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Chapter 11
Advance of Mercury's Perihelion

- *What does "advance of the perihelion" mean?*
- *Why pick out Mercury? Doesn't the perihelion of every planet change with time?*
- *The advance of Mercury's perihelion is really tiny. Who cares and why?*
- *You say Newton does not predict any advance of Mercury's perihelion. Why not?*
- *Why is Einstein's prediction about precession different from Newton's?*
- *You are always shouting at me to say whose time measures various motions. Why are you so sloppy about time in analyzing Mercury's orbit?*

CHAPTER

11 Advance of Mercury's Perihelion

Edmund Bertschinger & Edwin F. Taylor *

This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right. "For a few days, I was beside myself with joyous excitement." Later, he told Fokker that his discovery had given him palpitations of the heart. What he told de Haas is even more profoundly significant: when he saw that his calculations agreed with the unexplained astronomical observations, he had the feeling that something actually snapped in him.

—Abraham Pais

11.1 ■ JOYOUS EXCITEMENT

Tiny effect; large significance.

What discovery sent Einstein into "joyous excitement" in November 1915? It was his calculation showing that his brand new (actually, not quite completed) theory of general relativity gave the correct value for one detail of the orbit of the planet Mercury that had previously been unexplained, named the **precession of Mercury's perihelion**.

Mercury circulates around the Sun in a not-quite-circular orbit; like the other planets of the solar system, it oscillates in and out radially while it circles tangentially. The result is an elliptical orbit. Newton tells us that if we consider only the interaction between Mercury and the Sun, then the time for one 360-degree trip around the Sun is *exactly* the same as one in-and-out radial oscillation. Therefore the orbital point closest to the Sun, the so-called **perihelion,** stays in the same place; the elliptical orbit does not shift around with each revolution—according to Newton. In this project you will begin by verifying this nonrelativistic result for the Sun-Mercury system alone.

Newton: Sun-Mercury perihelion fixed.

However, observation shows that Mercury's orbit does, in fact, change. The innermost point, the perihelion, moves around the Sun *slowly*; it *advances* with each orbit (Figure 11.1). The long (major) axis of the ellipse rotates. We call this rotation of the axis the **precession of the perihelion**. The perihelion of Mercury actually precesses at the rate of 574 arcseconds (0.159

Observation shows: perihelion precesses.

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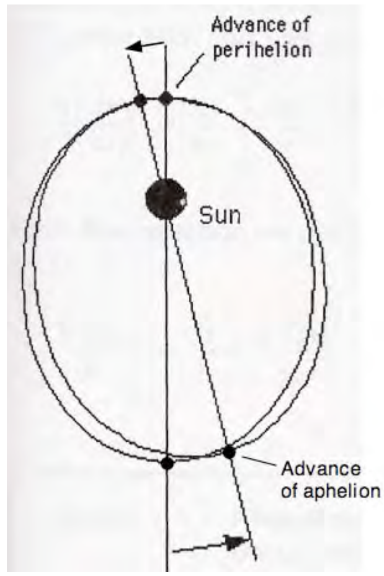

FIGURE 11.1 Exaggerated view of the advance during one century of Mercury's perihelion (and aphelion). The figure shows two elliptical orbits. One of these orbits is the one that Mercury traces over and over again almost exactly in the year, say 1900. The other elliptical orbit is the one that Mercury traces over and over again almost exactly in the year, say 2000. The two are shifted with respect to one another, a rotation called *the advance (or precession) of Mercury's perihelion.* The residual rotation (due to general relativity) is about 43 arcseconds, which is $43/3600 \approx 0.0119$ degree in that century, corresponding to less than the thickness of a line in this figure.

degree) *per century.* (One degree equals 3600 arcseconds.) The perihelion moves forward in the direction of rotation of Mercury; it *advances.* The **aphelion** is the point of the orbit farthest from the Sun; it advances at exactly the same angular rate as the perihelion (Figure 11.1).

Newton: Influence of other planets, predicts most of the perihelion advance . . .

. . . but not all of it.

Newtonian mechanics accounts for 531 seconds of arc of this advance by computing the perturbing influence of the other planets. But a stubborn 43 arcseconds (0.0119 degree) per century, called a **residual**, remains after all these effects are accounted for. This residual (though not its modern value) was computed from observations by Urbain Le Verrier as early as 1859and more accurately later by Simon Newcomb (Box 11.1). Le Verrier attributed the residual in Mercury's orbit to the presence of an unknown inner planet, tentatively named Vulcan. We know now that there is no planet Vulcan. (Sorry, Spock.)

