Quantum strings on $\mathrm{AdS}_4 \times \mathrm{CP}^3$: finite size spectrum vs. Bethe ansatz

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Based on

- arXiv:0807.1527
- arXiv:0912.2257
- arXiv:1101.0004
- arXiv:1111.6628

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The $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

AdS → Anti de Sitter

- Type IIA string theory on $\mathrm{AdS}_4 \times CP^3$
- 10 dimensional theory of gravity
- Bulk

CFT → Conformal Field Theory

- $\mathcal{N} = 6$ superconformal Chern-Simons theory (ABJM, arXiv:0806.1218) on $\mathbb{R}^3$
- 3 dimensional gauge theory
- Boundary

Parameters.

String theory

- String lenght: $\ell_S = \sqrt{\alpha'}$
- Radius of AdS: R

Gauge theory

- Gauge group: $SU(N)_k \times SU(N)_{-k}$
- Define a 't Hooft coupling: $\lambda = \frac{N}{k}$

- $2^5\pi^2\lambda = \frac{R^4}{(\alpha')^2} \Rightarrow \alpha' \to 0$
- Classical superstring

- $\lambda \to \infty$
- Strongly coupled

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- Description in terms of magnons.

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- Conjectured dispersion relation: $\Delta - J \simeq \frac{1}{2} + 4\lambda^2 \sin\left(\frac{p}{2}\right)^2$

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$\lambda \longrightarrow \infty$: the giant magnon

- **Giant magnon limit**: $SU(2) \times SU(2)$ giant magnons are classical ($\lambda \rightarrow \infty$) string solutions moving in the $\mathbb{R} \times S^2 \times S^2$ subspace of $AdS_4 \times CP^3$, carrying infinite energy and angular momentum, their difference being finite.
- Small giant magnons (arXiv:0806.4589) move in $\mathbb{R} \times S^2$, while big giant magnons (arXiv:0806.4959, 0807.0205) move in $\mathbb{R} \times S^2 \times S^2$ and are actually built of two elementary magnons attached.
- Geometric identification: the momentum of the magnon is the fixed angular separation of the endpoints in one S^2

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$\lambda \to \infty$: a preview of the pp-wave limit

- $SU(2) \times SU(2)$ **plane-wave limit**: proper $R \to \infty$ rescaling in the $AdS_4 \times CP^3$ metric as to single out the $SU(2) \times SU(2)$ subsector, in such a way that $\lambda \to \infty$, $J \to \infty$, $\lambda' = \frac{\lambda}{J^2}$ fixed and finite.
- The Green-Schwarz action becomes quadratic and can be diagonalized in light-cone gauge.
- $H_{\mathrm{lc}} = E - J = \sum_{i=1}^{4} \sum_{n \in \mathbb{Z}} \sqrt{1 + 2\pi^2 \lambda' n^2}\, \hat{N}_n^i + \sum_{a=1}^{2} \sum_{n \in \mathbb{Z}} \Big[\Big(\sqrt{\frac{1}{4} + 2\pi^2 \lambda' n^2} - \frac{1}{2} \Big) M_n^a + \Big(\sqrt{\frac{1}{4} + 2\pi^2 \lambda' n^2} + \frac{1}{2} \Big) N_n^a \Big]$, arXiv:0806.4959, 0807.1527
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From weak to strong coupling: guessing the dispersion relation

- $\mathrm{AdS}_5/\mathrm{CFT}_4$
- Dispersion relation
- $E - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin\left(\frac{p}{2}\right)^2}$

$$\Downarrow$$

- Both for $\lambda \to 0$ and $\lambda \to \infty$

- $\mathrm{AdS}_4/\mathrm{CFT}_3$
- Dispersion relation
- $E - J = \frac{1}{2}\left(\sqrt{1 + 16 h(\lambda)^2 \sin\left(\frac{p}{2}\right)^2} - 1\right)$

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- $h(\lambda) \simeq \lambda$ for $\lambda \to 0$
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Hints of integrability

- Integrability of the $SU(2) \times SU(2)$ subsector of the $\mathcal{N} = 6$ Chern Simons gauge theory.

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- Existence of elementary classical giant magnon solutions.

