Supersymmetry and Combinatorics*

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Abstract

We show how a recently proposed supersymmetric quantum mechanics model leads to non-trivial results/conjectures on the combinatorics of binary necklaces and linear-feedback shift-registers. Fermi statistics plays a crucial role through Pauli's famous exclusion principle: by projecting out certain states/necklaces it makes it possible to represent the supersymmetry algebra in the resulting Fock/Hilbert space. Some of our results can be rephrased in terms of generalizations of the well-known Witten index.

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1. Introduction

In a recent series of papers [1]–[3] two of us have introduced a supersymmetric quantum mechanical matrix model and studied some of its intriguing properties. The model is defined as the $N \to \infty$ limit of a quantum mechanical system whose degrees of freedom are bosonic and fermionic $N \times N$ creation and destruction operator matrices. The model's supersymmetry charges and Hamiltonian are explicitly given by:

$$\text{(1)} \qquad Q = \text{Tr}[f a^\dagger (1 + g a^\dagger)], \quad Q^\dagger = \text{Tr}[f^\dagger (1 + g a) a], \quad Q^2 = Q^{\dagger 2} = 0 \,,$$

$$\text{(2)} \qquad H = \{Q^\dagger, Q\} = H_B + H_F \,,$$

$$\text{(3)} \qquad H_B = \text{Tr}[a^\dagger a + g(a^{\dagger 2} a + a^\dagger a^2) + g^2 a^{\dagger 2} a^2] \,,$$

$$\text{(4)} \qquad \begin{aligned} H_F = \text{Tr}[&f^\dagger f + g(f^\dagger f(a^\dagger + a) + f^\dagger(a^\dagger + a)f) \\ &+ g^2(f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a)] \,, \end{aligned}$$

where bosonic and fermionic destruction and creation operators satisfy

$$\text{(5)} \qquad [a_{ij}, a^\dagger{}_{kl}] = \delta_{il}\delta_{jk} \; ; \; \{f_{ij} f^\dagger_{kl}\} = \delta_{il}\delta_{jk} \; ; \; i, j, k, l = 1, \ldots N \,,$$

all other (anti)commutators being zero. While taking the large-N limit, one keeps, as usual [4], the 't Hooft coupling, $\lambda \equiv g^2 N$, fixed. Note that the Hamiltonian (4) conserves (commutes with) the fermionic number $F = \text{Tr}[f^\dagger f]$. Hence the system can be studied separately for each eigenvalue of F. By contrast, H does not commute with the bosonic number operator $B = \text{Tr}[a^\dagger a]$ except in the trivial $g \to 0$ limit.

The model exhibits a number of interesting properties:

(*i*) It is exactly soluble in the $F = 0, 1$ sectors, i.e. the complete energy spectrum and the eigenstates are available in analytic form, in particular it exhibits a discontinuous phase transition at $\lambda = \lambda_{\text{c}} = 1$. At this point the otherwise discrete spectrum loses its energy gap and becomes continuous.

(*ii*) An exact weak-strong duality holds in the $F = 0, 1$ sectors relating spectra at λ and $1/\lambda$.

(*iii*) It exhibits unbroken supersymmetry, i.e. its $E \neq 0$ eigenstates consist of degenerate boson-fermion doublets.

(*iv*) In the weak coupling phase, $\lambda < 1$, there is only one (unpaired) zero-energy state (also referred to as a SUSY vacuum). It lies in the $F = 0$ sector and is nothing else but the empty Fock state $|0\rangle$ while for $\lambda > 1$ there are *two* zero-energy states in each bosonic (even F) sector of the model. For $F = 0$ the Fock vacuum continues to be a zero energy eigenstate, but it is joined by another, non-trivial, analytically known ground state. For each higher even F, the two non-trivial "vacua" appear suddenly at $\lambda > 1$. Some understanding of these unexpected states was obtained by considering the $\lambda \to \infty$ limit of the model [5]. In that same limit, the model can be connected to two interesting one-dimensional statistical mechanics quantum systems [5].

In the appropriate large-N limit defined above, the Hilbert space of the model can be restricted to the one corresponding to the action of single-traces of products of creation operators acting on the Fock vacuum. As such the vectors of the large-N Hilbert space can be put in one-to-one correspondence with binary necklaces, with the two beads representing bosonic and fermionic matrices. However, Fermi statistics provides a well-defined "Pauli razor", which projects out a subset of all binary necklaces. As we shall see, only after this projection is performed, the resulting space does allow for a faithful representation of supersymmetry[1].

