2\left(\log\left(1-\frac{1}{\left(\ell^2+1\right)^2}\right)+2\sqrt{\ell^2+1}\cot^{-1}\left(\sqrt{\ell^2+1}\right)+2\coth^{-1}\left(\sqrt{\ell^2+1}\right)-2\ell\cot^{-1}(\ell)\right)}{\pi\sqrt{\ell^2+1}}$$

$$c^{AdS_5}(\ell) = -\frac{\pi}{12\,(\ell^2+1)}$$

- Wrapping $\longrightarrow$ exponentially suppressed
- Dressing $\longrightarrow f_{10}(\ell),\, f_{11}(\ell)$
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The same pattern valid in ABJM
From the integral representation (coherent with the $-\frac{\mathcal{C}}{2}$ choice)

$$\begin{aligned} f_{10}^{AdS_4}(\ell) &= \frac{1}{2}\, f_{10}^{AdS_5}(\ell) \\ f_{11}^{AdS_4}(\ell) &= \frac{1}{2}\, f_{11}^{AdS_5}(\ell) \\ c^{AdS_4}(\ell) &= 2\, c^{AdS_5}(\ell) \end{aligned}$$

Expected !!

From the analysis of the central charge ($\propto \#$ massless modes) of the low energy effective theory at finite chemical potential ℓ

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- Algebraic curve → good to explore semi-classical string behavior
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- … but the long string limit is available as well (and useful for checks)

Prediction for short states at strong coupling in ABJM

$$E = (4\,\pi\,g)^{1/2}\sqrt{2\,S} - \frac{1}{2} + \frac{\sqrt{2\,S}}{(4\,\pi\,g)^{1/2}}\left(\frac{J\,(J+1)}{4\,S} + \frac{3\,S}{8} - \frac{1}{4} + \frac{1}{2}\log(2)\right) + \cdots$$

Ready to be tested with TBA techniques