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- Integrability of the classical superspace string sigma model: arXiv:0806.4940

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The Bethe equations/1

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k} x_{4,j}^+}{1 - 1/x_{1,k} x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{1 - 1/x_{1,k} x_{\bar{4},j}^+}{1 - 1/x_{1,k} x_{\bar{4},j}^-}$$

$$1 = \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \prod_{j=1}^{K_{\bar{4}}} \frac{x_{3,k} - x_{\bar{4},j}^+}{x_{3,k} - x_{\bar{4},j}^-}$$

The Bethe equations/2

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{K_1} \frac{1-1/x^-_{4,k}x_{1,j}}{1-1/x^+_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \times$$
$$\times \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma_{\mathrm{BES}}(u_{4,k}, u_{\bar{4},j})$$

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The Bethe equations/3

- Energy
- Conserved charges
- S-matrix
- Rapidity
- Dispersion relation

- $E = h^2(\lambda)\, \mathcal{Q}_2$
- $\mathcal{Q}_r(p_j) =$
  $\frac{\sin\left(\frac{r-1}{2} p_j\right)}{r-1} \left(\frac{\sqrt{\frac{1}{4}+4h(\lambda)^2 \sin^2 \frac{p_j}{2}} - \frac{1}{2}}{h(\lambda)^2 \sin \frac{p_j}{2}} \right)^{r-1}$
- $S(p_k, p_j) = \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}$
- $u_{4,j} = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1+16h(\lambda)^2 \sin^2 \frac{p_j}{2}}$
- $E - J =$
  $\frac{1}{2}\left(\sqrt{1+16h(\lambda)^2 \sin\left(\frac{p}{2}\right)^2} - 1\right)$

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  $$\frac{\sin\left(\frac{r-1}{2}p_j\right)}{r-1}\left(\frac{\sqrt{\frac{1}{4}+4h(\lambda)^2\sin^2\frac{p_j}{2}}-\frac{1}{2}}{h(\lambda)^2\sin\frac{p_l}{2}}\right)^{r-1}$$
- $S(p_k, p_j) = \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}$
- $u_{4,j} = \frac{1}{2}\cot\frac{p_j}{2}\sqrt{1+16h(\lambda)^2\sin^2\frac{p_j}{2}}$
- $E - J =$
  $$\frac{1}{2}\left(\sqrt{1+16h(\lambda)^2\sin\left(\frac{p}{2}\right)^2}-1\right)$$

The Bethe equations/3

- Energy
- Conserved charges
- S-matrix
- Rapidity
- Dispersion relation

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Evolution of the Bethe program

$\lambda \to 0$

- Complete test of two loop integrability: arXiv:0901.0411, 0901.1142
- Four loops dispersion relation: arXiv: 0908.2463, 0912.3460

$\lambda \to \infty$

- Quantum spinning strings: arXiv:0809.4038
- Further development on giant magnons
- Near plane-wave limit: arXiv:0807.1527, 0912.2257, 1101.0004, 1111.6628

Y-system is formulated for computation of energies/anomalous dimensions at any size. arXiv:0902.3930, 0902.4458

Open debate on subleading corrections to $h(\lambda)$ at strong coupling.

What do we learn from near plane wave limit?

- Plane wave limit involves the effective scaling in $\lambda' = \frac{\lambda}{J^2}$: it entails quantum with finite size corrections.
- Study of two impurities finite size spectrum allows comparison with Bethe states energies and their identification.
- Many impurities spectrum vs. Bethe ansatz provides evidence of integrability in the near plane wave limit.

Finite size spectrum sheds light on the nature of $h(\lambda) \simeq \sqrt{\frac{\lambda}{2}} + c_1 + \dots$. Is c_1 vanishing (arXiv 0903.1747) or is it $-\frac{\log 2}{2\pi}$ (arXiv 0807.3965, 0809.4038)?