The purpose of this paper is to illustrate how supersymmetry in our physical model gives non-trivial results on the combinatorics of binary necklaces and how, vice versa, known combinatorics results on the latter allow to determine the way supersymmetry is realized. In particular, combinatorics will allow us to understand where the null eigenstates lie and to compute the value of the Witten index [7] –and generalizations thereof– in different regions of λ.

The rest of the paper is organized as follows: in Sec. 2 we explain how single-trace states are connected to necklaces and describe how they can be enumerated taking into account Fermi statistics; the concept of Pauli *allowed* or *forbidden necklaces* is introduced together with some examples. We also introduce there the connection with "linear feedback shift registers" which helps in finding the correct answer. In Sec. 3 we provide a generalization of Polya's formula, by giving the number of forbidden necklaces with a prescribed number of bosonic and fermionic beads. In Sec. 4 we show how supersymmetry suggests combinatorial identities, which can also be proven by classical arguments, but are otherwise difficult to envisage. The Appendix provides a technical proof of a corollary of the main theorem.

2. Fock states, necklaces and linear feedback registers

The Hilbert space of our model (2) is spanned by states created by single trace operators, e.g.

$$\mathrm{Tr}[a^\dagger a^\dagger f^\dagger ... a^\dagger a^\dagger f^\dagger f^\dagger]|0\rangle .$$

These are also eigenstates of the above-mentioned fermion and boson number operators F and B. Such states too can be labeled by binary numbers, e.g.

$$(6) \qquad\qquad\qquad (001...0011),$$

with 0 (1) corresponding to bosonic (fermionic) creation operators. Because of cyclic property of a trace, all n binary sequences, related by cyclic shifts, describe the same state with n quanta, and consequently should be identified. Therefore our states correspond to what mathematicians define as necklaces - periodic chains made of different beads. In our model only two kinds of beads occur, hence only *binary* necklaces will be encountered here. From now on, if not specified otherwise, a term necklace will mean a binary necklace.

Since we are dealing with fermions, some of the above necklaces will not be allowed by the Pauli exclusion principle which will turn out to be crucial for supersymmetry as

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already mentioned in the introduction. We therefore define the *allowed* and *forbidden* necklaces as those which are allowed and forbidden by the Pauli principle. Hence the set of all necklaces is the union of allowed and forbidden ones.

2.1. **Counting states/necklaces.** We begin by recalling the classical results on counting all binary necklaces. The total number $N(n)$ of necklaces with n beads is given by MacMahon's formula

$$(7) \qquad N(n) = \frac{1}{n} \sum_{d|n} \varphi(d)\, 2^{n/d} \,,$$

where $d|n$ means that d divides n and $\varphi(d)$ is Euler's "totient" function, counting the numbers in $1, 2, ..., d-1$ relatively prime to d. The slightly "more differential" number number $N(B, F)$ of necklaces with B and F separate beads (B beads of type "0" and F of type "1") is given by Polya's formula [8, 9]:

$$(8) \qquad N(B, F) = \frac{1}{B+F} \sum_{d|B,F} \varphi(d) \binom{B/d + F/d}{F/d} \,,$$

where $d|B,F$ means that d is a divisor of both B and F. Of course the numbers given in (8) sum up to those in (7).

2.2. **Allowed vs. forbidden necklaces.** Let us now look in more detail how the anti-symmetry excludes some of the planar states/necklaces. For instance the sequence 0101 corresponding to the operator

$$\mathrm{Tr}[a^\dagger f^\dagger a^\dagger f^\dagger]$$

vanishes identically, since by anticommutation of f one has

$$\mathrm{Tr}[a^\dagger f^\dagger a^\dagger f^\dagger] = -\mathrm{Tr}[f^\dagger a^\dagger f^\dagger a^\dagger] = -\mathrm{Tr}[a^\dagger f^\dagger a^\dagger f^\dagger] = 0$$

On the other hand the sequence $aaff$ survives, since

$$\mathrm{Tr}[a^\dagger a^\dagger f^\dagger f^\dagger] = -\mathrm{Tr}[f^\dagger a^\dagger a^\dagger f^\dagger] = +\mathrm{Tr}[f^\dagger f^\dagger a^\dagger a^\dagger] = +\mathrm{Tr}[a^\dagger a^\dagger f^\dagger f^\dagger]$$