Einstein correctly predicts residual precession.

Newtonian mechanics says that there should be *no residual* advance of the perihelion of Mercury's orbit and so cannot account for the 43 seconds of arc per century which, though tiny, is nevertheless too large to be ignored or blamed on observational error. But Einstein's general relativity accounted for the extra 43 arcseconds on the button. Result: joyous excitement!

BOX 11.1. Simon Newcomb

FIGURE 11.2 Simon Newcomb
Born 12 March 1835, Wallace, Nova Scotia.
Died 11 July 1909, Washington, D.C.
(Photo courtesy of Yerkes Observatory)

From 1901 until 1959 and even later, the tables of locations of the planets (so-called **ephemerides**) used by most astronomers were those compiled by Simon Newcomb and his collaborator George W. Hill. By the age of five Newcomb was spending several hours a day making calculations, and before the age of seven was extracting cube roots by hand. He had little formal education but avidly explored many technical fields in the libraries of Washington, D. C. He discovered the *American Ephemeris and Nautical Almanac,* of which he said, "Its preparation seemed to me to embody the highest intellectual power to which man had ever attained."

Newcomb became a "computer" (a person who computes) in the American Nautical Almanac Office and, by stages, rose to become its head. He spent the greater part of the rest of his life calculating the motions of bodies in the solar system from the best existing data. Newcomb collaborated with Q. M. W. Downing to inaugurate a worldwide system of astronomical constants, which was adopted by many countries in 1896 and officially by all countries in 1950.

The advance of the perihelion of Mercury computed by Einstein in 1914 would have been compared to entries in the tables of Simon Newcomb and his collaborator.

Method: Compare in-and-out time with round-and-round time for Mercury.

Preview: In this chapter we develop an approximation that demonstrates Newton's no-precession conclusion, then carry out an approximate general-relativistic calculation that predicts precession. Both approximations assume that Mercury is in a near-circular orbit, from which we calculate the time for one orbit. The approximation also describes the small inward and outward radial motion of Mercury as if it were a harmonic oscillator moving back and forth radially about the minimum in a potential well centered at the radius of the circle (Figure 11.3). We calculate the time for one round-trip radial oscillation. The orbital and radial oscillation times are exactly equal, according to Newton, provided one considers only the Mercury-Sun interaction. In that case Mercury goes around once in the same time that it oscillates radially inward and back out again. The result is an elliptical orbit that closes on itself, so in the absence of other planets Mercury repeats its elliptical path forever—according to Newton. In contrast, our general relativity approximation shows that these two times—the orbital round-and-round time and the radial in-and-out time—are *not quite* equal. The radial oscillation takes place more slowly, so that by the time Mercury orbits once, the circular motion has carried it farther around the Sun than it was at the preceding maximum radius. From this difference we reckon the residual angular rate of advance of Mercury's perihelion around the Sun and show that this value is close to the residual advance derived from observation. Now for the details.

11.2 ■ SIMPLE HARMONIC OSCILLATOR

Assume radial oscillation is sinusoidal.

Why should the satellite oscillate in and out radially? Look at the effective potential for Newtonian motion, the heavy line in Figure 11.3. This heavy line has a minimum, the location at which a particle can ride around at constant radius, tracing out a circular orbit. But with a slightly higher energy, it can also oscillate radially in and out, as shown by the two-headed arrow.

How long will it take for one in-and-out oscillation? That depends on the shape of the effective potential curve near the minimum shown in Figure 11.3. If the amplitude of the oscillation is small, then the effective part of the curve is very close to this minimum, and we can use a well-known mathematical theorem: If a continuous, smooth curve has a local minimum, then near that minimum the curve can be approximated by a parabola with its vertex at the minimum point. Figure 11.3 shows such a parabola (thin curve) superimposed on the (heavy) effective potential curve. From the diagram it is apparent that the parabola is a good approximation of the potential near that local minimum. Actually, Mercury's orbit swings from a minimum radius (the perihelion) of 46.04 million kilometers to a maximum radius (the aphelion) of 69.86 million kilometers. The difference in radius is *not* a small fraction of the average radius of the orbit; nevertheless our approximate analysis yields a numerical result that nearly matches the observed residual precession.

In-and-out motion in parabolic potential . . .

. . . predicts simple harmonic motion.