Contents

The $SU(2) \times SU(2)$ plane-wave limit

$AdS_4 \times CP^3$ metric:

- $ds^2 = -\frac{R^2}{4} dt'^2 (\sin^2\psi + \sinh^2\rho) + \frac{R^2}{4}(d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2)$
  $ds^2_{CP^3} = \frac{1}{4} d\psi^2 + \frac{1-\sin\psi}{8} d\Omega_2^2 + \frac{1+\sin\psi}{8} d{\Omega'_2}^2 + \cos^2\psi (d\delta + \omega)^2$
  $d\Omega_2^2 = d\theta_1^2 + \cos^2\theta_1 d\varphi_1^2 \quad , \qquad d{\Omega'_2}^2 = d\theta_2^2 + \cos^2\theta_2 d\varphi_2^2$
- The $SU(2) \times SU(2)$ subsector corresponds to the two two-spheres in the CP^3 metric
- Cartan generators: $S_z^{(1)} = -i\partial_{\varphi_1}$, $S_z^{(2)} = -i\partial_{\varphi_2}$, $J = -\frac{i}{2}\partial_\delta$
- Coordinate transformations $t' = t$, $\chi = \delta - \frac{1}{2} t$
  $v = R^2\chi$, $x_1 = R\varphi_1$, $y_1 = R\theta_1$, $x_2 = R\varphi_2$, $y_2 = R\theta_2$, $u_4 = \frac{R}{2}\psi$
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Plane-wave limit

From the pp-wave metric to the classical light-cone Hamiltonian

- $ds^2 = dvdt' + \sum_{i=1}^{4}(du_i^2 - u_i^2 dt'^2) + \frac{1}{8}\sum_{i=1}^{2}(dx_i^2 + dy_i^2 + 2dt' y_i dx_i)$
- Bosonic string action
  $I = \frac{1}{2\pi}\int d\tau d\sigma \;\; \mathcal{L} = -\frac{1}{2}h^{\alpha\beta}G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu$
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Bosonic pp-wave spectrum

Quantizing:

- $H_{\mathrm{free}} = \sum_{i=1}^{4} \sum_{n\in\mathbb{Z}} \sqrt{1 + \frac{2\pi^2\lambda}{J^2} n^2} (\hat{a}^i_n)^\dagger \hat{a}^i_n + \sum_{a=1}^{2} \sum_{n\in\mathbb{Z}} \left[\left(\sqrt{\frac{1}{4} + \frac{2\pi^2\lambda}{J^2} n^2} - \frac{1}{2} \right) (a^a)^\dagger_n a^a_n + \left(\sqrt{\frac{1}{4} + \frac{2\pi^2\lambda}{J^2} n^2} + \frac{1}{2} \right) (\tilde{a}^a)^\dagger_n \tilde{a}^a_n \right]$
- Four bosons are light and four are heavy. Similarly for the fermions.
- Two of the light bosons further decouple and have energy of $\mathcal{O}(\lambda')$ in the $\lambda' \to 0$ limit.

- Compute near plane-wave spectrum (i.e. $\frac{1}{J}$ corrections) for two impurities states: focus on bosonic states with two light impurities.

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Cubic and quartic perturbation Hamiltonian

- Expand up to subleading $\frac{1}{R^2}$ order.
- A novel feature appears with respect to the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence: a $\frac{1}{R}$ cubic Hamiltonian, to be evaluated at second order in perturbation theory.

- $\mathcal{H}_{\text{int}}^{\text{lc}} = \mathcal{H}_3 + \mathcal{H}_4$
- $\mathcal{H}_3 = \frac{\sqrt{\pi} u_4}{2\sqrt{J}} \left[(\dot{x}_1)^2 - (\dot{x}_2)^2 + (\dot{y}_1)^2 - (\dot{y}_2)^2 - (x_1')^2 + (x_2')^2 - (y_1')^2 + (y_2')^2 \right]$
- $\mathcal{H}_4 = \frac{\lambda' \pi^2}{256 J} \left[4 \left(\dot{x}_a x_a' + \dot{y}_a y_a' \right)^2 - \left((x_a')^2 + (y_a')^2 + (\dot{x}_a)^2 + (\dot{y}_a)^2 \right)^2 \right] + \frac{1}{192 J} \left[3 \left(\left((\dot{x}_1)^2 - (x_1')^2 \right) y_1^2 + \left((\dot{x}_2)^2 - (x_2')^2 \right) y_2^2 \right) + \frac{1}{\sqrt{2\lambda' \pi}} \left(\dot{x}_1 y_1^3 + \dot{x}_2 y_2^3 \right) \right] + \ldots$