Our problem is to find in a systematic way which necklaces and how many of them survive the Pauli principle. The distinction into allowed and forbidden necklaces crucially depends whether a necklace has even or odd number of fermionic quanta. Therefore, from now on, we reserve the term *fermionic necklace* to one with an odd number of fermions (1's in its binary representation), while a *bosonic necklace* will denote a necklace with even number of fermionic quanta F. It follows that fermionic necklaces are always allowed, since a cyclic shift consists of even number of fermionic anticommutations, while some of the bosonic necklaces may be forbidden. To see this consider two longer necklaces

$$(9) \qquad\qquad (011011), \quad \text{and} \quad (01010101)$$

both of them are bosonic, however only the first one is allowed. Both states have the additional symmetry (Z_2 and Z_4 respectively) but the number of fermionic transpositions needed to shift them into themselves is different. A little thought allows now to identify the necessary and sufficient condition for a necklace to be forbidden:

A necklace with Z_k symmetry, k even, and $\frac{F}{k}$ odd, is forbidden and vice versa. Because of the cyclic invariance of a trace and of Fermi statistics, we find that these states are equal to their opposite and hence vanish.

The above condition splits the space of all necklaces in the way sketched in Fig. 1 where the necklaces are divided into four groups according to whether they contain even or odd number of fermionic and bosonic beads. The exclusion principle is only effective in the even-even group where some necklaces are Pauli-forbidden.

Supersymmetry manifests itself in terms of the existence of doublets of energy eigenstates (with non-vanishing eigenvalue) consisting of a boson (a bosonic necklace) and a fermion (a fermionic necklace). Precisely the removal of the forbidden necklaces from the even-even sector should give back the balance between even-even and odd-odd sectors required by supersymmetry.

$$n = B + F$$

$$N(n) = \frac{1}{n} \sum_{d|n} \varphi(d)\, 2^{n/d}$$

$$N_{allowed}(n) = \frac{1}{n} \sum_{\substack{d|n \\ d\,odd}} \varphi(d)\, 2^{n/d}$$

$$N_{forbidden}(n) = \frac{1}{n} \sum_{\substack{d|n \\ d\,even}} \varphi(d)\, 2^{n/d}$$

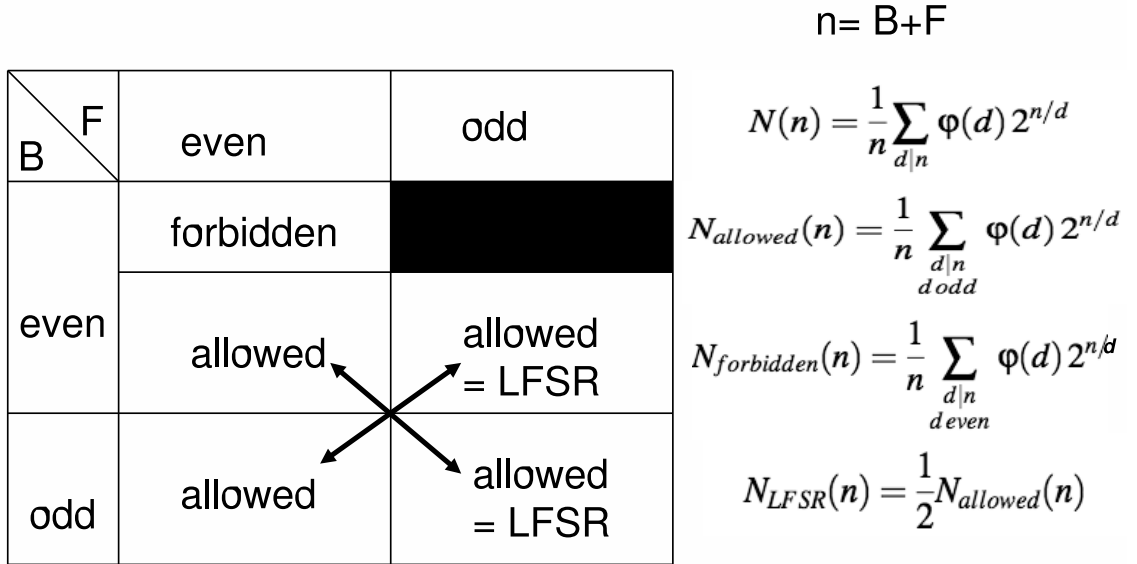$$N_{LFSR}(n) = \frac{1}{2} N_{allowed}(n)$$

FIGURE 1. A map of the space of all necklaces for even and odd B and F showing where Pauli-forbidden necklaces lie. The double arrows indicate the way supersymmetry connects allowed necklaces of opposite statistics at weak coupling. The connection with linear feedback shift registers (LFSR) is explained in subsection 2.3.