From introductory physics we know how a particle moves in a parabolic potential. The motion is called **simple harmonic oscillation**, described by the following expression:

$$x = \mathrm{A} \sin \omega t \tag{11.1}$$

Here A is the amplitude of the oscillation and ω (Greek lower case omega) tells us how rapidly the oscillation occurs in radians per unit time. The potential energy per unit mass, V/m, of a particle oscillating in a parabolic potential follows the formula

$$\frac{V}{m} = \frac{1}{2}\omega^2 x^2 \tag{11.2}$$

From (11.2) we find an expression for ω by taking the second derivative of both sides with respect to the displacement x:

$$\frac{d^2\left(V/m\right)}{dx^2} = \omega^2 \tag{11.3}$$

Taken together, (11.2) and (11.3) show that for the simple harmonic oscillator the oscillation frequency ω does not depend on the amplitude A. This is an example of a general result: From the expression for the potential, we can find the rate ω of harmonic oscillation around a local minimum by taking the second derivative of the curve and evaluating it at that minimum.

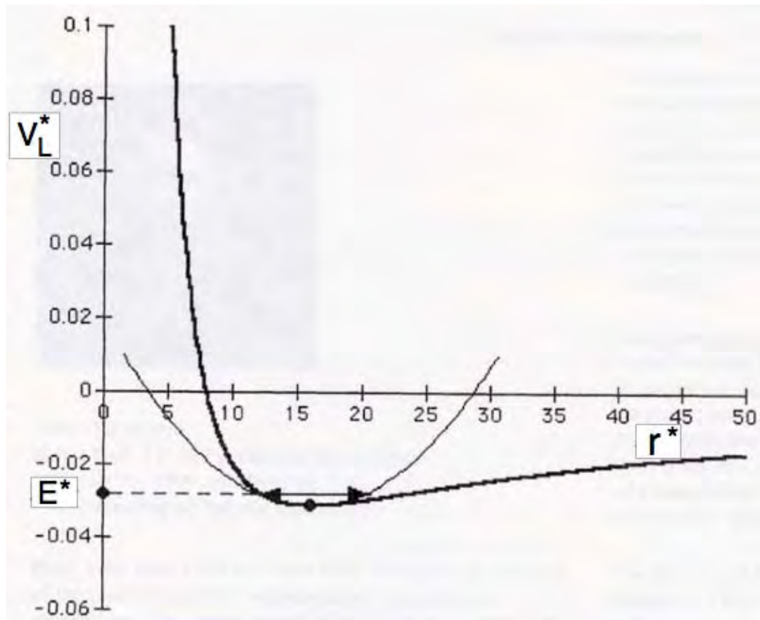

FIGURE 11.3 Newtonian effective potential (heavy curve), equation (11.5), on which is superimposed the parabolic potential of the simple harmonic oscillator (thin curve) with the shape of (11.3). The two curves conform to one another only near the minimum of the effective potential. We use a similar set of curves to approximate the oscillation of Mercury's orbital radius as an harmonic oscillation of small amplitude.

11.3 ■ HARMONIC OSCILLATION OF MERCURY'S ORBITAL RADIUS: NEWTON

Oscillating radially in and out about what center?

The in-and-out radial oscillation of Mercury does not take place around $x = 0$ but around the average radius r_0 of its orbit. What is the value of r_0? It is the radius at which the effective potential is minimum. To simplify the algebra, we shift over to unitless coordinates, described in equations (9.12) through (9.14). For Newtonian orbits, equation (9.27) describes the radial motion:

$$\frac{1}{2}\left(\frac{dr^*}{dt^*}\right)^2 = E^* - \left[-\frac{1}{r^*} + \frac{1}{2}\left(\frac{L^*}{r^*}\right)^2\right] = E^* - V^*_{\rm L}(r^*) \qquad \text{(Newton)} \qquad (11.4)$$

This equation defines the effective potential,

$$V^*_{\rm L}(r^*) \equiv -\frac{1}{r^*} + \frac{1}{2}\left(\frac{L^*}{r^*}\right)^2 \qquad \text{(Newton)} \qquad (11.5)$$

QUERY 11.1. Find the local minimum of Newton's potential
Take the derivative with respect to r^* of the potential per unit mass, $V_\mathrm{L}^*(r^*)$, given in (11.5). Set this first derivative aside for use in Query 11.2. As a separate calculation, equate this derivative to zero in order to determine the radius r_0^* at the local minimum of the effective potential. Use the result to write down an expression for the unknown quantity L^{*2} in terms of the known quantity r_0^*.

11.4 ■ ANGULAR VELOCITY OF MERCURY IN ITS ORBIT: NEWTON

Round and round how fast?

Newton: in-and-out time equals round-and-round time.