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- $\mathcal{H}^{\mathrm{lc}}_{\mathrm{int}} = \mathcal{H}_3 + \mathcal{H}_4$
- $\mathcal{H}_3 = \frac{\sqrt{\pi} u_4}{2\sqrt{J}} \left[(\dot{x}_1)^2 - (\dot{x}_2)^2 + (\dot{y}_1)^2 - (\dot{y}_2)^2 - (x_1')^2 + (x_2')^2 - (y_1')^2 + (y_2')^2 \right]$
- $\mathcal{H}_4 = \frac{\lambda' \pi^2}{256 J} \left[4 \left(\dot{x}_a x_a' + \dot{y}_a y_a' \right)^2 - \left((x_a')^2 + (y_a')^2 + (\dot{x}_a)^2 + (\dot{y}_a)^2 \right)^2 \right] + \frac{1}{192 J} \left[3 \left(\left((\dot{x}_1)^2 - (x_1')^2 \right) y_1^2 + \left((\dot{x}_2)^2 - (x_2')^2 \right) y_2^2 \right) + \frac{1}{\sqrt{2\lambda'} \pi} \left(\dot{x}_1 y_1^3 + \dot{x}_2 y_2^3 \right) \right] + \ldots$

Cubic and quartic perturbation Hamiltonian

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- $\mathcal{H}_4 = \frac{\lambda'\pi^2}{256J}\left[4\left(\dot{x}_a x_a' + \dot{y}_a y_a'\right)^2 - \left((x_a')^2 + (y_a')^2 + (\dot{x}_a)^2 + (\dot{y}_a)^2\right)^2\right] + \frac{1}{192J}\Big[3\left(\left((\dot{x}_1)^2 - (x_1')^2\right) y_1^2 + \left((\dot{x}_2)^2 - (x_2')^2\right) y_2^2\right) + \frac{1}{\sqrt{2\lambda'}\pi}\left(\dot{x}_1 y_1^3 + \dot{x}_2 y_2^3\right)\Big] + \ldots$

The spectrum: naif.

- For non degenerate states: $E_s^{(2)} = \langle s|H_4|s\rangle + \sum_{|i\rangle} \frac{|\langle i|H_3|s\rangle|^2}{E^{(0)}_{|s\rangle} - E^{(0)}_{|i\rangle}}$
- For degenerate states: diagonalize the mixing matrix.
- $\sum_{|i\rangle} \frac{|\langle i|H_3|s\rangle|^2}{E^{(0)}_{|s\rangle} - E^{(0)}_{|i\rangle}}$ contains infinite sums which diverge logaritmically!!
- In arXiv:0807.1527 ζ-function regularization: the infinite sum is set to 0. Spectra have only integer powers of λ', matching with Bethe ansatz predictions: one of the earliest test of the Bethe ansatz!!
- Example: take $|s\rangle = a_n^\dagger a_{-n}^\dagger|0\rangle$. It is non degenerate.

$$E_s^{(2)} = \frac{\lambda'}{J}\frac{4\pi^2n^2}{\frac{1}{4}+2\pi^2n^2\lambda'}\left(\sqrt{\frac{1}{4}+2\pi^2n^2\lambda'}-2\pi^2n^2\lambda'\right)$$

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The spectrum: smart!/1

The turning point

- The quartic Hamiltonian H_4 must be not normal ordered! Therefore...
- For whatever state, one needs the interacting Hamiltonian in all the subsectors. Achieved in arXiv:0912.2257.

Ordering schemes

- For a bosonic set:
$\left(a^{\dagger i}_m a^i_p\right)_Q \equiv \left(a^i_p a^{\dagger i}_m\right)_Q \equiv \alpha_i a^{\dagger i}_m a^i_p + (1-\alpha_i)\, a^i_p a^{\dagger i}_m.$
Different choices of α_i correspond to different ordering schemes. Similarly for quartic objects.
- Requests: spectra be finite and pp-wave algebra respected
- Unique choice: completely symmetrized!!!