2.3. **Allowed necklaces and linear feed-back registers.** It turns out that fermionic necklaces are closely related to linear feedback shift registers (LFSR) - yet another class of objects well known in combinatorics [10]. In order to avoid lengthy definitions we illustrate hereafter the concept of a LFSR, of length n, and its relation to odd necklaces of length n (giving $n = 4$ as an example):

i) Definition of a LFSR

- Take an arbitrary binary number with $n - 1$ (here 3) digits:

$$(000, 001, 010, 011, \ldots, 111);$$

- Start adding digits to its right by the following (linear feedback) rule: add a 0 if the sum of the three digits is odd and a 1 if the sum is even. This gives:

$$(0001, 0010, 0100, 0111, \ldots, 1110);$$

  By construction, the sum of the **4** digits is always odd.
- Repeat the procedure by applying the rule to the new last three figures. The result is

$$(00010, 00100, 01000, 01110, \ldots, 11101).$$

  Clearly the 5th figure coincides with the first. If we keep going, we get a series that is periodic with period **4**.

ii) Claim of equivalence

The claim is that the distinct LFSR thus obtained are in one-to-one correspondence with fermionic necklaces of length n.

Proof: we have already argued that elementary cells of length n have an odd sum. Also, if two cells of length n are related by a cyclic transformation, they lead to the same infinite periodic structure. Thus every inequivalent, odd necklace of length n gives a distinct infinite sequence of period n and vice versa.

This proof is illustrated by the following three infinite sequences:

$$010001000100010001\ldots$$
$$100010001000100010\ldots$$
(10) $$110111011101110111\ldots$$

generated by our rule out of three different initial "data". The first two infinite sequences are considered to be equivalent: they correspond to the same 4-digit periodic structure (up to a cyclic permutation) repeating itself indefinitely, and corresponds to the odd necklace of fig. 2 (left side), while the third sequence gives the only other inequivalent odd necklace of length four, also shown in fig. 2 (right side).

The number of linear feed-back registers of length n is catalogued as A000016(n) in Sloane's library (we shall often refer to this remarkable tool [11], which was very helpful at a certain stage of our work). In conclusion

$$(11) \qquad N_{\text{fermionic}}(n) = N_{\text{LFSR}}(n) = A000016(n) = \frac{1}{2n} \sum_{\substack{d|n \\ d\,\text{odd}}} \varphi(d)\, 2^{n/d}.$$
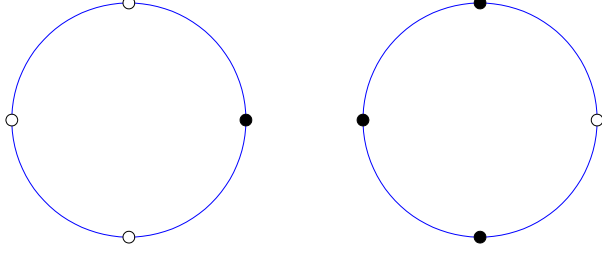
FIGURE 2. The two inequivalent necklaces corresponding to the infinite sequences (10).

2.4. **Separate, global counting of allowed and forbidden necklaces.** Supersymmetry requires that the numbers of allowed bosonic and fermionic necklaces are the same for given n. However this general condition has different consequences for even and odd n. If n is odd all necklaces are allowed (c.f. Fig.1) and consequently the number of bosonic necklaces coincides with the number of fermionic ones. In this case the restriction "d odd" in the last equation is superfluous and we get the total count of necklaces given by MacMahon's formula (7). On the contrary, when n *is even* we still have the value of $N_{\text{fermionic}}$ given by Eq. (11), but now supersymmetry requires this to match the number of *allowed* bosonic necklaces in the even-even sector. This yields the general (i.e. valid for all n) rule

$$(12) \qquad N_{allowed}(n) = \frac{1}{n} \sum_{\substack{d|n \\ d\,\text{odd}}} \varphi(d)\, 2^{n/d},$$

implying a total number of forbidden necklaces

$$(13) \qquad N_{forbidden}(n) = \frac{1}{n} \sum_{\substack{d|n \\ d\,\text{even}}} \varphi(d)\, 2^{n/d},$$

for any n.