We want to compare the rate ω_r of in-and-out radial motion of Mercury with its rate ω_ϕ of round-and-round tangential motion. Use the Newtonian definition of angular momentum, with increment dt of Newtonian universal time, similar to equation (9.11):

$$L^* = r^{*2}\frac{d\phi}{dt^*} = r^{*2}\omega_\phi^* \qquad \text{(Newton)} \tag{11.6}$$

where the unitless expression for angular velocity is $\omega_\phi^* \equiv d\phi/dt^*$. Equation (11.6) gives us the angular velocity of Mercury along its almost circular orbit.

QUERY 11.2. Newton's angular velocity of Mercury in orbit.
Into (11.6) substitute your value for L^* from Query 1.1 and set $r^* = r_0^*$. Find an expression for ω_ϕ^* in terms of r_0^*.

QUERY 11.3. Newton's oscillation rate $\omega_{r^*}^*$ for radial motion
We want to use (11.3) to find the rate of radial oscillation. Accordingly, continue by taking a second derivative of V_L^* in (11.5) with respect to r^*. Set $r^* = r_0^*$ in the resulting expression and substitute your value for L^{*2} from Query 11.1. Use (11.3) to find an expression for the rate $\omega_{r^*}^*$ at which Mercury oscillates in and out radially—according to Newton!

QUERY 11.4. Compare radial oscillation rate with orbital angular velocity: Newton
Compare your value of angular velocity ω_ϕ^* from Query 11.3 with your value for radial oscillation rate $\omega_{r^*}^*$ from Query 11.2. Will there be an advance of the perihelion of Mercury's orbit around the Sun (when only the Sun-Mercury interaction is considered)—according to Newton?

11.5 ■ EFFECTIVE POTENTIAL: EINSTEIN

Extra effective potential term advances perihelion.

Now we repeat the analysis for the general relativistic case, using the Newtonian analysis as our model. Chapter 10 predicts the radial motion of an orbiting satellite. Multiply equations (10.33) and (10.34) through by 1/2 to obtain an equation similar to (11.4) above for the Newtonian case:

$$\begin{aligned}\frac{1}{2}\left(\frac{dr^*}{d\tau^*}\right)^2 &= \frac{1}{2}E^{*2} - \frac{1}{2}\left(1-\frac{2}{r^*}\right)\left(1+\frac{L^{*2}}{r^{*2}}\right) \\ &= \frac{1}{2}E^{*2} - \frac{1}{2}V_{\mathrm{L}}^{*2} \qquad \text{(Einstein)}\end{aligned} \tag{11.7}$$

Set up general relativity effective potential.

Equations (11.4) and (11.7) are of similar form, and we use this similarity to make a general relativistic harmonic analysis of the radial motion of Mercury in orbit. In this process we adopt the *algebraic manipulations* of the Newtonian analysis of Sections 11.3 and 11.4 but apply them to the general relativistic expression (11.7).

Before proceeding, note three characteristics of equation (11.7):

Different times on different clocks do not matter.

First, the time on the left side of (11.7) is the proper time τ^*, the wristwatch time of the planet Mercury, not Newton's universal time t^*. This different reference time is not necessarily fatal, since we have not yet decided which relativistic time should replace Newton's universal time t^*. You will show in Section 11 that for Mercury the choice of which time to use (wristwatch time, Schwarzschild map time, or even shell time at the radius of the orbit) makes a negligible difference in our predictions about the rate of advance of the perihelion.

Note, second, that in equation (11.7) the relativistic expression $(1/2)E^{*2}$ stands in the place of the Newtonian expression E^* in (11.4). However, both are constant quantities, which is all that matters in carrying out the analysis. Evidence that we are on the right track follows from multiplying out the second term of the middle equality in (11.7), which is the effective potential (9.22) with the factor one-half. Note that we have assigned the symbol $(1/2)V_{\mathrm{L}}^{*2}$ to this second term.

$$\begin{aligned}\frac{1}{2}V_{\mathrm{L}}^{*2} &= \frac{1}{2}\left(1-\frac{2}{r^*}\right)\left(1+\frac{L^{*2}}{r^{*2}}\right) \qquad \text{(Einstein)} \\ &= \frac{1}{2} - \frac{1}{r^*} + \frac{L^{*2}}{2r^{*2}} - \frac{L^{*2}}{r^{*3}}\end{aligned} \tag{11.8}$$

