Scattering Amplitudes at the Integrand Level

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Cortona 2012
May 30-31, June 1

based on:

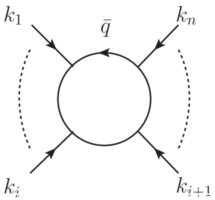
arXiv:1203.0291: P. Mastrolia, E. Mirabella, T. P.
work in progress: G. Ossola, P. Mastrolia, E. Mirabella, T. P.

Outline

Introduction: Scattering amplitudes at one-loop

- A generic n-point one-loop amplitude

$$\mathcal{M}_n \equiv \int \mathcal{A}_n(\bar{q})\, d^d\bar{q} \equiv \int \frac{N(\bar{q})}{\bar{D}_1 \dots \bar{D}_n} d^d\bar{q}$$



  - we split the loop momentum $\bar{q}$ in a 4-dimensional part q and a $(d-4)$-dimensional part $\vec{\mu}$

$$\bar{q} = q + \vec{\mu} \qquad \bar{q}^2 = q^2 - \mu^2$$

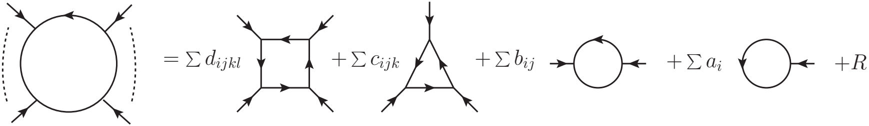
  - the numerator of the integrand

$$N(\bar{q}) = N(q, \mu^2)$$

  - the denominators

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 = (q + p_i)^2 - m_i^2 - \mu^2$$

Introduction: Scattering amplitudes at one-loop



- Every one-loop amplitude in $d = 4 - 2\epsilon$ can be decomposed as

$$\mathcal{M}_n = \sum_{ijkl} d_{ijkl}\, I_{ijkl} + \sum_{ijk} c_{ijk}\, I_{ijk} + \sum_{ij} b_{ij}\, I_{ij} + \sum_{i} a_i\, I_i + R + \mathcal{O}(\epsilon)$$

$$I_{ijk\ldots} = \int \frac{d\bar{q}}{\bar{D}_i \bar{D}_j \bar{D}_k \ldots}$$

  - the basis of Master Integrals (MIs) $I_{ijk\ldots}$ is known
  - the computation of the amplitude can be reduced to the problem of computing the coefficients of this decomposition and the rational part R

Integrand-level decomposition: OPP

$$\int \mathcal{A}_n(\bar{q}) d\bar{q} = \sum_{ijkl} d_{ijkl} \int \frac{d\bar{q}}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} + \sum_{ijk} c_{ijk} \int \frac{d\bar{q}}{\bar{D}_i \bar{D}_j \bar{D}_k} + \sum_{ij} b_{ij} \int \frac{d\bar{q}}{\bar{D}_i \bar{D}_j} + \sum_i a_i \int \frac{d\bar{q}}{\bar{D}_i} + R$$

- The previous decomposition holds at the integral-level

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- The previous decomposition holds at the integral-level
- An analogous decomposition holds at the integrand-level

  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]

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- The previous decomposition holds at the integral-level
- An analogous decomposition holds at the integrand-level

  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]
- in $d = 4 - 2\epsilon$ dimensions

$$\mathcal{A}(\bar{q}) = \frac{N(q,\mu^2)}{\bar{D}_1\ldots\bar{D}_n} = \sum_{ijklm} \frac{\Delta_{ijklm}(\mu^2)}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l\bar{D}_m} + \sum_{ijkl} \frac{\Delta_{ijkl}(q,\mu^2)}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} + \sum_{ijk} \frac{\Delta_{ijk}(q,\mu^2)}{\bar{D}_i\bar{D}_j\bar{D}_k} + \sum_{ij} \frac{\Delta_{ij}(q,\mu^2)}{\bar{D}_i\bar{D}_j} + \sum_{i} \frac{\Delta_{i}(q,\mu^2)}{\bar{D}_i}$$