This result follows by supersymmetry; it will be also derived by traditional combinatorial arguments later on (see Appendix). It is perhaps amusing that the obvious algebraic fact that odd n has only odd divisors while even n admits both (i.e. even and odd) divisors, directly corresponds to the existence of the necklaces allowed and forbidden by the Pauli principle!

3. GENERALIZATION OF POLYA'S FORMULA FOR FORBIDDEN NECKLACES

We would like to find the counterpart of Polya's formula eq. (8) which holds for the bosonic/fermionic necklaces with fixed numbers of separate beads . Equivalence between classical necklaces and susy necklaces when F is odd tells us that, in this case, we simply

have:

$$N_{\text{allowed}}(B, F) = \frac{1}{B+F} \sum_{d|B,F} \varphi(d) \binom{B/d + F/d}{F/d}$$

(14)
$$= \frac{1}{B+F} \sum_{\substack{d|B,F \\ d \text{ odd}}} \varphi(d) \binom{B/d + F/d}{F/d}, \, F \text{ odd}.$$

By an obvious symmetry, the same formula holds if B is odd and F is even. The only tricky case, again, is the one where both B and F are even: here we want to distinguish allowed from forbidden necklaces and count them separately for given values of B and F.

It turns out to be easier to find first the general formula for the number of forbidden necklaces, which, when combined with Polya's eq. (8), will produce as a corollary also the number of allowed necklaces. Our claim is as follows:

Theorem 1. *Let r be the unique positive integer (if it exists) for which $f = F/2^r$ is odd and $b = B/2^r$ is an integer. The number of Pauli Forbidden Necklaces is given by*

(15)
$$N_{\text{forbidden}}(B, F) = N(B/2^r, F/2^r) = \frac{1}{b+f} \sum_{d|b,f} \varphi(d) \binom{b/d + f/d}{f/d}.$$

If such an r does not exist then $N_{\text{forbidden}}(B, F) = 0$.

Proof. Pauli principle is active in deleting necklaces which are Z_p-symmetric with p even and F/p odd; then it is clear that, by writing $p = 2^r q$ with q odd, $F/2^r = f$ must be odd and $B/2^r = b$ must be an integer. If we now consider any sequence of length $b + f$ (a cell repeated 2^r times along the whole necklace), we see that such a cell is itself an arbitrary necklace with b bosons, f fermions and symmetry Z_q with q any odd number. Since f is odd, such a Z_q symmetry covers all possible cases, and therefore the number of inequivalent cells is indeed given by Polya's formula; notice that a different cyclic permutation of the elementary cell gives the same necklace, because a cyclic permutation of the cell is equivalent to a cyclic permutation of the whole necklace. $\square$

As an example, consider the string $\text{Tr}[f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger a^\dagger]$ corresponding to the necklace of Fig.3.
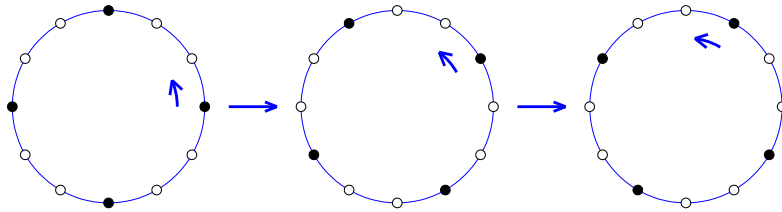
FIGURE 3. Counterclockwise rotating the necklace until it matches the initial configuration (from left to right): $(f\,a\,a\,f\,a\,a\,f\,a\,a\,f\,a\,a), (a\,f\,a\,a\,f\,a\,a\,f\,a\,a\,f\,a), (a\,a\,f\,a\,a\,f\,a\,a\,f\,a\,a\,f)$

The elementary cell in this case is (faa) and the number of forbidden necklaces coincides with $N(2, 1) = 1$, which corresponds to the necklaces depicted in Fig.3. In fact, rotating

the necklace until it matches the initial configuration, one has an odd number of fermionic commutations yielding the minus sign which kills the necklace.