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The spectrum: smart!/2

The results

$$
\begin{aligned}
E_s^{(2)} &= -\frac{4\pi^2 n^2 \lambda'}{J}\left(1 - \frac{1}{\sqrt{2n^2\pi^2\lambda' + \frac{1}{4}}}\right) \\
&\quad - \frac{2n^2}{J\sqrt{n^2 + \frac{1}{8\pi^2\lambda'}}}\sum_{q=1}^{\infty}\left[1 - (-1)^q\right] K_0\left(\frac{q}{\sqrt{2\lambda'}}\right) \simeq \\
&\quad \frac{1}{J}\left(8n^2\pi^2\lambda' - 64n^4\pi^4\lambda'^2 + 448n^6\pi^6\lambda'^3 + \dots\right)
\end{aligned}
$$

K_0 is the modified Bessel of the second kind.

The infinite sum above is exponentially suppressed.

What can we say about $h(\lambda)$?

Single magnon dispersion relation

- For a single magnon the infinite sum is the only contribution to the spectrum. Let $|s_1\rangle = a_n^\dagger|0\rangle$. Level matching condition lifted.
- $E_{s_1}^{(2)} = -\frac{n^2}{J\sqrt{n^2+\frac{1}{8\pi^2\lambda'}}}\sum_{q=1}^{\infty}\left[1-(-1)^q\right]K_0\left(\frac{q}{\sqrt{2\lambda'}}\right) \simeq \mathcal{O}\left(e^{-\frac{1}{\sqrt{\lambda'}}}\right)$.
- Compare with the dispersion relation from the Bethe ansatz: $a_1 = 0$!!!!

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\begin{aligned}
E &= \sqrt{\frac{1}{4}+4\left(\sqrt{\frac{\lambda}{2}}+a_1\right)^2\sin^2\frac{p}{2}} \\
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The dispersion relation

- Our results are compatible with the following dispersion relation for the single magnon:

$$E = \sqrt{\frac{1}{4} + 4\left(\sqrt{\frac{\lambda}{2}} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right)^2 \frac{n^2\pi^2}{J^2}} - \frac{\pi n^2 \sqrt{2\lambda'}}{J\sqrt{\frac{1}{4} + 2\lambda' n^2 \pi^2}} \sum_{q=1}^{\infty} \left[1 - (-1)^q\right] K_0\left(\frac{q}{\sqrt{2\lambda'}}\right)$$

- The dispersion relation from the Bethe ansatz is asymptotic, i.e. obtained in the limit $J \to \infty$.
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Classification of the states/1

- Define $c = \frac{1}{\pi\sqrt{2\lambda'}}$, $\omega_n = \sqrt{n^2 + \frac{c^2}{4}}$.
- 32 light two oscillator bosonic states.
- 4 (and 4) have pp-wave energy $2\left(\omega_n \pm \frac{c}{2}\right)$ and are not mixed by the interactions. Addressed in arXiv:0807.1527.
- 24 have pp-wave energy $2\omega_n$. Addressed in arXiv:1111.6628.

Two bosonic oscillators

$$\begin{aligned} v_n^1 &= a_n^{1\dagger}\tilde{a}_{-n}^{1\dagger}|0\rangle, \\ v_n^2 &= a_n^{1\dagger}\tilde{a}_{-n}^{2\dagger}|0\rangle, \\ v_n^3 &= a_n^{2\dagger}\tilde{a}_{-n}^{1\dagger}|0\rangle, \\ v_n^4 &= a_n^{2\dagger}\tilde{a}_{-n}^{2\dagger}|0\rangle. \end{aligned}$$

Symmetry between the $SU(2)$'s

$$\begin{aligned} s_n &= \frac{1}{\sqrt{2}}(v_n^1 + v_n^4) \\ p_n &= \frac{1}{\sqrt{2}}(-v_n^1 + v_n^4) \\ q_n &= \frac{1}{\sqrt{2}}(v_n^2 + v_n^3) \\ r_n &= \frac{1}{\sqrt{2}}(-v_n^2 + v_n^3). \end{aligned}$$

Classification of the states/2

Two bosonic oscillators

Impose symmetry for $n \to -n$ and $a_i \to \tilde{a}_i$.