  - pentagons $\Delta_{ijklm}(\mu^2)$ vanish upon integration

Residues

- The residues $\Delta_{ijk\dots}$ have a known parametric form in terms of
  - residue-dependent four vectors $e_i, v_\perp$
  - unknown coefficients f_0, d_i, c_i, b_i, a_i

$$
\begin{aligned}
\Delta_{ijklm}(\mu^2) &= f_0\, \mu^2 \\
\Delta_{ijkl}(q,\mu^2) &= d_0 + d_2\mu^2 + d_4\mu^4 + (d_1 + d_3\mu^2)(q\cdot v_\perp) \\
\Delta_{ijk}(q,\mu^2) &= c_0 + c_7\,\mu^2 \\
&\quad + \left(c_1 + c_8\mu^2\right)(q\cdot e_3) + c_2\,(q\cdot e_3)^2 + c_3\,(q\cdot e_3)^3 \\
&\quad + \left(c_4 + c_9\mu^2\right)(q\cdot e_4) + c_5\,(q\cdot e_4)^2 + c_6\,(q\cdot e_4)^3 \\
\Delta_{ij}(q,\mu^2) &= b_0 + b_9\,\mu^2 + b_1(q\cdot e_2) + b_2(q\cdot e_2)^2 \\
&\quad + b_3(q\cdot e_3) + b_4(q\cdot e_3)^2 + b_5(q\cdot e_4) + b_6(q\cdot e_4)^2 \\
&\quad + b_7(q\cdot e_2)(q\cdot e_3) + b_8(q\cdot e_2)(q\cdot e_4) \\
\Delta_i(q) &= a_0 + a_1(q\cdot e_1) + a_1(q\cdot e_1) + a_1(q\cdot e_1) + a_1(q\cdot e_1)
\end{aligned}
$$

  - the red terms give Master Integrals
  - the blu terms determine the rational part
  - the other terms (spurious) vanish upon integration

Extended decomposition

- The previous decomposition holds for renormalizable theories
  - the rank of the numerator can not be greater that the number n of denominators
- It can be extended to non-renormalizable theories
  [P. Mastrolia, E. Mirabella, T. P. (2012)]
  - if, for instance, the rank of the numerator is equal $n+1$ we get

$$
\begin{aligned}
\tilde{\Delta}_{ijk\ell m}(q,\mu^2) &= \Delta_{ijk\ell m}(q,\mu^2)\\
\tilde{\Delta}_{ijk\ell}(q,\mu^2) &= \Delta_{ijk\ell}(q,\mu^2) + d_{4,5}\,\mu^4\, q\cdot v_\perp\\
\tilde{\Delta}_{ijk}(q,\mu^2) &= \Delta_{ijk}(q,\mu^2) + c_{3,14}\,\mu^4 + c_{3,10}\,\mu^2\,(q\cdot e_3)^2 + c_{3,11}\,\mu^2\,(q\cdot e_4)^2 + c_{3,12}(q\cdot e_3)^4 + c_{3,13}(q\cdot e_4)^4\\
\tilde{\Delta}_{ij}(q,\mu^2) &= \Delta_{ij}(q,\mu^2) + \mu^2\Big(b_{10}^{(ij)}\,(q\cdot e_2) + b_{11}^{(ij)}\,(q\cdot e_3) + b_{12}(q\cdot e_4)\Big) + b_{13}\,(q\cdot e_2)^3 + b_{14}(q\cdot e_3)^3\\
&\quad + b_{15}(q\cdot e_4)^3 + b_{16}(q\cdot e_2)^2(q\cdot e_3) + b_{17}(q\cdot e_2)^2(q\cdot e_4) + b_{18}(q\cdot e_2)(q\cdot e_3)^2 + b_{19}(q\cdot e_2)(q\cdot e_4)^2\\
\tilde{\Delta}_{i}(q,\mu^2) &= \Delta_{i}(q,\mu^2) + a_5(q\cdot e_1)^2 + c_{1,6}(q\cdot e_2)^2 + c_{1,7}(q\cdot e_3)^2 + c_{1,8}(q\cdot e_4)^2 + a_{10}(q\cdot e_1)(q\cdot e_3)\\
&\quad + a_{11}(q\cdot e_1)(q\cdot e_4) + a_{12}(q\cdot e_2)(q\cdot e_3) + a_{13}(q\cdot e_2)(q\cdot e_4) + a_{14}\,\mu^2 + a_{15}(q\cdot e_3)(q\cdot e_4)\\
\tilde{\Delta}_{Q}(q,\mu^2) &= c_{Q,0}\,.
\end{aligned}
$$