Details of relativistic effective potential

The heavy curve in Figure 4 plots this function. The second line in (11.8) contains the two effective potential terms that made up the Newtonian expression (11.5). In addition, the first term assures that far from the center of attraction the radial speed in (11.7) will have the correct value. For example, let the energy equal the rest energy, $E^* = 1$. Then for large r^*, the radial speed $dr^*/d\tau^*$ in (11.7) goes to zero, as it must. The final term on the right of the second line of (11.8) describes an attractive potential arising from general

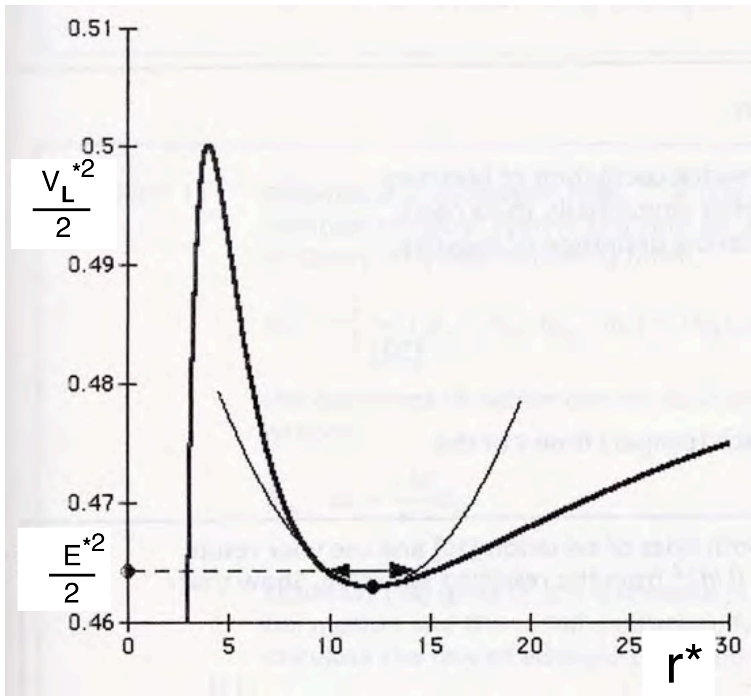

FIGURE 11.4 General-relativistic effective potential $V_{\rm L}^{*2}/2$ (heavy curve) and its approximation at the local minimum by a parabola (light curve) in order to analyse the radial excursion (double-headed arrow) of Mercury as simple harmonic motion. The effective potential curve is for a black hole, not for the Sun, whose effective potential near the potential minimum would be indistinguishable from the Newtonian function on the scale of this diagram. However, this minute difference accounts for the tiny precession of Mercury's orbit.

relativity. For the Sun-Mercury case, at the radius of Mercury's orbit this term results in the slight precession of the elliptical orbit. In the case of the black hole, quite close to its center the r^{*3} in the denominator causes this term to overwhelm all other terms in (11.8), leading to the downward plunge in the effective potential at the left side of Figure 11.4.

The third comment about equation (11.7): The final term $(1/2)V_{\rm L}^{*2}$ takes the place of the effective potential $V_{\rm L}^{*}$ in equation (11.4) of the Newtonian analysis.

In summary, we can manipulate general relativistic expressions (11.7) and (11.8) in almost exactly the same way that we manipulated Newtonian expressions (11.4) and (11.5) in order to analyze the radial component of Mercury's motion and the small perturbations of the Newtonian elliptical orbit brought about by general relativity.

11.6 ■ HARMONIC OSCILLATION OF MERCURY'S RADIUS: EINSTEIN

Einstein tweaks Newton's solution.

Now analyze the radial oscillation of Mercury according to Einstein.

QUERY 11.5. Finding the local minimum of the effective potential
Take the derivative of the effective potential (11.8) with respect to r^*, that is find $d[(1/2)V_{\mathrm{L}}^{*2}]/dr$. Set this first derivative aside for use in Query 11.6. As a separate calculation, equate this derivative to zero, set $r = r_0$, and solve the resulting equation for the unknown quantity $(L/m)^2$ in terms of the known quantities M and r_0.

