Deforming Gravity

L. Pilo[1]

[1] Department of Physics
University of L'Aquila

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Einstein's GR

A 95 year-long successful theory
a single free parameter and it works great

- Weak Equivalence principle (10^{-13})
- Solar system tests (weak field) ($10^{-3} - 10^{-5}$)
- Binary pulsar (nonlinear) (10^{-3})
- Newton's Law tested between 10^{-1}mm and 10^{16}mm

however .....

- CMB + Supernovae data require Dark energy
  $p = w\rho$, $w < 0$. Expanded acceleration

  Perhaps just a tiny (??) cosmological constant, $w = -1$,
  $\Lambda \sim (10^{-4}\,\text{eV})^4$ or a bizarre fluid?
- Is GR an isolated theory ?
  Can we modify GR at large distances?

Massive Deformed GR

- Add to GR an extra piece such that when $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$(\sqrt{g}\,R + \mathcal{L}_{def}) = \mathcal{L}_{\text{spin 2}} + m^2 \left(a\,h_{\mu\nu}h^{\mu\nu} + b\,h^2\right) + \cdots$$

- To build a mass term we need an extra tensor field: with $g_{\mu\nu}$ and $g^{\mu\nu}$ there is no polynomial of g
- Introduce a new tensor field $G_{\mu\nu}$, then scalar objects can be constructed from the metric using

$$X^\mu_\nu = g^{\mu\alpha} G_{\alpha\nu} \qquad \tau_n = \mathrm{Tr}(X^n)$$

- Example: $G_{\alpha\nu} = \eta_{\alpha\nu}$

$$g^{\mu\nu} G_{\mu\nu} = 4 - h^{\mu\nu}\eta_{\mu\nu} + h^{\mu\nu}h_{\mu\nu} + \cdots$$

$$a\,(\tau_1 - 4)^2 + b\,(\tau_2 - 2\tau_1 + 4) = \left(a\,h_{\mu\nu}h^{\mu\nu} + b\,h^2\right) + \cdots$$

- The metric $G_{\mu\nu}$ can be dynamical or a priori given: two different formulations of massive gravity

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The Stuckelberg Trick in Massive GR

The extra metric is non-dynamical flat given metric

- To recover diff (gauge) invariance: 4 Stuckelberg fields to recast

$$G_{\mu\nu} = \frac{\partial \Phi^A}{\partial x^\mu} \frac{\partial \Phi^B}{\partial x^\nu} \eta_{AB}$$

  $G_{\mu\nu}$ and $X = g^{-1} G$ are tensors and $\tau_n = \mathrm{Tr}(X^n)$ scalars
- Φ^A: coordinates of a fictitious flat space $\mathcal{M}$ point-wise identified with spacetime with a tetrad basis $e^A = d\Phi^A$
- One can chose coordinates such that (Unitary gauge)

$$\frac{\partial \Phi^A}{\partial x^\mu} = \delta^A_\mu \Rightarrow G_{\mu\nu} = \eta_{\mu\nu}$$

- $G_{\mu\nu}$ can be also dynamical $\rightarrow$ bigravity, not in this talk !

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Degrees of Freedom: Linearized Level

$$S_{mGR} = \int d^4x \sqrt{g}\, M_{pl}^2 \left[R(g) - 4m^2\, V(X)\right]$$

- GR $\quad M_{pl}^2\, E_{\mu\nu}^{(1)} = T_{\mu\nu}^{(1)}\,, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

  DOF $10 - 2 \times 4 = 2 \qquad$ 4 gauge modes $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

- Linearized massive GR (unitary gauge) Fierz-Pauli theory (1939)

  $\mathcal{L}_{FP} = M_{pl}^2\, \mathcal{L}_{\text{spin2}}^{(2)} + M_{pl}^2 m^2\, \left(a\, h_{\mu\nu} h^{\mu\nu} + b\, h^2\right)$

  $E_{\mu\nu}^{(1)} - \frac{1}{4} m^2\, \left(a\, h_{\mu\nu} + b\, h\, \eta_{\mu\nu}\right) = M_{pl}^{-2}\, T_{\mu\nu}^{(1)} \qquad \partial^\nu E_{\mu\nu}^{(1)} = 0$

  4 constraints $\qquad$ DOF $10 - 4 = 6 = 5 + 1$

- The sixth mode is a ghost (Boulware-Deser).
  Absent in flat space when $a + b = 0$ (FP theory)
- When the ghost is projected out, light bending badly contradicts experiments van Dam, Veltman, Zakharov discontinuity