Result for the integrated amplitude

- A generic integrand can be decomposed as

$$\mathcal{A}_n = \sum_{ijklm} \frac{\Delta_{ijklm}}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l\bar{D}_m} + \sum_{ijkl} \frac{\Delta_{ijkl}}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} + \sum_{ijk} \frac{\Delta_{ijk}}{\bar{D}_i\bar{D}_j\bar{D}_k} + \sum_{ij} \frac{\Delta_{ij}}{\bar{D}_i\bar{D}_j} + \sum_{i} \frac{\Delta_{i}}{\bar{D}_i}$$

  - the residues $\Delta_{ij\ldots}$
    - have a known parametric form
    - contain the coefficients of the MIs
- After integration

$$\int \mathcal{A}_n(\bar{q})d\bar{q} = \sum_{ijkl} \int \frac{d_0^{(ijkl)}}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} + \sum_{ijk} \int \frac{c_0^{(ijk)}}{\bar{D}_i\bar{D}_j\bar{D}_k} + \sum_{ij} \int \frac{b_0^{(ij)}}{\bar{D}_i\bar{D}_j} + \sum_{i} \int \frac{a_0^{i}}{\bar{D}_i} + R$$

  - several terms vanish (they are called spurious)
  - the rational part R is determined by integrals in μ^2, e.g.

$$\int \frac{\mu^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{1}{6} + \mathcal{O}(\epsilon), \qquad \int \frac{\mu^2}{\bar{D}_i\bar{D}_j\bar{D}_k} = \frac{1}{2} + \mathcal{O}(\epsilon), \qquad \ldots$$

$N = N$ decomposition

$$\frac{N(q,\mu^2)}{D_1 \dots D_n} = \sum_{ijklm} \frac{\Delta_{ijklm}}{D_i D_j D_k D_l D_m} + \sum_{ijkl} \frac{\Delta_{ijkl}}{D_i D_j D_k D_l} + \sum_{ijk} \frac{\Delta_{ijk}}{D_i D_j D_k} + \sum_{ij} \frac{\Delta_{ij}}{D_i D_j} + \sum_{i} \frac{\Delta_i}{D_i}$$

- The former decomposition can be rewritten as

$$N(q,\mu^2) = \sum_{ijklm} \Delta_{ijklm} \prod_{h \neq i,j,k,l} D_h + \sum_{ijkl} \Delta_{ijkl} \prod_{h \neq i,j,k,l} D_h + \sum_{ijk} \Delta_{ijk} \prod_{h \neq i,j,k} D_h$$
$$+ \sum_{ij} \Delta_{ij} \prod_{h \neq i,j} D_h + \sum_{i} \Delta_i \prod_{h \neq i} D_h$$

- the residues $\Delta_{ijk\dots}$ are polynomials in the components of q and μ^2
- the coefficients which parametrize the residues can be found by polynomial fitting
- an efficient strategy is to evaluate the integrand on solutions of multiple cuts i.e. on values of q and μ^2 such that some denominators D_i vanish

Integrand reduction and multiple cuts

- On the solutions of the 5-cut: $D_i = D_j = D_k = D_l = D_m = 0$

$$\left. \frac{N(q)}{\prod_{h \neq i,j,k,l,m} D_h} \right|_{\text{cut}} = \Delta_{ijklm}$$

- On the solutions of the 4-cut: $D_i = D_j = D_k = D_l = 0$

$$\left[\frac{N(q)}{\prod_{h \neq i,j,k,l} D_h} - \sum_m \frac{\Delta_{ijklm}}{D_m} \right]_{\text{cut}} = \Delta_{ijkl}$$