QUERY 11.6. Radial oscillation rate.
We want to use (11.3) to find the rate of oscillation in the radial direction. Accordingly, continue to the second derivative of $(1/2)V_{\mathrm{L}}^{*2}$ from (11.8) with respect to r^*. Set $r^* = r_0^*$ in the result and substitute the expression for $(L/m)^2$ from Query 5 to obtain

$$\left[\frac{d^2}{dr^{*2}}\left(\frac{1}{2}V_{\mathrm{L}}^{*2}\right)\right]_{r=r_0} = \omega_{r^*}^{*2} = \frac{r_0^* - 6}{r_0^{*3}\,(r_0^* - 3)} \qquad \text{(Einstein)} \qquad (11.9)$$

QUERY 11.7. Newtonian limit of radial oscillation
The average radius of Mercury's orbit around the Sun: $r_0 = 5.80 \times 10^{10}$ meters; recall that $r_0^* \equiv r_0/M$, where M is the Sun's mass. From the resulting value of r_0^*, show that you are entitled to use an approximation in (11.9) that leads to an expression for $\omega_{r^*}^*$ equal to that of Newton derived in Query 11.3.

11.7 ■ ANGULAR VELOCITY IN ORBIT: EINSTEIN

In-out motion not in step with round-and-round motion.

We want to compare the rate of in-and-out oscillation of Mercury's orbital radius with the angular rate at which Mercury moves tangentially in its orbit. The rate of change of azimuth ϕ springs from the definition of angular momentum in equation (9.11):

$$L^* = r^{*2}\frac{d\phi}{d\tau^*} \qquad (11.10)$$

Note that the time here, too, is the wristwatch (proper) time τ^* of the satellite.

QUERY 11.8. Angular velocity. Square both sides of (11.10) and use your result from Query 11.5 to eliminate L^{*2} from the resulting equation. Show that the result can be written

$$\omega_\phi^{*2} \equiv \left(\frac{d\phi}{d\tau^*}\right)^2 = \frac{1}{r_0^{*2}\,(r_0^* - 3)} \qquad (11.11)$$

Does general relativity predict that the round-and-round tangential motion of Mercury take place exactly in step with the in-and-out radial oscillation, as it does in the Newtonian analysis?

QUERY 11.9. Newtonian limit of angular velocity.
Make the same kind of approximation as in Query 11.7 and demonstrate that the resulting value of ω_ϕ^* is the same as your Newtonian expression derived in Query 2.

11.8 ■ PREDICTING ADVANCE OF THE PERIHELION

Simple outcome

Einstein: in-out rate differs from round-round rate.

The advance of the perihelion of Mercury springs from the difference between the frequency at which the planet sweeps around in its orbit and the frequency at which it oscillates in and out radially. In the Newtonian analysis these two frequencies are equal, provided one considers only the interaction between Mercury and the Sun. But Einstein's theory shows that these two frequencies are *not quite* equal, so Mercury reaches its maximum (and also its minimum) radius at a slightly different angular position in each successive orbit. This results in the advance of the perihelion of Mercury.

NOTE: For the final prediction, we stop using unitless coordinates.

The time to make a complete in-and-back-out radial oscillation is

$$T_\mathrm{r} \equiv \frac{2\pi}{\omega_\mathrm{r}} \qquad \text{(period of radial oscillation)} \qquad (11.12)$$

In this time Mercury goes around the Sun through an angle, in radians:

$$\omega_\phi T_\mathrm{r} = \frac{2\pi\omega_\phi}{\omega_\mathrm{r}} = \text{(Mercury rotation angle in time } T_\mathrm{r}) \qquad (11.13)$$

which exceeds one complete revolution by (radians):

$$\omega_\phi T_\mathrm{r} - 2\pi = T_\mathrm{r}\left(\omega_\phi - \omega_\mathrm{r}\right) = \text{(excess angle per revolution)} \qquad (11.14)$$

QUERY 11.10. Difference in oscillation rates.
The two angular rates ω_ϕ and ω_r are *almost* identical in value, even in the Einstein analysis. Therefore write the result of equations (11.9) and (11.11) in the following form:

$$\omega_\phi^2 - \omega_\mathrm{r}^2 = \left(\omega_\phi + \omega_\mathrm{r}\right)\left(\omega_\phi - \omega_\mathrm{r}\right) \approx 2\omega_\phi\left(\omega_\phi - \omega_\mathrm{r}\right) \qquad (11.15)$$

Complete the derivation to show that in this approximation

$$\omega_\phi - \omega_\mathrm{r} \approx \frac{3M}{r_0}\omega_\phi \qquad (11.16)$$

Equation (11.16) gives us the difference in angular rate between the tangential motion and the radial oscillation. From this rate difference we can calculate the rate of advance of the perihelion of Mercury.

UNITS: The symbol ω in (11.16) expresses rotation rate in radians per unit of time. What unit of time? It does not matter, as long as the unit of time is the *same* on both sides of the equation. In the following queries you use (11.16) to calculate the precession rate of Mercury in radians per Earth century, then convert the result first to degrees per century and finally to arcseconds per century.