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Hamiltonian Analysis

ADM decompositions

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N_j h^{ij} & N_i \\ N_i & \gamma_{ij} \end{pmatrix}$$

Hamiltonian of GR and mGR in the unitary gauge

$$H = M_{\rm pl}^2 \int d^3x \left[N^A \mathcal{H}_A + m^2 N \sqrt{\gamma}\, V \right] \qquad \mathcal{H}_A = (\mathcal{H}, \mathcal{H}_i)$$

$\Pi^{ij} \rightarrow$ Conj. momenta of γ_{ij}

$P^A = (P^0, P^i)$ Conjugate momenta of $N^A = (N, N^i)$

$$\mathcal{H}_i = -2\gamma_{ij} D_k \Pi^{jk}\,, \quad \mathcal{H} = -\gamma^{1/2} R^{(3)} + \gamma^{-1/2} \left(\Pi_{ij}\Pi^{ij} - \frac{1}{2}(\Pi^i_i)^2 \right)$$

No time derivatives of $N^A \rightarrow P_A = 0$ Constrained theory !

Constrained Theory: Dirac treatment in a nutshell

1. Momenta are not all independent → introduce Lagrange multipliers (LMs) to enforce the constraints
2. Time evolution us generated by the the total Hamiltonian: canonical + constraints + LMs

$$H_T = H + \int d^3x \, \lambda^A \Pi_A \,,$$

   EoMs: dynamical + time evolution of primary ($P^A = 0$) constraints
3. enforcing the consistency of constrs. with time evolution produces new constraints or determine some of the LMs

The a set of constraints $\{\mathcal{C}_s \,, i = 1, 2, \cdots c\}$ is conserved in time that reduces the number of DoF from 10 down to $(10 + 10 - c)/2$

If some of the LMs are not determined → gauge invariance

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Example: GR

- Time evolution of $P_A = 0$ via Poisson brackets are just the Eqs. of N^A, being H linear in N^A

$$\{P^A(t,x),\, H_T(t)\} = \{P^A(t,x),\, H\} = \mathcal{H}_A = 0$$

- Thanks to the GR algebra the four secondary constraints are conserved and no LM is determined (Diff invariance)

$$\{\mathcal{H}(x),\, \mathcal{H}(y) = \mathcal{H}^i(x)\, \partial_i^{(x)}\, \delta^{(3)}(x-y) - \mathcal{H}^i(y)\, \partial_i^{(y)}\, \delta^{(3)}(x-y)$$

$$\{\mathcal{H}(x),\, \mathcal{H}_j(y)\} = \mathcal{H}(y)\, \partial_j^{(x)}\, \delta^{(3)}(x-y)$$

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$\text{DoF} = [20 - (2 \times 4 + 2 \times 4)]/2 = 2$

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mGR

- When V deforming potential is turned on, the time evolution of $P_A = 0$ still gives N^A Eqs

$$\{P_A(t,x),\, H_T(t)\} = \mathcal{S}_A = \mathcal{H}_A + \mathcal{V}_A$$ 4 new secondary constraints

$$\mathcal{V} = m^2\, N\, \gamma^{1/2}\, V \qquad \frac{\partial \mathcal{V}}{\partial N^A} = \mathcal{V}_A$$

- Is time evolution consistent with $\mathcal{S}_A$? $\mathcal{V}_{AB} = \mathcal{V}_{,AB} = \partial^2\mathcal{V}/\partial N^A \partial N^B$

$$\mathcal{T}_A \equiv \{\mathcal{S}_A,\, H_T\} = \{\mathcal{S}_A,\, H\} - \mathcal{V}_{AB}\,\lambda^B = 0$$

If the $r = \text{Rank}(\mathcal{V}_{AB}) = 4$: $\mathcal{V}$ non degenerate Hessian
all LMs λ^A are determined and we are done
DoF=10 - (4 + 4)/2 = 6 = 5 + 1
Around Minkowski: massive spin 2 (5) plus a ghost scalar (1)
Boulware-Deser mode

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The Hessian $\mathcal{V}_{AB}$ of $\mathcal{V}$ has a single zero mode χ^A, $r = 3$

$$\mathcal{V}_{AB}\,\chi^B = 0\,, \qquad \mathcal{V}_{AB}\,E_n^B = \kappa_n\,E_n^A$$

$$\lambda^A = z\,\chi^A + \sum_{n=1}^{3} d_n\,E_n^A \stackrel{\text{def}}{\equiv} z\,\chi^A + \bar{\lambda}^A\,.$$