- On the solutions of the 3-cut: $D_i = D_j = D_k = 0$

$$\left[\frac{N(q)}{\prod_{h \neq i,j,k} D_h} - \sum_{l,m} \frac{\Delta_{ijklm}}{D_l D_m} - \sum_l \frac{\Delta_{ijkl}}{D_l} \right]_{\text{cut}} = \Delta_{ijk}$$

Integrand reduction and multiple cuts

- On the solutions of the 2-cut: $D_i = D_j = 0$

$$\left[\frac{N(q)}{\prod_{h\neq i,j} D_h} - \sum_{k,l,m} \frac{\Delta_{ijklm}}{D_k D_l D_m} - \sum_{k,l} \frac{\Delta_{ijkl}}{D_k D_l} - \sum_{k} \frac{\Delta_{ijk}}{D_k} \right]_{\text{cut}} = \Delta_{ij}$$

- On the solutions of the 1-cut: $D_i = 0$

$$\left[\frac{N(q)}{\prod_{h\neq i} D_h} - \sum_{j,k,l,m} \frac{\Delta_{ijklm}}{D_j D_k D_l D_m} - \sum_{j,k,l} \frac{\Delta_{ijkl}}{D_j D_k D_l} - \sum_{j,k} \frac{\Delta_{ijk}}{D_j D_k} - \sum_{j} \frac{\Delta_{ij}}{D_j} \right]_{\text{cut}} = \Delta_i$$

Integrand reduction and multiple cuts

- On the solutions of the 2-cut: $D_i = D_j = 0$

$$\left[\frac{N(q)}{\prod_{h\neq i,j} D_h} - \sum_{k,l,m} \frac{\Delta_{ijklm}}{D_k D_l D_m} - \sum_{k,l} \frac{\Delta_{ijkl}}{D_k D_l} - \sum_{k} \frac{\Delta_{ijk}}{D_k} \right]_{\text{cut}} = \Delta_{ij}$$

- On the solutions of the 1-cut: $D_i = 0$

$$\left[\frac{N(q)}{\prod_{h\neq i} D_h} - \sum_{j,k,l,m} \frac{\Delta_{ijklm}}{D_j D_k D_l D_m} - \sum_{j,k,l} \frac{\Delta_{ijkl}}{D_j D_k D_l} - \sum_{j,k} \frac{\Delta_{ijk}}{D_j D_k} - \sum_{j} \frac{\Delta_{ij}}{D_j} \right]_{\text{cut}} = \Delta_{i}$$

- by sampling the integrand on cut-solutions one can recursively fit the coefficients of each residue – from the 5-point ones to the 1-point ones – by solving smaller systems of equations
- higher-point residues and computed first and then subtracted from the integrand in order to find the lower-point ones

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- The coefficients of the pentagons and the spurious coefficients of the boxes do not have to be computed
- No subtraction is needed for the computation of 4 and 3-point coefficients
- The computation of 3, 2, and 1-point residues is completely disentangled from the higher-point ones

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- No systems of equations appear
- The coefficients of the pentagons and the spurious coefficients of the boxes do not have to be computed
- No subtraction is needed for the computation of 4 and 3-point coefficients
- The computation of 3, 2, and 1-point residues is completely disentangled from the higher-point ones
- In the case of 2 and 1-point residues, the subtractions at integrand level are replaced by corrections to the coefficients
- The parametric form of this corrections is known as a function of the 3 and 2-point coefficients

P. Mastrolia, E. Mirabella, T. P. (2012)

The coefficients of the boxes

- The residue of a box reads

$$\Delta_{ijkl}(q,\mu^2) = d_0 + d_2\mu^2 + d_4\,\mu^4 + (d_1 + d_3\mu^2)(q\cdot v_\perp)$$

- d_0 can be computed via 4-dimensional 4ple cuts [Britto, Cachazo, Feng (2004)]
- d_4 can be computed from d-dimensional 4ple cuts in the limit $\mu^2 \to \infty$ [S. Badger (2008)]
  - the d-dimensional solutions of a quadruple cut are

$$q_\pm = a^\mu \pm \sqrt{\alpha + \frac{\mu^2}{\beta^2}}\; v_\perp^\mu = \pm\frac{\sqrt{\mu^2}}{\beta}\; v_\perp^\mu + \mathcal{O}(1)$$

    with $a^\mu, v_\perp^\mu, \alpha, \beta$ fixed by the cut conditions
  - the integrand in the asymptotic limit $\mu^2 \to \infty$ of the cut-solutions

$$\left.\frac{N(q,\mu^2)}{\prod_{m\neq i,j,k,l} D_m}\right|_{\text{cut}} = d_4\,\mu^4 + \mathcal{O}(\mu^3)$$