state	definition	P_n	$\tilde{P}$	P_a
u_1	$\frac{1}{\sqrt{2}}(s_n + s_{-n})$	1	1	1
u_2	$\frac{1}{\sqrt{2}}(-s_n + s_{-n})$	−1	−1	1
u_3	$\frac{1}{\sqrt{2}}(p_n + p_{-n})$	1	1	−1
u_4	$\frac{1}{\sqrt{2}}(-p_n + p_{-n})$	−1	−1	−1
u_5	$\frac{1}{\sqrt{2}}(q_n + q_{-n})$	1	1	1
u_6	$\frac{1}{\sqrt{2}}(-q_n + q_{-n})$	−1	−1	1
u_7	$\frac{1}{\sqrt{2}}(r_n + r_{-n})$	1	−1	−1
u_8	$\frac{1}{\sqrt{2}}(-r_n + r_{-n})$	−1	1	−1

Classification of the states/3

Two fermionic oscillators

- They are of the type $d^\dagger_\alpha A_{\alpha\beta} d^\dagger_\beta |0\rangle$. Projecting onto physical states reveals there are 16 such states.
- Choice of the basis:

$$\begin{aligned} A_{i=9\ldots24} &= \{\tfrac{1}{2}, \tfrac{1}{2}\Gamma_{56}, \tfrac{1}{2}\Gamma_{12}, \tfrac{1}{2}\Gamma_{13}, \tfrac{1}{2}\Gamma_{23}, \\ &\quad \tfrac{1}{2}\Gamma_{1256}, \tfrac{1}{2}\Gamma_{1356}, \tfrac{1}{2}\Gamma_{2356}, \tfrac{1}{2}\Gamma_{1457}, \tfrac{1}{2}\Gamma_{2457}, \\ &\quad \tfrac{1}{2}\Gamma_{3457}, \tfrac{1}{2}\Gamma_{1467}, \tfrac{1}{2}\Gamma_{2467}, \tfrac{1}{2}\Gamma_{3467}, \tfrac{1}{2}\Gamma_{0579}, \tfrac{1}{2}\Gamma_{0679}\}. \end{aligned}$$

- We have an orthonormal basis $u_1, \ldots, u_{24}$.

Spectrum of the basis $u_1, \ldots, u_{24}/1$

Two bosonic oscillators: spectrum

The basis $u_1, \ldots, u_{24}$ diagonalizes the perturbation Hamiltonian: it is the physical basis!

state	expansion of the spectrum
u_1	$\frac{1}{J}\left(-16n^2\pi^2\lambda' + 96n^4\pi^4\lambda'^2 - 768n^6\pi^6\lambda'^3 + \ldots\right)$
u_2	$\frac{1}{J}\left(-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \ldots\right)$
u_3	$\frac{1}{J}\left(-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \ldots\right)$
u_4	$\frac{1}{J}\left(-16n^2\pi^2\lambda' + 96n^4\pi^4\lambda'^2 - 768n^6\pi^6\lambda'^3 + \ldots\right)$
u_5	$\frac{1}{J}\left(-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \ldots\right)$
u_6	$\frac{1}{J}\left(-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \ldots\right)$
u_7	$\frac{1}{J}\left(-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \ldots\right)$
u_8	$\frac{1}{J}\left(-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \ldots\right)$

Spectrum of the basis $u_1, \ldots, u_{24}/2$

Two fermionic oscillators: spectrum

state	expansion of the spectrum
$u_{9,10}$	$\frac{1}{J}\left(-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \ldots\right)$
$u_{11,\ldots,16}$	$\frac{1}{J}\left(-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \ldots\right)$
$u_{17,\ldots,22}$	$\frac{1}{J}\left(-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \ldots\right)$
$u_{23,24}$	$\frac{1}{J}\left(8n^2\pi^2\lambda' - 96n^4\pi^4\lambda'^2 + 768n^6\pi^6\lambda'^3 + \ldots\right)$

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Single magnon states

Bosonic

K_{u_4}	$K_{u_{\bar{4}}}$	K_{u_1}	K_{u_2}	K_{u_3}
1	0	0	0	0
0	1	0	0	0
1	0	1	1	1
0	1	1	1	1

Fermionic

K_{u_4}	$K_{u_{\bar{4}}}$	K_{u_1}	K_{u_2}	K_{u_3}
1	0	1	0	0
0	1	1	0	0
1	0	1	1	0
0	1	1	1	0