If $\det(\mathcal{V}_{ij}) \neq 0$, then $\chi^A = (1, -\mathcal{V}_{ij}^{-1}\,\mathcal{V}_{0j})$

Projection of $\dot{\mathcal{T}}_A = 0$ along χ^A is a **single** new constraint

Projection on the remaining eigenvectors gives Three out ($\bar{\lambda}^A$) of the four LMs

$$\chi^A\{\mathcal{S}_A,\, H\} = \mathcal{T}_A\,\chi^A = \mathcal{T} = 0$$

$$E_n^A\{\mathcal{S}_A,\, H\} - d_n\,\kappa_n\,z \;= 0 \qquad \text{No sum in } n$$

Time evolution of $\mathcal{T}$

$$\begin{aligned}\mathcal{Q}(x) &= \{\mathcal{T}(x), H_T\} \\ &= \{\mathcal{T}(x), H\} + \int d^3y \,\{\mathcal{T}(x), \lambda^A(y)\Pi_A(y)\} = 0\end{aligned}$$

1. If $\mathcal{Q}$ does not depend on z, the last LM, we have a new constraint

   z is determinate by the time evolution of $\mathcal{Q}$. We are done.

   Total # of constraints $4\,(P_A) + 4\,(\mathcal{S}_A) + 1\,(\mathcal{T}) + 1\,(\mathcal{Q}) = 10$

   DoF: $10 - 10/2 = 5$

2. If $\mathcal{Q} = 0$ determines z we are done and there is no additional constraints
   Total # of constraints $4\,(P_A) + 4\,(\mathcal{S}_A) + 1\,(\mathcal{T}) = 8 + 1$

   DoF: $10 - 9/2 = 5 + \mathbf{1/2}$

Time evolution of $\mathcal{T}$

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   z is determinate by the time evolution of $\mathcal{Q}$. We are done.

   Total # of constraints $4\,(P_A) + 4\,(\mathcal{S}_A) + 1\,(\mathcal{T}) + 1\,(\mathcal{Q}) = 10$

   DoF: $10 - 10/2 = 5$

2. If $\mathcal{Q} = 0$ determines z we are done and there is no additional constraints
   Total # of constraints $4\,(P_A) + 4\,(\mathcal{S}_A) + 1\,(\mathcal{T}) = 8 + 1$

   DoF: $10 - 9/2 = 5 + \mathbf{1/2}$

mGR

Time evolution of $\mathcal{T}$

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mGR

$$\begin{aligned} \{\mathcal{T}, \lambda^A \cdot \Pi_A\} &= \text{terms z indep.} - \int d^3y\, \Theta(x,y)\, z(y) = \cdots - I[z] \\ \Theta(x,y) &= \chi^A(x)\, \{S_A(x), S_B(y)\}\, \chi^A(y) = A^i(x,y)\partial_i \delta^{(3)}(x-y) \\ A(x,y) &= A(y,x) \end{aligned}$$

Only in field theory Θ can be non zero !

$$I[z] = -\frac{1}{2z(x)}\partial_i \left[z(x)^2 A^i(x,x) \right]$$

Q is free from z if $A^i(x,x) = 0$, which consists in the following condition

$$\chi^{0^2}\ \tilde{\mathcal{V}}_i + 2\,\chi^A \chi^j \frac{\partial \tilde{\mathcal{V}}_A}{\partial \gamma^{ij}} = 0\,, \qquad \mathcal{V} = \gamma^{1/2}\tilde{\mathcal{V}}$$

mGR: Summary of the Canonical Analysis

Necessary and sufficient conditions for having 5 DoF in mGR

$$\text{Rank}(\tilde{\mathcal{V}}_{AB}) = 3 \Rightarrow \tilde{\mathcal{V}}_{00} - \tilde{\mathcal{V}}_{0i}(\tilde{\mathcal{V}}_{ij})^{-1}\tilde{\mathcal{V}}_{j0} = 0 \qquad (1)$$

$$\chi^{0^2}\,\tilde{\mathcal{V}}_i + 2\,\chi^A\chi^j\,\frac{\partial \tilde{\mathcal{V}}_A}{\partial \gamma^{ij}} = 0 \qquad \chi^A = (1, -\tilde{\mathcal{V}}_{ij}^{-1}\,\tilde{\mathcal{V}}_{0j}) \qquad (2)$$

Notice: If only (1) holds 5+1/2 DoF propagate

A theory with 5+1/2 DoF is physically acceptable ?