Finally, by taking the difference between eq. (8) and (15), we conclude that:

$$(16) \qquad N_{\text{allowed}}(B, F) = N(B, F) - N_{\text{forbidden}}(B/2^r, F/2^r),$$

where r is as defined in Theorem 1. The values of $N_{\text{allowed}}(B, F)$ for $F + B \leq 26$ are reported[2] in the form of a (B, F) array in table 1. At this point, if our supersymmetry-based argument is correct, summing over the number of allowed necklaces given by eq. (16) for fixed n should reproduce exactly the same number as twice the sum over the fermionic supersymmetric partners, i.e. just eq. (12). We have verified numerically that this is the case up to $n = 3000$ and later found a direct mathematical proof reported in the appendix. The existence of such a proof confirms the solidity of the supersymmetry-based arguments, as well as their considerable heuristic value.

4. Witten-like indices

The formula for N_{allowed} verifies a number of checks coming from the properties of the supersymmetric model at weak and strong coupling. In the first, weak-coupling regime, which is fully under control, supersymmetry tells us that the allowed necklaces with a given $n = B + F$ should organize themselves in supersymmetry doublets, each of which consists of a necklace with some B and F and one with $B' = B \pm 1$ and $F' = F \mp 1$. Since the number of such pairs is always non-negative, we obtain the following inequalities for graded partial sums (a kind of generalization of Witten's index [7]):

$$(17) \qquad W(n; m) \equiv \sum_{\substack{B+F=n \\ 0 \leq F \leq m}} (-1)^{F-m} \, N_{\text{allowed}}(B, F) \geq 0 \,, \ W(n; n) = 0 \,,$$

where the last equality corresponds to that between even and odd allowed necklaces with a given n. The above consequences of supersymmetry have been explicitly checked up to $n \sim 5000$), while, so far, we have not been able to construct a direct proof of them by more standard techniques.

The strong ('t Hooft) coupling limit of the model of [1] can be shown [5] to imply instead that PANs must also organize in supersymmetry doublets whose partners have the same value of $B + 2F$ (and again differ by one, positive or negative, unit of F). A look at Table 1 shows that, along diagonals at fixed $B + 2F$, the balance between even and odd PANs is not always satisfied. This implies that, along those diagonals, there must be, at large coupling, (unpaired) $E = 0$ states.

The large-coupling limit unfortunately is not fully under control yet. Therefore, in this case, the connection between eigenstates and allowed necklaces can be used in either direction to infer properties of one in terms of known properties of the other. For instance, some evidence has been accumulated on where zero-energy states lie in the B, F plane. On the basis of this evidence we can conjecture new checks on our formulae for N_{allowed}

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[2]This table and the following one were produced by a *sieve* method for $B + F \leq 26$, independently of Theorem 1, and were used to check the general formulae.

by the following property of a second Witten-like index:
(18)
$$\tilde{W}(n;m) \equiv \sum_{\substack{B+2F=n \\ F \le m/2}} (-1)^{F-[m/2]} \left(N_{\text{allowed}}(B,F) - \frac{\delta_{F,B+1} + \delta_{F,B-1}}{2} (1 + (-1)^F) \right) \ge 0\,,$$

(with $m \le n$) and, in particular,

$$(19) \qquad \tilde{W}(n;n) = 0 \Rightarrow \sum_F (-1)^F N_{\text{allowed}}(n-2F, F) = \delta_{n \equiv 1 (\text{mod } 6)} + \delta_{n \equiv -1 (\text{mod } 6)}\,.$$

Our formulae passed the test of these (in)equalities for $n \le 5000$.

Actually, the validity of eq. (19) for all values of n follows from an explicit expression for the generating function of $N_{\text{allowed}}(B,F)$ recently obtained by D. Zagier:

$$\Phi_{\text{allowed}}(x,y;n) \equiv \sum_{F=0}^n N_{\text{allowed}}(n-F,F)\, x^{n-F} y^F = \frac{1}{n} \sum_{d|n} \varphi(d)\, \left(x^d - (-y)^d \right)^{n/d}\,,$$

after setting $y = -x^2$ and summing over n [3].

When $B+2F$ is small, the zero-energy eigenstates causing the imbalance can be uniquely identified in table 1, while, for the moment, their identification can only be guessed at (and verified later) for $B+2F$ large. This is how we arrived at the conjecture [5] that, at strong coupling, there is one and only one zero-energy eigenstate for each even value of F and $B = F \pm 1$, a conjecture leading precisely to eqs. (18) and (19).