Two magnon states corresponding to the $u_1, \ldots, u_{24}$ basis

K_{u_4}	$K_{u_{\bar{4}}}$	K_{u_1}	K_{u_2}	K_{u_3}	Multiplicity
1	1	1	1	1	4
2	0	1	1	1	2
0	2	1	1	1	2
1	1	2	0	0	2
1	1	2	1	0	4
1	1	2	2	0	2
2	0	2	0	0	1
0	2	2	0	0	1
2	0	2	1	0	2
0	2	2	1	0	2
2	0	2	2	0	1
0	2	2	2	0	1

- Multiplicities:
- $p \to -p$ symmetry
- double branchings.
- 24 states

Spectrum of the two magnon configurations

State	Expansion of the spectrum
$(1,1,1,1,1)_{\mathrm{branch1}}$	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \dots$
$(2,0,1,1,1)_{\mathrm{branch1}}$	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \dots$
$(1,1,1,1,1)_{\mathrm{branch2}}$	$-16n^2\pi^2\lambda' + 96n^4\pi^4\lambda'^2 - 768n^6\pi^6\lambda'^3 + \dots$
$(2,0,1,1,1)_{\mathrm{branch2}}$	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \dots$
$(1,1,2,2,0)$	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \dots$
$(2,0,2,2,0)$	$8n^2\pi^2\lambda' - 96n^4\pi^4\lambda'^2 + 768n^6\pi^6\lambda'^3 + \dots$
$(1,1,2,1,0)_{\mathrm{branch1}}$	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \dots$
$(2,0,2,1,0)_{\mathrm{branch1}}$	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \dots$
$(1,1,2,1,0)_{\mathrm{branch2}}$	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \dots$
$(2,0,2,1,0)_{\mathrm{branch2}}$	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \dots$
$(1,1,2,0,0)$	$-8n^2\pi^2\lambda' + 32n^4\pi^4\lambda'^2 - 256n^6\pi^6\lambda'^3 + \dots$
$(2,0,2,0,0)$	$-32n^4\pi^4\lambda'^2 + 256n^6\pi^6\lambda'^3 + \dots$

String theory vs. Bethe ansatz/1

Boson-boson spectrum comparison

Write the spectrum in this form

$$E = \frac{1}{J} \sum_{i=1} c_i \lambda'^i \left(8\pi^2 n^2\right)^i$$

$(c_1, \ldots, c_5)$	Multiplicity	BA state	String state
$(0, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	2	$(2,0,1,1,1)_{branch1}$	5, 7
$(-1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	4	$(2,0,1,1,1)_{branch2}$ $(1,1,1,1,1)_{branch1}$	2, 3, 6, 8
$(-2, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2})$	2	$(1,1,1,1,1)_{branch2}$	1, 4

String theory vs. Bethe ansatz/1

Fermion-fermion spectrum comparison

$(c_1, \ldots, c_5)$	Multiplicity	BA state	String state
$(1, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$	2	$(2,0,2,2,0)$	$23, 24$
$(0, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	8	$(1,1,2,2,0)$	$9, 10$
		$(2,0,2,1,0)_{branch1}$	$17, 18$
		$(2,0,2,1,0)_{branch2}$	$19, 20$
		$(2,0,2,0,0)$	$21, 22$
$(-1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	6	$(1,1,2,1,0)_{branch1}$	$11, 12$
		$(1,1,2,1,0)_{branch2}$	$13, 14$
		$(1,1,2,0,0)$	$15, 16$

Perfect agreement between String theory and Bethe ansatz: in magnitude and multiplicity

Summary and perspectives/1

Summary..

- Finite size spectrum vs. Bethe ansatz in the $SU(2) \times SU(2)$ subsector: one of the earliest tests of Bethe ansatz! arXiv:0807.1527
- Further analysis in the $SU(2) \times SU(2)$ subsector: dispersion relation at finite size and $h(\lambda)$. arXiv:1101.004
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- Penrose limit treats light and heavy on equal footing: it looks like 8B+8F degrees of freedom.
- Bethe ansatz has 4B+4F degrees of freedom.
- Complete the comparison: finite size vs. Bethe ansatz and thus inquire the nature of heavy modes.

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