- d_1, d_2, d_3 are spurious and do not need to be computed

The coefficients of the triangles

- The residue of a triangle reads (4-dim for brevity)

$$\Delta_{ijk}(q) = c_0 + c_1\,(q\cdot e_3) + c_2\,(q\cdot e_3)^2 + c_3\,(q\cdot e_3)^3 + c_4\,(q\cdot e_4) + c_5\,(q\cdot e_4)^2 + c_6\,(q\cdot e_4)^3$$

- the solutions of a triple cut can be parametrized by a variable t

$$q_+^\mu = a^\mu + t\, e_3^\mu + \frac{\alpha}{t}\, e_4^\mu, \qquad q_- = a^\mu + \frac{\alpha}{t}\, e_3^\mu + t\, e_4^\mu$$

- in the limit $t \to \infty$ uncut denominators are linear in t, hence
  - pentagons vanish as $1/t^2$
  - boxes are constant but they vanish in the average over $q_\pm$
  - the integrand

$$\left.\frac{N(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = \Delta_{ijk} + \sum_l \frac{\Delta_{ijkl}}{D_l} + \sum_{lm} \frac{\Delta_{ijklm}}{D_l D_m}$$
$$= \Delta_{ijk} + \quad d_\pm + \mathcal{O}(1/t)$$

with $d_+ + d_- = 0$

[Forde (2007)]

The coefficients of the triangles

- In the asymptotic limit $t \to \infty$

$$\left. \frac{N(q_\pm)}{\prod_{m \neq i,j,k} D_m} \right|_{\text{cut}} = d_\pm + \Delta_{ijk} + \mathcal{O}(1/t) \qquad \text{with } d_+ + d_- = 0$$

  - the integrand

$$\left. \frac{N(q_\pm)}{\prod_{m \neq i,j,k} D_m} \right|_{\text{cut}} = n_0^\pm + n_1^\pm \, t + n_2^\pm \, t^2 + n_3^\pm \, t^3 + \mathcal{O}(1/t)$$

  - the residue

$$\Delta_{ijk}(q_+) = c_0 + c_4 \, (e_3 \cdot e_4) \, t + c_5 \, (e_3 \cdot e_4) \, t^2 + c_6 \, (e_3 \cdot e_4) \, t^3 + \mathcal{O}(1/t)$$
$$\Delta_{ijk}(q_-) = c_0 + c_1 \, (e_3 \cdot e_4) \, t + c_2 \, (e_3 \cdot e_4) \, t^2 + c_3 \, (e_3 \cdot e_4) \, t^3 + \mathcal{O}(1/t)$$

  - by comparison we get

$$c_1 = \frac{n_1^-}{(e_3 \cdot e_4)}, \quad c_2 = \frac{n_2^-}{(e_3 \cdot e_4)^2}, \quad c_3 = \frac{n_3^-}{(e_3 \cdot e_4)^3}, \quad c_4 = \frac{n_1^+}{(e_3 \cdot e_4)}, \quad \ldots$$
$$c_0 = \frac{n_0^+ + n_0^-}{2}$$

The coefficients of the bubbles

- The residue of a bubble reads (4-dim for brevity)

$$\Delta_{ij}(q) = b_0 + b_1\,(q\cdot e_2) + b_2\,(q\cdot e_2)^2 + b_3\,(q\cdot e_3) + b_4\,(q\cdot e_3)^2 + b_5\,(q\cdot e_4) + b_6\,(q\cdot e_4)^2 + b_7\,(q\cdot e_2)(q\cdot e_3) + b_8\,(q\cdot e_2)(q\cdot e_4)$$