5+1/2 DoF found also in a class of Horava-Lifshitz modified gravity theory
(1) is a homogeneous Monge-Ampere equation
many solutions are know
(2) is much more restrictive

Solutions

Strategy

1. Find a solution of Monge-Ampere equation (rank$(\mathcal{V}) = 3$
2. Check that the candidate satisfies the additional equation to get rid of 1/2 DoF

$$\text{Rank}(\tilde{\mathcal{V}}_{AB}) = 3 \Rightarrow \tilde{\mathcal{V}}_{00} - \tilde{\mathcal{V}}_{0i}(\tilde{\mathcal{V}}_{ij})^{-1}\tilde{\mathcal{V}}_{j0} = 0$$

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2D Lorentz Invariant case

To simplify things: Eqs in 2D where $\tilde{\mathcal{V}}(N, N^1, \gamma)$ and $\gamma_{11} \equiv \gamma$
Lorentz Invariant case: V depends on the eiegenvalues $\lambda_1, \lambda_2, \cdots$ of $X = g^{-1}\eta$

After expressing N, N^1 in terms of $\lambda_{1/2}$, $\det \tilde{\mathcal{V}}_{AB}$ must hold for any γ !
The resulting equation is cubic and splits into two branches of three differential equations

$$\tilde{\mathcal{V}}^{(2,0)} = -\frac{3}{2\,\lambda_1}\,\tilde{\mathcal{V}}^{(1,0)}\,, \quad \tilde{\mathcal{V}}^{(0,2)} = -\frac{3}{2\,\lambda_2}\,\tilde{\mathcal{V}}^{(0,1)}\,,$$

$$\tilde{\mathcal{V}}^{(1,1)} = -\frac{\lambda_1^{3/2}\,\tilde{\mathcal{V}}^{(1,0)} \pm \lambda_2^{3/2}\,\tilde{\mathcal{V}}^{(0,1)}}{2\,\lambda_1\,\lambda_2\,(\lambda_1^{1/2} \pm \lambda_2^{1/2})}.$$

Solutions, (all !)

$$\mathcal{V}_{I,II} = \frac{\alpha_1\sqrt{\lambda_1\lambda_2} + \alpha_2\,(\sqrt{\lambda_1} \pm \sqrt{\lambda_2}) + \alpha_3}{\sqrt{\lambda_1\lambda_2}},$$

with $\alpha_{1,2,3}$ integration constants.

2D: Lorentz Invariant case

- Both I and II satisfies also the second equation that kills 1/2 DoF
- In terms of X

$$\mathcal{V}_I = \alpha_1 + \alpha_2 \frac{\mathrm{Tr}(X^{1/2})}{\sqrt{\det X}} + \frac{\alpha_3}{\sqrt{\det X}} ,$$

  2D version of the ghost free potential found by de Rham-Gabadadze-Tolley
- The second solution is different but does not admits Minkowski as a background

$$\mathcal{V}_{II} = \alpha_1 + \alpha_2 \frac{\sqrt{\mathrm{Tr}(X) - 2\sqrt{\det X}}}{\sqrt{\det X}} + \frac{\alpha_3}{\sqrt{\det X}} .$$

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2D: Loentz Breaking Case

One can generalize the previous solutions to the case of Lorentz breaking solutions

- A class of potential singular Hessian

$$\mathcal{V} = \beta_1 \left[(x+\beta_2)^2 - (y^{1/2}+\beta_3)^2\right]^{1/2} + \beta_4 \, x \,,$$

$y = N^i N^j \gamma_{ij}$ and $\beta_{n=1,\dots,4}$ scalar functions of γ_{ij}

- Also the second equation is satisfied when

$$\beta_2 = \text{ constant} \qquad \beta_4 = \gamma^{1/2} \bar{\beta}_4$$

Unfortunately the previous solutions does not generalizes to 4D

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Conclusions

- Deforming GR is very difficult
- A randomly picked deforming potential propagates 5+1 DOF; one is a ghost around Minkowski space
- The condition for having 5 DoF can can be encoded in a set differential equations
- In 2D, for the the Lorentz invariant case the solutions is unique
- There is no known underlying symmetry to get the very special form of V required for having 5 DoF
- V is likely to be destabilized by matter's quantum corrections
- Phenomenology (original motivation) is difficult