11.9 ■ COMPARISON WITH OBSERVATION

You check out Einstein.

Now you can compare our approximate relativistic prediction with observation.

QUERY 11.11. Mercury's orbital period.
The period of Mercury's orbit is 7.602×10^6 seconds and that of Earth is 3.157×10^7seconds. What is the value of Mercury's orbital period in Earth-years?

QUERY 11.12. Number of Mercury's orbital revolutions in one century.
How many revolutions around the Sun does Mercury make in one century (in 100 Earth-years)?

QUERY 11.13. Correction factor
The mass M of the Sun is 1.477×10^3 meters and the radius r_0 of Mercury's orbit is 5.80×10^{10} meters. Calculate the value of the correction factor $3M/r_0$ in (11.16).

QUERY 11.14. Advance angle in degrees per century.
From equation (11.16) derive a numerical prediction of the advance of the perihelion of Mercury's orbit in degrees per century (assuming only Mercury and the Sun are interacting).

QUERY 11.15. Advance angle in arcseconds per century.
There are 60 minutes of arc per degree and 60 arcseconds per minute of arc. Multiply your result from Query 14 by $60 \times 60 = 3600$ to obtain your prediction of the advance of the perihelion of Mercury's orbit in arcseconds per century.

Observation and careful calculation match for Mercury.

A more careful relativistic analysis predicts a value of 42.980 arcseconds (0.0119 degrees) per century (see Table 11.1). The observed rate of advance of the perihelion is in perfect agreement with this value: 42.98 ± 0.1 arcseconds per century. (See references.) How close was your prediction?

For comparison, the precession rate exceeds 4 degrees per year for the binary neutron star systems called B1913+16 and J0737-3039. In these cases, each neutron star in the binary system orbits the center of mass in a nearly elliptical orbit whose point of closest approach (technically called **periastron** for a center of attraction to a star other than our Sun) shifts in angular position due to the same effects as those present in the Mercury-Sun system, only thousands of times stronger. See the references.

Neutron star binary: faster precession.

11.10 ■ ADVANCE OF THE PERIHELIA OF THE INNER PLANETS

Help from a supercomputer.

Do the *perihelia* (plural of *perihelion*) of other planets in the solar system also advance as described by general relativity? Yes, but these planets are farther from the Sun, so the magnitude of the predicted advance is less than that for Mercury. In this section we compare our estimated advance of the perihelia of the inner planets Mercury, Venus, Earth, and Mars with results of an accurate calculation.

All planet orbits precess.

The Jet Propulsion Laboratory (JPL) in Pasadena, California, supports an active effort to improve our knowledge of the positions and velocities of the major bodies in the solar system. For the major planets and the moon, JPL maintains a database and set of computer programs known as the Solar System Data Processing System (SSDPS). The input database contains the observational data measurements for current locations of the planets. Working together, more than 100 interrelated computer programs use these data and the relativistic laws of motion to compute locations of planets at times in the past and future. The equations of motion take into account not only the gravitational interaction between each planet and the Sun but also interactions among all planets, Earth's moon, and 300 of the most massive asteroids, as well as interactions between Earth and Moon due to nonsphericity and tidal effects.

To help us with our project on perihelion advance, Myles Standish, Principal Member of the Technical Staff at JPL, kindly used the numerical integration program of the SSDPS to calculate orbits of the four inner planets over four centuries, from A.D. 1800 to A.D. 2200. In an overnight run he carried out this calculation twice, once with the full program including relativistic effects and a second time "with relativity turned off." Standish "turned off relativity" by setting the speed of light to 10^{10} times its measured value, effectively making light speed infinite. The coefficient of dt^2 in the Schwarzschild metric is written in conventional units as $1 - 2GM_{\text{conv}}/(rc^2)$. The value of this expression approaches unity for a large value of c.

JPL multi-program computation.

For each of the two runs, the perihelia of the four inner planets were computed for a series of points in time covering the four centuries. The results from the nonrelativistic run were subtracted from those of the relativistic run, revealing advances of the perihelia per century accounted for only by general

TABLE 11.1 Advance of the perihelia of the inner planets

Planet	Advance of perihelion in seconds of arc per century (JPL calculation)	Radius of orbit in AU*	Period of orbit in years
Mercury	42.980 ± 0.001	0.38710	0.24085
Venus	8.618 ± 0.041	0.72333	0.61521
Earth	3.846 ± 0.012	1.00000	1.00000
Mars	1.351 ± 0.001	1.52368	1.88089

*Astronomical Unit (AU): average radius of Earth's orbit; inside front cover.

relativity. The second column of Table 11.1 shows the results, together with the estimated computational error.