Finally, it is amusing to notice that the total number of strong-coupling eigenstates at these special locations (forming a kind of magic staircase in table 1) is given by the sequence

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$$

which is easily recognized as being that of Catalan's numbers:

$$N_{\text{Catalan}} = \frac{1}{n+1} \binom{2n}{n}\,.$$

Catalan's numbers are ubiquitous, 66 appearances of them being listed in Stanley's treatise [9]. It is easy to convince oneself that to every necklace with $|B-F| = 1$ one can associate an infinite sequence of *ups* and *downs* describing a mountain profile, the number of which is precisely given by Catalan numbers [12]. These entries belong to the subset with $N_{\text{forbidden}} = 0$, since either B or F is odd. Other diagonals can be identified with known sequences; for instance, $N_{\text{allowed}}(F \pm 2, F)$ is identical to the number of plane trees with odd/even number of leaves (A071684, A071688 [11]).

To summarize our main results:

- We have been able to divide all binary necklaces in two disjoint classes, which we termed (Pauli)-allowed and (Pauli)-forbidden .

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[3]We are very grateful to Professor Zagier for informing us of this result, and for giving us permission to report it here.

- At the most "inclusive" level, the number of binary necklaces with a total number n of beads, as given by MacMahon's formula (7), is split into allowed and forbidden necklaces by restricting the divisor d in (7) to odd and even values, respectively.
- At a more "differential" level, the number of necklaces with B bosonic and F fermionic beads is rewritten in terms of allowed necklaces with different values of B and F via eq. (16), which can also be rewritten as:

$$(20) \qquad N(B,F) = N_{\text{allowed}}(B,F) \ + \ N(B/2^r, F/2^r),$$

where r is as defined in Theorem 1. We have verified numerically (and then proved directly, see appendix) that the appropriate sum performed on (20) reproduces the above-mentioned relation at fixed $n = B + F$.

- Supersymmetry implies several non-trivial constraints on $N_{\text{allowed}}(B,F)$ and thus, through (20), also on $N(B,F)$. Examples have been given in Section 4, but we stress that, by suitably extending the supersymmetric model under consideration, it is quite conceivable that many more constraints will emerge, not only for binary necklaces, but also for their generalization to more than two kinds of beads.

This new game (that we may dub "super-combinatorics") should reserve further surprises both for physicists and for mathematicians.

Appendix: Proof of consistency between eqs. (13) and (15).

Theorem 2.

$$\sum_{F=0}^{n} N_{\text{forbidden}}(n-F, F) = \frac{1}{n} \sum_{\substack{d|n \\ d \, \text{even}}} \phi(d) 2^{n/d}.$$

Proof. Let $n = 2^r q$, with q odd. We have

$$\frac{1}{2^r q} \sum_{\substack{d|2^r q \\ d \, \text{even}}} \phi(d) 2^{2^r q/d} = \frac{1}{2^r q} \sum_{m=1}^{r} \sum_{d|q} \phi(2^m d) 2^{2^{r-m} q/d}$$

$$= \frac{1}{2^r q} \sum_{m=1}^{r} 2^{m-1} \sum_{d|q} \phi(d) 2^{2^{-m} n/d} = \frac{1}{2} \sum_{m=1}^{r} \frac{2^m}{n} \sum_{d|q} \phi(d) 2^{2^{-m} n/d} = \sum_{m=1}^{r} N_{\text{LFSR}}(2^{-m} n)$$

(we use $\phi(2^m) = 2^{m-1}$ and in the second step we put $d = d_1 d, d_1|2^r, d|q$).

On the other hand the only non-vanishing contributions to $\sum_{F=0}^{n} N_{\text{forbidden}}(n-F,F)$ come from $F = 2^m q'$, where $m = 1, 2, ..., r$ and $q' = 1, 3, 5, ..., q$, so that we have:

$$\sum_{F=0}^{n} N_{\text{allowed}}(n-F,F) = \sum_{F=0}^{n} N_{\text{forbidden}}(2^r q - F, F) = \sum_{m=1}^{r} \sum_{\substack{q'=1 \\ q' \text{ odd}}}^{q} N_{\text{forbidden}}(2^r q - 2^m q', 2^m q')$$

$$= \sum_{m=1}^{r} \sum_{\substack{q'=1 \\ q' \text{ odd}}}^{q} N_{\text{BNL}}(2^{r-m} q - q', q') = \sum_{m=1}^{r} N_{\text{LFSR}}(2^{-m} n) \,.$$