- the solutions of a double cut can be parametrized by two variables t, x

$$q_+ = x\,e_1 + (\alpha_0 + x\,\alpha_1)e_2 + t\,e_3 + \frac{\beta_0 + \beta_1 x + \beta_2 x^2}{t}\,e_4$$

$$q_- = x\,e_1 + (\alpha_0 + x\,\alpha_1)e_2 + \frac{\beta_0 + \beta_1 x + \beta_2 x^2}{t}\,e_3 + t\,e_4$$

- in the limit $t \to \infty$ uncut denominators are linear in t, hence
  - pentagons and boxes vanish as $1/t^3$ and $1/t$ respectively
  - the integrand

$$\left.\frac{N(q_\pm)}{\prod_{m\neq i,j} D_m}\right|_{\text{cut}} = \Delta_{ij} + \sum_k \frac{\Delta_{ijk}}{D_k} + \sum_{kl} \frac{\Delta_{ijkl}}{D_k D_l} + \sum_{klm} \frac{\Delta_{ijklm}}{D_k D_l D_m} = \Delta_{ij} + \sum_k \frac{\Delta_{ijk}}{D_k} + \mathcal{O}(1/t)$$

The coefficients of the bubbles

- In the asymptotic limit $t \to \infty$
  - the integrand

$$\left.\frac{N(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = n_0^\pm + n_1^\pm\, x + n_2^\pm\, x^2 + \left(n_3^\pm + n_4^\pm x\right) t + n_5^\pm\, t^2 + \mathcal{O}(1/t)$$

  - the subtraction term

$$\frac{\Delta_{ijk}(q_\pm)}{D_k} = \tilde{b}_0^{k,\pm} + \tilde{b}_1^{k,\pm}\, x + \tilde{b}_2^{k,\pm}\, x^2 + \left(\tilde{b}_3^{k,\pm} + \tilde{b}_4^{k,\pm} x\right) t + \tilde{b}_5^{k,\pm}\, t^2 + \mathcal{O}(1/t)$$

  where $\tilde{b}_i^{k,\pm}$ are known functions of the triangle coefficients
  - the residue

$$\Delta_{ij}(q_+) = b_0 + b_1\,(e_1\cdot e_2)\, x + b_2\,(e_1\cdot e_2)^2 x^2 + \\ + \left(b_5 + b_7\,(e_1\cdot e_2) x\right)(e_3\cdot e_4)\, t + b_6\,(e_3\cdot e_4)^2\, t^2 + \mathcal{O}(1/t)$$

  - by comparison

$$b_0 = n_0^\pm - \sum_k \tilde{b}_0^{k,\pm}, \qquad b_1 = \frac{n_1^\pm}{e_1\cdot e_2} - \sum_k \tilde{b}_1^{k,\pm} \qquad \ldots$$

Semi-numeric implementation

- The loop momentum on a triple, double or single cut

$$q^\mu = \eta^\mu_{-1}\,\frac{1}{t} + \eta^\mu_0 + \eta^\mu_1\, t$$

- The integrand

$$\left.\frac{N(q)}{D_i D_j \ldots}\right|_{\text{cut}} = \frac{\sum n_k\, t^k}{\left(\sum d_{i,k}\, t^k\right)\left(\sum d_{j,k}\, t^k\right)\ldots}$$

  - the coefficients n_k are functions of the vectors η^μ_k

$$n_k = n_k(\eta_{-1}, \eta_0, \eta_1)$$

  - these functions can be easily obtained from either the analytic expression of the numerator or the tensor structure of the integrand
  - the Laurent expansion of the ratio of two rational functions can be computed (analytically or numerically) via polynomial division neglecting the remainder

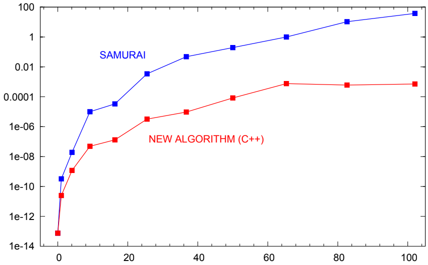
Stability of the algorithm

- For a 6-point amplitude of rank 6: $N(q) = \prod_{i=1}^{6}(q \cdot r_i)$
  - we only need to compute 386 coefficients out of 461 (16% less)
  - the reconstruction of the integrand is a couple of digits more accurate than the one of Samurai