QUERY 11.16. Approximate advances of the perihelia of the inner planets
Compare the JPL-computed advances of the perihelia of Venus, Earth, and Mars with results of the approximate formula developed in this project.

11.11 ■ CHECKING THE STANDARD OF TIME

Whose clock?

We have been casual about whose time tracks the advance of the perihelion of Mercury and other planets. Does this invalidate our approximations?

QUERY 11.17. Difference between shell time and Mercury wristwatch time.
Use special relativity to find the fractional difference between planet Mercury's wristwatch time increment $\Delta\tau$ and the time increment $\Delta t_{\mathrm{shell}}$ read on shell clocks at the same average radius r_0 at which Mercury moves in its orbit at the average velocity 4.8×10^4 meters/second. By what fraction does a change of time from $\Delta\tau$ to $\Delta t_{\mathrm{shell}}$ change the total angle covered in the orbital motion of Mercury in one century? Therefore by what fraction does it change the predicted angle of advance of the perihelion in that century?

QUERY 11.18. Difference between shell time and Schwarzschild map time.
Find the fractional difference between shell time increment $\Delta t_{\mathrm{shell}}$ at radius r_0 and Schwarzschild map time increment Δt for r_0 equal to the average radius of the orbit of Mercury. By what fraction does a change from $\Delta t_{\mathrm{shell}}$ to a lapse in global time t alter the predicted angle of advance of the perihelion in that century?

QUERY 11.19. Does the time standard matter?
From your results in Queries 11.17 and 11.18, say whether or not the choice of a time standard—wristwatch time of Mercury, shell time, or map time—makes a detectable difference in the numerical prediction of the advance of the perihelion of Mercury in one century. Would your answer differ if the time were measured with clocks on Earth's surface?

DEEP INSIGHTS FROM THREE CENTURIES AGO

> *Newton himself was better aware of the weaknesses inherent in his intellectual edifice than the generations that followed him. This fact has always roused my admiration.*
>
> —Albert Einstein

We agree with Einstein. In the following quote from the end of his great work *Principia*, Isaac Newton tells us what he knows and what he does not know about gravity. We find breathtaking the scope of what Newton says—and the integrity of what he refuses to say.

"I do not 'feign' hypotheses."

> *Thus far I have explained the phenomena of the heavens and of our sea by the force of gravity, but I have not yet assigned a cause to gravity. Indeed, this force arises from some cause that penetrates as far as the centers of the sun and planets without any diminution of its power to act, and that acts not in proportion to the quantity of the* surfaces *of the particles on which it acts (as mechanical causes are wont to do) but in proportion to the quantity of* solid *matter, and whose action is extended everywhere to immense distances, always decreasing as the squares of the distances. Gravity toward the sun is compounded of the gravities toward the individual particles of the sun, and at increasing distances from the sun decreases exactly as the squares of the distances as far as the orbit of Saturn, as is manifest from the fact that the aphelia of the planets are at rest, and even as far as the farthest aphelia of the comets, provided that those aphelia are at rest. I have not as yet been able to deduce from phenomena the reason for these properties of gravity, and I do not "feign" hypotheses. For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy. In this experimental philosophy, propositions are deduced from the phenomena and are made general by induction. The*

impenetrability, mobility, and impetus of bodies, and the laws of motion and the law of gravity have been found by this method. And it is enough that gravity really exists and acts according to the laws that we have set forth and is sufficient to explain all the motions of the heavenly bodies and of our sea.

—Isaac Newton

11.12 ■ REFERENCES

Our analysis in this project is influenced by that of Robert M. Wald (*General Relativity,* University of Chicago Press, 1984, pages 142–143)

Initial quote: Abraham Pais, *Subtle Is the Lord: The Science and the Life of Albert Einstein*, Oxford University Press, New York, 1982, page 253.

Myles Standish of the Jet Propulsion Laboratory ran the programs on the inner planets presented in Section 10. He also made useful comments on the project as a whole for the first edition.

Precession in binary neutron star systems: For B1913+16: J. M. Weisberg and J. H. Taylor, ApJ, 576, 942 (2002). For J0737-3039: R. P. Breton et al, Science, Vol. 321, 104 (2008)

Final Newton quote from I. Bernard Cohen and Anne Whitman, *Isaac Newton, the Principia, A New Translation*, University of California Press, Berkeley CA, 1999 , page 943