$\square$

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$B \downarrow$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	2	3	3	3	4	5	5	5	6	7	7	7	8	9	9	9	10	11
3	1	1	2	4	5	7	10	12	15	19	22	26	31	35	40	46	51	57	64	70	77
4	1	1	2	5	9	14	20	30	43	55	70	91	115	140	168	204	245	285	330	385	445
5	1	1	3	7	14	26	42	66	99	143	201	273	364	476	612	776	969	1197	1463	1771	2126
6	1	1	3	10	22	42	76	132	217	335	497	728	1038	1428	1932	2586	3399	4389	5601	7084	8866
7	1	1	4	12	30	66	132	246	429	715	1144	1768	2652	3876	5538	7752	10659	14421	19228	25300	
8	1	1	4	15	42	99	212	429	809	1430	2424	3978	6308	9690	14520	21318	30667	43263	60060		
9	1	1	5	19	55	143	335	715	1430	2704	4862	8398	14000	22610	35530	54484	81719	120175			
10	1	1	5	22	73	201	497	1144	2438	4862	9226	16796	29414	49742	81686	130752	204347				
11	1	1	6	26	91	273	728	1768	3978	8398	16796	32066	58786	104006	178296	297160					
12	1	1	6	31	115	364	1028	2652	6310	14000	29372	58786	112716	208012	371384						
13	1	1	7	35	140	476	1428	3876	9690	22610	49742	104006	208012	400024							
14	1	1	7	40	172	612	1932	5538	14550	35530	81686	178296	371516								
15	1	1	8	46	204	776	2586	7752	21318	54484	130752	297160									
16	1	1	8	51	244	969	3384	10659	30666	81719	204248										
17	1	1	9	57	285	1197	4389	14421	43263	120175											
18	1	1	9	64	335	1463	5601	19228	60115												
19	1	1	10	70	385	1771	7084	25300													
20	1	1	10	77	445	2126	8844														
21	1	1	11	85	506	2530															
22	1	1	11	92	578																
23	1	1	12	100																	
24	1	1	12																		
25	1	1																			
26	1																				

TABLE 1. $N_{\mathrm{PAN}}(B, F)$ as generated with the sieve method.

$B \downarrow$	$F \rightarrow$ 0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
4	0	1	1	2	0	3	1	4	0	5	1	6	0	7	1	8	0	9	1
6	0	1	0	4	0	7	0	12	0	19	0	26	0	35	0	46	0	57	0
8	0	1	1	5	1	14	2	30	0	55	3	91	1	140	4	204	0	285	5
10	0	1	0	7	0	26	0	66	0	143	0	273	0	476	0	776	0	1197	0
12	0	1	1	10	0	42	4	132	0	335	7	728	0	1428	12	2586	0	4389	19
14	0	1	0	12	0	66	0	246	0	715	0	1768	0	3876	0	7752	0	14421	0
16	0	1	1	15	1	99	5	429	1	1430	14	3978	2	9690	30	21318	0	43263	55
18	0	1	0	19	0	143	0	715	0	2704	0	8398	0	22610	0	54484	0	120175	0
20	0	1	1	22	0	201	7	1144	0	4862	26	16796	0	49742	66	130752	0	312455	143
22	0	1	0	26	0	273	0	1768	0	8398	0	32066	0	104006	0	297160	0	766935	0
24	0	1	1	31	1	364	10	2652	0	14000	42	58786	4	208012	132	643856	0	1789515	335
26	0	1	0	35	0	476	0	3876	0	22610	0	104006	0	400024	0	1337220	0	3991995	0
28	0	1	1	40	0	612	12	5538	0	35530	66	178296	0	742900	246	2674440	0	8554275	715
30	0	1	0	46	0	776	0	7752	0	54484	0	297160	0	1337220	0	5170604	0	17678835	0
32	0	1	1	51	1	969	15	10659	1	81719	99	482885	5	2340135	429	9694845	1	35357670	1430
34	0	1	0	57	0	1197	0	14421	0	120175	0	766935	0	3991995	0	17678835	0	68635478	0
36	0	1	1	64	0	1463	19	19228	0	173593	143	1193010	0	6653325	715	31429068	0	129644790	2704
38	0	1	0	70	0	1771	0	25300	0	246675	0	1820910	0	10855425	0	54587280	0	238819350	0
40	0	1	1	77	1	2126	22	32890	0	345345	201	2731365	7	17368680	1144	92798380	0	429874830	4862

TABLE 2. N_{PFN}, the number of Pauli-forbidden necklaces calculated from eq. (15)(entries with odd F and/or odd B vanish identically).