- A simple example: $0 \to 4\gamma$
  - Plotting the relative error

$$\left| \frac{\mathcal{A}_{\text{analytic}} - \mathcal{A}_{\text{numeric}}}{\mathcal{A}_{\text{analytic}}} \right|$$

as a function of m^2/s



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- the traditional OPP approach performs a full reconstruction of the integrand in terms of Pentagons, Boxes, Triangles, Bubbles and Tadpoles
    - the computation of lower point residues requires the knowledge of all the higher point residues
    - at every step in the reduction we must subtract all the higher point residues and solve a system of equations

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- The coefficients of the MIs and the rational part of a 1-loop amplitude can be computed with operations at integrand level
- the traditional OPP approach performs a full reconstruction of the integrand in terms of Pentagons, Boxes, Triangles, Bubbles and Tadpoles
  - the computation of lower point residues requires the knowledge of all the higher point residues
  - at every step in the reduction we must subtract all the higher point residues and solve a system of equations
- By exploiting the analytic information about the integrand we can construct a simplified reduction algorithm with
  - no system of equations to be solved
  - no subtraction of pentagons and boxes
  - subtractions of 3-point and 2-point residues are replaced by corrections at coefficient level
  - successfully implemented in C++ and MATHEMATICA

Higher loops

How does this extend to higher loops?

- few papers on the subject

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- . . . but there are important differences
  - a complete basis of master integrals is not known
  - the reduction tells us which MIs we need
  - the form of the residues must be worked out for every different topology

Integrand reduction at 2 loops

At 2 loops:

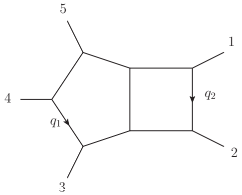
- the decomposition in $d = 4$ dimensions is

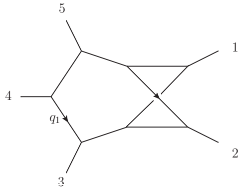
$$\frac{N(q_1,q_2)}{D_1\dots D_n} = \sum_{i_1,\dots,i_8} \frac{\Delta_{i_1\dots i_8}}{D_{i_1}\dots D_{i_8}} + \sum_{i_1,\dots,i_7} \frac{\Delta_{i_1\dots i_7}}{D_{i_1}\dots D_{i_7}} + \dots + \sum_{i_1,i_2} \frac{\Delta_{i_1\, i_2}}{D_{i_1} D_{i_2}}$$

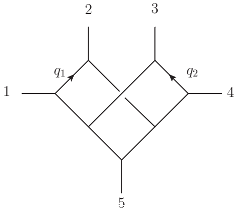
- the residues can sit over 8 or less denominators
- their parametric form can be found with several techniques
- the unknown coefficients which appear in this parametrization can be found by evaluating the integrand on solutions of multiple cuts
  - we start from 8-cuts
  - we subtract their residues and proceed with 7-cuts
  - ...

5-point amplitude in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG

G. Ossola, P. Mastrolia, E. Mirabella, T. P. (to be published)







- 5-point amplitude in $\mathcal{N} = 4$ SYM
  - the numerator has rank 1 [Carrasco, Johansson (2011)]
  - can be decomposed in terms of 8-cut and 7-cut residues
  - simple analytic expressions for the coefficients found with a generalization of the Lorentz-expansion technique
- 5-point amplitude in $\mathcal{N} = 8$ SG
  - the numerator has rank 2 [Carrasco, Johansson (2011)]
  - can be decomposed in terms of 8-cut, 7-cut and 6-cut residues
  - performed complete numerical reduction

Summary and conclusions

Scattering amplitudes at the integrand level

- At one-loop, the reduction at the integrand level
  - has been implemented in several codes, some of which publicly available (e.g. Samurai, CutTools, NGluon,. . .)
  - simplified reduction via Laurent expansion can provide improved stability
- At higher loops
  - the first results look promising
  - applied to both planar and non-planar diagrams
  - analytic techniques such as the Laurent expansion and polynomial division of the integrand can also simplify the computation at two (and more?) loops
  - . . . work is in progress!